UNIT 13 THICK AND THIN CYCLINDERS

Structure

13.1 INTRODUCTION

This unit presents the analysis of thin and thick cylindrical shells subjected to fluid pressure. Steam boilers, reservoirs, reactors, nuclear containers tanks, working chambers of engines, etc. are the common examples. In this unit, stresses and strains induced in the walls of the cylinder will be found out based on the geometry of the shell and equilibrium of the forces involved. We shall begin by defining a thin cylinder identifying the assumptions made in the analysis. After finding the stresses in the material of the cylinder, strains will be calculated. Stresses in wire bound pipes will then be considered.

We shall then see the limitations for treating a shell to be thin and look for the differences in the behaviour of a thick shell as against a thin shell. The stresses in a thick cylinder will be obtained based on a standard method involving certain assumptions.

Objectives

After studying this unit, you should be able to

- define a cylindrical shell and distinguish between thin and thick cylinders,
- identify the assumptions involved in the analysis of a thin cylinder,
- determine the stresses in a thin cylinder,
- find the strains and deformation in thin cylinder,
- find stresses in a wire bound pipe,
- makeout the assumptions for analysing a thick cylinder,
- derive the standard expressions for strresses in thick shell, and
- find the stress distribution across a compound cylinder.

13.2 THIN CYLINDERS

In this section, we shall derive expressions for the stresses and strains in thin cylinders and use them for working out related problems.

13.2.1 Assumptions

The following assumptions are made in order to derive the expressions for the stresses and strains in thin cylinders :

(i) The diameter of the cylinder is more than 20 times the thickness of the shell.

(ii) The stresses are uniformly distributed through the thickness of the wall.

(iii) The ends of the cylindrical shell are not supported from sides.

(iv) The weight of the cylinder and that of the fluid contained inside are not taken into account.

(iv) The atmoshpheric pressure is taken as the reference pressure.

13.2.2 Stresses

Consider a thin seamless cylindrical shell of nominal diameter d, and shell thickness which is containing some fluid at an internal pressure of p. The two ends of the cy linder are closed with walls perpendicular to the shell (Figure 13.1).

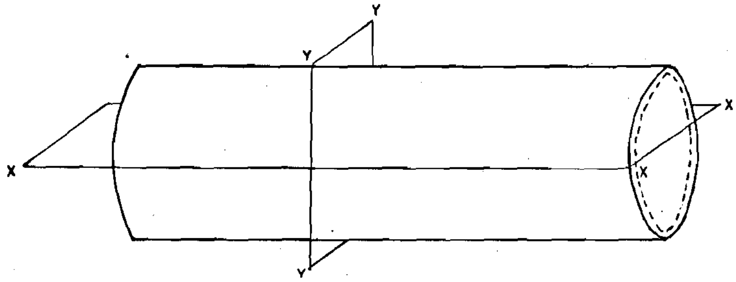

Figure 13.1

We shall consider a vertical plane YY which cuts the cylinder anywhere along the length. We shall consider the left portion of the cylinder and see the nature and magnitude of the internal stresses acting on the section. The stresses will be as shown in Figure 13.2.

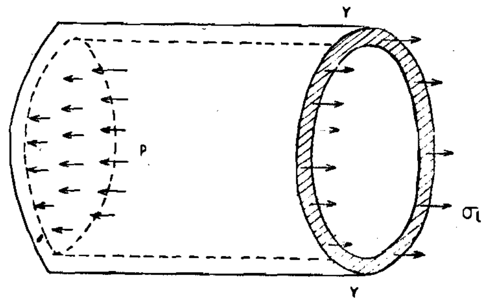

Figure 13.2

It can be seen that the internal stresses act over the shaded annular portion of the cylinder, which is the wall area exposed due to the cutting the by plane Y-Y. The direction of these internal stresses will be clearly longitudinal as the exposed area is in the vertical plane. In addition it can also be seen that these stresses develop owing to the unbalanced horizontal force acting on the left vertical wall of the cylinder, since the pressure acting on the curved walls balance each other. Thus the stresses will be tensile in nature so as to maintain equilibrium. The unbalanced force acting on the left wall is called the bursting force and the force due to internal stresses acting on the wall thickness of the cylinder is called the resisting force.

We can write the expressions for the bursting and resisting forces as below.

It can be clearly seen that the bursting force is caused due to the internal pressure acting on the vertical circular wall of nominal diameter d.

Hence, the bursting force $= p \times \frac{\pi d^2}{4}$

The resisting force is generated by the longitudinal tensile stress σ_l acting on the vertical area exposed, of thickness t and diameter d.

Hence, the resisting force $= \sigma_l \times \pi dt$

For equilibrium, the resisting force should be equal to the bursting force.

Thus, we get, $$\sigma_l \times \pi dt = p \times \frac{\pi d^2}{4}$$

or $$\sigma_l = \frac{pd}{4t}$$

This internal stress is called **longitudinal stress**, indicating the direction in which it is acting and its nature will be tensile.

We shall now consider a horizontal diametrical plane *XX* which cuts the cylindrical shell in two halves. The stresses have been shown in Figure 13.3.

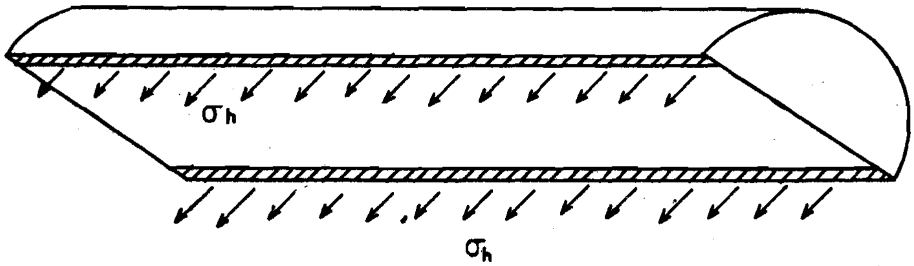

Figure 13.3

We shall consider the equilibrium of the top portion of the cylinder. The horizontal pressure acting on the two end walls will balance each other and hence, there will be no longitudinal stress in the wall of the cylinder. The pressure acting on the curved surface of the shell creates the bursting force for this free body diagram which should be balanced by the reacting force caused by the development of internal stresses along with wall thickness of the cylinder. Since the plane *XX* is horizontal, the cylinder's wall exposed by the cutting, will also be horizontal and it will be in the form of two rectangular strips, along with longitudinal direction of length l and thickness t. Thus, the stress acting on this strip will be in the vertical direction.

The pressure acting on the curved surface acts normal to the surface and hence, it will be acting in different direction at different points along the surface. The vertical component of the bursting force is obtained by considering an element at angle θ to the horizontal which is subtended by an angle $d\theta$ at the centre as shown in Figure 13.3. The length of this element may be the total length of the cylinder itself. The revolution of the elemental force in the vertical direction is shown in Figure 13.4.

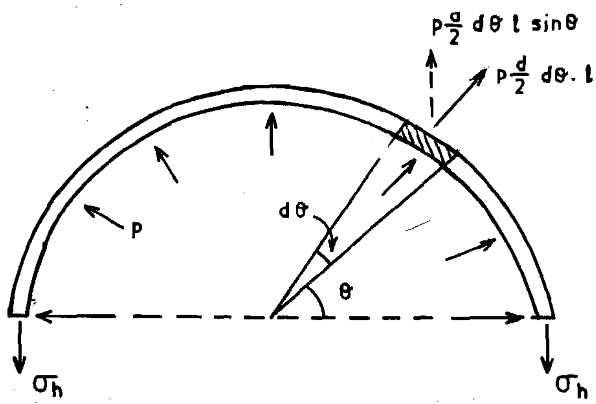

Figure 13.4

The radial force acting on the element,

$$p \times \frac{d}{2} \times d\theta \times l.$$

The vertical component of the elemental force,

$$p \times \frac{d}{2} \times d\theta \times l \sin \theta.$$

Hence, the total vertical component of the bursting force in the top portion of the cylindrical shell will be

$$P_v = \int_0^{\pi} p \times \frac{d}{2} \times l \sin \theta \; d\theta.$$

$$= p.d.l$$

Since this resultant vertical component is acting upwards, the internal stress on the horizontal strip of dimensions l and t will be acting downwards indicating that the nature of this stress is tensile.

$$\text{The resisting force} = \sigma_n \times l \times t \times 2.$$

Since the equilibrium is maintained by the action of bursting and resisting forces only, they must be equal.

$$\sigma_n \times l \times t \times 2 = p\,d\,l$$

$$\sigma_n = \frac{pd}{2t}$$

This stress is called the **hoop stress** acting in the circumferential direction and it will be tensile in nature.

It can be stated at this stage that the internal stresses in a thin cylindrical shell are acting in the longitudinal and circumferential directions and hence they are named as longitudinal stress and hoop stress and both are tensile in nature. Let us see some example for finding stresses in thin cylindrical shells.

Example 13.1

A cylindrical boiler is 2.5 m in diameter and 20 mm in thickness and it carries steam at a pressure of 1.0 N/mm^2. Find the stresses in the shell.

Solution

Diameter of the shell, d = 2.5 m = 2500 mm.

Thickness of the shell, t = 20 mm.

Internal pressure, p = 1.0 N/mm^2

$$\therefore \text{ Longitudinal stress } \frac{pd}{4t} = \frac{1.0 \times 2500}{4 \times 20} = 31.25 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$\therefore \text{ Hoop stress } \frac{pd}{2t} = \frac{1.0 \times 2500}{2 \times 20} = 62.50 \text{ N/mm}^2 \quad \text{(tensile)}$$

Example 13.2

A thin cylindrical vessel of 2 m diameter and 4 m length contains a particulr gas at a pressure of 1.65 N/mm^2. If the permissible tensile stress of the material of the shell is 150 N/mm^2, find the minimum thickness required.

Solution

In a thin cylindrical shell, stress will be higher, since it is double that of longitudinal stress. Hence, maximum stress is reached in the circumferential direction.

Diameter of the shell, d = 2 m = 2000 mm

Internal presssure, p = 1.65 N/mm^2

Permissible tensile stress = 150 N/mm^2

If thickness required is t, then

$$\frac{pd}{2t} = 150$$

$$\frac{1.65 \times 2000}{2 \times t} = 150$$

$$t = 11 \text{ mm}$$

Thus, minimum thickness required is 11 mm.

Example 13.3

A cylindrical compressed air drum is 2 m in diameter with plates 12.5 mm thick. The efficiencies of the longitudinal (η_l) and circumferential (η_c) joints are 85% and 45% respectively. If the tensile stress in the plating is to be limited to 100 MN/m^2, find the maximum safe air pressure.

Solution

The efficiency of the joint influences the stresses induced. For a seamless shell (with no joints), efficiency is 100%. When the efficiency of joint is less than 100%, the stressess are increased accordingly.

Hence, if η is the efficiency of a joint in the longitudinal direction, influencing the hoop stress, then the stress will be given as,

$$\sigma_n = \frac{pd}{4t \times \eta_l}$$

Here, the diameter $d = 2$ m = 2000 mm.

Thickness, $t = 12.5$ mm.

Limiting tensile stress = 100 MN/m^2 = 100 N/mm^2.

Considering the circumferential joint which influences the longitudinal stress,

$$\frac{pd}{4t \times \eta_l} = 100$$

$$\frac{p \times 2000}{4 \times 12.5 \times 0.45} = 100$$

$$p = 1.125 \text{ N/mm}^2$$

Similarly, considering the longitudinal joint which influences the hoop stress,

$$\frac{pd}{2t \times \eta_c} = 100$$

$$\frac{p \times 2000}{2 \times 12.5 \times 0.85} = 100$$

$$p = 1.063 \text{ N/mm}^2$$

Evidently, safe pressure is governed by hoop stress.

Hence, maximum safe air pressure = 1.063 N/mm^2

SAQ 1

A closed cylindrical vessel with plane ends is made of steel plates 3 mm thick, the internal dimensions being 600 mm and 250 mm for length and diameter respectively. Determine the stresses in the material of the vessel, when the internal pressure is 2.0 N/mm^2.

SAQ 2

A copper tube of 50 mm internal diameter, 1.2 m long and 1.25 mm thickness has closed ends and filled with a gas under pressure. If the safe tensile stress for copper is 60 N/mm^2, find maximum pressure that can be applied.

SAQ 3

A boiler of 2 m diameter is to be made from mild steel plates. Taking the efficiencies of longitudinal and circumferential joints as 75% and 50% respectively, find the thickness of plate required if the internal pressure, that will develop in the boiler is 1.5 N/mm^2 and the permissible tensile stress for mild steel is 150 N/mm^2.

13.2.3 Strains

At any point in a thin cylindrical shell with an internal pressure p, we have obtained the expressions for stresses along longitudinal direction and circumferential direction. In order to obtain the strain along any direction, we have to see the state of stress at any point. In the three mutually perpendicular directions, the stresses are as follows :

Stress along the radial direction = p (compressive)

Stresss along the circumferential direction = $\sigma_n = \frac{pd}{2t}$ (tensile)

Stress along the longitudinal direction = $\sigma_l = \frac{pd}{4t}$ (tensile)

The state of stress is shown in Figure 13.5.

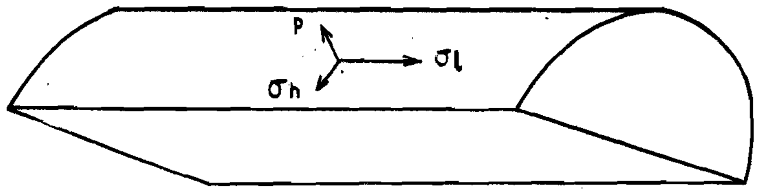

Figure 13.5

These are the principal stresses acting at the point considered. However, when $\frac{d}{t}$ is very large making the shell thin, the radial pressure p will be very small compared to the longitudinal and hoop stresses. Hence, this compressive stress can be neglected at any point for the purpose of working out the strain, which is going to be still smaller. This assumption leaves only the two tensile stresses at any point, mutually perpendicular to each other. If E is the Young's Modulus of the material of the shell and ν, its Poisson's ratio, then the expression for the strains in the two direction are obtained as follows :

$$\text{Longitudinal strain } \varepsilon_l = \frac{\sigma_l}{E} - \nu \frac{\sigma_n}{E}$$

$$= \frac{pd}{4tE} - \nu \frac{pd}{2tE}$$

$$= \frac{pd}{4tE}(1 - 2\nu)$$

$$\text{Hoop strain} \quad \varepsilon_n = \frac{\sigma_n}{E} - \nu \frac{\sigma_l}{E}$$

$$= \frac{pd}{2tE} - \nu \frac{pd}{4tE}$$

$$= \frac{pd}{4tE}(2 - \nu)$$

Using these expression, we may proceed to obtain the changes in length and diameter of the cylinder.

Change in length = longitudinal strain × original length = $\varepsilon_l \times l$

However, it can be noted that since the circumference is a constant product of diameter, i.e. $C = \pi d$, the diametrical strain will be the same as the circumferential strain.

Thus, change in diameter = Hoop stain × original diameter

Volumetric Strain

To find the volumetric strain (ratio of change in volume to the original volume) the expression for the volume of the cylinder will be considered.

Volume is given by,

$$V = \frac{\pi d^2}{4} \times l$$

On differentiating, we get change in volume, $\delta V = \frac{\pi d^2}{4}\delta l + \frac{\pi}{4} \times l \times 2d \times \delta d$

Hence, volumetric strain, $\varepsilon_v = \frac{\delta V}{V}$

$$= \frac{\frac{\pi d^2}{4} . \delta l + \frac{\pi}{4} . l . 2d . \delta d}{\frac{\pi d^2}{4} . l}$$

$$= \frac{\delta l}{l} + 2\frac{\delta d}{d} = \varepsilon_l + 2\varepsilon_n$$

Thus, $\varepsilon_v = \varepsilon_l + 2\varepsilon_n$

$$= \frac{pd}{4tE}(1 - 2\nu) + 2\frac{pd}{4tE}(2 - \nu)$$

$$= \frac{pd}{4tE}(1 - 2\nu + 4 - 2\nu)$$

$$= \frac{pd}{4tE}(5 - 4\nu)$$

Thus, the volumetric strain is obtained as the sum of longitudinal strain and twice the hoop strains.

In terms of the pressure, diameter and thickness volumetric strain can be expressed as,

$$\varepsilon_v = \frac{pd}{4tE}(5 - 4\nu)$$

We shall now see some examples for finding out the strains and deformations in thin cylindrical shells.

Example 13.4

A cylindrical shell, 0.8 m in diameter and 3 m long is having 10 mm wall thickness. If the shell is subjected to an internal pressure of 2.5 N/mm^2, determine (a) change in diameter (b) change in length and (c) change in volume. Take E = 200 GPa and Poisson's ratio = 0 .25.

Solution

Diameter of the shell d = 0.8 m = 800 mm

Thickness of the shell, t = 10 mm

Internal pressure p = 2.5 N/mm^2

Hoop stress, $\sigma_n = \frac{pd}{2t}$

$$= \frac{2.5 \times 800}{2 \times 10} = 100 \text{ N/mm}^2$$

Longitudinal stress, $\sigma_l = \frac{pd}{4t}$

$$= \frac{2.5 \times 800}{4 \times 10} = 50 \text{ N/mm}^2$$

Hoop strain, $\varepsilon_n = \frac{1}{E}(\sigma_n - \varepsilon\sigma_l) = \frac{1}{2 \times 10^5}(100 - 0.25 \times 50)$

$$= 4.375 \times 10^{-4}$$

Longitudinal strain, $\varepsilon_l = \frac{1}{E}(\sigma_l - \varepsilon\sigma_n) = \frac{1}{2 \times 10^5}(50 - 0.25 \times 100)$

$$= 1.25 \times 10^{-4}$$

Volumetric strain, = $2\varepsilon_n + \varepsilon_l = 2 \times 4.375 \times 10^{-4} + 1.25 \times 10^{-4}$

$$= 10 \times 10^{-4} = 10^{-3}$$

Increase in diameter = Hoop strain × original diameter

$$4.375 \times 10^{-4} \times 800 = 0.35 \text{ mm}$$

Increase in length = Longitudinal strain × original length

$$1.25 \times 10^{-4} \times 3000 = 0.375 \text{ mm}$$

Increase in volume = Volumetric strain × original volume

$$\text{Original volume} = \frac{\pi d^2}{4} \times l = \frac{\pi}{4} \times 800^2 \times 3000 = 1507 \times 10^6 \text{ mm}^3$$

$$\text{Increase in volume} = 10^{-3} \times 1507 \times 10^6 = 1507 \times 10^3 \text{ mm}^3$$

Example 13.5

A copper tube of 50 mm diameter and 1200 mm length has a thickness of 1.2 mm with closed ends. It is filled with water at atmospheric pressure. Find the increase in pressure when an additional volume of 32 cc of water is pumped into the tube. Take E for copper = 100 GPa, Poisson's ratio = 0.3 and K for water = 2000 N/mm^2.

Solution

The additional quantity of water pumped in accounts for the change in volume of the shell as well as the compression of the water in it.

Hence, if p is the increase in pressure in water, then,

$$\text{Hoop stress} = \frac{pd}{2t} = \frac{p \times 50}{2 \times 1.2} = 20.83\, p$$

$$\text{Longitudinal stress} = \frac{pd}{4t} = \frac{p \times 50}{4 \times 1.2} = 10.42\, p$$

$$\text{Hoop strain} = \frac{1}{E}[\,\sigma_n - \nu\sigma_l\,] = \frac{1}{E}[\,20.83\, p - 0.3 \times 10.42\,] = \frac{17.7p}{E}$$

$$\text{Longitudinal strain,} = \frac{1}{E}[\,\sigma_l - \nu\sigma_n\,] = \frac{1}{E}[\,10.42\, p - 0.3 \times 20.83\,] = \frac{4.17p}{E}$$

$$\text{Volumetric strain,} = 2\,\varepsilon_n + \varepsilon_l = \frac{2 \times 17.7p}{E} + \frac{4.17p}{E} = \frac{39.57p}{E} \quad \text{(increase)}$$

Due to compression in water, its volumetric strain $\frac{p}{K}$ (decrease)

∴ Additional volume pumped

= increase in volume of cylinder + decrease in volume of water

$$\text{i.e.} \qquad 32 \times 10^3 = \frac{39.57p}{E} \times V + \frac{p}{K} V$$

$$= \left(\frac{39.57}{1 \times 10^5} + \frac{1}{2000} \right) \times p \times \frac{\pi d^2}{4} \times l$$

$$= 87.57 \times 10^{-5} \times p \times \frac{\pi}{4} \times 50^2 \times 1200$$

$$\therefore \quad p = 15.51 \text{ N/mm}^2$$

SAQ 4

A cylindrical shell 3 m long and 1 m in diameter has 10 mm of metal thickness. Calculate the changes in dimensions and the volume of the cylinder when it is subjected to an internal pressure of 1.5 N/mm^2.

Take E = 204 GPa and Poisson's ratio = 0.3.

SAQ 5

A copper cylinder 900 mm long, 400 mm in diameter and 6 mm thick, with flat ends, is initially full of oil at atmospheric pressure. Calculate the volume of the oil which must be pumped into the cylinder in order to raise the pressure to 5 above atmospheric pressure. Take Young's modulus of copper = 1×10^5 N/mm^2, Poission's ratio = 0.33 and Bulk modulus of oil = 2600 N/mm^2.

SAQ 6

A thin cylindrical shell is subjected to internal fluid pressure, the ends being closed by (a) two water tight pistons attached to a common piston rod and (b) flanged ends. Find the increase in diameter in the two cases for the following data :

Diameter = 200 mm; thickness = 5 mm, internal pressure =3.5 N/mm^2;
Young's Modulus = 210 GPa and Poisson's ratio = 0.3.

13.2.4 Wire Bound Pipes

In order to resist large internal pressures in a thin cylinder, it will be wound closely with a wire. These are called wire bound pipes. The tension in the wire, binding the cylinder will create initial compressive stresses in the pipe before applying the internal pressure. Thus, on application of the internal pressure, the cylinder and the wire jointly resisted the bursting force. The final stresses in the wire or the cylinder will be the sum of the initial stresses due to winding and the stresses induced due to the application of internal pressure. The relation between the stresses in the wire and cylinder are obtained by considering the strain at the common surface of the shell and the wire which should be the same for both.

This is illustrated in the examples given below.

Example 13.6

A cast iron pipe of 200 mm internal diameter and 13 mm metal thickness is closely wound with a layer of 5 mm diameter steel wire under a tensile stress of 40 N/mm^2. Calculate the stresses, set up in the pipe and the wire, if water under a pressure of 2 N/mm^2 is admitted into the pipe. For cast iron, Young's Modulus is 101 GPa and Poisson's ratio is 0.3. For steel, Young's Modulus is 204 GPa.

Solution

Consider 10 mm length of the pipe. Let the initial stresses in the wire and the pipe be f_w and f_p respectively, as shown in Figure 13.6.

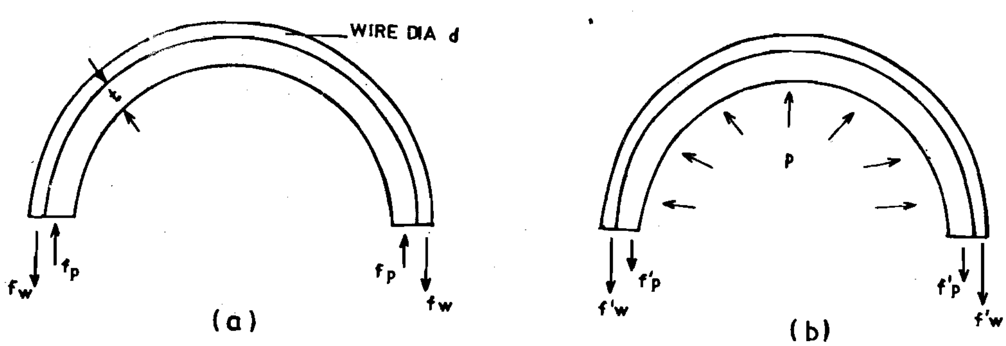

Figure 13.6 : (a) Due to Wire Tension (b) Due to Internal Pressure

For Equilibrium

Tensile force in the wire = compressive force in the pipe

For considered length of 10 mm, two cross sections of wire will be contributing the tensile forces since the diameter of the wire is 5 mm.

Here, $f_w = 40 \text{ N/mm}^2$

$$\therefore \quad 2 \times \frac{10}{5} \times \frac{\pi}{4} \times 5^2 \times 40 = f_p \times 2 \times 13 \times 10$$

$$f_p = 12.8 \text{ N/mm}^2$$

After water is admitted,

let the stresses in wire and pipe be f_w' and f_p' respectively.

Bursting force per 10 mm length = $p \, . \, d \, . \, l$

$$= 3 \times 200 \times 10 = 6000 \text{ N}$$

Resisting force due to pipe $= f_p' \times 2 \times t \times l$

$$= f_p' \times 2 \times 13 \times 10$$

$$= 260 f_p'$$

Resisting force due to wire $= f_w' \times 2 \times \frac{10}{5} \times \frac{\pi}{4} \times 5^2$

$$= 78.54 f_w'$$

Since, resisting force should be equal to bursting force, we get

$$78.54 f_w' + 260 f_p' = 6000 \qquad (1)$$

Longitudinal Stress = $\frac{pd}{4t}$

$$f_{pl}' = \frac{3 \times 20000}{4 \times 13} = 11.54 \text{ N/mm}^2$$

Strain in the circumferential direction for the pipe and wire should be the same.

Strain in the wire $\frac{f_w'}{E_w} = \frac{f_w'}{101 \times 10^3}$ Here, $E_w = 101 \times 10^3 \text{ N/mm}^2$

Strain in the pipe $\frac{f_p'}{E} - \nu \frac{f_{pl}'}{E}$

Thus,

$$\frac{f_p'}{E} - \nu \frac{f_{pl}'}{E} = \frac{f_p'}{204 \times 10^3} - 0.3 \times \frac{11.54}{204 \times 10^3}$$

$$= \frac{1}{204 \times 10^3} (f_p' - 3.462)$$

$$\frac{f_w'}{101 \times 10^3} = \frac{1}{204 \times 10^3} (f_p' - 3.462)$$

$$f_w' = 0.495 f_p' - 1.714 \qquad (2)$$

Substituting (2) in (1), we get,

$$f_p' = 15.65 \text{ N/mm}^2 \quad \text{(tensile)}$$

$$f_w' = 24.60 \text{ N/mm}^2 \quad \text{(tensile)}$$

Thus, Final stress in pipe = $-12.08 + 15.65 = 3.57 \text{ N/mm}^2$ (tensile)

Final stress in wire = $40 + 24.60 = 64.60 \text{ N/mm}^2$ (tensile)

Example 13.7

A copper tube 38 mm external diameter and 35 mm internal diameter, is closely wound with a steel wire of 0.8 mm diameter. Estimate the tension at which the wire must have been wound if an internal pressure of 2 N/mm^2 produces a tensile circumferential stress of 7 N/mm^2 in the tube. Young's Modulus of steel is 1.6 times that of copper. Take Poisson's ratio of copper is 0.3.

Solution

Let f_p and f_w be the stresses in the pipe and the wire respectively, before applying the internal pressure.

Considering for 1 mm length of pipe, we know,

Compressive force in pipe = tensile force in wire.

$$f_p \times 2 \times 1.5 = f_w \times \frac{1}{0.8} \times 2 \times \frac{\pi}{4} \times 0.8^2$$

$$f_p = 0.419 f_w$$

Due to internal pressure alone, let the stresses be f_p' and f_w'. Then, we know,

Tensile force in pipe + Tensile force in wire = Bursting force

$$f_p' \times 2 \times 1.5 + f_w' \times \frac{1}{0.8} \times 2 \times \frac{\pi}{4} \times 0.8^2 = 2 \times 35$$

$$3 f_p' + 1.257 f_w' = 70$$

Longitudinal stress $\frac{pd}{4t} = \frac{2 \times 35}{4 \times 1.5} = 11.67$ N/mm^2

Hoop strain in the pipe = strain in the wire at the junction

$$\frac{f_p'}{E_c} - \nu \frac{f_e}{E_c} = \frac{f_w'}{E_s}$$

$$\frac{f_p'}{E_c} - 0.3 \times \frac{11.67}{E_c} = \frac{f_w'}{1.6\, E_s}$$

$$f_p' - 3.5 = 0.625 f_w'$$

$$f_p' - 0.625 f_w' = 3.5$$

Solving for f_p' and f_w',

$$f_p' = 15.44 \text{ N/mm}^2$$

$$f_w' = 19.11 \text{ N/mm}^2$$

$\therefore$ Final tensile stress in the pipe, $15.44 - f_p = 7$,

Using the relation between initial stresses, $f_p = 8.44$ N/mm^2

$\therefore$ Tension in the wire = 20.15 N/mm^2

SAQ 7

A cast iron pipe having an internal diameter of 300 mm has wall thickness of 6 mm and is closely wound with a single layer of steel wire of 3 mm diameter under a tensile stress of 8 N/mm^2. Calculate the stresses in the pipe and the wire, when the internal pressure inside the pipe is 1 N/mm^2. Take E for steel = 200 GPa, E for cast iron = 101 GPa and Poisson's ratio = 0.3.

SAQ 8

A cylindrical thin walled vessel with closed ends made of an alloy has internal diameter of 200 mm and a wall thickness of 5 mm. The cylinder is strengthened by surrounding it with a single layer of steel wire of diameter 1.25 mm closely wound under tension. Determine the minimum tension under which the wire must be wound if the hoop tension in the cylinder is not to exceed 50 N/mm^2 when the vessel is subjected to an internal pressure of 4 N/mm^2.

Take E for steel is 200 GPa and for the alloy is 100 GPa and Poisson's ratio is 0.33.

13.3 THICK CYLINDERS

In this section, we shall analyse cylindrical shells whose wall thickness is large such that they can no longer be considered as thin. After listing the assumptions involved, we shall derive expressions for the stresses in thick cylinders. We shall then workout related problems.

13.3.1 Assumptions

(i) The diameter-thickness ratio is less than 20.

(ii) The material of the cylinder is homogeneous and isotropic.

(iii) Plane section perpendicular to the longitudinal axis of the cylinder remain plane even after the application of the internal pressure. This implies that the longitudinal strain is same at all points of the cylinder.

(iv) All fibres of the material are free to expand or contract independently without being confined by the adjacent fibers.

13.3.2 Stresses

Figure. 13.7 shows a thick cylinder subjected to an internal pressure, P_1 and external pressure P_2. The internal and external radii are r_1 and r_2 respectively.

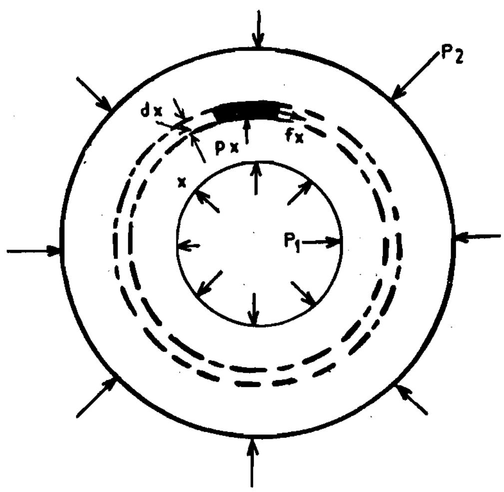

Figure 13.7

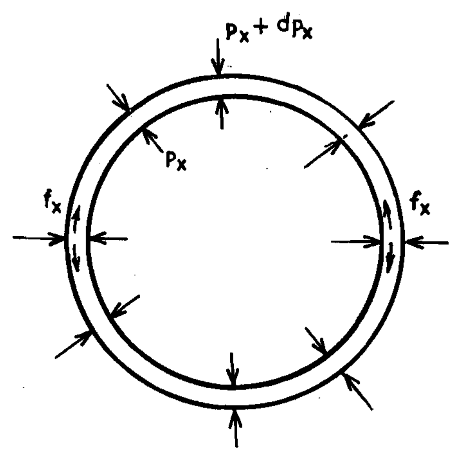

Figure 13.8

In order to derive expressions for the internal stresses, consider an annular cylinder of radius x and radial thickness dx. On any small element of this ring, p_x will be the radial stress and f_x will be the hoop stress. Considering the equilibrium of the ring similar to a thin cylinder as shown in Figure 13.8.

Bursting force in the vertical direction,

$$p_x \times 2x \times l - \left[(p_x + dp_x) \times 2\,(x + dx) \times l \right]$$

Neglecting the second order terms,

$$\text{Bursting force} = -\,2l\,(p_x\,dx + x\,dp_x)$$

$$\text{Resisting force} = 2 \times dx \times l \times f_x$$

Equating the two, $$2 \times dx \times l \times f_x = -2l\,(p_x\,dx + x\,dp_x)$$

$$f_x = -p_x - x\frac{dp_x}{dx}$$

Thus, we get, $$f_x + p_x + x\frac{dp_x}{dx} = 0$$

Considering the assumption that their longitudinal streain is constant, we can write,

$$\frac{\sigma_l}{E} - \nu\frac{f_x}{E} + \nu\frac{p_x}{E} = \text{constant}$$

As longitudinal strain is constant, $\frac{\sigma_l}{E}$ constant. Thus, grouping all the constant terms to the right hand side, we can write,

$$f_x - p_x = \text{some constant, say } 2a$$

Subsituting for f_x,

$$-p_x - x\frac{dp_x}{dx} - p_x = 2a$$

$$-x\frac{dp_x}{dx} = 2p_x + 2a = 2\,(p_x + a)$$

$$\frac{dp_x}{dx} = \frac{-2\,(p_x + a)}{x}$$

or $$\frac{dp_x}{p_x + a} = -\frac{2dx}{x}$$

Integrating, $\log_e (p_x + a) = -2\log_e x + \text{constant}$

Taking the constant as $\log_e b$,

$$\log_e (p_x + a) = \log_e b - 2\log_e x$$

$$\log_e (p_x + a) = \log_e b - \log_e x^2$$

$$\log_e (p_x + a) = \log_e \frac{b}{x^2}$$

Thus, $$p_x + a = \frac{b}{x^2}$$

or $$p_x = \frac{b}{x^2} - a$$

But, we know that $f_x - p_x = 2a$, therefore, $f_x = p_x + 2a$.

Thus, $$f_x = \frac{b}{x^2} + a$$

These expressions for the radial stress and hoop stress are called Lames expressions. The constants a and b will be found from the known external pressure and the internal pressure.

13.3.3 Compound Cylinders

In order to reduce the hoop stresses developed in thick cylinders subjected to large internal pressure, compound cylinders with one thick cylinder shrunk over the other are used. With calculated junction pressure between the two cylinders, it is possible to reduce the hoop stress and to make it more or less uniform over the thickness. Lame's expressions are applied to both the inner and outer cylinders before and after the introduction of the internal fluid pressure, and the stresses in the two stages are superposed to obtain the final values. When the outer cylinder is shrunk over the inner cylinder, the difference between the inner diameter of the outer cylinder and the outer diameter of the inner cylinder determine the shrinkage pressure developed at the junction. This differerence can be worked out as given in subsequent paragraphs.

Let r_j be the common radius at the junction after stringing on. Let Sr_1 be the difference between the outer radius of inner cylinder and r_j and Sr_2 be the difference between r_j and inner radius of the outer cylinder .

If Sr be the difference in the radii before shrinking on, then we have,

$$Sr = Sr_1 + Sr_2$$

For the inner tube, the circumferential strains at the common radius r_j is given by

$$\frac{Sr_1}{r_j} = \frac{1}{E}\left[\left(\frac{b}{r_j^2}+a\right) + \nu\, p_j\right] \quad \text{(compressive)}$$

whre p_j is the junction pressure due to shrinkage. Similarly, for the outer cylinder,

$$\frac{Sr_2}{r_j} = \frac{1}{E}\left[\left(\frac{b'}{r_j^2}+a'\right) + \nu\, p_j\right] \quad \text{(tensile)}$$

Thus, $$\frac{Sr}{r_j} = \frac{Sr_1+Sr_2}{r_j} = \frac{1}{E}\left[\left(\frac{b'}{r_j^2}+a'\right) - \left(\frac{b}{r_j^2}+a\right)\right]$$

i.e. the original difference of radii at the junction will be given by the algebraic difference between the hoop stresses for the tubes at the junction multiplied by the junction radius divided by E.

Now, we shall see some examples of problems relating to thick cylinders.

Example 13.8

The internal and external diameters of a thick hollow cylinder are 80 mm and 120 mm respectively. It is subjected to an external pressure of 40 N/mm^2 and an internal pressure of 120 N/mm^2. Calculate the circumferential stress at the external and internal surfaces and determine the radial and circumferential stresses at the mean radius.

Solution

We know that $$p_x = \frac{b}{x^2} - a$$

Thus, at $x = 40$, $p_x = 120$ N/mm^2, and

at $x = 60$, $p_x = 40$ N/mm^2.

Substituting these values, we get,

$$120 = \frac{b}{40^2} - a \quad \text{and}$$

$$40 = \frac{b}{60^2} - a$$

On solving, we get, $a = 24$ and $b = 230400$.

Circumferential stress is given by, $$f_x = \frac{b}{x^2} + a$$

Thus, at $x = 40$, $$f_x = \frac{230400}{40^2} + 24 = 168 \text{ N/mm}^2$$

at $x = 60$, $$f_x = \frac{230400}{60^2} + 24 = 88 \text{ N/mm}^2$$

At the mean radius, i.e. $\frac{(40+60)}{2} = 50$ mm,

Radial stress $= \frac{230400}{50^2} - 24 = 68.16$ N/mm^2, and

Circumferential stress $= \frac{230400}{50^2} + 24 = 116.16$ N/mm^2.

Example 13.9

A thick cylinder of 0.5 m external diameter and 0.4 m internal diameter is subjected simultaneously to internal and external pressures. If the internal pressure is 25 MN/m^2 and the hoop stress at the inside of the cylinder is 45 MN/m^2 (tensile), determine the intensity of the external pressure.

Solution

Using Lame's expression for hoop stress and radial stress, which are

$$f_x = \frac{b}{x^2} + a \quad \text{and}$$

$$p_x = \frac{b}{x^2} - a.$$

For the internal surface, i.e. $x = 0.2$ m,

$$25 = \frac{b}{0.2^2} - a$$

$$45 = \frac{b}{0.2^2} + a$$

On solving, we get, $b = 1.4$ and $a = 10$.

Thus, the intensity of external pressure at $x = 0.25$ will be,

$$p_x = \frac{1.4}{0.25^2} - 10$$

$$= 12.4 \text{ MN/m}^2$$

Example 13.10

The cylinder of a hydraulic press has an internal diameter of 0.3 m and is to be designed to withstand a pressure of 10 MN/m^2 without the material being stressed over 20 MN/m^2. Determine the thickness of the metal and the hoop stress on the outer side of the cylinder.

Sotution

Internal radius = $\frac{0.3}{2}$ = 0.15 m.

Let the external radius be R.

Hoop stress will be maximum at the inner side.

Thus, $$20 = \frac{b}{0.15^2} + a$$

Radial pressure at the inner side = 10 MN/m^2.

Therefore $$10 = \frac{b}{0.15^2} - a$$

On solving, we get, $b = 0.3375$ and $a = 5$.

For external pressure = 0, we get,

$$0 = \frac{0.3375}{R^2} - 5$$

$$R = 0.26 \text{ m}.$$

Thus, metal thickness = 0.26 – 0.15 = 0.11 m = 110 mm

Finally, hoop stress at the outside of the cylinder will be,

$$= \frac{0.3375}{0.26^2} + 5$$

$$= 10.00 \text{ MN/m}^2$$

Example 13.11

A thick cylinder of steel having an internal diameter of 100 mm and external diameter of 200 mm is subjected to an internal pressure of 80 N/mm^2. Find the maximum stress induced in the material and the change in the external diameter.

Take Young's modulus = 2×105 N/mm^2 and Poisson's ratio = 0.3.

Solution

Using Lames expressions for the inside and outside pressure, we have,

$$80 = \frac{b}{50^2} - a \quad \text{and}$$

$$0 = \frac{b}{100^2} - a$$

On solving, we get, $b = 266666.67$ and $a = 26.667$.

Thus, the maximum stress (Hoop stress at the inner surface of the cylinder)

$$= \frac{b}{50^2} + a = \frac{266666.67}{50^2} + 26.667$$

$$= 133.33 \text{ N/mm}^2$$

To find the strain at the outer surface,

$$\text{Hoop stress at surface} = \frac{266666.67}{100^2} + 26.667 = 53.33 \text{ N/mm}^2$$

Since, pressure outside is zero,

$$\text{Hoop strain} = \frac{53.33}{E} = \frac{53.33}{2 \times 10^5} = 2.666 \times 10^{-4}$$

Thus, increse in external diameter = $2.666 \times 10^{-4} \times 200 = 0.053$ mm.

Example 13.12

A compound tube is made by shrinking one tube on another, the final dimension being, 80 mm internal diameter, 160 mm external diameter and 120 mm being the diameter at the junction.

If the radial pressure at the junction due to shrinking is 15 N/mm^2, find the greatest tensile and compressive stresses induced in the material of the cylinder. What difference must there be in the external diameter of the inner cylinder and the internal diameter of the outer cylinder before shrinking ?

Take Young's modulus as 200 GPa.

Solution

For outer cylinder,

at $x = 60$, $p_x = 15$, and

at $x = 80$, $p_x = 0$.

We get,
$$15 = \frac{b_1}{60^2} - a_1 \quad \text{and}$$

$$0 = \frac{b_1}{80^2} - a_1.$$

On solving, we get, $b_1 = \frac{864000}{7}$ and $a_1 = \frac{135}{7}$.

Circumferential stress at the inner surface,

$$f_x = \frac{864000}{7 \times 60^2} + \frac{135}{7}$$

$$= 53.6 \text{ N/mm}^2 \quad \text{(tensile)}$$

For inner cylinder,

at $x = 60$, $p_x = 15$, and

at $x = 40$, $p_x = 0$.

We get, $$15 = \frac{b_2}{60^2} - a_2 \quad \text{and}$$

$$0 = \frac{b_2}{40^2} - a_2$$

On solving, we get, $b_2 = -43200$ and $a_2 = -27$.

Circumferential stress at the inner surface,

$$f_x = -\frac{43200}{40^2} + (-27)$$

$$= -54 \text{ N/mm}^2 \quad \text{(compressive)}$$

Circumferential stress at the junction,

$$f_x = -\frac{43200}{60^2} + (-27)$$

$$= -39 \text{ N/mm}^2 \quad \text{(compressive)}$$

Required difference in diameter,

$$= \frac{\text{Junction diameter}}{E} \times \text{(Algebraic difference between hoop stresses at the junction)}$$

$$= \frac{120}{2 \times 10^5} (53.6 + 39) = 0.0556 \text{ mm}$$

SAQ 9

A steel pipe 100 mm external diameter and 75 mm internal diameter is subjected to an internal pressure of 14 MN/m^2 and an external pressure of 5.5 MN/m^2. Find the distribution of hoop stress across the wall of the pipe.

SAQ 10

A steel cylinder 160 mm external diameter and 120 mm internal diameter has another cylinder 200 mm external diameter shrunk on it. If the maximum tensile stress induced in the outer cylinder is 80 N/mm^2. Find the radial compressive stress between the cylinders. Determine the circumferential stress at inner and outer diameters of both cylinders. Find also the initial difference in the common diameter of the two cylinders required.

Take Young's modulus = 2×10^5 N/mm^2.

13.4 SUMMARY

In this unit, we have seen the assumptions made in the analysis of thin cylinders. The hoop stress and longitudinal stress for a thin cylinders have been found to be $\frac{pd}{2t}$ and $\frac{pd}{4t}$ respectively. The expressions for hoop strain, circumferential strain and the volumetric strain were derived. Examples for finding stresses, strains and deformations in the thin cylinders were worked out. Stresses in wire bound pipes were then found through examples. Lame's expressions for the stresses in a thick cylindrical shell were derived after considering the assumptions involved. Compound cylinders involving shrinkage pressure at the junction were considered. Expression to find the initial difference in junction radii was deduced. Examples have been worked out in illustrating these expressions.

13.5 ANSWERS TO SAQs

SAQ 1

200 N/mm^2; 100 N/mm^2.

SAQ 2

0.6 N/mm^2.

SAQ 3

13.33 mm.

SAQ 4

0.208 mm, 0.147 mm, 1095 × 103 mm^3.

SAQ 5

520 × 103 mm^3.

SAQ 6

0.067 mm, 0.057 mm.

SAQ 7

12.51 N/mm^2; 31.8 N/mm^2.

SAQ 8

153.8 N/mm^2.

SAQ 9

24.86 MN/m^2; 16.36 MN/m^2.

SAQ 10

17.56 N/mm^2; 62.4 N/mm^2; 80.3 N/mm^2; 0.114 mm.

Chapter

8 THICK CYLINDRICAL SHELLS

8.1 INTRODUCTION :

When the thickness of the shell is comparably large, the cylindrical shell is called a *thick cylinder*. Thick cylinders are used to withstand high pressures. In case of thin walled cylindrical shell (d/t > 20) subjected to internal pressure, the stress is assumed uniform throughout the thickness of the wall. This assumption is not accepted when the thickness of shell is not small, and is comparable to the diameter of the shell. In case of thick shells, the stress is not uniform, but varies with maximum at inner surface to a minimum at outer surface.

In case of thick shells the following assumption are made.

1. Material is homogeneous and isotropic, and stresses are within proportional limit.
2. The longitudinal strain at any point in the thickness of the metal is constant and is independent of position. This means that sections perpendicular to the longitudinal axis remain plane before and after the pressure applied.

8.2 STRESSES IN A THICK CYLINDRICAL SHELL-LAME'S EQUATIONS :

Consider a thick cylinder of *unit* length with outer and inner radii r_1 and r_2 respectively. Consider an elementary ring of the shell of radius, *r* and thickness dr.

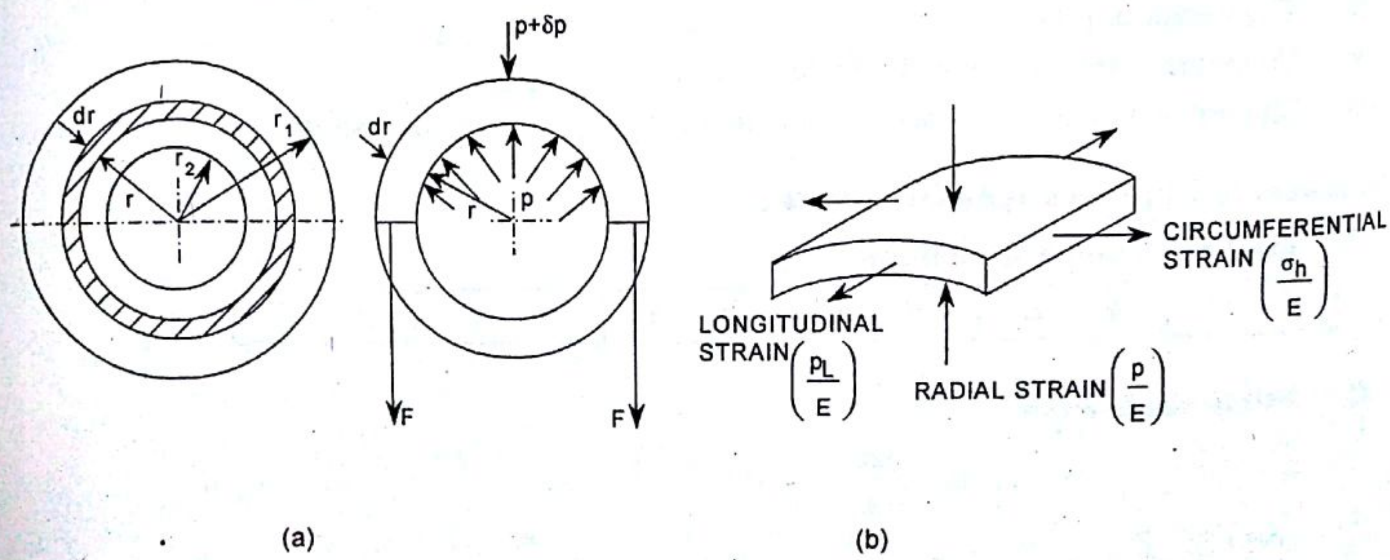

Fig. 8.1

Considering the equilibrium of thin cylindrical element subject to internal pressure (p),

Bursting force at AB = $p.2r - (p + \delta p)\, 2\, (r + \delta r)$

Resisting force, $2F = 2\sigma_h.\delta r$

Equating bursting force to resisting force

$$2\sigma_h.\delta r = 2r.p - (p + \delta p)\, 2(r + \delta r)$$

Neglecting small quantities,

$$\sigma_h.\delta r = -p\delta r - r\delta p$$

$$\sigma_h = -p - \frac{r\,\delta p}{\delta r} \quad \text{... (i)}$$

$$\sigma_h = -\frac{\delta(r\delta p)}{\delta r}$$

At any point, in the section of elementary ring, the three principal stresses are :

1. Radial (compressive) stress, p
2. Circumferential (tensile) stress, σ_h
3. Longitudinal (tensile) stress, p_L

Longitudinal tensile stress is assumed to be uniform across the entire thickness of the shell and given by the equation.

$$p_L = \frac{p.\pi r_2^2}{\pi(r_1^2 - r_2^2)} = \frac{p.r_2^2}{r_1^2 - r_2^2}$$

It may be assumed that longitudinal strain (e) is constant, which means that cross-sections remain plane after straining. Longitudinal strain at any point in the ring,

$$e = \frac{p_L}{E} - \frac{\sigma_h}{mE} + \frac{p}{mE}$$

$$= \frac{1}{E}\left[p_L - \frac{(\sigma_h - p)}{m}\right]$$

Assuming stress in axial direction (p_L) is uniform over the cross section, and for given material of shell E and $\frac{1}{m}$ are constant. Therefore $(\sigma_h - p)$ must be constant.

Let $\sigma_h - p = 2a$ where a is constant. ... (ii)

From relations (i) and (ii)

$$p + 2a = -p - \frac{r\delta p}{\delta r}$$

$$\therefore \quad \frac{\delta p}{\delta r} = -\frac{2(p+a)}{r}$$

$$\text{or} \quad \frac{\delta p}{(p+a)} = -\frac{2\delta r}{r}$$

Integrating,

$$\log_e(p + a) = -2 \log_e r + \log_e b$$

Where $\log_e b =$ constant of integration

$$\therefore \qquad p + a = \frac{b}{r^2}$$

i.e.,
$$p = \frac{b}{r^2} - a \qquad \text{... (iii)}$$

and from equations (ii) and (iii)

$$\sigma_h = \frac{b}{r^2} + a \qquad \text{... (iv)}$$

The above relations for radial stress and circumferential stress are called Lame's equations.

The constants *a* and *b* can be evaluated for given conditions ;

Let radial pressure inside is p_i, reducing to atmospheric pressure on the outside,

$$p_i = \frac{b}{r_2^2} - a$$

$$0 = \frac{b}{r_1^2} - a$$

Solving the above equations for *a* and *b*

$$a = \frac{p_i r_1^2}{(r_1^2 - r_2^2)} \quad \text{and} \quad b = \frac{p_i r_1^2 . r_2^2}{(r_1^2 - r_2^2)}$$

The max. circumferential stress on the inside surface

$$\sigma_h = \frac{b}{r_2^2} + a = \frac{p(r_1^2 + r_2^2)}{(r_1^2 - r_2^2)}$$

Consider pressures are p_1 and p_2 at outer and inner surface i.e., at radius r_1 and r_2 respectively.

$$p_2 = \frac{b}{r_2^2} - a \quad \text{and} \quad p_1 = \frac{b}{r_1^2} - a$$

$$\therefore \qquad b = \frac{(p_2 - p_1) r_1^2 r_2^2}{(r_1^2 - r_2^2)} \quad \text{and } a = \frac{p_2 r_2^2 - p_1 r_1^2}{(r_1^2 - r_2^2)}$$

Example 8.1 :

A 300 mm internal diameter water pipe 75 mm thick carries water under a pressure of 6 N/mm². Calculate the maximum and minimum intensities of hoop stress.

Solution :

Internal pressure at r_2 (150 mm), $p_i = 6$ N/mm² at r_1 (150 + 75 mm), $p_0 = 0$

$$a = \frac{p_i . r_2^2}{(r_1^2 - r_2^2)} = \frac{6 \times 150^2}{225^2 - 150^2} = 4.8$$

and $$b = \frac{p_i . r_1^2 . r_2^2}{(r_1^2 - r_2^2)} = \frac{6 \times 225^2 \times 150^2}{(225^2 - 150^2)} = 243000$$

Max. hoop stress, $$\sigma_{h, max} = \frac{b}{r_2^2} + a = \frac{243000}{150^2} + 4.8$$

$$= \mathbf{15.6 \ N/mm^2} \text{ Ans.}$$

Min. hoop stress, $$\sigma_{h,min} = \frac{b}{r_1^2} + a = \frac{243000}{225^2} + 4.8$$

$$= \mathbf{9.6 \ N/mm^2} \text{ Ans.}$$

Example 8.2 :

A thick cylinder of internal diameter 160 mm is subjected to an internal pressure 40 N/mm². If the allowable stress in the material is 120 N/mm², Find the thickness required.

[JNTUH - May/June 2012]

Solution :

Internal pressure, $p_i = 40$ N/mm²

Allowable stress, $\sigma_h = 120$ N/mm²

Internal diameter, $d_2 = 160$ mm

$\therefore$ $$r_2 = \frac{160}{2} = 80 \text{ mm}$$

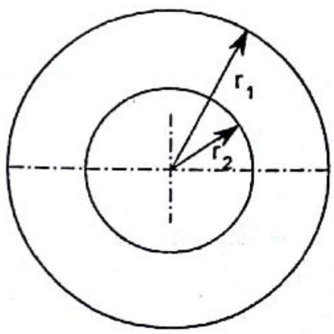

Fig. 8.2

$$a = \frac{p_i . r_2^2}{(r_1^2 - r_2^2)} = \frac{40 \times 80^2}{(r_1^2 - 80^2)} = \frac{256000}{(r_1^2 - 80^2)}$$

$$b = \frac{p_i . r_1^2 . r_2^2}{(r_1^2 - r_2^2)} = \frac{40 \times r_1^2 \times 80^2}{(r_1^2 - 80^2)} = \frac{256000 r_1^2}{(r_1^2 - 80^2)}$$

$$\sigma_{h,max} = \frac{b}{r_2^2} + a$$

$$= \frac{256000 r_1^2}{80^2(r_1^2 - 80^2)} + \frac{256000}{(r_1^2 - 80^2)}$$

$$120 = \frac{40 r_1^2}{(r_1^2 - 6400)} + \frac{256000}{(r_1^2 - 6400)}$$

$$120(r_1{}^2 - 6400) = 40 r_1{}^2 + 256000$$

$$80 r_1{}^2 = 1024000$$

$$\therefore \quad r_1 = \sqrt{\frac{1024000}{80}} = 113.14 \text{ mm}$$

$$\text{Thickness, } t = r_1 - r_2$$

$$= 113.14 - 80 = \mathbf{33.14\ mm\ Ans.}$$

Example 8.3 :

A thick cylindrical pipe of outside diameter 300 mm and internal diameter of 200 mm is subjected to an internal fluid pressure of 20 N/mm² and external fluid pressure of 5 N/mm². Determine the maximum hoop stress developed and draw the variation of hoop stress and radial stress across the thickness. Indicate values at every 25 mm interval. **[JNTU H - May/June, 2012]**

Solution :

$$r_1 = \frac{d_1}{2} = \frac{300}{2} = 150 \text{ mm}, \qquad r_2 = \frac{d_2}{2} = \frac{200}{2} = 100 \text{ mm}$$

$$p_2 = 20 \text{ N/mm}^2, \ p_1 = 5 \text{ N/mm}^2$$

$$a = \frac{p_2 r_2^2 - p_1 r_1^2}{(r_1^2 - r_2^2)} = \frac{20\times100^2 - 5\times150^2}{(150^2 - 100^2)} = 7$$

$$\text{and} \quad b = \frac{(p_2 - p_1) r_1^2 . r_2^2}{(r_1^2 - r_2^2)} = \frac{(20-5)150^2\times100^2}{150^2 - 100^2} = 270000$$

$$\text{Max. hoop stress, } \sigma_{max} = \frac{b}{r_2^2} + a$$

$$= \frac{270000}{100^2} + 7 = \mathbf{34\ N/mm^2\ Ans.}$$

Variation of hoop stress :

$$\sigma_r = 125\text{ mm}, \quad \sigma = \frac{270000}{125^2} + 7 = 24.28\text{ N/mm}^2$$

$$r = 150\text{ mm}, \quad \sigma_{min} = \frac{270000}{150^2} + 7 = 19\text{ N/mm}^2$$

Variation of radial stress :

$$r = 100\text{ mm}, \quad p = 20\text{ N/mm}^2\text{ (given)}$$

$$r = 125\text{ mm}, \quad p = \frac{b}{r^2} - a$$

$$= \frac{270000}{125^2} - 7 = 10.28\text{ N/mm}^2$$

$$r = 150\text{ mm}, \quad p = 5\text{ N/mm}^2\text{ (given)}$$

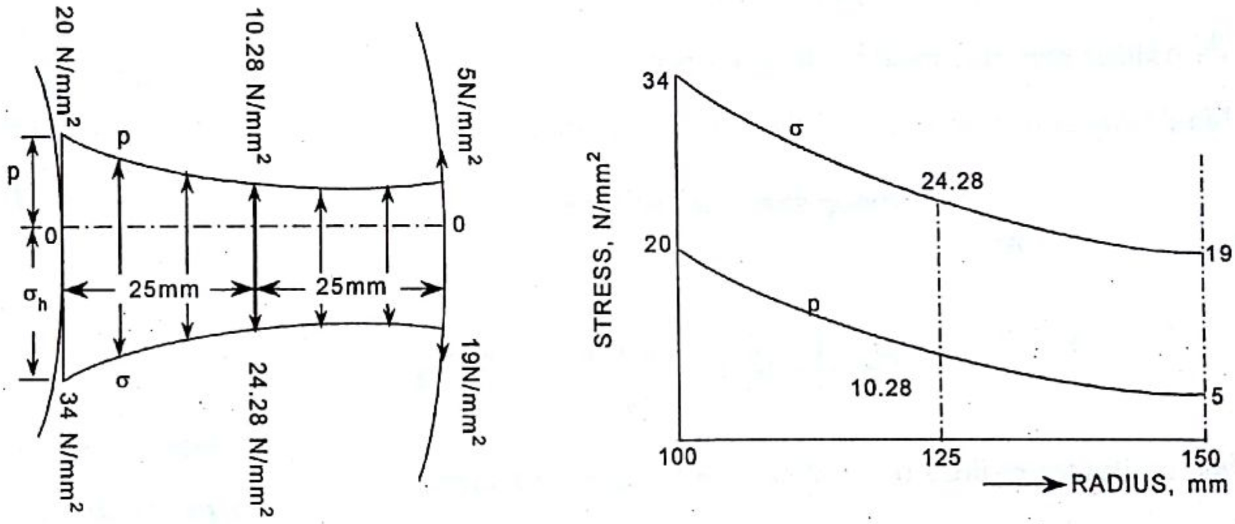

Fig. 8.3

Example 8.4 :

The cylinder of a hydraulic ram is 60 mm inter diameter. Find the thickness to withstand an internal pressure of 40 N/mm², if the maximum tensile stress is limited to 60 N/mm² and the maximum shear stress to 50 N/mm². **[AMIE]**

Solution :

$$\sigma_{h,\,max} = 60\text{ N/mm}^2,$$

$$p_i = 40\text{ N/mm}^2,$$

$$r_2 = \frac{60}{2} = 30\text{ mm}$$

Considering the maximum tensile stress :

The maximum tensile stress is the hoop stress at the inside.

$$\sigma_h = \frac{p_i(r_1^2 + r_2^2)}{(r_1^2 - r_2^2)}$$

$$60 = \frac{40(r_1^2 + 30^2)}{(r_1^2 - 30^2)}$$

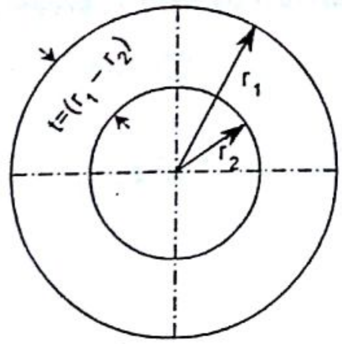

Fig. 8.4

$$60(r_1^2 - 900) = 40(r_1^2 + 900)$$

$$60r_1^2 - 54000 = 40r_1^2 + 36000$$

$$\therefore \quad r_1^2 = \frac{36000 + 54000}{20} = 4500$$

$$\therefore \quad r_1 = \sqrt{4500} = 67.08 \text{ mm}$$

Consider the maximum shear stress :

Maximum shear stress is half the stress difference,

$$\tau_{max} = \frac{(\text{hoop stress} - \text{radial stress})}{2}$$

$$= \frac{[\sigma_h - (-p_i)]}{2} = \frac{[\sigma_h + p_i]}{2} = \frac{p_i . r_1^2}{r_1^2 - r_2^2}.$$

Minus sign for radial stress indicates a compression stress.

$$\therefore \quad \tau_{max} = \frac{p_i . r_1^2}{r_1^2 - r_2^2}$$

$$50 = \frac{40 \times r_1^2}{(r_1^2 - 30^2)}$$

$$50\, r_1^2 - 45000 = 40r_1^2$$

$$r_1^2 = \frac{45000}{10} = 4500$$

$r_1 = \sqrt{4500} = 67.08$ mm same as before.

$\therefore$ Thickness, $t = (r_1 - r_2) = (67.08 - 30) =$ **37.08 mm Ans.**

8.3 COMPOUND CYLINDERS :

When the cylindrical shell is subjected to internal pressure the hoop stress across the section is not uniform.

Let hoop stress $= \sigma_x = \dfrac{b}{x^2} + a$

The maximum hoop stress occurs at the inner circumference and the hoop stress decreases towards the outer circumference.

So the maximum pressure inside the shell is limited corresponding to the condition that hoop stress at the inner circumference reaches the permissible value.

The compound tube as shown in Fig. 8.5 is subjected to internal pressure, and both the inner and outer tubes will be subjected to hoop tensile stress due to the internal pressure alone.

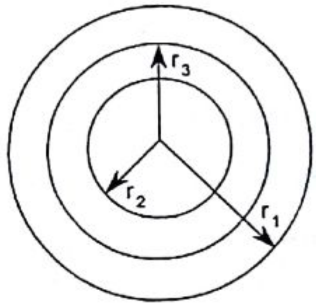

Fig. 8.5

Adding the internal stresses caused while shrinking and the stresses due to pressure alone the final hoop stresses in both the tubes can be determined. By this arrangement the hoop stresses throughout the metal will be nearly uniform.

Let r_1 and r_2 = outer and inner radii of the compound tube

r_3 = radius at the junction of the two tubes.

p_j = radial pressure intensity at the junction of the two tubes after shrinking the outer tubes over the inner tube.

as per Lame's relation.

For outer tube

$$p_x = \frac{b_1}{x^2} - a_1 \quad \text{and} \quad \sigma_x = \frac{b_1}{x^2} + a_1$$

At $x = r_1 ; p_x = 0$

$$\therefore \quad \frac{b_1}{x^2} - a_1 = 0 \qquad \text{... (1)}$$

and at $x = r^3, p_j = \dfrac{b_1}{x^2} - a_1$ ... (2)

The constants a_1 and b_1 are determined from equations (1) and (2).

Lame's relation for inner tube :

$$p_x = \frac{b_2}{x^2} - a_2 \text{ and } \sigma_x = \frac{b_2}{x^2} + a_2$$

at $x = r_2$; $p_x = 0$

$$\therefore \quad \frac{b_2}{x^2} - a_2 = 0 \qquad \text{... (3)}$$

and at $x = r_3$; $p_x = p_j$

$$p_j = \frac{b_2}{x^2} - a_2 \qquad \text{... (4)}$$

by solving equations (3) and (4) b_2 and a_2 can be determined.

Now, the hoop stresses for outer and inner tubes can be determined.

Note : If compound tube is subjected to an internal pressure p_i, the inner and outer tubes will together be considered as one thick shell.

Example 8.5 :

A compound cylinder has inner radius 200 mm, radius at common surface 260 mm and outer radius is 300 mm. Initial pressure at common surface is 6 N/mm². What are the final hoop stresses after a fluid is admitted at a pressure of 80 N/mm² ? Sketch the variation of hoop and radial stresses. **[JNTUH - May/June, 2012]**

Solution :

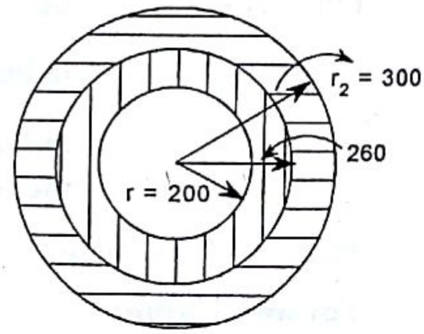

Fig. 8.6

For outer cylinder :

External radius (r_2) 300 mm

Internal radius (r) = 260 mm

For internal cylinder :

Internal radius (r_1) = 200 mm

Radial pressure due to shrinkage at junction (p) = 6 N/m²

Fluid pressure in the compound cylinder (p) = 80 N/mm²

1. Stresses due to shrinking in the outer and inner cylinder before the fluid pressure is admitted :

- Lame's equations for outer cylinders

$$p_x = \frac{b_1}{x^2} - a_1 \qquad \text{... (1)}$$

$$\text{and} \quad \sigma_x = \frac{b_1}{x^2} + a_1 \qquad \text{... (2)}$$

at x = 300 mm, $p_x = 0$

$$\frac{b_1}{(300)^2} - a_1 = 0 \quad \dots (3)$$

at $x = r = 260$ mm $p_x = p = 6$ N/mm^2

$$\frac{b_1}{(260)^2} - a_1 = 6 \quad \dots (4)$$

from eqn. (3) & (4); ((4) – (3))

$$b_1 (1.478 \times 10^{-5} - 1.11 \times 10^{-5}) = 6$$

$$b_1 = 1.626 \times 10^6, \text{ and}$$

$$a_1 = 18.07$$

Substituting b_1 and a_1 values is eqn. (2)

$$\sigma_x = \frac{1.626 \times 10^6}{x^2} + 18.07$$

The stresses at the outer and inner surface of the outer cylinder is obtained by substituting

$$\text{at } x = 300 \text{ mm}, \; \sigma_{300} = \frac{1.626 \times 10^6}{(300)^2} + 18.07 = 36.14 \text{ N/mm}^2\text{(T)}$$

$$\text{and } x = 260 \text{ mm}, \; \sigma_{260} = \frac{1.626 \times 10^6}{(260)^2} + 18.07 = 42.12 \text{ N/mm}^2\text{(T)}$$

b. Lame's equation for the inner cylinder :

$$p_x = \frac{b_2}{x^2} - a_2 \quad \dots (5)$$

and

$$\sigma_x = \frac{b_2}{x^2} + a_2 \quad \dots (6)$$

at $r_1 = 200$; $p_x = 0$ (There is no fluid under pressure)

$$0 = \frac{b_2}{(200)^2} - a_2 \quad \dots (7)$$

at $r_2 = 200$ mm, $p_x = 6$ N/mm^2

$$6 = \frac{b_2}{(260)^2} - a_2 \quad \dots (8)$$

From eqns. (7) and (8), (8) – (7)

$$b_2 (1.479 \times 10^5 - 2.5 \times 10^{-5}) = 6$$

$$b_2 = -587659.16$$

and $\quad a_2 = -14.69$

By substituting the value of b_2 and a_2 in eqn. (6)

$$\sigma_x = \frac{587659.16}{x^2} - 14.69$$

Hence, the hoop stress for the inner cylinder is obtained by substituting x = 260 mm and 200 mm

$$\sigma_{260} = \frac{-587659.16}{260^2} - 14.69$$

$$= -23.38 \text{ N/mm}^2 \text{ (comp)}$$

$$\sigma_{200} = \frac{-587659.16}{(200)^2} - 14.49$$

$$= -29.38 \text{ N/mm}^2 \text{ (comp)}$$

2. Stresses due to fluid pressure alone :

When the fluid under pressure is admitted inside the compound cylinder, the two cylinders together will be considered as one single unit. The hoop stresses are calculated by Lame's equations

$$p_x = \frac{B}{x_2} - A \quad \text{... (9)}$$

$$\sigma_x = \frac{B}{x_2} + A \quad \text{... (10)}$$

at x = 200 mm $\quad p_x = p = 80 \text{ N/mm}^2$

$$80 = \frac{B}{(200)^2} - A \quad \text{... (11)}$$

at x = 260 mm $\quad p_x = 0$

$$0 = \frac{B}{(260)^2} - A \quad \text{... (12)}$$

From equation (12) & (11) [(12) – (11)]

$$B[1.479 \times 10^{-5} - 2.5 \times 10^{-5}] = -80$$

$$\therefore \quad B = 7835453.44$$

$$A = 115.909$$

Substituting (A) and (B) values in eqn. (11)

$$\sigma_x = \frac{7.835\times10^6}{x^2} + 115.909$$

Hence, the hoop stresses due to internal fluid pressure alone are given by,

$$\sigma_{200} = \frac{7.835\times10^6}{(200)^2} + 115.909 = 311.784 \text{ N/mm}^2 \text{ (T)}$$

$$\sigma_{260} = \frac{7.835\times10^6}{(260)^2} + 115.909 = 231.81 \text{ N/mm}^2 \text{ (T)}$$

$$\sigma_{300} = \frac{7.835\times10^6}{(300)^2} + 115.909$$

$$= 202.97 \text{ N/mm}^2 \text{ (T)}$$

The resultant stresses will be the algebraic sum of the initial stresses due to shrinking and those due to internal fluid pressure.

Inner cylinder $F_{200} = -29.38 + 311.784 = 282.404 \text{ N/mm}^2$

$F_{260} = -23.38 + 231.81 = 208.43 \text{ N/mm}^2$

Outer cylinder $F_{260} = 42.12 + 231.81 = 273.93 \text{ N/mm}^2$

$F_{300} = 36.14 + 202.97 = 239.11 \text{ N/mm}^2$

Example 8.6 :

A compound cylinder formed by shrinking one tube on to another, is subjected to an internal pressure of 60 N/mm^2. Before the fluid is admitted, the internal and external diameter of compound cylinder are 120 mm and 200 mm and diameter at the junction is 160 mm. If after shrinking on, the radial pressure at the common surface is 8.0 MPa. Calculate the final stress setup by the section.

[JNTU H - May/June, 2012]

Solution :

For outer cylinder :

External radius $(r_2) = \frac{200}{2} = 100$ mm

Internal radius $(r) = \frac{160}{2} = 80$ mm

For internal cylinder :

$$\text{Internal radius } (r_1) = \frac{120}{2} = 60 \text{ mm}$$

Radial pressure due to shrinkage at junction = 8 N/mm^2

Fluid pressure in the compound cylinder = 60 N/mm^2

1. Stress due to shrinking in the outer and inner cylinder before the fluid pressure is admitted :

Lame's equations for out cylinder :

$$p_x = \frac{b_1}{x^2} - a_1 \qquad \text{... (1)}$$

and $$\sigma_x = \frac{b_1}{x^2} + a_1 \qquad \text{... (2)}$$

at $x = 100$ mm, $p_x = 0$

$$0 = \frac{b_1}{x^2} - a \qquad \text{... (3)}$$

$$x = 80, p_x = 8$$

$$8 = \frac{b_1}{x^2} + a \qquad \text{... (4)}$$

From eqns. (3) & (4) [(4) – (3)]

$$b_1 [1.5625 \times 10^{-4} - 1 \times 10^{-4}] = 8$$

$$b_1 = 142222.22$$

$$a_1 = 14.22$$

Substituting the values for b_1 and a_1 in eqn. 4,

$$\sigma_x = \frac{142222.22}{x^2} + 14.22$$

$$\sigma_{100} = \frac{142222.22}{(100)^2} + 14.22 = 28.44 \text{ N/mm}^2 \text{ (T)}$$

$$\sigma_{80} = \frac{142222.22}{(80)^2} + 14.22 = 36.44 \text{ N/mm}^2 \text{ (T)}$$

b. Lame's equation for the inner cylinder :

$$p_x = \frac{b_2}{x^2} - a_2 \qquad \text{... (5)}$$

and $$\sigma_x = \frac{b_2}{x^2} + a_2 \qquad \text{... (6)}$$

at 80 mm ; $p_x = 8$ N/mm^2

at 60 mm ; $p_x = 0$ N/mm^2 (There is no fluid under pressure)

$$0 = \frac{b_2}{(60)^2} - a_2 \quad \text{... (7)}$$

$$8 = \frac{b_2}{(80)^2} - a_2 \quad \text{... (8)}$$

From equations (7) and (8), [(8) – (7)]

$$b_2 [1.5625 \times 10^{-4} - 2.77 \times 10^{-4}] = 8$$

$$\therefore \quad b_2 = -66252.59$$

$$a_2 = -18.352$$

Substituting the values of a_2 and b_2 in equation (6)

$$\sigma_x = -\frac{66252.59}{x^2} - 18.352$$

Hoop stress at x = 80 mm, $\sigma_{80} = -28.704$ N/mm^2 (comp)

at x = 60 mm, $\sigma_{60} = -36.755$ N/mm^2 (comp)

2. Stresses due to fluid pressure alone :

When the fluid under pressure is admitted inside the compound cylinder, the two cylinders together will be considered as one single unit. The hoop stresses are calculated by Lame's equations,

$$p_x = \frac{B}{x^2} - A$$

and $$\sigma_x = \frac{B}{x^2} + A$$

At x = 60 mm, $p_x = p = 60$ N/mm^2

$$80 = \frac{B}{(60)^2} - A \quad \text{... (11)}$$

at x = 80 mm, $p_x = 0$ N/mm^2

$$0 = \frac{B}{(80)^2} - A \quad \text{... (12)}$$

From equations (11) and (12), [(12) – (11)]

$$B\,[1.5625 \times 10^{-4} - 2.778 \times 10^{-4}] = -80$$

$$B = 658285.71$$

$$A = 102.824$$

Substituting the values ofA and B in equation (12)

$$\sigma_x = \frac{658285.71}{x^2} + 102.824$$

Hoop stress due to internal fluid pressure alone are given by

$$\sigma_{100} = \frac{658285.71}{(100)^2} + 102.824 = 168.652 \text{ N/mm}^2$$

$$\sigma_{80} = \frac{658285.71}{(80)^2} + 102.824 = 205.681 \text{ N/mm}^2$$

$$\sigma_{60} = 285.68 \text{ N/mm}^2$$

The resultant stress :

Inner cylinder :

$$F_{60} = -36.755 + 285.68 = 248.925 \text{ N/mm}^2$$

$$F_{80} = -28.704 + 205.681 = 176.977 \text{ N/mm}^2$$

Outer cylinder :

$$F_{80} = 36.44 + 205.681 = 242.121 \text{ N/mm}^2$$

$$F_{100} = 28.44 + 168.652 = 197.065 \text{ N/mm}^2$$

SUMMARY

- Pressure vessels with $\frac{d}{t} < 20$ are called thick shells.
- Longitudinal strain at any point,

$$e = \frac{1}{E}\left[p_L - \frac{\sigma_h}{m} + \frac{p}{m}\right]$$

$$= \frac{1}{E}\left[p_L - \left(\frac{\sigma_h - p}{m}\right)\right]$$

- Radial stress, $p = \frac{b}{r^2} - a$, and

Hoop (circumferential) stress

$$\sigma_h = \frac{b}{r^2} + a$$

where *a* and *b* are constants, depend on given conditions.

When pressure $p = p_i$ at inside, and

$= 0$ at outside

$$a = \frac{p_i . r_2^2}{(r_1^2 - r_2^2)} \text{ and } b = \frac{p_i . r_1^2 r_2^2}{(r_1^2 - r_2^2)}$$

If p_1 and p_2 are the pressures at outer and inner surface,

$$a = \frac{p_2 r_2^2 - p_1 r_1^2}{(r_1^2 - r_2^2)} \text{ and } b = \frac{(p_2 - p_1)(r_1^2 . r_2^2)}{(r_1^2 - r_2^2)}$$