chp-3

* Replacement of items that Deteriorate with time when many value is not counted & counted.

1)

year	1	2	3	4	5
Resale value	84,000	60,000	40,800	28,000	19,300
maintainance cost	8,000	8,540	9,760	11,400	13,600
cost of labour spares	28,000	32,000	36,000	42,000	50,000

A m/c cost is RS 1,20,000 ?

sol

Replacement of end of year (n)	maintainance cost (m)	cost of labour spares (l)	Running cost = (m+l)	Cummlative Running cost (C.R.C)	Capital cost (C)	Resale value (or) scrap value (S)	Depreciation cost = C−S	Total cost T.C = CRC + DC	Avg. annual cost T.C/n
1	8000	28,000	36,000	36,000	1,20,000	84,000	36000	72000	72000
2	8540	32000	40,540	76,540	1,20,000	60,000	60,000	136540	68270
3	9760	36,000	45760	122300	1,20,000	40,800	79200	201500	67166.6
4	11400	42,000	53400	175700	1,20,000	28,000	92000	267700	66925
5	13600	50,000	63600	239300	1,20,000	19,300	100700	340000	68000

At the end of 4th year the machine was replaced.

\#

2) A taxi owner estimates from his past records that the cost per year for opperating a taxi whose purchase price is 60,000 is are given below:

Age in years	1	2	3	4	5
operating cost	10,000	12,000	15,000	18,000	20,000

After 5 years the operating cost 60,000 k where k = 6, 7, 8, 9, 10 that is age in years. If the resale value decreases by 10% of the purchase price each year what is the optimum replacement policy when the cost of money is zero.

Sol operating cost = 6000 k

k = 6, 7, 8, 9, 10 years

operating cost 6^{th} year = 6000 × 6 = 36,000

" " 7^{th} " = 6000 × 7 = 42,000

" " 8^{th} " = 6000 × 8 = 48,000

" " 9^{th} " = 6000 × 9 = 54,000

" " 10^{th} " = 6000 × 10 = 60,000

Replacement of the End of year (n)	Running cost (R.C)	Cummulative Running cost (C.R.C)	Capital cost (C)	Resalve value (or) scrap value (S)	Depreciation cost (D.C) = C − S	Total cost (T.C) = CRC + DC	AVG annual cost (T.C ÷ n)
1	10,000	10,000	60000	54,000	6000	16,000	16000
2	12,000	22,000	60000	48,000	12,000	34,000	17000
3	15,000	37,000	60000	42,000	18,000	55,000	18333.33
4	18,000	55,000	60000	36,000	24,000	79,000	19750
5	20,000	75,000	60000	30,000	30,000	105000	21000
6	36,000	111000	60000	24,000	36,000	147,000	245000
7	42,000	153000	60000	18,000	42000	1,95,000	27857.1
8	48,000	201000	60000	12,000	48000	249000	31125
9	54,000	255000	60000	6,000	54000	309000	34333.3
10	60,000	315000	60000	0	60000	375000	37500

At the end of the first year the m/c was replaced.

3) a) m/c A cost Rs 9000 Annual operating cost are Rs 200 for the first year & then increase by Rs 2000 every year. Determine the best age at which we replace the machine.

If the optimum replacement policy followed what will be the avg yearly cost of owning & operating the m/c.

b) m/c B cost Rs, 10000 Annual operating cost are Rs 400 for the first year & then increase by Rs 800 every year.

You know have a m/c of type A which is one year old. should you replaced with B. If so when?

Sol a) For machine A :-

$C = 9000$ Rs : Rc_1 = 1st year = 200

every year increase by 2000 Rs

$Rc_2 = 200 + 2000 = 2200$ → 2nd year.

$Rc_3 = 2200 + 2000 = 4200$ → 3rd year

$Rc_4 = 4200 + 2000 = 6200$ → 4th year

$Rc_5 = 6200 + 2000 = 8200$ → 5th year

we take only 5 years.

Replacement of end of year	Running cost	Cumm Running cost	capital cost	Dec cost = C − S	Total cost = CRC + DC	Avg. Annual cost
1	200	200	9000	9000	9200	9200
2	2200	2400	9000	9000	11400	5700
3	4200	6600	9000	9000	15600	5200
4	6200	12800	9000	9000	21800	5450
5	8200	21,000	9000	9000	30,000	6000

Lowest m/c A for Avg. annual cost = 5200 Rs

b) For m/c B :-

C = 10,000 Rs ; R_{c1} → 1^{st} year = 400

Every year increased by 800 Rs

R_{c2} → 2^{nd} year = 1200

R_{c3} → 3^{rd} year = 2000

R_{c4} → 4^{th} year = 2800

R_{c5} → 5^{th} year = 3600

R_{c6} → 6^{th} year = 4400

we take 6 years.

n	R.C	C.R.C	C	D.C = C−S	T.C = CRC + D.C	Avg An cost
1	400	400	10,000	10,000	10,400	10,400
2	1200	1600	10,000	10,000	11,600	5,800
3	2000	3600	10,000	10,000	13,600	4533
4	2800	6400	10,000	10,000	16,400	4100
5	3600	10,000	10,000	10,000	20000	4000
6	4400	14,400	10,000	10 000	24,400	4066.7

Lowes for m/c B Avg. Annual cost = 4000 Rs.

when we can replace m/c 'A' by M/c 'B'

Lowest Avg. annual cost for A = 5200

Lowest Avg. Annual cost for B = 4000

Lowest Avg. annual cost for m/c A > Lowest Avg. Annual cost for m/c B

Note :- only this type of problems we satisfy this equation.

Replacement at which year :-

→ Compare the differential total cost value for m/c A with lowest Avg Annual cost for m/c B.

→ If differential total cost exceeds lowest Avg Annual cost replace at ~~at~~ that end of the previous year.

machine - A

S.NO	Total cost for current year (a)	Total cost for previous year (b)	Differential T.C = a-b
1	9200	-	4000
2	11400	9200	2200 < 4000
3	15600	11400	4200 > 4000
4	21800	15600	6200 > 4000
5	30,000	21,800	8200 > 4000

Differential total cost values for m/c A exceeds lowest Avg. Annual cost for m/c B from 3rd year onwards. so replace the m/c A by m/c B at the end of 2nd year itself.

* Present worth factor :

1) The initial cost of an item is 15,000 Rs & maintain -nace or running cost for d/f years are given below.

year	1	2	3	4	5	6	7
running	2500	3000	4000	5000	6500	8000	10,000

what is the replacement policy to be adopted if the capital cost is work 10% & their is no scrap value (salvage & Resale value).

Sol/

year (n)	Running cost	Present worth factor	Present value of Running cost	Comm R.V.R.C	Depression cost = C-S	Total cost = D.C + CPVRC	Comm present worth factor	Avg. Annual cost
1	2500	1	2500	2500	15000	17500	1	17500
2	3000	0.909	2727	5227	15000	20227	1.909	10595
3	4000	0.826	3305	8532	15000	23532	2.735	8603
4	5000	0.7513	3755	12286	15000	27287	3.486	7826
5	6500	0.6830	4439	16725	15000	31727	4.169	7609
6	8000	0.620	4960	21685	15000	36687	4.789	7660
7	10,000	0.564	5640	27325	15000	42327	5.353	7906

at the end of 5th year the m/c was replaced.

2) The cost of a new m/c is 5000 the running cost of the n^{th} year is given by $R_n = 500(n-1)$ where $n = 1, 2, 3, ----$ suppose money is worth 5% per year. After how many years will it be economical to replace the m/c with a new one.

Sol

year (n)	Running cost	Present worth factor	Present value of Running cost	Cummlative value of Running cost	Depression cost = C−S	Total cost = CPRC+DC	cummulative Present worth factor	weighted Avg annual cost
1	0	1	0	0	5000	5000	1	5000
2	500	0.952	475	475	5000	5475	1.952	2804.8
3	1000	0.906	902	1377	5000	6377	2.854	2234.4
4	1500	0.863	1285.5	2662.5	5000	7662.5	3.71	2064.8
5	2000	0.822	1629	4291.5	5000	9291.5	4.525	2053.1
6	2500	0.783	1932.5	6224	5000	11224	5.298	2118.5
7	3000	0.745	–	–	5000	–	–	–
8	3500	0.710	–	–	5000	–	–	–

$$P.W.F = \frac{1}{1+r} = \frac{1}{1+\frac{5}{100}} = \frac{1}{\frac{100+5}{100}} = \frac{100}{105} = 0.952$$

At the end of 5th year the m/c w[illegible] replaced.

3) Let the value of money assume to the 10% per year & suppose m/c A is replaced after every 3 years where as m/c B is replaced every 6 years the yearly cost of both m/c's is as given below.

year	1	2	3	4	5	6
m/c A	1000	200	400	1000	200	400
m/c B	1700	100	200	300	400	500

Determine which m/c should be purchased.

Sol: Given data:-

interest rate = 10%.

present worth factor, $= \frac{1}{1+\cancel{r}} = \frac{1}{1+10\%}$

$= \frac{1}{1+\frac{10}{100}} \Rightarrow \frac{1}{\frac{100+10}{100}} = \frac{100}{110} = 0.909.$

for m/c A :-

Total cost after discount $= (1000 \times 1) + (200 \times \frac{10}{11}) + \left[400 \times \left(\frac{10}{11}\right)^2\right] + \left(1000 \times \left\{\frac{10}{11}\right\}^3\right) + \left[200 \times \left(\left(\frac{10}{11}\right)^4\right)\right] + \left[400 \times \left(\frac{10}{11}\right)^5\right] = 2648.1$ RS

for m/c B :-

Total cost after discount $= (1700 \times 1) + \left[(100 \times \left(\frac{10}{11}\right)\right]$ $\left[200 \times \left(\frac{10}{11}\right)^2\right] + \left[300 \times \left(\frac{10}{4}\right)^3\right] + \left[400 \times \left(\frac{10}{11}\right)^4\right] + \left[500 \times \left(\frac{10}{11}\right)^5\right] = 2764.7$ RS.

∴ we take the m/c A.

* Replacement of items which falls suddenly & complete break down :-

a) Individual replacement b) Group replacement :-

1) The following failure rate have been observed for a certain type of light bulbs.

week	1	2	3	4	5
% failed by the end of week	10	25	50	80	100

They are 1000 bulbs is used & its cost RS 2 to replace an individual bulb which has burnt out. If all bulbs replaced simulataneously it would cost 50 ps/bulb. It is proposed to replace all the bulbs at fixed interval of time wheather they have burnt out or not. And to continue replacing burnt out bulbs as and when the fail. At what interval should bulbs as and when the fail. At what interval should all the bulbs be replaced. At what ~~interval should~~ all the bulbs be replaced. At what replacement price per bulb would ~~all the~~ group replacement of strictly individual replacement become a policy of strictly individual replacement preparable to be adopted policy.

Sol Individual replacement :-

Let P_i be the probability of failure in the i^{th} week.

P_1 = Probability of failure in first week.

$$P_1 = \frac{10}{100} = 0.1$$

P_2 = probability of failure in second week

$$P_2 = \frac{25-10}{100} = \frac{15}{100} = 0.15.$$

P_3 = probability of failure in the third week

$$P_3 = \frac{50-25}{100} = 0.25$$

P_4 = probability of failure in the fourth week.

$$P_4 = \frac{80-50}{100} = 0.30$$

P_5 = probability of failure in the fifth week.

$$P_5 = \frac{100-80}{100} = 0.20$$

$$\sum_{i=1}^{5} P_i = P_1 + P_2 + P_3 + P_4 + P_5$$

$$P_1 = 0.1 + 0.15 + 0.25 + 0.30 + 0.20 = 1.$$

$$\sum_{i=1}^{5} P_i = 1$$

expected life of bulbs $= \sum_{i=1}^{5} i\, P_i$

$= 1 \times P_1 + 2 \times P_2 + 3 \times P_3 + 4 \times P_4 + 5 \times P_5$

$= (1 \times 0.1) + (2 \times 0.15) + (3 \times 0.25) + (4 \times 0.30) + (5 \times 0.20)$

$= 3.35$ weeks.

Avg. no. of bulbs per week $= \dfrac{\text{Total no. of bulbs}}{\text{Expected life bulbs}}$

$$= \frac{1000}{3.35} = 298.6 \simeq 299 \text{ bulbs.}$$

Total cost in individual replacement

$= 299 \times 2 = 598$ Rs.

Group replacement :-

let N_i be the no. of replacement in i^{th} week.

N_0 be the total no. of bulbs at the begining $= 1000$

N_1 = no. of replacement at the end of 1^{st} week

$= N_0 P_1 = 1000 \times 0.1 = 100$ bulbs.

2)

N_2 = No. of replacements at the end of 2nd week.

$= N_0 P_2 + N_1 P_1 = (1000 \times 0.15) + (100 \times 0.1) = 160$ bulbs

N_3 = No. of replacements at the end of 3rd week

$= N_0 P_3 + N_1 P_2 + N_2 P_1$

$= (1000 \times 0.25) + (100 \times 0.15) + (160 \times 0.1) = 281$ bulbs

N_4 = No. of replacements at the end of 4th week

$= N_0 P_4 + N_1 P_3 + N_2 P_2 + N_3 P_1$

$= (1000 \times 0.3) + (100 \times 0.25) + (160 \times 0.15) + (281 \times 0.1)$

$= 377.1$ bulbs.

Sol

N_5 = No. of replacements at the end of 5th week

$= N_0 P_5 + N_1 P_4 + N_2 P_3 + N_3 P_2 + N_4 P_1$

$= (1000 \times 0.20) + (100 \times 0.3) + (160 \times 0.25) + (281 \times 0.15)$

$+ (377.1 \times 0.1) = 350$ bulbs.

End of the week	Total cost of group replacement	Avg. cost per week
1	$(100 \times 2) + (1000 \times 0.5) = 700$	$700 \div 1 = 700$
2	$(100 \times 2) + (160 \times 2) + (1000 \times 0.5) = 1020$	$1020 \div 2 = 510$
3	$(100 \times 2) + (160 \times 2) + (281 \times 2) + (1000 \times 0.5) = 1582$	$1582 \div 3 = 527.3$
4	$(100 \times 2) + (160 \times 2) + (281 \times 2) + (377.1 \times 2) + (1000 \times 0.5) = 2338$	$2338 \div 4 = 584.3$
5	$(100 \times 2) + (160 \times 2) + (281 \times 2) + (377.1 \times 2) + (350 \times 2) + (1000 \times 0.5) = 3038$	$3038 \div 5 = 607.5$

At the end of 2nd week we need to go for group replacement ##

UNIT - III (1)

REPLACEMENT

Introduction:-

The study of replacement is concerned with situations that arise when some items such as machines, men, electric-light bulbs, etc., need replacement due to their deteriorating efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. For example, an electric light bulb fails all of a sudden, pipeline get blocked, parts of machines become faulty. These are some situations that need most economic replacement policy for replacing faulty units or to take some remedial to restore efficiency.

Suppose an item goes on performing and with decreasing efficiency, then it requires more money to be spent in order to increase the operating cost, repair cost and so on. In such a situation the replacement of an old item with new one is the only alternative. Thus, the problem of replacement is to decide the best policy to determine an age at which the replacement is most economical instead of continuous increase in cost.

The need for replacement arises in:

(i) We may decide either to wait for comp failure of the item or to replace earlier due to high expense.

(ii) The expensive items may be considered individually to decide whether we should replace now or when it should be next replaced.

(iii) Whether the replaced item is of same type or of different type of the item.

The main objective of replacement is to direct the organisation for maximising its profit.

Replacement problems can be classified as:

(i) When the equipment deteriorates with time due to constant use and needs increased operating and maintenance cost.

(ii) When equipments such as light bulbs, tubes, radio and television parts, and so on, do not give any indication of deterioration with time, but fail completely all of a sudden.

(iii) Existing working staff in an organisation reduces gradually due to retirement, death and so on.

Failure Mechanism of Items:-

(i) Gradual Failure:- It is progressive in nature, that is, the lifetime increases, but its efficiency deteriorates causing
(a) increased maintenance & operating costs.
(b) decreased productivity
(c) decrease in the value of the equipment, that is, resale or salvage value.

(ii) Sudden Failure:- This type of failure occurs after some period of service rather than deterioration distribution which may be progressive, retrogressive or random in nature.

(a) Progressive Failure:- If the probability of failure increases with the increase in its life, then the failure is said to be progressive.
Eg:- Electric light bulbs, automobile tubes.

(b) Retrogressive failure:- If the probability of failure in the beginning of the life of an item is more and due to change of time, the chances of failure decreases, then the failure is said to be retrogressive.

(c) Random Failure:- The constant probability of failure is associated with items that fail from random causes like physical shocks not related to age. In such a case, virtually all items fail before ageing has any effect.

Replacement of Items that deteriorate with time

Generally, the cost of maintenance and repair of certain items increases with time. When years go by, these costs become so high that it is more economical to replace the item by a new one.

Value of Money Does not change with time:-

The cost of maintenance of a machine is given as a function increasing with time and its scrap value is constant.

(a) If a time is continuous variable, then the average annual cost will be minimised by replacing the machine when the average cost to date becomes equal to the current maintenance cost.

(b) If time is a discrete variable, then the average annual cost will be minimised by replacing the machine when the next period's maintenance cost becomes greater than the current average cost.

Proof:- Let

C - Capital cost of the item

S - Scrap value of the item.

n - number of years that the equipment would be in use.

f(t) - Maintenance cost function

A(n) - Average total annual cost.

(a) When t is a continuous variable: If the item is used for n years, then the total cost used during this period is:

Total cost = Capital cost − Scrap value + Maintenance cost

$$= C - S + \int_0^n f(t)\,dt.$$

Average annual total cost is

$$A(n) = \frac{\text{Total cost}}{n} = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt.$$

Now, we find such time n for which A(n) is minimum. therefore, differentiating A(n) w.r.t 'n'

$$\frac{dA(n)}{dn} = \frac{1}{n} f(n) - \frac{1}{n^2}\int_0^n f(t)\,dt - \frac{C-S}{n^2}$$

$= 0$ for minimum of $A(n)$.

$$\Rightarrow f(n) = \frac{C-S}{n} + \frac{1}{n}\int_0^n f(t)\,dt.$$

$$= A(n)$$

$$\therefore \frac{d^2}{dn^2}[A(n)] \geq 0 \text{ at } f(n) = A(n).$$

(b) When time t is a discrete variable: Since the item time t is taken as discrete, it can take the values 1, 2, 3, ...

Then,

$$A(n) = \frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)$$

By using finite differences $A(n)$ will be minimum if the relationship is satisfied:

$$A(n+1) - A(n) \geq 0 \text{ and } A(n) - A(n-1) \leq 0.$$

$$A(n+1) - A(n) = \left[\frac{C-S}{n+1} + \frac{1}{n+1}\sum_{t=1}^{n+1} f(t)\right] - \left[\frac{C-S}{n} + \frac{1}{n}\sum_{t=1}^{n} f(t)\right].$$

$$= \frac{1}{n+1}\left[C-S+\sum_{t=1}^{n} f(t) + f(n+1)\right] - \frac{1}{n}\left[C-S+\sum_{t=1}^{n} f(t)\right]$$

$$= \frac{f(n+1)}{n+1} + \sum_{t=1}^{n} f(t)\left[\frac{1}{n+1} - \frac{1}{n}\right] + (C-S)\left[\frac{1}{n+1} - \frac{1}{n}\right]$$

$$= \frac{f(n+1)}{n+1} - \frac{1}{n(n+1)}\left[\sum_{t=1}^{n} f(t) + (C-S)\right].$$

Since $A(n+1) - A(n) \geq 0$,

$$\frac{f(n+1)}{n+1} \geq \frac{1}{n(n+1)}\left[\sum_{t=1}^{n} f(t) + (C-S)\right]$$

$$\Rightarrow f(n+1) \geq A(n).$$

Similarly, $A(n) - A(n-1) \leq 0 \Rightarrow f(n) \leq A(n-1)$.

∴ Replace the machine at the end of n years when the maintenance cost in the $(n+1)^{th}$ year is more than the average total cost in the n^{th} year & the n^{th} year maintenance cost is less than the previous year's average total cost.

Problem:- The cost of a machine is 6,100/- and its scrap value is only 100/-. The maintenance costs are found from experience to be:

Year	1	2	3	4	5	6	7	8
Maintenance cost in Rs.	100	250	400	600	900	1250	1600	2000

When shoud machine be replaced?

Sol:-

Total cost in a year = Capital cost − scrap value + Maintenance cost.

Replacement at the end of year (n).	Maintenance cost $f(n)$.	Total Maintenance cost $\Sigma f(n)$.	Difference b/w price value & scrap value $C-S$.	Total cost $\Sigma f(n) + C-S$	Average cost $\frac{\Sigma f(n)}{n} + \frac{C-S}{n}$
1	100	100	6000	6100	6100
2	250	350	6000	6350	3175
3	400	750	6000	6750	2250
4	600	1350	6000	7350	1837
5	900	2250	6000	8250	1650
6	1250	3500	6000	9500	1583
7	1600	5100	6000	11,100	1586
8	2000	7700	6000	13,100	1638

Here, it may be observed that the average cost per year is minimum in the 6th year and the maintenance cost in the 7th year becomes greater than average cost for six years. So, machine should be replaced at the end of the 6^{th} year.

Value of Money changes with constant rate during the period:- As money value changes with time, calculate the present value or present worth of the money to be spent in a few years. One rupee a year from ~~we~~ now is equivalent to $(1.1)^{-1}$ rupee at the interest rate of 10% per year. One rupee spent two years from now is equivalent to $(1.1)^{-2}$ today. Hence one rupee spent n years from now is equivalent to $(1.1)^{-n}$ today. The quantity $(1.1)^{-n}$ is called the present worth factor of one rupee spent in n years from now.

Generally, if r is the rate of interest per year, then $(1+r)^{-n}$ is called the present worth factor of one rupee spent in n years time from now. The expression $(1+r)^{n}$ is known as the payment compound amount factor of one rupee spent in n years time.

Discount rate or Depreciation value: The present worth factor of unit amount to be spent after one year is given by $v = (1+r)^{-1}$, where r is the interest rate. Then, v is called discount rate or depreciation value.

Problem:- Let the value of money be assumed to be 10% per year and suppose that machine A is replaced after every 3 years whereas machine B is replaced after every 6 years. The yearly costs of both the machines are given below:

Year	1	2	3	4	5	6
Machine A	1000	200	400	1000	200	400
Machine B	1700	100	200	300	400	500.

Determine which machine should be purchased.

Sol:- Since money carries the rate of interest, the present worth factor is:

$$v = (1+r)^{-1} = \left(1 + \frac{10}{100}\right)^{-1} = \left(\frac{11}{10}\right)^{-1} = \frac{10}{11}.$$

Total discount cost of A for 3 years is:

$$= Rs\left[1000 + 200 \times \frac{10}{11} + 400 \times \left(\frac{10}{11}\right)^2\right]$$

$= Rs.\ 1,512.$

The total discount cost of B for 6 years is:

$$= Rs\left[1700 + 100 \times \left(\frac{10}{11}\right) + 200 \times \left(\frac{10}{11}\right)^2 + 300\left(\frac{10}{11}\right)^3 + 400 \times \left(\frac{10}{11}\right)^4 + 500 \times \left(\frac{10}{11}\right)^5\right].$$

$= Rs.\ 2,765.$

Average yearly cost of $A = \frac{1513}{3} = Rs.\ 504$

Average yearly cost of $B = \frac{2765}{6} = Rs.\ 461.$

Average yearly

This shows that apparent advantage with B, the comparison is unfair because the periods of consideration are different. So, if we consider 6-year period for machine A, then the total discounted cost of A is

$$= 1000 + 200 \times \left(\frac{10}{11}\right) + 400 \times \left(\frac{10}{11}\right)^2 + 1000 \times \left(\frac{10}{11}\right)^3 + 200 \times \left(\frac{10}{11}\right)^4 + 400 \times \left(\frac{10}{11}\right)^5$$

$= \text{Rs. } 2,647.$

Hence, the average yearly cost of A is $\frac{2647}{6} = 441$ which is lesser than the average yearly cost of B. Hence, machine A should be purchased.

Replacement of Items that fail Completely and Suddenly :- It is not easy to predict when a particular equipment will fail and its time of failure. This difficulty can be overcome by determining the probability distribution of failures. Assume that the failures occur only at the end of the period, t. The objective is to find the value of t for which the total cost after replacement of an equipment is minimum. Replacement policies are:

1) Individual Replacement policy

2) Group Replacement Policy.

Individual Replacement Policy:- Under this policy, an item is replaced immediately on its failure.

Mortality Theorem:- A large population is subject to a given mortality law for a very long period of time. All deaths are immediately replaced by births and there are no other entries or exits. Then, the age distribution ultimately becomes stable and that the number of deaths per unit time becomes constant.

Proof:- Assume that death occurs just before the age of $(k+1)$ years, where k is an integer. That is, the lifespan of any item is between $t=0$ to $t=k$. Define,

$f(t)$ = number of births at time t.

$p(x)$ = probability of death just before the age $x+1$, that is, failure at the age x.

$$\Rightarrow \sum_{x=0}^{k} p(x) = 1.$$

Group Replacement Policy:- It is concerned with those items that either work or fail completely. It happens that a system contains a large number of items that are increasingly liable to failure with age. In this case it is advisable to replace all items irrespective of the fact that the items have failed or not failed, with a provision that if any item fails before optimal time, it may be individually replaced. Such a policy is called group replacement policy and is best when the value of any individual item is so small that the cost of keeping records of individual ages cannot be justied.

(a) Group replacement should be made at the end of the period if the cost of individual replacements for the period t is greater than the average cost per period through the end of period, t.

(b) Group replacement is not advisable at the end of period t if the cost of individual replacements at the end of period t-1 is less than the average cost per period through the end of ith period.

Problem:- Let p(t) be the probability that a machine in a group of 30 machines would breakdown in period t. The cost of repairing a breaken machine is Rs. 200. Preventive maintenance is performed by the serving team on all the 30 machines at the end T units of time. Preventive maintenance cost is Rs. 15 per machine. Find optimal T which will minimise the expected total cost per period of serviing, given

$$p(t) = \begin{cases} 0.03 & \text{for } t=1 \\ p(t-1)+0.01 & \text{for } t=2,3,\dots,10 \\ 0.13 & \text{for } t=11,12,\dots \end{cases}$$

Sol:-

t	1	2	3	4	5	6	7	8	9	10	11	12
$p(t)$	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	0.11	0.12	0.13	0

as $p(1)+p(2)+\dots p(11) = 0.88 < 1.$

if we add $p(12) = 0.13$,

then $p(1)+\dots+p(12) = 1.01 > 1$, where the sum of all probabilities can never be greater than 1, so consider $p_{12}=0$, $p_{13}=0$, and so on.

This means that a machine which has already lasted up to the 11th period is sure to fail in the 12th period. Let N_i be the number of machines at the end of ith period. Then,

$$N_0 = 30$$

$$N_1 = N_0 p_1 = 30 \times 0.03 = 0.9 \approx 1$$

$$N_2 = N_0 p_2 + N_1 p_1 = (30 \times 0.04) + (1 \times 0.03) \approx 1.23 \approx 1$$

$$N_3 = N_0 p_3 + N_1 p_2 + N_2 p_1 = 30 \times 0.05 + 1 \times 0.04 + 1 \times 0.03 \approx 2$$

similarly, $N_4 = 2$, $N_5 = 2$, $N_6 = 3$, $N_7 = 3$, $N_8 = 4$, $N_9 = 4$, $N_{10} = 5$, $N_{11} = 6$.

As the expected life of each machine;

$\sum_{i=1}^{11} i p_i = 6.41$ time units, the average number of machines failed per period is $\frac{30}{6.41} \simeq 5$.

Hence, cost of individual replacement =

Rs. 5 × 200 = Rs. 1,000.

Group maintenance cost is:

End of period	Cost of group maintenance (in Rs)	Average cost of group maintenance per period (in Rs)
1	(30×15)+(1×200) = 650	650
2	(30×15)+(2×200) = 850	~~420~~ 425
3	(30×15)+(4×200) = 1250	4~~17~~
4	(30×15)+(6×200) = 1650	412
5	(30×15)+(8×200) = 2050	$\boxed{410}$
6	(30×15)+(11×200) = 2650	442

Since the minimum cost occurs in the 5th period it is optimal to maintain all the machines upto the 5th period.

UNIT - IV

Theory of Games :-

Game theory is a type of decision theory in which one's choice of action is determined after taking into account all possible alternatives available to an opponent playing the same game, rather than just by the possibilities of several outcomes.

The mathematical analysis of competitive problems is fundamentally based upon the 'minimax (maximin) criterion' of J. Von Neumann. This criterion implies the assumption of rationality from which it is argued that each player will act so as to maximize his minimum gain or minimize his maximum loss.

Game is defined as an activity between two or more persons involving activities by each person according to a set of rules, at the end of which each person receives some benifit or satisfaction or suffers loss.

Characteristics of Game Theory :-

(1) Chance of Strategy :- If in a game, activities are determined by skill, it is said to be a game of strategy, if they are determined by chance, it is a game of chance.

(2) Number of Persons :- A game is called an n-person game if the number of persons playing is n.

The person means an individual or a group aiming at a particular objective.

(3) Number of activities:- These may be finite or infinite.

(4) Number of alternatives available to each person:- It may also be finite or infinite. A finite game has a finite number of activities, each involving a finite number of alternatives, otherwise the game is

(5) Information to the players about the past activities of other players:- is completely available, partly available or not available at all.

(6) Payoff:- A quantitative measure of satisfaction a person gets at the end of each play is called a payoff. It is a real valued function of variables in the game. Let v_i be the payoff to the player P_i, $1 \leq i \leq n$, in an n-person game. If $\sum_{i=1}^{n} v_i = 0$, then the game is said to be a zero-sum game.

Basic Terminologies:-

Game:- A competitive situation is called as a game if it has following properties:

(i) There are finite number of participants called players.

(ii) Each player has finite number of strategies available to him.

(iii) Every game results in an outcome.

a group **Number of Players**:- If a game involves only two players, then it is called a two-person game. However, if the number of players are more than two, the game is known as n-person game.

Sum of gains and losses:- If in a game the gains of one player are exactly the losses to another player, such that sum of gains and losses equal to zero, then the game is said to be a zero-sum game. Otherwise it is said to be non-zero sum game.

Strategy:- The strategy for a player is the list of all possible actions that he will take for every pay off the might arise. It is assumed that the rules governing the choices are known in advance to the players. The outcome resulting from a particular choice is also known to the players in advance and is expressed in terms of numerical values. Here, it is not necessary that players have a definite information about each other strategies.

Optimal strategy:- The particular strategy by which a player optimises his gains or losses without knowing the competitor's strategies is called optimal strategy.

Value of the game:- The expected outcome pay. play when players follow their optimal strategies is called the value of the game.

Pure strategy:- It is a decision rule which is always used by the player to select the particular course of action. Thus, each player knows in advance of all the strategies out of which he always selects only one particular strategy, irrespective of the strategy others may choose and the objective of the players is to maximise gains or minimise losses.

Mixed strategy:- When both players are guessing as to which course of action is to be selected on a particular occasion with some fixed probability, it is a mixed strategic game. Thus, there is a probabilistic situation and objective of the players is to maximise expected gains or to minimise expected losses by making a solution among pure strategies with fixed probabilities.

Two person zero sum game:- A game with only two persons is said to be two-person zero-sum game if the gain of one player is equal to the loss of the other.

Pay off matrix:- The pay offs in terms of gains or losses, when players selected their particular ~~start~~ strategies, can be represented in the form of a matrix, called the pay off matrix.

$$\begin{array}{cc} & \text{Player B's strategies} \\ & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{array}{c} \text{Player A's strategies} \end{array} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Games with saddle point:-

Minimax and Maximin Principle:- Consider the pay off matrix of a game which represents pay off of player A. Now, the objective of the study is to know how these players must elect their respective strategies so that they may optimise their pay off. Such a decision-making criterion is referred to as the minimax-maximin principle.

For player A minimum value in each row represents the least gain (pay off) to him if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives largest gain among the row minimum values. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game denoted by $\underline{v}$.

For player B (loser), the maximum value in each column represents the maximum loss to him if he chooses his particular strategy. These are written in the pay off matrix by column minima. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game denoted by $\bar{v}$.

Saddle point:- A saddle point of a pay off matrix is that position in the pay off matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game:- The pay off of the saddle point is called the value of the game denoted by v.

Fair game:- A game is said to be fair if $\underline{v} = 0 = \bar{v}$.

Strictly determinable game :- A game is said to be strictly determinable if $\underline{v} = v = \bar{v}$.

Procedure to determine saddle point :-

1) Select the minimum ~~va~~ element in each row and enclose it in a rectangle. ($\square$)
2) Select the maximum element in each column and enclose it in a circle (o).
3) Find out the element which is enclosed by the rectangle as well as the circle. Such element is the value of the game and that position is called as the saddle point.

Problem:- Solve the game whose payoff matrix is given by

Player B

		I	II	III	IV	V
Player A	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6
	IV	5	3	-4	2	-6

Sol:- Select the row minimum and enclose it in a rectangle. Then, select the column maximum and enclose it in a circle.

Player B.

	I	II	III	IV	V
I	$\boxed{-2}$	0	0	5	3
II	3	2	$\boxed{1}$	2	2
III	$\boxed{-4}$	-3	0	-2	6
IV	5	3	-4	2	$\boxed{-6}$

Player A

It is clear that saddle point is (II, III) and the value of game $v=1$.

Player A uses his course of action II throughout.

Player B uses his course of action III throughout.

Problem:- Find the range of values of p & q which will render the entry (2,2) a saddle point for the game.

Player B.

2	4	5
10	7	q
4	p	6

Player A

Sol:- First ignoring the values of p & q determine the maximin and minimax values of the pay off matrix.

Player B.

	B_1	B_2	B_3
A_1	$\boxed{2}$	4	5
A_2	10	$\boxed{7}$	q
A_3	$\boxed{4}$	p	6

Maximin value $\underline{v} = 7 =$ minimax value.

This imposes the condition on p as $p \leq 7$ and on q as $q \geq 7$. Hence, the range of p and q will be $p \leq 7$, $q \leq 7$.

Games without Saddle Point Mixed Strategies:-

There are some games for which no saddle point exists. In such cases both the players must determine an optimal combination of strategies to find a saddle point. The optimal strategy combination for each player may be determined by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combination of available choices of strategy.

The value of game obtained by the use of mixed strategies represents least pay off which player A can expect to win and the least which player B can lose. The expected pay off to a player in a game with arbitrary pay off matrix $[a_{ij}]$ of order $m \times n$ is defined as

$$E(P, Q) = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

$$= P^T A Q.$$

where, $P = (p_1, p_2, \ldots, p_m)$ and $Q = (q_1, \ldots, q_n)$ denotes the mixed strategies for players A & B. Also, $p_1 + p_2 + \cdots + p_m = 1$ and $q_1 + q_2 + \cdots + q_n = 1$.

A particular strategy with particular probability a player chooses can also be interpreted as the relative frequency with which a strategy is chosen from the number of strategies of the game.

A mixed strategy game can be solved by different solution methods such as

(1) Algebraic method.

(2) Analytical or calculus method.

(3) matrix method.

(4) Graphical method

5. Linear programming method.

Dominance Property :- Some times reduce the size of a game's pay off matrix by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. The concept of dominance is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

General Rule :-

1. If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, then k^{th} row is dominated by the r^{th} row. ~~then k^{th} row~~ is
2. If all the elements of a column, say k^{th} are greater than or equal to the corresponding elements of any other column, say r^{th}, then k^{th} column is dominated by r^{th} column.
3. Omit dominated rows or columns.
4. If some linear combination of some rows dominates i^{th} row, then i^{th} row will be deleted. Similar argument follows for columns.

Problem :- Reduce the size of the game whose matrix is given by:

Player B.

	I	II	III
I	-4	6	3
II	-3	-3	4
III	2	-3	4

Sol:-

	I	II	III
I	$\boxed{-4}$	6	3
II	$\boxed{-3}$	$\boxed{-3}$	4
III	2	$\boxed{-3}$	4

No saddle point exists. Consider Ist & IIIrd columns from the player B's point of view. observe that each pay off in the IIIrd column is greater than the corresponding element in the 1st column regardless of player A's strategy. Evidently, the choice of IIIrd strategy by the player B will always result in the greater loss compared to that of selecting the 1st strategy. Column III is inferior to I and is never to be used. Hence deleting the III^rd column which is dominated by I, the reduced size pay off matrix is: (7)

		B I	B II
A	I	-4	6
	II	-3	-3
	III	2	-3

Again, if the reduced matrix is looked at from player A's point of view, it is seen that the player A will never use the II^nd strategy which is dominated by III. Hence, the size of the matrix can be reduced further by deleting the II^nd row. Hence, the reduced matrix is

		B I	B II
A	I	-4	6
	III	2	-3

Graphical Method (for $2\times n$ or $m\times 2$ games):-

The graphical method is useful for the game where the pay off matrix is of the size $2\times n$ or $m\times 2$. That is, the game with mixed strategies that has only two pure strategies for one of the players in the two person zero-sum game. optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Consider the $2\times n$ pay off matrix of a game without a saddle point.

Player B.

$$\text{Player A} \quad \begin{array}{c} \\ A_1 \\ A_2 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \end{array}$$

Let the mixed strategy for player A be given by $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$, where $p_1+p_2=1$ and $p_1 \geq 0$, $p_2 \geq 0$.

Now, for each of the pure strategies available to B, expected pay off for player A would be as follows:

B's pure move	A's expected pay off $E(P)$
B_1	$E_1(P) = a_{11}P_1 + a_{21}P_2$
B_2	$E_2(P) = a_{12}P_1 + a_{22}P_2$
$\vdots$	$\vdots$
B_n.	$E_n(P) = a_{1n}P_1 + a_{2n}P_2$

The player B would like to choose that pure move B_j against S_A for which $E_j(P)$ is a minimum for $j = 1, \ldots, n$. Let us denote this minimum expected payoff for A by

$$v = \min\{E_j(p), j = 1, \ldots, n\}.$$

The objective of player A is select p_1 and hence p_2 in such a way that v is as large as possible. This may be done by the plotting straight lines.

$$E_j(P) = a_{1j}p_1 + a_{2j}p_2 = (a_{1j} - a_{2j})p_1 + a_{2j} \ (j = 1, 2, \ldots, n)$$

as linear function of p_1.

The highest point on the lower boundary of these lines will give maximum expected pay off among the minimum expected pay offs on the lower boundary and the optimum value of probability p_1 and p_2.

Now, the two strategies of player B corresponding to those lines which pass through the maximum point can be determined.

The (m×2) games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected pay off will give the maximum expected pay off (minimax value) and the optimum value of probability q_1 and q_2.

Problem:- Obtain the optimal strategies for both persons and the value of the game for zero-sum two-person game whose pay off matrix is as follows:

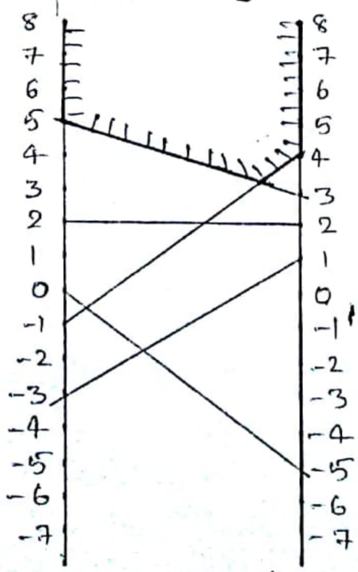
Player B

$$\text{Player A}\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$$

Sol:- Observe that the given problem does not possess any saddle point. So, let the player B play the mixed strategy $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ with $q_2 = 1 - q_1$ against player A. Then, B's expected pay offs against A's pure moves are given by:

A's pure move	B's expected pay off $E(q_1)$
A_1	$q_1 - 3q_2$
A_2	$3q_1 + 5q_2$
A_3	$-q_1 + 6q_2$
A_4	$4q_1 + q_2$
A_5	$2q_1 + 2q_2$
A_6	$-5q_1 + 0q_2$

The expected pay off equations are then plotted as functions of q_1 in the graph:



Since the player B wishes to minimise his maximum expected pay off, we consider the minimax point on the upper envelope of B's expected payoff equations. Hence, the given pay off matrix of the game is reduced to

Player B

$$\text{Player A}\quad \begin{array}{c} \\ A_1 \\ A_4 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Let $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & p_1 & 0 & p_2 \end{bmatrix}$ be the optimal strategy of player A.

Then, $p_1 = \dfrac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \dfrac{(1-4)}{(3+1)-(4+5)} = \dfrac{3}{5}$

Hence, $p_2 = 1 - p_1 = 1 - \frac{3}{5} = \frac{2}{5}$.

Let $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ be the optimal strategy of player B.

Then $q_1 = \dfrac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \dfrac{(1-5)}{(3+1)-(4+5)} = \dfrac{4}{5}$.

So, $q_2 = 1 - q_1 = 1 - \frac{4}{5} = \frac{1}{5}$

value of the game,

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$= \frac{(3\times1)-(4\times5)}{(3+1)-(4+5)} = \frac{17}{5}$$