UNIT 3

Replacement Analysis

UNIT- 3
Replacement Problems

Introduction:

The replacement problems are concerned with the situations that arise when some items such as men, machines and usable things etc need replacement due to their decreased efficiency, failure or breakdown. Such decreased efficiency or complete breakdown may either be gradual or all of a sudden.

If a firm wants to survive the competition it has to decide on whether to replace the out dated equipment or to retain it, by taking the cost of maintenance and operation into account. There are two basic reasons for considering the replacement of an equipment.

They are (i) Physical impairment or malfunctioning of various parts.

(ii) Obsolescence of the equipment.

The physical impairment refers only to changes in the physical condition of the equipment itself. This will lead to decline in the value of service rendered by the equipment, increased operating cost of the equipments, increased maintenance cost of the equipment or the combination of these costs. Obsolescence is caused due to improvement in the existing Tools and machinery mainly when the technology becomes advanced therefore, it becomes uneconomical to continue production with the same equipment under any of the above situations. Hence the equipments are to be periodically replaced.

Sometimes, the capacity of existing facilities may be in adequate to meet the current demand. Under such cases, the following two alternatives will be considered.

1. Replacement of the existing equipment with a new one
2. Argument the existing one with an additional equipments.

Type of Maintenance:

Maintenance activity can be classified into two types

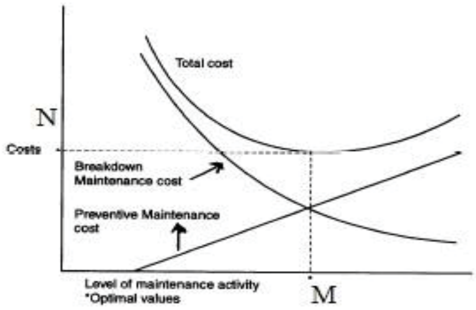
i) Preventive Maintenance

ii) Breakdown Maintenance

Preventive maintenance (PN) is the periodical inspection and service which are aimed to detect potential failures and perform minor adjustments a requires which will prevent major operating problem in future. Breakdown maintenance is the repair which is generally done after the equipment breaks down. It is offer an emergency which will have an associated penalty in terms of increasing the cost of maintenance and downtime cost of equipment, Preventive maintenance will reduce such costs up-to a certain extent . Beyond that the cost of preventive maintenance will be more when compared to the cost of the breakdown maintenance.

Total cost = Preventive maintenance cost + Breakdown maintenance cost.

This total cost will go on decreasing up-to P with an increase in the level of maintenance up-to apoint, beyond which the total cost will start increasing from P. The level of maintenance corresponding to the minimum total cost at P is the Optional level of maintenance this concept is illustrated in the follows diagram



The points M and N denote optimal level of maintenance and optimal cost respectively
Types of replacement problem : The replacement problem can be classified into two categories.
i) Replacement of assets that deteriorate with time (replacement due to gradual failure, due to wear and tear of the components of the machines) This can be further classified into the following types.
a) Determination of economic type of an asset.
b) Replacement of an existing asset with a new asset.
ii) Simple probabilistic model for assets which will fail completely (replacement due to sudden failure).

Determination of Economic Life of an asset
Any asset will have the following cost components
i) Capital recovery cost (average first cost), Computed form the first cost (Purchase price) of the asset.
ii) Average operating and maintenance cost.
iii) Total cost which is the sum of capital recovery cost (average first cost) and average operating and maintenance cost.
A typical shape of each of the above cost with respect to life of the asset is shown below.

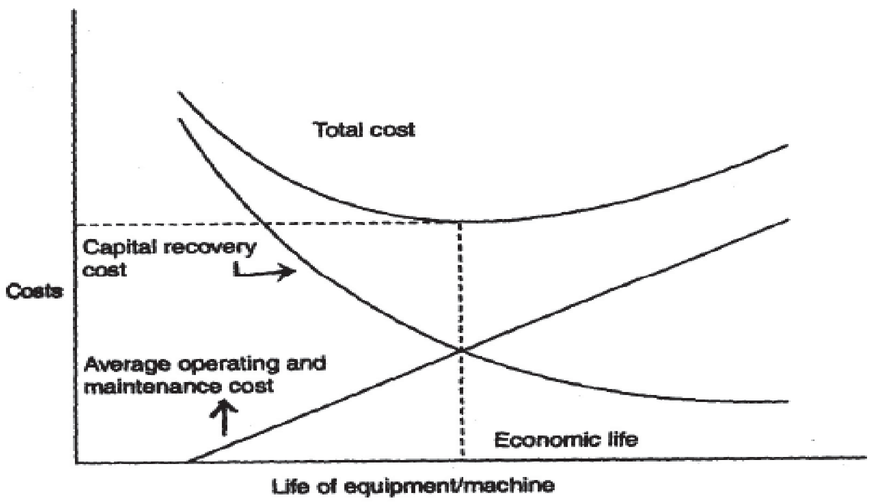


Chart showing economic life.

From figure, when the life of the machine increases, it is clear that the capital recovery cost (average first cost) goes on decreasing and the average operating and maintenance cost goes on increasing. From the beginning the total cost goes on decreasing upto a particular life of the asset and then it starts increasing. The point P were the total cost in the minimum is called the Economic life of the asset. To solve problems under replacement, we consider the basics of interest formula.

Time value of money does not change

If the value of money does not change with time, then the user of the equipment does not need to pay interest on his investments. We wish to determine the optimal time to replace the equipment. If the value of money does not change with time, then the user of the equipment does not need to pay interest on his investments. We wish to determine the optimal time to replace the equipment.

We make use of the following notations:

C = Capital cost of the equipment

S = Scrap value of the equipment

n = Number of years that the equipment would be in use

Cm = Maintenance cost function.

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ATC = Average total annual cost

Two possibilities are there

(i) Time *t* is a continuous random variable In this case the deterioration of the equipment is being monitored continuously. The

total cost of the equipment during n years of use is given by

TC = Capital cost - Scrap value + Maintenance cost

$$= C - S + \int_0^n C_m(t)\,dt$$

$$\therefore A(n) = \frac{1}{n}TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n C_m(t)\,dt$$

$$\text{For minimum cost, } \frac{d}{dn}A(n) = 0$$

$$\therefore -\frac{C-S}{n^2} - \frac{1}{n^2}\int_0^n C_m(t)dt + \frac{1}{n}C_m(n) = 0$$

$$\therefore C_m = \frac{C-S}{n^2} + \frac{1}{n^2}\int_0^n C_m(n)\,dt = A(n)$$

$$\text{And } \frac{d^2A(n)}{dn^2} \geq 0 \text{ at } C_m(n) = A(n)$$

i.e., when the maintenance cost becomes equal to the average annual cost, the decision should be to replace the equipment.

(ii) Time *t* is a discrete random variable

In this case

$$A(n) = \frac{1}{n} TC = \frac{C - S}{n} + \frac{1}{n} \sum_{0}^{n} C_m$$

A (*n*) is Minimum when

$$A(n+1) \geq A(n) \text{ and } A(n-1) \geq A(n)$$

$$\text{Or, } A(n+1) - A(n) \geq 0 \; and \; A(n) - A(n-1) \leq 0$$

$$A(n+1) - A(n) \frac{1}{n+1}\left(C - S + \sum_{0}^{n} C_m(t)\right) + \frac{1}{n+1} C_m(n+1) - A(n)$$

$$\frac{n}{n+1} A(n) + \frac{1}{n+1} C_m(n+1) - A(n) \geq 0$$

$$\therefore A(n+1) \geq A(n)$$

Similarly

$$A(n) - A(n-1) \leq 0$$

$$\therefore C_m(n) \leq C_m(n-1)$$

Thus the optimal policy is Replace the equipment at the end of n years if the maintenance cost in the (n+1) th year is more than the average total cost in the n^{th} year and the n^{th} year's maintenance cost is less than previous year's average total cost.

Present worth factor denoted by (P/F, i,n). If an amount P is invested now with amount earning interest at the rate i per year, then the future sum (F) accumulated after n years can be obtained.

P - Principal sum at year Zero

F - Future sum of P at the end of the nth year

i - Annual interest rate

n - Number of interest periods.

Then the formula for future sum $F = P(1 + i)^n$

$P = F/(1 + i)^n$ = Fx (present worth factor)

If A is the annual equivalent amount which occurs at the end of every year from year one through n years is given by

$$A = \frac{P \times i(1+i)^n}{(1+i)^n - 1}$$

$$= P(A/P, i, n)$$

= P x equal payment series capital recovery factor

Problem;

The cost of equipment is Rs. 62,000 and its scrap value is Rs. 2,000. The life of the equipment is 8 years. The maintenance costs for each year are as given below:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in Rs.	1000	2000	3500	5000	8000	11000	16000	24000

When the equipment should be replaced?

Solution:-

C = 62,000/-

As the avg. yearly cost is minimum for 6th year the equipment should be replace after 6 year.

Year **n**	Resale Price **S**	Maintenance Cost **C_m**	Cumulative Maintenance Cost **$\sum C_m$**	Total Cost **$TC = C - S + \sum C_m$**	Annual Total Cost **$ATC = \frac{TC}{n}$**
1	2000	1000	1000	61000	61000
2	2000	2000	3000	63000	31500
3	2000	3500	6500	65000	21666.6
4	2000	5000	11500	71500	17875
5	2000	8000	19500	79500	15900
6	**2000**	**11000**	**30500**	**90500**	**15083.3**
7	2000	16000	46500	106500	15214.2
8	24000				

Problem:

(a) Machine A cost Rs. 36,000. Annual operating costs are Rs. 800 for the first year, and then increase by Rs. 8000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy if followed, what will be the yearly cost of owning and operating the machine?
(b) Machine B costs Rs. 40,000. Annual operating costs Rs. 1,600 for the first year, and then increase by Rs. 3,200 every year. You now have a machine of type A which is one year old. Should you replace it with B, if so when? Assume that both machines have no resale value

Solution:-

(a) Machine A

C = 36,000/- □As the avg. yearly cost is minimum for 3rd year for machine A, machine A should be replace after 3 year.

Avg. yearly cost for operating & owing the machine A is Rs. 20,800.

□The avg. cost per year of operating & owing the machine B is less that of machine A.

Year **n**	Resale Price **S**	Maintenance Cost **Cm**	Cumulative Maintenance Cost **∑ Cm**	Total Cost **TC=C-S+∑ Cm**	Annual Total Cost **ATC =** $\frac{TC}{n}$
1	0	800	800	36800	36800
2	0	8800	9600	45600	22800
3	**0**	**16800**	**26400**	**62400**	**20800**
4	0	24800	51200	87200	21800

(b) Machine B

C = 40,000/- Machine A should be replaced with machine B. As the cost of using machine A in 3rd year is more than avg. yearly cost of operating & owing the machine.

□ Machine A should be replaced machine B after 2 years. i.e. 1 year from now because of machine A is already 1 year old.

Year n	Resale Price S	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC=C-S+\sum C_m$	Annual Total Cost $ATC = \frac{TC}{n}$
1	0	1600	1600	41600	41600
2	0	4800	6400	46400	23200
3	0	8000	14400	54400	18133.3
4	0	11230	25600	65600	16400
5	**0**	**14400**	**40000**	**80000**	**16000**
6	0	17600	57600	97600	16266.6

N^{th} year	Cost of N^{th} year (Rs.)
2	45600-36800=8800
3	62400-45600=16800

Problem :
A firm pays Rs. 10,000 for its equipment. Their operating and maintenance costs are about Rs. 2500 per year for the first two years and then go up by approximately Rs. 1,500 per year. When such equipment replaced? The discount rate is 10% per year.

Solution:-

$$d = \frac{1}{1+i} = \frac{1}{1+0.1} = 0.909$$

C = 10,00 0/- i = 0.10

Year n	C_m	Discount Factor d^{n-1}	Discounted Maintenance Cost $C_m * d^{n-1}$	Discounted Cumulative Maintenance Cost $\sum C_m * d^{n-1}$	$TC=C-S+\sum C_m * d^{n-1}$	$\sum d^{n-1}$	$ATC = \frac{TC}{\sum d^{n-1}}$
1	2500	1	2500	2500	12500	1	12500
2	2500	0.909	2272.5	4772.5	14772.5	1.909	7738.3
3	4000	0.826	3304	8076.5	18076.5	2.735	6609.3
4	**5500**	**0.751**	**4130.5**	**12207**	**22207**	**3.486**	**6370.3**
5	7000	0.683	4781	16988	26988	4.169	6473.4

As the avg. yearly cost is minimum for 4th year the equipment should be replace after 4 year

Problem :

The following mortality rates have been observation for certain type of light bulbs There are 1000 bulbs in use and it costs Rs 10 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it would cost Rs 2.5 per bulbs. It is proposed to replace all the bulbs at fixed interval, and individually those which fail between the intervals. What would be the best policy to adopt?

Month	1	2	3	4	5
Percent failing by month end	10	25	50	80	100

Solution:

Month i	**Cumulative % failure up to the end of month**	**% failure during the month**	**Probability P_i that a new bulb shall fail during the month**
1	10	10	0.10
2	25	15	0.15
3	50	25	0.25
4	80	30	0.30
5	100	20	0.20

Month i	Bulbs failing during i^{th} month	Bulbs replaced until i^{th} month	Cost of Individual Replacement TCI	Cost of Group Replacement TCG	Total Cost TC=TCI+TCG	Average Cost per month $ATC = \frac{TC}{n}$
1	100	100	1000	2500	3500	3500
2	160	260	2600	2500	5100	2550
3	281	541	5410	2500	7910	2636.6
4	377	918	9180	2500	11680	2920
5	349	1267	12670	2500	15170	3034

$$N_0 = 1000$$

$$N_1 = N_0 \times P_1$$

$$= \frac{10}{100} \times 1000$$

$$= 100$$

$$N_2 = N_0 \times P_2 + N_1 \times P_1$$
$$= \frac{15}{100} \times 1000 + \frac{10}{100} \times 100$$
$$= 160$$
$$N_3 = N_0 \times P_3 + N_1 \times P_2 + N_2 \times P_3$$
$$= \frac{25}{100} \times 1000 + \frac{15}{100} \times 100 + \frac{10}{100} \times 160$$
$$= 281$$
$$N_4 = N_0 \times P_4 + N_1 \times P_3 + N_2 \times P_2 + N_3 \times P_1$$
$$= \frac{30}{100} \times 1000 + \frac{25}{100} \times 100 + \frac{15}{100} \times 160 + \frac{10}{100} \times 281$$
$$= 377$$
$$N_5 = N_0 \times P_5 + N_1 \times P_4 + N_2 \times P_3 + N_3 \times P_2 + N_4 \times P_1$$
$$= \frac{20}{100} \times 1000 + \frac{30}{100} \times 100 + \frac{25}{100} \times 160 + \frac{15}{100} \times 281 + \frac{10}{100} \times 377$$
$$= 349$$

$$\text{Avg life} = \sum i \times P_i$$
$$= 1(P_1) + 2(P_2) + 3(P_3) + 4(P_4) + 5(P_5)$$
$$= 1(0.1) + 2(0.15) + 3(0.25) + 4(0.3) + 5(0.2)$$
$$= 3.35 \text{ months}$$

$$\text{No. of bulbs replaced per months} = \frac{1000}{3.35}$$
$$= 298 \text{ bulbs}$$
$$\text{Cost of individual replacement} = 298 \times 10$$
$$= 2980 \text{ Rs.}$$

As cost of group replacement after every 2nd month is less than cost of individual replacement. Group replacement policy after every 2 months is better.

Problem:
A firm is considering replacement of an equipment whose first cost is Rs. 1750 and the scrap value is negligible at any year. Based on experience, it is found that maintenance cost is zero during the first year and it increases by Rs. 100 every year thereafter.
(i) When should be the equipment replaced if
a) i = 0%
b) i = 12%

Solution :

Given the first cost = Rs 1750 and the maintenance cost is Rs. Zero during the first years and then increases by Rs. 100 every year thereafter. Then the following table shows the calculation.
(a.) Calculations to determine Economic life (a) First cost Rs. 1750. Interest rate = 0%

End of year (n)	Maintenance cost at end of year	Summation of maintenance Cost	Average cost of maintenance through the given year	Average first cost if replaced at the given year and	Average total cost through the given year
A	B (Rs)	C (Rs)	D (in Rs)	E (Rs)	F (Rs)
		C = ΣB	C/A	$\frac{1750}{A}$	D + E
1	0	0	0	1750	1750
2	100	100	50	875	925
3	200	300	100	583	683
4	300	600	150	438	588
5	400	1000	200	350	550
6	**500**	**1500**	**250**	**292**	**542**
7	600	2100	300	250	550
8	700	2800	350	219	569

The value corresponding to any end-of-year (n) in Column F represents the average total cost of using the equipment till the end of that particulars year.
In this problem, the average total cost decreases till the end of the year 6 and then it increases. Hence the optimal replacement period is 6 years ie the economic life of the equipment is 6 years .
When interest rate i = 12% When the interest rate is more than 0% the steps to get the economic life are summarized in the following table.
(b.)Calculation to determine Economic life First Cost = Rs. 1750 Interest rate = 12%

End of year (n)	Maintenance cost at end of years	(P/F,12v,n)	Present worth as beginning of years 1 of maintenance costs	Summation of present worth of maintenance costs through the given year	Present simulator maintenance cost and first cost	(A/P, 12%,n) $=\frac{i(1+i)^n}{(1+i)^n-1}$ G	Annual equipment total cost through the giver year
A	B	C	D	E	F	G	H
	B (iR)	C = $\frac{1}{(1+12/100)^n}$	BxC	ΣD	E+ Rs. 1750		FxG
1	0	0.8929	0	0	1750	1.1200	1960
2	100	0.7972	79.72	79.72	1829.72	0.5917	1082.6
3	200	0.7118	142.36	222.08	1972.08	0.4163	820.9
4	300	0.6355	190.65	412.73	2162.73	0.3292	711.9
5	400	0.5674	226.96	639.69	2389.69	0.2774	662.9
6	500	0.5066	253.30	892.99	2642.99	0.2432	642.7
7	600	0.4524	271.44	1164.43	2914.430	0.2191	638.5
8	700	0.4039	282.73	1447.16	3197.16	0.2013	680.7

Identify the end of year for which the annual equivalent total cost is minimum in column. In this problem the annual equivalent total cost is minimum at the end of year hence the economics life of the equipment is 7 years.

Simple probabilistic model for items which completely fail

Electronic items like bulbs, resistors, tube lights etc. generally fail all of a sudden, instead of gradual failure. The sudden failure of the item results in complete breakdown of the system. The system may contain a collection of such items or just an item like a single tube-light. Hence we use some replacement policy for such items which would minimize the possibility of complete breakdown. The following are the replacement policies which are applicable in these cases.

i) Individual replacement policy : Under this policy, each item is replaced immediately after failure.

ii) Group replacement policy : Under group replacement policy, a decision is made with regard the replacement at what equal internals, all the item are to be replaced simultaneously with a provision to replace the items individually which fail during the fixed group replacement period. Among the two types of replacement polices, we have to decide which replacement policy we have to follow. Whether individual replacement policy is better than group replacement policy. with regard to economic point of view. To decide this, each of the replacement policy is calculated and the most economic one is selected for implementation.

UNIT 3

Theory of Games

UNIT III
Theory of Games

Introduction : The definition given by William G. Nelson runs as follows: "Game theory, more properly the theory of games of strategy, is a mathematical method of analyzing a conflict. The alternative is not between this decision or that decision, but between this strategy or that strategy to be used against the conflicting interest".

In the perception of Robert Mockler, "Game theory is a mathematical technique helpful in making decisions in situations of conflicts, where the success of one part depends at the expense of others, and where the individual decision maker is not in complete control of the factors influencing the outcome".

According to von Neumann and Morgenstern, "The 'Game' is simply the totality of the rules which describe it. Every particular instance at which the game is played – in a particular way – from beginning to end is a 'play'. The game consists of a sequence of moves, and the play of a sequence of choices".

According to Edwin Mansfield, "A game is a competitive situation where two or more persons pursue their own interests and no person can dictate the outcome. Each player, an entity with the same interests, make his own decisions. A player can be an individual or a group".

Assumptions for a Competitive Game:

Game theory helps in finding out the best course of action for a firm in view of the anticipated countermoves from the competing organizations. A competitive situation is a competitive game if the following properties hold,

1. The number of competitors is finite, say N.
2. A finite set of possible courses of action is available to each of the N competitors.
3. A play of the game results when each competitor selects a course of action from the set of courses available to him. In game theory we make an important assumption that all the players select their courses of action simultaneously. As a result, nocompetitor will be in a position to know the choices of his competitors.
4. The outcome of a play consists of the particular courses of action chosen by the individual players. Each outcome leads to a set of payments, one to each player, which may be either positive, or negative, or zero.

Managerial Applications of the Theory of Games:

The techniques of game theory can be effectively applied to various managerial problems as detailed below:

1) Analysis of the market strategies of a business organization in the long run.
2) Evaluation of the responses of the consumers to a new product.
3) Resolving the conflict between two groups in a business organization.
4) Decision making on the techniques to increase market share.
5) Material procurement process.
6) Decision making for transportation problem.
7) Evaluation of the distribution system.

8) Evaluation of the location of the facilities.
9) Examination of new business ventures and
10) Competitive economic environment.

Concepts in the Theory of Games:

Players: The competitors or decision makers in a game are called the players of the game.

Strategies: The alternative courses of action available to a player are referred to as his strategies.

Pay off: The outcome of playing a game is called the pay off to the concerned player.

Optimal Strategy: A strategy by which a player can achieve the best pay off is called the optimal strategy for him.

Zero-sum game: A game in which the total payoffs to all the players at the end of the game is zero is referred to as a zero-sum game.

Non-zero sum game: Games with "less than complete conflict of interest" are called non-zero sum games. The problems faced by a large number of business organizations come under this category. In such games, the gain of one player in terms of his success need not be completely at the expense of the other player.

Payoff matrix: The tabular display of the payoffs to players under various alternatives is called the payoff matrix of the game.

Pure strategy: If the game is such that each player can identify one and only one strategy as the optimal strategy in each play of the game, then that strategy is referred to as the best strategy for that Player and the game is referred to as a game of pure strategy or a pure game
.
Mixed strategy: If there is no one specific strategy as the 'best strategy' for any player in a game, then the game is referred to as a game of mixed strategy or a mixed game. In such a game, each player has to choose different alternative courses of action from time to time.

N-person game: A game in which N-players take part is called an N-person game.

Maxi min-Mini max Principle: The maximum of the minimum gains is called the maxi min value of the game and the corresponding strategy is called the maxi min strategy. Similarly the minimum of the maximum losses is called the mini max value of the game and the corresponding strategy is called the mini max strategy. If both the values are equal, then that would guarantee the best of the worst results.

Negotiable or cooperative game: If the game is such that the players are taken to cooperate on any or every action which may increase the payoff of either player, then we call it a negotiable or cooperative game.

Non-negotiable or non-cooperative game: If the players are not permitted for coalition then we refer to the game as a non-negotiable or non-cooperative game.

Saddle point: A saddle point of a game is that place in the payoff matrix where the maximum of the row minima is equal to the minimum of the column maxima. The payoff at the saddle point is called **the value of the game** and the corresponding strategies are called the **pure strategies.**

Dominance: One of the strategies of either player may be inferior to at least one of the remaining ones. The superior strategies are said to dominate the inferior ones.

Types of Games:

There are several classifications of a game. The classification may be based on various factors such as the number of participants, the gain or loss to each participant, the number of Strategies available to each participant, etc. Some of the important types of games are enumerated below.

Two person games and n-person games : In two person games, there are exactly two players and each competitor will have a finite number of strategies. If the number of players in a game exceeds two, then we refer to the game as n-person game.

Zero sum game and non-zero sum game: If the sum of the payments to all the players in a game is zero for every possible outcome of the game, then we refer to the game as a zero sum game. If the sum of the payoffs from any play of the game is either positive or negative but not zero, then the game is called a non-zero sum game

Games of perfect information and games of imperfect information: A game of perfect information is the one in which each player can find out the strategy that would be followed by his opponent. On the other hand, a game of imperfect information is the one in which no player can know in advance what strategy would be adopted by the competitor and a player has to proceed in his game with his guess works only.

Games with finite number of moves / players and games with unlimited number of moves : A game with a finite number of moves is the one in which the number of moves for each player is limited before the start of the play. On the other hand, if the game can be continued over an extended period of time and the number of moves for any player has no restriction, then we call it a game with unlimited number of moves.

Constant-sum games: If the sum of the game is not zero but the sum of the payoffs to both players in each case is constant, then we call it a constant sum game. It is possible to reduce such a game to a zero sum game.

2x2 two person game and 2xn and mx2 games: When the number of players in a game is two and each player has exactly two strategies, the game is referred to as 2x2 two person game. A game in which the first player has precisely two strategies and the second player has three or more strategies is called an 2xn game. A game in which the first player has three or more strategies and the second player has exactly two strategies is called an mx2 game.

3x3 and large games: When the number of players in a game is two and each player has exactly three strategies, we call it a 3x3 two person game. Two-person zero sum games are said to be larger if each of the two players has 3 or more choices. The examination of 3x3 and larger games is involves difficulties. For such games, the technique of linear programming can be used as a method of solution to identify the optimum strategies for the two players.

Non-constant games: Consider a game with two players. If the sum of the payoffs to the two players is not constraint in all the plays of the game, then we call it a non-constant game. Such games are divided into negotiable or cooperative games and non-negotiable or non-cooperative games.

Two-person zero sum games: A game with only two players, say player A and player B, is called a two-person zero sum game if the gain of the player A is equal to the loss of the player B, so that the total sum is zero.

Payoff matrix: When players select their particular strategies, the payoffs (gains or losses) can be represented in the form of a payoff matrix..Since the game is zero sum, the gain of one player is equal to the loss of other and vice-versa. Suppose A has m strategies and B has n strategies. Consider the following payoff matrix. Player A wishes to gain as large a payoff **a**ij as possible while player B will do his best to reach as small a value ij a as possible where the gain to player B and loss to player A be (-**a**ij).

The amount of payoff, i.e., V at an equilibrium point is known as the **value of the game**. The optimal strategies can be identified by the players in the long run.

Fair game: The game is said to be fair if the value of the game V = 0.

Player B's strategies

$$\text{Player A's strategies} \quad \begin{array}{c} \\ A_1 \\ A_1 \\ \vdots \\ Am \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Assumptions for two-person zero sum game:

For building any model, certain reasonable assumptions are quite necessary. Some assumptions for building a model of two-person zero sum game are listed below.
a) Each player has available to him a finite number of possible courses of action. Sometimes the set of courses of action may be the same for each player. Or, certain courses of action may be available to both players while each player may have certain specific courses of action which are not available to the other player.
b) Player A attempts to maximize gains to himself. Player B tries to minimize losses to himself.
c) The decisions of both players are made individually prior to the play with no communication between them.
d) The decisions are made and announced simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
e) Both players know the possible payoffs of themselves and their opponents. Mini max and Maxi min Principles

The selection of an optimal strategy by each player without the knowledge of the competitor's strategy is the basic problem of playing games. The objective of game theory is to know how these players must select their respective strategies, so that they may optimize their payoffs. Such a criterion of decision making is referred to as mini max-maxi min principle. This principle in games of pure strategies leads to the best possible selection of a strategy for both players.

For example, if player A chooses his **i^th^** strategy, then he gains at least the payoff min **a ij** , which is minimum of the **i^th^** row elements in the payoff matrix. Since his objective is to Maximize his payoff, he can choose strategy i so as to make his payoff as large as possible.
i.e., a payoff which is not less than

$$\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij}$$

Similarly player B can choose j^{th} column elements so as to make his loss not greater than

$$\min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij} .$$

If the maxi min value for a player is equal to the mini max value for another player, i.e.

$$\max_{1 \le i \le m} \min_{1 \le j \le n} a_{ij} = V = \min_{1 \le j \le n} \max_{1 \le i \le m} a_{ij}$$

then the game is said to have a saddle point (equilibrium point) and the corresponding strategies are called optimal strategies. If there are two or more saddle points, they must be equal.

Problem:

Solve the game with the following pay-off matrix.

Player B

Strategies

Player A Strategies		*I*	*II*	*III*	*IV*	*V*
	1	–2	5	– 3	6	7
	2	4	6	8	–1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of {-3, -1, 2, 12} = 12

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of {15, 14, 18, 12, 20}= 12

We see that the maximum of row minima = the minimum of the column maxima. So the

game has a saddle point. The common value is 12. Therefore the value V of the game = 12.
Interpretation: In the long run, the following best strategies will be identified by the two players

The best strategy for player A is strategy 4.
The best strategy for player B is strategy IV.
The game is favorable to player A.

Problem 2: Solve the game with the following pay-off matrix

Strategies

		I	*II*	*III*	*IV*	*V*
Player X Strategies	1	9	12	7	14	26
	2	25	35	20	28	30
	3	7	6	–8	3	2
	4	8	11	13	–2	1

Solution: First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

Maximum of {7, 20, –8, -2} = 20
Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

Minimum of {25, 35, 20, 28, 30}= 20

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20.This indicates that the value V of the game is 20.

Interpretation: The best strategy for player X is strategy 2.
The best strategy for player Y is strategy III.
The game is favorable to player A.

Problem :

Solve the following game:

Player B

Strategies

Player A Strategies		I	II	III	IV
	1	1	−6	8	4
	2	3	−7	2	−8
	3	5	−5	−1	0
	4	3	−4	5	7

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-6
2	-8
3	-5
4	-4

Maximum of {-6, -8, -5, -4} = -4

Next consider the maximum of each column

Column	Maximum Value
1	5
2	-4
3	8
4	7

Minimum of {5, -4, 8, 7}= - 4

Since the max {row minima} = min {column maxima}, the game under consideration has a saddle point. The common value is –4. Hence the value of the game is –4.

Interpretation.

The best strategy for player A is strategy 4.

The best strategy for player B is strategy II. Since the value of the game is negative, it is concluded that the game is favorable to player B.

Games with no Saddle point:

2 x 2 zero-sum game When each one of the first player A and the second player B has exactly two strategies, we have a 2 x 2 game.

Motivating point First let us consider an illustrative example**.**

Problem :

Examine whether the following 2 x 2 game has a saddle point

Player B

Player A	3	5
	4	2

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	3
2	2

Maximum of {3, 2} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	5

Minimum of {4, 5}= 4

We see that max {row minima} and min {column maxima} are not equal. Hence thegame has no saddle point

Method of solution of a 2x2 zero-sum game without saddle point**:** Suppose that a 2x2 game has no saddle point. Suppose the game has the following pay-off matrix.

Player B

Strategy

Player A Strategy $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since this game has no saddle point, the following condition shall hold:

$$Max\{Min\{a,b\},Min\{c,d\}\} \neq Min\{Max\{a,c\},Max\{b,d\}\}$$

In this case, the game is called a mixed game. No strategy of Player A can be called the best strategy for him. Therefore A has to use both of his strategies. Similarly no strategy of Player B can be called the best strategy for him and he has to use both of his strategies.

Let p be the probability that Player A will use his first strategy. Then the probability that Player A will use his second strategy is 1-p.If Player B follows his first strategy. Expected value of the pay-off to Player A.

Expected value of the pay-off to Player A

$$= \left\{ \begin{matrix} \text{Expected value of the pay-off to Player A} \\ \text{arising from his first strategy} \end{matrix} \right\} + \left\{ \begin{matrix} \text{Expected value of the pay-off to Player A} \\ \text{arising from his second strategy} \end{matrix} \right\}$$

$$= ap + c(1-p) \longrightarrow (1)$$

In the above equation, note that the expected value is got as the product of the corresponding values

of the pay-off and the probability.

If Player B follows his second strategy

$$\left. \begin{matrix} \text{Expected value of the} \\ \text{pay-off to Player A} \end{matrix} \right\} = bp + d(1-p) \qquad (2)$$

If the expected values in equations (1) and (2) are different, Player B will prefer the minimum of the two expected values that he has to give to player A. Thus B will have a pure strategy.

This contradicts our assumption that the game is a mixed one. Therefore the expected values of the pay-offs to Player A in equations (1) and (2) should be equal. Thus we have the condition

$$
\begin{aligned}
ap + c(1-p) &= bp + d(1-p) \\
ap - bp &= (1-p)[d-c] \\
p(a-b) &= (d-c) - p(d-c) \\
p(a-b) + p(d-c) &= d-c \\
p(a-b+d-c) &= d-c \\
p &= \frac{d-c}{(a+d)-(b+c)} \\
1-p &= \frac{a+d-b-c-d+c}{(a+d)-(b+c)} \\
&= \frac{a-b}{(a+d)-(b+c)}
\end{aligned}
$$

$$
\left\{\begin{array}{c}\text{The number of times A} \\ \text{will use first strategy}\end{array}\right\} : \left\{\begin{array}{c}\text{The number of times A} \\ \text{will use second strategy}\end{array}\right\} = \frac{d-c}{(a+d)-(b+c)} : \frac{a-b}{(a+d)-(b+c)}
$$

The expected pay-off to Player A

$$
\begin{aligned}
&= ap + c(1-p) \\
&= c + p(a-c) \\
&= c + \frac{(d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{c\{(a+d)-(b+c)\} + (d-c)(a-c)}{(a+d)-(b+c)} \\
&= \frac{ac + cd - bc - c^2 + ad - cd - ac + c^2)}{(a+d)-(b+c)} \\
&= \frac{ad - bc}{(a+d)-(b+c)}
\end{aligned}
$$

Therefore, the value V of the game is

$$
\frac{ad - bc}{(a+d)-(b+c)}
$$

To find the number of times that B will use his first strategy and second strategy:

Let the probability that B will use his first strategy be *r*. Then the probability that B will use his second strategy is *1-r*.

When A use his first strategy

The expected value of loss to Player B with his first strategy = *ar*

The expected value of loss to Player B with his second strategy = *b(1-r)*

Therefore the expected value of loss to B = ar + b(1-r) ⟶ (3)

When A use his second strategy

The expected value of loss to Player B with his first strategy = cr

The expected value of loss to Player B with his second strategy = d(1-r)

Therefore the expected value of loss to B = cr + d(1-r) $\longrightarrow$ (4)
If the two expected values are different then it results in a pure game, which is a contradiction. Therefore the expected values of loss to Player B in equations (3) and (4) should be equal. Hence we have the condition

$$ar + b(1-r) = cr + d(1-r)$$
$$ar + b - br = cr + d - dr$$
$$ar - br - cr + dr = d - b$$
$$r(a - b - c + d) = d - b$$
$$r = \frac{d-b}{a-b-c+d}$$
$$= \frac{d-b}{(a+d)-(b+c)}$$

Problem:
Solve the following game

$$\begin{array}{cc} & Y \\ X & \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

Solution:
First consider the row minima

Row	Minimum Value
1	2
2	1

Maximum of {2, 1} = 2
Next consider the maximum of each column

Column	Maximum Value
1	4
2	5

We see that Max {row minima} $\neq$ min {column maxima}
So the game has no saddle point. Therefore it is a mixed game.
We have a = 2, b = 5, c = 4 and d = 1.
Let *p* be the probability that player X will use his first strategy. We have

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{1-4}{(2+1)-(5+4)}$$
$$= \frac{-3}{3-9}$$
$$= \frac{-3}{-6}$$
$$= \frac{1}{2}$$

The probability that player *X* will use his second strategy is $1\text{-}p = 1\text{-}\frac{1}{2} = \frac{1}{2}$.

Value of the game V = $\frac{ad-bc}{(a+d)-(b+c)} = \frac{2-20}{3-9} = \frac{-18}{-6} = 3$.

Let *r* be the probability that Player Y will use his first strategy. Then the probability that Y will use his second strategy is *(1-r)*. We have

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{1-5}{(2+1)-(5+4)}$$
$$= \frac{-4}{3-9}$$
$$= \frac{-4}{-6}$$
$$= \frac{2}{3}$$
$$1-r = 1-\frac{2}{3} = \frac{1}{3}$$

Interpretation.

$p : (1\text{-}p) = \frac{1}{2} : \frac{1}{2}$

Therefore, out of 2 trials, player X will use his first strategy once and his second strategy once.

$r : (1\text{-}r) = \frac{2}{3} : \frac{1}{3}$

Therefore, out of 3 trials, player Y will use his first strategy twice and his second strategy once.

The Principle of Dominance:

In the previous lesson, we have discussed the method of solution of a game without a saddle point. While solving a game without a saddle point, one comes across the phenomenon of the dominance of a row over another row or a column over another column in the pay-off matrix of the game. Such a situation is discussed in the sequel.In a given pay-off matrix A, we say that the ith row dominates the kth row if

$a_{ij} \geq a_{kj}$ for all j = 1,2,…,n and $a_{ij} > a_{kj}$ for at least one j.

In this case, the player B will loose more by choosing the strategy for the qth column than by choosing the strategy for the pth column. So he will never use the strategy corresponding to the qth column. When dominance of a row (or a column) in the pay-off matrix occurs, we can delete a row (or a column) from that matrix and arrive at a reduced matrix. This principle of dominance can be used in the determination of the solution for a given game.

Let us consider an illustrative example involving the phenomenon of dominance in a game.

Problem :

Solve the game with the following pay-off matrix:

Player B

Player A	I	II	III	IV
1	4	2	3	6
2	3	4	7	5
3	6	3	5	4

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	2
2	3
3	3

Maximum of {2, 3, 3} = 3

Next consider the maximum of each column.

Column	Maximum Value
1	6
2	4
3	7
4	6

Minimum of {6, 4, 7, 6}= 4

The following condition holds:
Max {row minima}$\neq$= min {column maxima}
Therefore we see that there is no saddle point for the game under consideration.
Compare columns II and III.

Column II	Column III
2	3
4	7
3	5

We see that each element in column III is greater than the corresponding element in column II. The choice is for player B. Since column II dominates column III, player B will discard his strategy 3.Now we have the reduced game

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} I & II & IV \end{matrix} \\ \begin{bmatrix} 4 & 2 & 6 \\ 3 & 4 & 5 \\ 6 & 3 & 4 \end{bmatrix} \end{array}$$

For this matrix again, there is no saddle point. Column II dominates column IV. The choice is for player B. So player B will give up his strategy 4
The game reduces to the following:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{matrix} I & II \end{matrix} \\ \begin{bmatrix} 4 & 2 \\ 3 & 4 \\ 6 & 3 \end{bmatrix} \end{array}$$

This matrix has no saddle point.
The third row dominates the first row. The choice is for player A. He will give up his strategy 1 and retain strategy 3. The game reduces to the following

$$\begin{bmatrix} 3 & 4 \\ 6 & 3 \end{bmatrix}$$

Again, there is no saddle point. We have a 2x2 matrix. Take this matrix as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then we have a = 3, b = 4, c = 6 and d = 3. Use the formulae for p, 1-p, r, 1-r and V.

$$p = \frac{d-c}{(a+d)-(b+c)}$$
$$= \frac{3-6}{(3+3)-(6+4)}$$
$$= \frac{-3}{6-10}$$
$$= \frac{-3}{-4}$$
$$= \frac{3}{4}$$

$$1-p = 1-\frac{3}{4} = \frac{1}{4}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$
$$= \frac{3-4}{(3+3)-(6+4)}$$
$$= \frac{-1}{6-10}$$
$$= \frac{-1}{-4}$$
$$= \frac{1}{4}$$

$$1-r = 1-\frac{1}{4} = \frac{3}{4}$$

The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$
$$= \frac{3x3-4x6}{-4}$$
$$= \frac{-15}{-4}$$
$$= \frac{15}{4}$$

Thus, $X = \left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$ and $Y = \left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$ are the optimal strategies.

Method of convex linear combination :

A strategy, say s, can also be dominated if it is inferior to a convex linear combination of several other pure strategies. In this case if the domination is strict, then the strategy s can be deleted. If strategy s dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination will be decided as per the above rules. Let us consider an example to illustrate this case.

Problem:

Solve the game with the following pay-off matrix for firm A:

$$\text{Firm A} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \overset{\text{Firm B}}{\begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \\ \hline 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \\ 4 & -1 & 5 & 7 & 3 \end{array}}$$

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-2
2	0
3	-6
4	-3
5	-1

Maximum of {-2, 0, -6, -3, -1} = 0

Next consider the maximum of each column.

Column	Maximum Value
1	4
2	8
3	6
4	8
5	6

Minimum of { 4, 8, 6, 8, 6}= 4
Hence,
Maximum of {row minima} $\neq$ minimum of {column maxima}.
So we see that there is no saddle point. Compare the second row with the fifth row. Each element in the second row exceeds the corresponding element in the fifth row. Therefore, A_2 dominates A_5 . The choice is for firm A. It will retain strategy A_2 and give up strategy A_5 .Therefore the game reduces to the following.

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \\ 4 & -3 & 5 & 6 & 3 \end{bmatrix} \end{array}$$

Compare the second and fourth rows. We see that A_2 dominates A_4 . So, firm A will retain the strategy A_2 and give up the strategy A_4 . Thus the game reduces to the following:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 & 6 \\ 4 & 0 & 6 & 8 & 5 \\ -2 & -6 & -4 & 4 & 2 \end{bmatrix} \end{array}$$

Compare the first and fifth columns. It is observed that B1 dominates B5. The choice is for firm B. It will retain the strategy B_1 and give up the strategy B_5 . Thus the game reduces to the Following

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 & 5 \\ 4 & 0 & 6 & 8 \\ -2 & -6 & -4 & 4 \end{bmatrix} \end{array}$$

Compare the first and fourth columns. We notice that B1 dominates B4. So firm B will discard the strategy B_4 and retain the strategy B_1 . Thus the game reduces to the following

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 4 & 8 & -2 \\ 4 & 0 & 6 \\ -2 & -6 & -4 \end{bmatrix} \end{array}$$

For this reduced game, we check that there is no saddle point. Now none of the pure strategies of firms A and B is inferior to any of their other strategies. But, we observe that convex linear combination of the strategies B_2 and B_3 dominates B_1 , i.e. the averages of payoffs due to strategies B_2 and B_3,

$$\left\{\frac{8-2}{2}, \frac{0+6}{2}, \frac{-6-4}{2}\right\} = \{3, 3, -5\}$$

dominate B_1 . Thus B_1 may be omitted from consideration. So we have the reduced matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \\ -6 & -4 \end{bmatrix} \end{array}$$

Here, the average of the pay-offs due to strategies A_1 and A_2 of firm A, i.e.

$\left\{\frac{8+0}{2}, \frac{-2+6}{2}\right\} = \{4, 2\}$ dominates the pay-off due to A_3. So we get a new reduced 2x2 pay-off matrix

Firm B's strategy

Firm A's strategy $\quad \begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{bmatrix} 8 & -2 \\ 0 & 6 \end{bmatrix} \end{array}$

We have a = 8, b = -2, c = 0 and d = 6.

$$p = \frac{d-c}{(a+d)-(b+c)}$$

$$= \frac{6-0}{(6+8)-(-2+0)}$$

$$= \frac{6}{16}$$

$$= \frac{3}{8}$$

$$1-p = 1-\frac{3}{8} = \frac{5}{8}$$

$$r = \frac{d-b}{(a+d)-(b+c)}$$

$$= \frac{6-(-2)}{16}$$

$$= \frac{8}{16}$$

$$= \frac{1}{2}$$

$$1-r = 1-\frac{1}{2} = \frac{1}{2}$$

. Value of the game

$$V = \frac{ad - bc}{(a+d)-(b+c)}$$
$$= \frac{6x8 - 0x(-2)}{16}$$
$$= \frac{48}{16} = 3$$

So the optimal strategies are

$$A = \left\{\frac{3}{8}, \frac{5}{8}, 0, 0, 0\right\} \text{ and } B = \left\{0, \frac{1}{2}, \frac{1}{2}, 0, 0\right\}.$$

The value of the game = 3. Thus the game is favourable to firm A.

Problem:

For the game with the following pay-off matrix, determine the saddle point

$$\begin{array}{c} \\ \\ \text{Player A} \end{array} \quad \begin{array}{c} \text{Player B} \\ \begin{array}{cc} & \begin{array}{cccc} I & II & III & IV \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & 0 & -3 \\ 1 & 0 & 3 & 2 \\ -3 & -2 & -1 & 4 \end{bmatrix} \end{array} \end{array}$$

Solution:

$$\begin{array}{c} \textit{Column II} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} \end{array} \qquad \begin{array}{c} \textit{Column III} \\ \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \end{array} \qquad \begin{array}{r} 0 > -1 \\ 3 > 0 \\ -1 > -2 \end{array}$$

The choice is with the player B. He has to choose between strategies II and III. He will lose more in strategy III than in strategy II, irrespective of what strategy is followed by A. So he will drop strategy III and retain strategy II. Now the given game reduces to the following game.

$$\begin{array}{cc} & \begin{array}{ccc} I & II & IV \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -1 & -3 \\ 1 & 0 & 2 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Consider the rows and columns of this matrix

Row minimum:

I Row	:	-3	
II Row	:	0	Maximum of {-3, 0, -3} = 0
III Row	:	-3	

Column maximum:

I Column	:	2	
II Column	:	0	Minimum of {2, 0, 4} = 0
III Column	:	4	

Interpretation: No player gains and no player loses. i.e., The game is not favourable to any player. i.e. It is a fair game.

Problem:
Solve the game

$$\text{Player A} \overset{\text{Player B}}{\begin{bmatrix} 4 & 8 & 6 \\ 6 & 2 & 10 \\ 4 & 5 & 7 \end{bmatrix}}$$

Solution:
First consider the minimum of each row

Row	Minimum
1	4
2	2
3	4

Maximum of {4, 2, 4} = 4

Next, consider the maximum of each column.

Column	Maximum
1	6
2	8
3	10

Minimum of {6, 8, 10} = 6

Since Maximum of { Row Minima} and Minimum of { Column Maxima } are different, it follows that the given game has no saddle point.
Denote the strategies of player A by A_1, A_2 , A_3 . Denote the strategies of player B by B_1 ,B_2 ,B_3 .

Compare the first and third columns of the given matrix.

$$\begin{array}{cc} B_1 & B_3 \\ \begin{bmatrix} 4 \\ 6 \\ 7 \end{bmatrix} & \begin{bmatrix} 6 \\ 10 \\ 7 \end{bmatrix} \end{array}$$

The pay-offs in B_3 are greater than or equal to the corresponding pay-offs in B_1 . The player B has to make a choice between his strategies 1 and 3. He will lose more if he follows strategy 3 rather than strategy 1. Therefore he will give up strategy 3 and retain strategy 1. Consequently, the given game is transformed into the following game:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 4 & 5 \end{bmatrix} \end{array}$$

Compare the first and third rows of the above matrix.

$$\begin{array}{c} \\ A_1 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 4 & 5 \end{bmatrix} \end{array}$$

The pay-offs in A_1 are greater than or equal to the corresponding pay-offs in A_2 . The player A has to make a choice between his strategies 1 and 3. He will gain more if he follows strategy 1 rather than strategy 3. Therefore he will retain strategy 1 and give up strategy 3.Now the given game is transformed into the following game.

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix} \end{array}$$

It is a 2x2 game. Consider the row minima

Row	Minimum
1 2	4 2

Maximum of {4, 2} = 4

Next, consider the maximum of each column

Column	Maximum
1 2	6 8

Minimum of {6, 8} = 6

Maximum {row minima} and Minimum {column maxima } are not equal.Therefore, the reduced game has no saddle point. So, it is a mixed game

Take $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$. We have a = 4, b = 8, c = 6 and d = 2.

The probability that player A will use his first strategy is p. This is calculated as

$$p = \frac{d-c}{(a+d)-(b+c)}$$

$$= \frac{2-6}{(4+2)-(8+6)}$$

$$= \frac{-4}{6-14}$$

$$= \frac{-4}{-8} = \frac{1}{2}$$

The probability that player B will use his first strategy is r. This is calculated as

$$r = \frac{d-b}{(a+d)-(b+c)}$$

$$= \frac{2-8}{-8}$$

$$= \frac{-6}{-8}$$

$$= \frac{3}{4}$$

Value of the game is V. This is calculated as

$$V = \frac{ad-bc}{(a+d)-(b+c)}$$

$$= \frac{4x2-8x6}{-8}$$

$$= \frac{8-48}{-8}$$

$$= \frac{-40}{-8} = 5$$

Interpretation

Out of 3 trials, player A will use strategy 1 once and strategy 2 once. Out of 4 trials, player B will use strategy 1 thrice and strategy 2 once. The game is favorable to player A.

Problem:

Solve the game with the following pay-off matrix.**(Dividing a game into sub-games)**

Player B

$$\text{Player A}\quad \begin{matrix} & 1 & 2 & 3 \\ I & -4 & 6 & 3 \\ II & -3 & 3 & 4 \\ III & 2 & -3 & 4 \end{matrix}$$

Solution:

First, consider the row minima.

Row	Minimum
1	-4
2	-3
3	-3

Maximum of {-4, -3, -3} = -3

Next, consider the column maxima.

Column	Maximum
1	2
2	6
3	4

Minimum of {2, 6, 4} = 2

We see that Maximum of { row minima} ≠ Minimum of { column maxima}.

So the game has no saddle point. Hence it is a mixed game. Compare the first and third Columns.

$$\begin{matrix} I\ Column & III\ Column & \\ -4 & 3 & -4 \le 3 \\ -3 & 4 & -3 \le 4 \\ 2 & 4 & 2 \le 4 \end{matrix}$$

We assert that Player B will retain the first strategy and give up the third strategy. We get the following reduced matrix

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \\ 2 & -3 \end{bmatrix}$$

We check that it is a game with no saddle point.

Sub games : Let us consider the 2x2 sub games. They are:

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$$

First, take the sub game

$$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$$

Compare the first and second columns. We see that $-4 \le 6$ and $-3 \le 3$. Therefore, the game reduces to $\begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Since $-4 < -3$, it further reduces to -3.

Next, consider the sub game

$$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$$

We see that it is a game with no saddle point. Take a = -4, b = 6, c = 2, d = -3. Then the value of the game is

$$V = \frac{ad - bc}{(a+d)-(b+c)}$$

$$= \frac{(-4)(-3)-(6)(2)}{(-4+3)-(6+2)}$$

$$= 0$$

Next, take the sub game $\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$. In this case we have a = -3, b = 3, c = 2 and d = -3. The value of the game is obtained as

$$V = \frac{ad - bc}{(a+d)-(b+c)}$$

$$= \frac{(-3)(-3)-(3)(2)}{(-3-3)-(3+2)}$$

$$= \frac{9-6}{-6-5} = -\frac{3}{11}$$

Let us tabulate the results as follows:

Sub game	Value
$\begin{bmatrix} -4 & 6 \\ -3 & 3 \end{bmatrix}$	-3
$\begin{bmatrix} -4 & 6 \\ 2 & -3 \end{bmatrix}$	0
$\begin{bmatrix} -3 & 3 \\ 2 & -3 \end{bmatrix}$	$-\frac{3}{11}$

The value of 0 will be preferred by the player A. For this value, the first and third strategies of A correspond while the first and second strategies of the player B correspond to the value 0 of the game. So it is a fair game.

Graphical solution of a 2x2 game with no saddle point:

Problem:
Consider the game with the following pay-off matrix.

Player B

$$\text{Player A} \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

Solution: First consider the row minima.

Row	Minimum
1	2
2	1

Maximum of {2, 1} = 2.

Next, consider the column maxima.

Column	Maximum
1	4
2	5

Minimum of {4, 5} = 4.

We see that Maximum { row minima} ≠ Minimum { column maxima }

So, the game has no saddle point. It is a mixed game.

Equations involving probability and expected value:

Let p be the probability that player A will use his first strategy.

Then the probability that A will use his second strategy is **1--p**.

Let E be the expected value of pay-off to player A.

When B uses his first strategy

The expected value of pay-off to player A is given by

$$E = 2p + 4(1-p)$$
$$= 2p + 4 - 4p$$
$$= 4 - 2p \longrightarrow (1)$$

When B uses his second strategy

The expected value of pay-off to player A is given by

$$E = 5p + 1(1-p)$$
$$= 5p + 1 - p$$
$$= 4p + 1 \longrightarrow (2)$$

Consider equations (1) and (2). For plotting the two equations on a graph sheet, get some points on them as follows: E = -2p+4

p	0	1	0.5
E	4	2	3

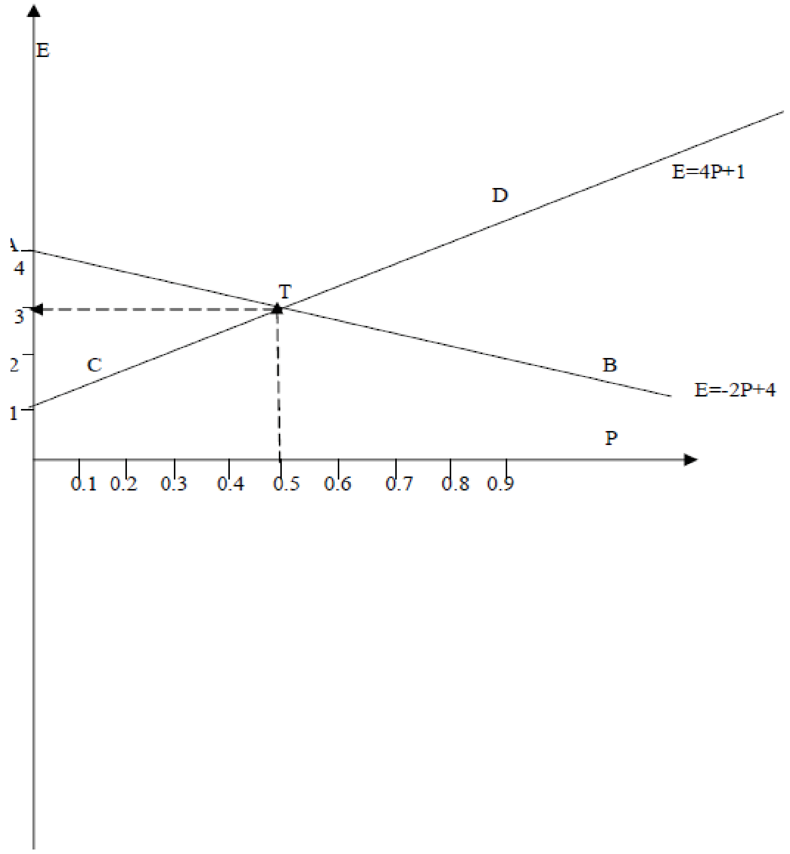
E = 4p+1

p	0	1	0.5
E	1	5	3

Graphical solution:

Procedure: Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will give the common solution of the two equations (1) and (2). Thus we would obtain the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = 0.5 and E = 3. Therefore, the value V of the game is 3.



Problem:

Solve the following game by graphical method.

$$\text{Player A} \overset{\text{Player B}}{\begin{bmatrix} -18 & 2 \\ 6 & -4 \end{bmatrix}}$$

Solution:

First consider the row minima.

Row	Minimum
1	- 18
2	- 4

Maximum of {-18, - 4} = - 4.

Next, consider the column maxima.

Column	Maximum
1	6
2	2

Minimum of {6, 2} = 2.

We see that Maximum {row minima} ≠ Minimum { column maxima }So, the game has no saddle point. It is a mixed game. Let p be the probability that player A will use his first strategy. Then the probability that A will use his second strategy is 1- p.

When B uses his first strategy. The expected value of pay-off to player A is given by

$$E = -18p + 6(1-p)$$
$$= -18p + 6 - 6p$$
$$= -24p + 6$$

E = -24 p + 6

p	0	1	0.5
E	6	-18	-6

$$E = 2p - 4(1-p)$$
$$= 2p - 4 + 4p$$
$$= 6p - 4$$

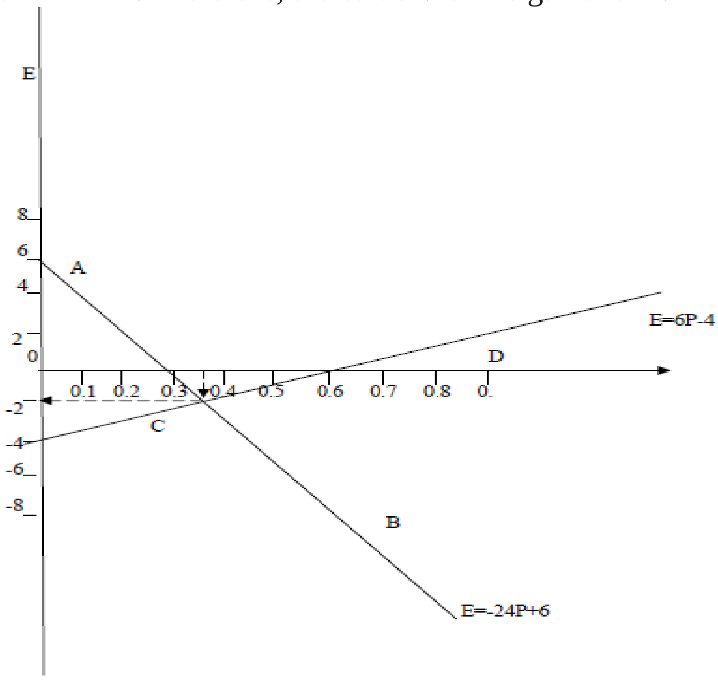
E = 6p-4

p	0	1	0.5
E	-4	2	-1

Graphical solution:

Take probability and expected value along two rectangular axes in a graph sheet. Draw two straight lines given by the two equations (1) and (2). Determine the point of intersection of the two straight lines in the graph. This will provide the common solution of the two equations (1) and (2). Thus we would get the value of the game.

Represent the two equations by the two straight lines AB and CD on the graph sheet. Take the point of intersection of AB and CD as T. For this point, we have p = 1/ 3 and E = -2. Therefore, the value V of the game is -2.



Tutorial Questions

1. a)Explain the terms i)Rectangular games ii) type of Strategies

   b) Solve the following game graphically where pay off matrix for player A has been prepared

1	5	-7	4	2
2	4	9	-3	1

2. a)Explain the terms
   i) Maxmin criteria and Minimax criteria ii) Strategies: Pure and mixed strategies.

   b) Solve the following game graphically

	Player B		
Player A	B_1	B_2	B_3
A_1	1	3	11
A_2	8	5	2

3. a)What are characteristics of a game?
   b Reduce the following Game by dominance and the fit the game value

PlayerA					
		I	II	III	IV
	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

4. a) Obtain the optimal strategies for both pensions and the value of the game for two persons zero sum game whose payoff matrix is as follows.

		player-B	
		B1	B2
Player-A	A1	1	-3
	A2	3	5
	A3	-1	6
	A4	4	1
	A5	2	2
	A6	-5	0

   b) Explain pay of matrix and types of strategy in game theory?

Assignment Questions

1 a)What are characteristics of a game?

b) Reduce the following Game by dominance and the fid the game value

PlayerA					
		I	II	III	IV
	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

2. a)Explain the terms i)Rectangular games ii) type of Strategies

b) Solve the following game graphically where pay off matrix for player A has been prepared

1	5	-7	4	2
2	4	9	-3	1