Deflection of Beams

When a beam is subjected to some type of loading it deflects from its initial/original position. The amount of deflection depends upon its cross-section and B.M. Generally the beams are designed basing upon the two criteria.

1. Strength criterion 2. Stiffness Criterion.

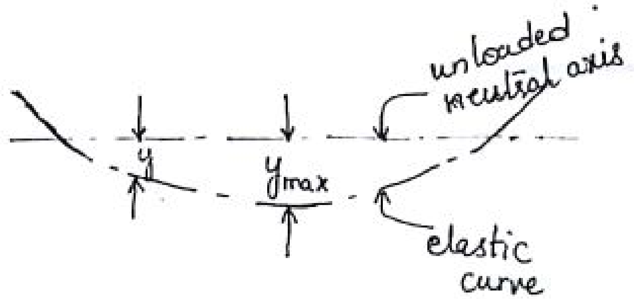
→ According to strength criterion of the beam design, the beam should be strong to resist shear force and bending moment.

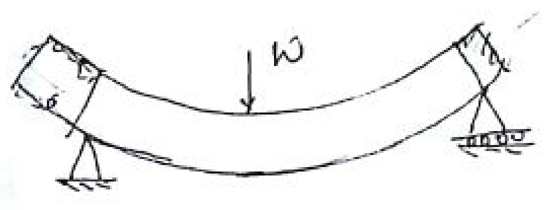
→ According to stiffness criterion of the beam design, the beam should be stiff to resist deflection.

The beam should be stiff enough not to deflect more than the permissible limit.

Under load the neutral axis becomes a curved line and is called the <u>elastic curve</u>. The deflection 'y' is vertical distance between a point on the elastic curve and the unloaded neutral axis.

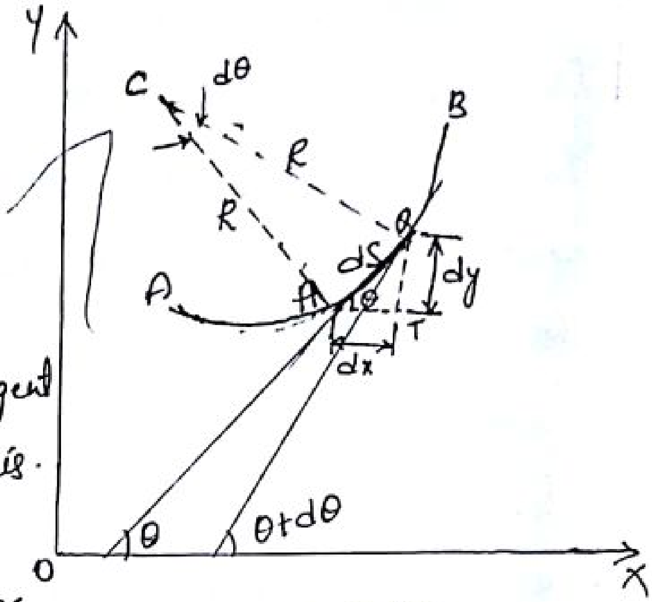
(Beam without load)





(Beam with load)

Relation Between Slope, Deflection & Radius of Curvature :-



Consider a small portion PQ of a beam bent into an arc.

Let ds = length of beam PQ.

C : centre of arc.

θ : Angle which the tangent at P makes with x-axis.

θ + dθ = Angle which the tangent at Q makes with x-axis.

$ds = R\,d\theta$

$\frac{1}{R} = \frac{d\theta}{ds}$

from figure $PQ = ds = R(d\theta)$

from the ΔPQT, $\tan\theta = dy/dx$

$\sec\theta = \frac{ds}{dx}$

$\frac{d\theta}{ds} = \frac{1}{R}$

Differentiating ~~Integrating~~ w.r.to. 'x'.

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2}$$

$\frac{1}{\cos\theta} = \frac{dx}{ds}$

$\Rightarrow \sec^2\theta \cdot \frac{d\theta}{ds} \times \frac{ds}{dx} = \frac{d^2y}{dx^2}$

$\sec\theta = \frac{1}{\cos\theta}$

$\Rightarrow \sec^2\theta \cdot \frac{1}{R} \cdot \sec\theta = \frac{d^2y}{dx^2}$

$\sec\theta = \frac{ds}{dx}$

$$\frac{1}{R}\sec^3\theta = \frac{d^2y}{dx^2}$$

$$\rightarrow \frac{1}{R} = \frac{(d^2y/dx^2)}{(\sec^2\theta)^{3/2}} = \frac{d^2y/dx^2}{(1+\tan^2\theta)^{3/2}}$$

$$\boxed{(\text{or})\quad \frac{1}{R} = \frac{d^2y/dx^2}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}}$$

For a practical beam the slope $\left(\frac{dy}{dx}\right)$ is very small, so it may be neglected.

But from bending eqn.

$$\therefore \frac{1}{R} = \frac{d^2y}{dx^2}$$

$$\frac{M}{I} = \frac{E}{R}$$

$$\Rightarrow M = \frac{EI}{R}$$

$$\Rightarrow \frac{1}{R} = \frac{M}{EI}$$

$$\Rightarrow \frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\boxed{\Rightarrow M = EI\frac{d^2y}{dx^2}}$$

↳ Differential equation for ~~bending~~ deflection.

$EI \Rightarrow$ Flexural Rigidity.

- → Deflection $= y$
- → Slope $= \frac{dy}{dx}$
- → Bending Moment $= EI.\frac{d^2y}{dx^2}$
- → Shear force $= EI.\frac{d^3y}{dx^3}$
- → Load intensity $\Rightarrow w = EI.\frac{d^4y}{dx^4}$

Methods for Slope & Deflection at a section.

1. Double Integration method
2. Moment Area method

} Suitable for single load.

3. Macaulay's method → Suitable for several loads.

Double Integration method.

Bending moment at a point:

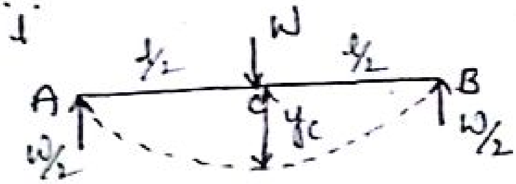
$$M = EI\frac{d^2y}{dx^2}$$

Integrating the above eqn $EI\frac{dy}{dx} = \int M$

$EIy = \iint M$

Simply supported beam with a central load.

Consider a SSB AB of length 'l' carrying a point load 'W' at the centre.



$\therefore R_a = R_b = \frac{W}{2}$

The B.M at any section at a distance x from B.

$$B.M \Rightarrow M_x = R_b \cdot x = \frac{W}{2}x$$

$$\therefore EI\frac{d^2y}{dx^2} = M$$

$$\therefore EI\frac{d^2y}{dx^2} = \frac{W}{2}x \quad \text{——— (1)}$$

Integrating the above eqn

$$EI \cdot \frac{dy}{dx} = \frac{W}{2} \cdot \frac{x^2}{2} + C_1$$

$$= \frac{Wx^2}{4} + C_1 \quad \text{——— (2)}$$

where C_1 is the first constant of integration.

We know that when $x = l/2$, then $\frac{dy}{dx} = 0$

Substituting the values in eqn (2)

$$EI \cdot (0) = \frac{W}{4} \cdot \left(\frac{l}{2}\right)^2 + C_1$$

$$= \frac{Wl^2}{16} + C_1$$

$$\therefore C_1 = -\frac{Wl^2}{16}$$

Substituting this value C_1 in eqn (2)

$$\boxed{EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{Wl^2}{16}} \quad \text{——— (3)}$$

This is the required eqn for the slope at any section. 28

The max. slope occurs at A & B. Thus for max. slope at B. substituting $x = 0$ in eqn ③

$$EI \cdot i_B = -\frac{wl^2}{16}$$

$$i_B = -\frac{wl^2}{16EI}$$

$$\therefore i_B = \frac{wl^2}{16EI} \text{ radians.}$$

{Minus sign means that the tangent at B makes an angle with AB in the -ve direction or anticlockwise direction}

By symmetry $i_A = \frac{wl^2}{16EI}$ radians.

Integrating the above eqn once ③

$$EI \cdot y = \frac{wx^3}{12} - \frac{wl^2}{16}x + C_2 \quad \text{——— ④}$$

$\therefore C_2$ is the second constant of integration.

when $x = 0$, then $y = 0$. Substituting the values in eqn ④

we get $C_2 = 0$

$$\boxed{\therefore \; EI \cdot y = \frac{wx^3}{12} - \frac{wl^2}{16}x}$$

This is the required eqn for deflection, at any section.

For max. deflection substituting $x = l/2$ in eqn ④

$$EI\, y_C = \frac{w}{12}\left(\frac{l}{2}\right)^3 - \frac{wl^2}{16}\left(\frac{l}{2}\right)$$

$$= \frac{wl^3}{96} - \frac{wl^3}{32} = -\frac{wl^3}{48}$$

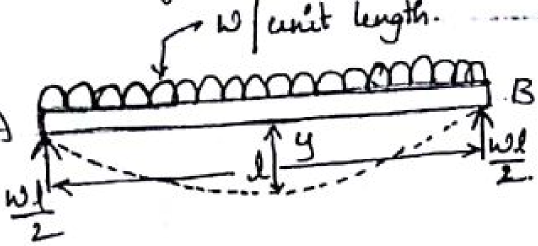
$$\therefore y_C = -\frac{wl^3}{48EI}$$

{Minus means deflection is downwards}

$$y_C = \frac{wl^3}{48EI}$$

Simply Supported Beam with Uniformly Distributed load.

Consider a simply supported beam of length and carrying a UDL per unit length.



$$R_A = R_B = \frac{wl}{2}$$

Consider a section x at a distance x from B.

$$M_x = R_b x - w.x.\frac{x}{2}$$

$$= \frac{wl}{2}x - \frac{wx^2}{2}$$

$$\therefore \quad M = EI\frac{d^2y}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = \frac{wl}{2}x - \frac{wx^2}{2} \quad \text{——— (1)}$$

Integrating the above equation.

$$EI.\frac{dy}{dx} = \frac{wl}{2}.\frac{x^2}{2} - \frac{w}{2}.\frac{x^3}{3} + C_1$$

$$= \frac{wl}{4}x^2 - \frac{wx^3}{6} + C_1 \quad \text{——— (2)}$$

$\therefore$ C_1 is the first constant of integration.

when $x = l/2$, then $\frac{dy}{dx} = 0$

$$EI(0) = \frac{wl}{4}\left(\frac{l}{2}\right)^2 - \frac{w}{6}\left(\frac{l}{2}\right)^3 + C_1$$

$$= \frac{wl^3}{16} - \frac{wl^3}{48} + C_1$$

$$\therefore C_1 = -\frac{wl^3}{24} \quad \text{——— (*)}$$

Substitute this value in (2)

$$\boxed{EI.\frac{dy}{dx} = \frac{wl}{4}x^2 - \frac{wx^3}{6} - \frac{wl^3}{24}} \quad \text{——— (3)}$$

This is the required eqn for slope.

The maximum slope occurs at A & B. For max. slope, substituting $x=0$ in eqn ③.

$$EI.\, i_B = -\frac{wl^3}{24} \quad \text{(or)} \quad i_B = \frac{wl^3}{24EI}$$

$$i_B = \frac{-wl^3}{24EI}$$

{Minus sign means that the tangent at A makes an angle in the -ve direction}

By symmetry $i_A = \frac{wl^3}{24EI}$

Integrating the equation once again.

$$EI.y = \frac{wl}{12}x^3 - \frac{wx^4}{24} - \frac{wl^3}{24}x + C_2 \quad \text{——— ④}$$

C_2 is the second constant of integration. Substituting these values when $x=0$, then $y=0$. Substituting these values in equation ④, we get $C_2=0$.

$$\boxed{EI.y = \frac{wl}{12}x^3 - \frac{wx^4}{24} - \frac{wl^3}{24}x} \quad \text{——— ⑤}$$

This is the required eqn for deflection.

The max. deflection occurs at the mid point.

at $x = \frac{l}{2}$ » $EI.y_c = \frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^3}{24}\left(\frac{l}{2}\right)$

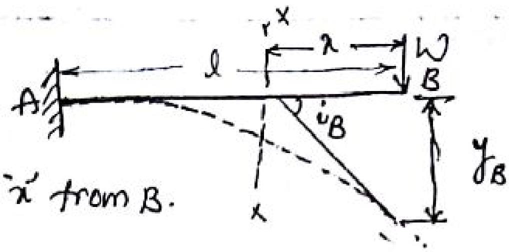
$$EI.y_c = -\frac{5wl^4}{384\cancel{EI}}.$$

{-ve means deflection is downwards}

$$\boxed{\therefore y_c = \frac{5wl^4}{384EI}.}$$

Cantilever with a point load at its free end.

Consider a cantilever AB of length 'l' carrying a point load W at free end.



consider a section X at a distance 'x' from B.

$$M_x = -Wx$$

$$EI \cdot \frac{d^2y}{dx^2} = -Wx \text{ ——— (1)}$$

Integrating the above eqn.

$$EI \cdot \frac{dy}{dx} = -W \cdot \frac{x^2}{2} + C_1 \text{ ——— (2)}$$

where C_1 is the first constant of integration.

We know that when $x = l$, $\frac{dy}{dx} = 0$

Substituting these values in the above eqn (2).

$$0 = -W \cdot \frac{l^2}{2} + C_1$$

$$\therefore C_1 = \frac{Wl^2}{2}$$

Substitute this value in eqn (2).

$$EI \cdot \frac{dy}{dx} = -W \cdot \frac{x^2}{2} + \frac{Wl^2}{2} \text{ ——— (3)}$$

This is the required eqn for slope, we get the slope at any point on the cantilever. So the maximum slope occurs at the free end.

For maximum slope, substituting $x = 0$ eqn (3).

$$EI \cdot i_B = \frac{Wl^2}{2}$$

$$i_B = \frac{Wl^2}{2EI}$$

+ve sign means the tangent at B makes an angle in the +ve direction.

Integrating once again

$$EI \cdot y = -\frac{W}{2} \cdot \frac{x^3}{3} + \frac{Wl^2}{2}x + C_2 \text{ ——— (4)}$$

Where C_2 is the second constant of integration.

when $x = l$, $y = 0$

$$0 = -\frac{Wl^3}{6} + \frac{Wl^3}{2} + C_2$$

(–ve meant that the deflection is downwards.)

$$\therefore C_2 = -\frac{Wl^3}{3}$$

Substituting this value of C_2 in eqn 4.

$$EI \cdot y = -\frac{Wx^3}{6} + \frac{Wl^2}{2}x - \frac{Wl^3}{3}$$

$$\boxed{EI \cdot y = \frac{Wl^2}{2}x - \frac{Wx^3}{6} - \frac{Wl^3}{3}}$$

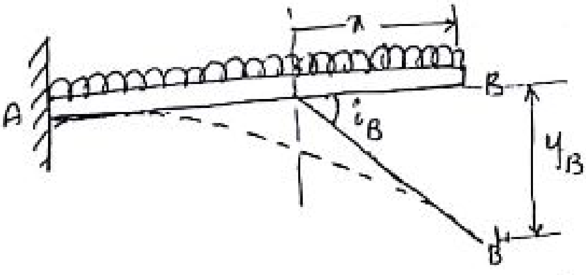
This is the required eqn for the deflection at any section. The max. deflection occurs at the free end.

For max. deflection substituting $x = 0$ in eqn 4

$$EI \cdot y_B = -\frac{Wl^3}{3}$$

$$\boxed{y_B = \frac{-Wl^3}{3EI}}$$

Cantilever with Uniformly Distributed Load.



Consider a cantilever AB of length 'l' carrying a UDL. w/unit length as shown in fig.

B.M. at any section $= -\frac{wx^2}{2}$

$$EI \cdot \frac{d^2y}{dx^2} = M$$

$$EI \frac{d^2y}{dx^2} = -\frac{Wx^2}{2} \quad \text{—— ①}$$

Integrating the equation ①

$$EI \cdot \frac{dy}{dx} = -\frac{W}{2} \cdot \frac{x^3}{3} + C_1$$

$$EI \frac{dy}{dx} = -\frac{Wx^3}{6} + C_1 \quad \text{—— ②}$$

Where C_1 is the first const. of integration.

when $x = l$, then $\frac{dy}{dx} = 0$

Substitute these values in eqn ②.

$$0 = -\frac{W}{6}(l)^3 + C_1$$

$$C_1 = \frac{Wl^3}{6}$$

Substituting this value of C_1 in eqn ②

$$\boxed{EI \cdot \frac{dy}{dx} = -\frac{W}{6}x^3 + \frac{Wl^3}{6}} \quad \text{—— ③}$$

This is the required eqn for slope at any point.

For max. slope, substituting $x = 0$. in eqn ③.

$$EI \cdot i_B = \frac{Wl^3}{6}$$

$$i_B = \frac{Wl^3}{6EI}$$

Integrating the above eqn ③ once again.

$$EI \cdot y = -\frac{W}{6} \cdot \frac{x^4}{4} + \frac{Wl^3}{6}x + C_2$$

$$= \frac{-W}{24}x^4 + \frac{Wl^3}{6}x + C_2 \quad \text{—— ④}$$

∴ C_2 is second const. of integration.

when $x = l$, then $y = 0$

Substituting these values in eqn (4)

$$0 = -\frac{wl^4}{24} + \frac{wl^4}{6} + C_2$$

$$\therefore C_2 = -\frac{wl^4}{8}$$

Substituting this value of C_2 in eqn (4).

$$EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x - \frac{wl^4}{8}$$

$$\boxed{EI \cdot y = \frac{wl^3}{6}x - \frac{wx^4}{24} - \frac{wl^4}{8}}$$

This is the required eqn for the deflection.

For max. deflection, substitute $x = 0$ in eqn (5)

$$EI \cdot y_B = \frac{wl^4}{8}$$

$$\boxed{y_B = \frac{wl^4}{8EI}}$$

⇛ A wooden beam 140mm wide and 240mm deep has a span of 4m. Determine the load, that can be placed at its centre to cause the beam a deflection of 10mm. Take E as 6 GPa.

Sol:- $b = 140mm$ central deflection $y_c = 10mm$

$d = 240mm$ $E = 6\,GPa$

$l = 4m \Rightarrow 4\times10^3 mm.$ $\Rightarrow 6\times10^3 N/mm^2.$

$$I = \frac{bd^3}{12} = \frac{140\times(240)^3}{12}$$

$$= 161.3\times10^6\ mm^4.$$

deflection of the beam at its centre.

$$y_c = \frac{Wl^3}{48EI}$$

$$10 = \frac{W \times 4 \times 10^3}{48 \times 6 \times 10^3 \times 161.3 \times 10^6}$$

$$W = 7.25 \times 10^3 \text{ N}$$

⇒ A timber beam of rectangular section has a span of 4.8 mts and is simply supported at its ends. It is required to carry a total load of 45 KN/m uniformly distributed over the whole span. find the values of breadth (b) and depth (d) of the beam, if max. bending stress is not to exceed 7 MPa. and max. deflection is limited to 9.5mm. Take E for timber as 10.5 GPa.

Sol:- Given:-

$l = 4.8m \Rightarrow 4.8 \times 10^3$ mm

Total load $W \Rightarrow wl \Rightarrow 45 \text{ kN/m} = 45 \times 10^3$ N.

Max. bending stress $\sigma_{b\,max} = 7$ MPa

$y_c = 9.5$ mm

$E = 10.5$ GPa $\Rightarrow 10.5 \times 10^3$ N/mm²

$$M = \frac{Wl^2}{8}$$

$$= \frac{wl \times (l)}{8} \Rightarrow \frac{W \times l}{8}$$

$$= \frac{45 \times 10^3 \times 4.8}{8}$$

$$M = 27 \times 10^6 \text{ N-mm}$$

5

(3.1)(3.2)

$$I = \frac{bd^3}{12}$$

$$y = \frac{d}{2}$$

$\therefore$ Max. bending stress $[\sigma_{b_{max}}]$

$$7 = \frac{M}{I} \times y$$

$$= \frac{27 \times 10^6}{\frac{bd^3}{12}} \times \frac{d}{2}$$

$$7 = \frac{162 \times 10^6}{bd^2}$$

$$bd^2 = \frac{162 \times 10^6}{7} = 23.14 \times 10^6$$

$\therefore$ Max deflection (y_c)

$$9.5 = \frac{5 wl^4}{384 EI}$$

$$= \frac{5(wl)l^3}{384 EI}$$

$$= \frac{5(45 \times 10^3) \times (4.8 \times 10^3)^3}{384 \times (10.5 \times 10^3) \times \frac{bd^3}{12}}$$

$$9.5 = \frac{74.1 \times 10^9}{bd^3} \longrightarrow (2)$$

$$bd^3 = 7.8 \times 10^9 \longrightarrow (2)$$

dividing the eqn (2) by (1)

$$d = \frac{7.8 \times 10^9}{23.14 \times 10^6} = 337 \text{mm}$$

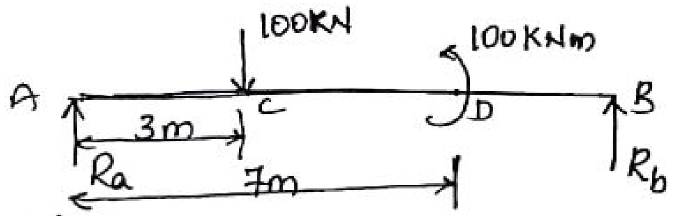
$$b(337)^2 = 23.14 \times 10^6$$

$$b = \frac{23.14 \times 10^6}{(337)^2}$$

$$= \underline{\underline{204 \text{ mm}}}$$

- ⇒ A simply supported beam of 10m length carries a point load of 100 KN. and a pure moment of 100 KN-m. at 3m and 7m respectively from the left end. find the slopes at simply supported ends and the deflection under the point load. Also find the position and magnitude of maximum deflection. Take $E = 210$ GPa. and $I = 180 \times 10^6$ mm⁴.

Sol:-



$\Sigma F_v = 0$

$\therefore \quad R_a + R_b = 100$

$\Sigma M_A = 0$

$$-R_b \times 10 - 100 + 100 \times 3 = 0$$

$$-R_b \times 10 - 100 = -100 \times 3$$

$$R_b \times 10 = -300 + 100$$

$$-R_b \times 10 = -200$$

$$R_b = \cancel{100}\ 20 \text{ KN}$$

$$\therefore R_a = 100 - 20$$

$$\therefore R_a = 80 \text{ KN}.$$

Using Macauly's Method, we have.

$$EI\frac{d^2y}{dx^2} = 80x \,\vdots\, -100(x-3) \,\vdots\, -100(x-7)^0 \quad \text{—— ①}$$

Integrating the above equ ①

$$EI\frac{dy}{dx} = 80\frac{x^2}{2} + C_1 \,\vdots\, -100\frac{(x-3)^2}{2} \,\vdots\, -100\frac{(x-7)^2}{2}$$

$$\therefore EI\frac{dy}{dx} = 40x^2 + C_1 \,\vdots\, -50(x-3)^2 \,\vdots\, -50(x-7)^2 \quad \text{—— ②}$$

Integrating once again

$$EI \cdot y = 40\frac{x^3}{3} + C_1 x + C_2 \,\vdots\, -50\frac{(x-3)^3}{3} \,\vdots\, -100\frac{(x-7)^2}{2}$$

Now at $x=0$, $y=0$, giving $C_2 = 0$

Also at $x=10$, then $y=0$

$$0 = \frac{40(10)^3}{6} + 10C_1 \,\vdots\, -50\frac{(10-3)^3}{3} \,\vdots\, -100\frac{(10-7)^2}{2}$$

$$= \frac{40000}{6} + 10C_1 - \frac{17150}{3} - 450$$

$$10C_1 = -7166.66$$

$$\therefore C_1 = -716.66.$$

Hence $EI\frac{dy}{dx} = 40x^2 + 716.66 \,\vdots\, -50(x-3)^2 \,\vdots\, \cancel{-50(x-7)^2}$ $-100(x-7)$

and $EI\, y = \frac{40x^3}{3} - 716.66x \,\vdots\, -50\frac{(x-3)^3}{3} - 50(x-7)^2$

at $x=0$, slope

$$EI\frac{dy}{dx} = 40\times 0 - 716.66$$

Now $EI = 210\times10^9 \times 180\times10^6 \times 10^{-12}\ Nm^2$

$$= 210 \times 180 \times 10^3$$
$$= 3.78 \times 10^7$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=0} = \frac{-716.66}{3.78 \times 10^7} = -1.9 \times 10^{-5}$$

Slope at $x = 10$

$$EI \frac{dy}{dx} = 40 \times 10^2 - 50(10-3)^2 - 100(10-7) - 716.66$$
$$= 4000 - 2450 - 300 - 716.66$$
$$= 533.34$$

$$\left.\frac{dy}{dx}\right|_{x=10} = \frac{533.34}{3.78 \times 10^7} = 1.47 \times 10^{-5}$$

When there is maximum deflection, $\frac{dy}{dx} = 0$

$$\therefore 40x^2 - 716.66 - 50(x-3)^2 - 100(x-7)$$
$$40x^2 - 716.66 - 50(x^2 - 6x + 9) - 100x + 700 = 0$$
(or) $$40x^2 - 50x^2 + 300x - 450 - 100x + 700 - 717 = 0$$
$$10x^2 - 200x + 467 = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{200 \pm \sqrt{(200)^2 - (4 \times 10 \times 467)}}{2 \times 10}$$

$x = 17.3\,m$ which is greater than $10m$

$\therefore$ So not acceptable.

Again $40x^2 - 50(x-3)^2 - 717 = 0$

$$40x^2 - 50x^2 + 300x - 450 - 717 = 0$$

$$10x^2 - 300x + 1167 = 0$$

$$x = \frac{300 \pm \sqrt{(300)^2 - 4 \times 10 \times 1167}}{2 \times 10}$$

$$= 4.6\,m.$$

This is with in 3 & 7m. so acceptable.

$\therefore$ Maximum deflection is at $x = 4.6m$.

$$EIy = \frac{40(4.6)^3}{3} - 716.6 \times 4.6 - \frac{50(4.6-3)^3}{3}$$

$$= 1297.81 - 3298.2 - 68.26$$

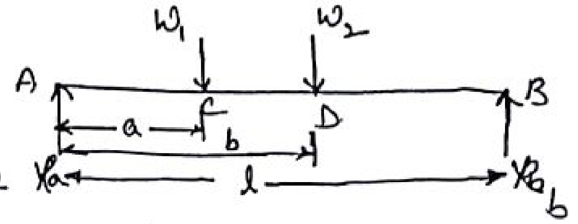
$$= -2068.65.$$

$$\therefore y_{max} = \frac{-2068.65}{3.78 \times 10^7} = 5.47 \times 10^{-5} m$$

Macaulay's Method.

This is a convenient method for determining the deflections of a beam subjected to point loads. or in general discontinuous loads. This method mainly consists the bending moment at any section is expressed and in the manner in which the integration is carried out.

fig shows a simply supported beam AB of span l and carrying the loads W_1 & W_2 at C and D ~~at~~ a distances a & b from the end A.



Let R_a & R_b be the vertical reactions at A & B.

At any section b/n A & C distant 'x' from A.

$$B.M = R_a x$$

This expression for the B.M holds good for all values of x b/n $x=0$ & $x=a$.

At any section b/n C & D distant 'x' from A.

$$B.M = R_a x - W_1(x-a)$$

This expression for B.M holds good for all values of x b/n $x=a$ & $x=b$.

At any section b/n D & B distance 'x' from A.

$$B.M = R_a x - W_1(x-a) - W_2(x-b)$$

This expression holds good for all values b/n $x=b$ & $x=l$.

In general at any section the B.M is

$$M_x = EI\frac{d^2y}{dx^2}$$

$$= R_a x \,|\, - w_1(x-a) \,|\, - w_2(x-b) \quad \text{--- (1)}$$

The manner in which the above expression should be noted. The magnitude of x goes on increasing so that the law of loading changes, additional expressions appears.

For values of x b/n $x>0$ & $x=a$, only first term should be considered.

For values of x b/n $x=a$ & $x=b$, only first two terms of the above expression should be considered.

For values of x b/n $x=b$ & $x=l$, all the terms should be considered.

Integrating the eqn (1) we get, slope eqn.

$$EI\frac{dy}{dx} = R_a\cdot\frac{x^2}{2} + C_1 \,\Big|\, - w_1\frac{(x-a)^2}{2} \,\Big|\, - w_2\frac{(x-b)^2}{2} \quad \text{--- (2)}$$

IMP

* The constant of integration C_1 should be written after the first term of the expression.
* The quantity $(x-a)$ should be integrated as $\frac{(x-a)^2}{2}$ and not as $\frac{x^2}{2} - ax$.

  Ily $(x-b)$ should be integrated as $\frac{(x-b)^2}{2}$ not as $\frac{x^2}{2} - bx$.
* The constant C_1 is valid for all values of x.

Integrating the eqn (2), we get ~~slope~~ deflection eqn.

$$EI\cdot y = R_a\frac{x^3}{6} + C_1x + C_2 \,\Big|\, - w_1\frac{(x-a)^3}{6} \,\Big|\, w_3\frac{(x-b)^3}{6} \quad \text{--- (3)}$$

$(x-a)^2$ has been integrated to $\frac{(x-a)^3}{3}$.

$(x-b)^2$ " " " " $\frac{(x-b)^3}{3}$.

The constant C_2 is written after $C_1 x$.

" " C_2 is valid for all values of x.

The C_1 & C_2 can be evaluated if the end conditions are known.

For SSB the deflection is zero at A & B.

i.e. at $x=0$, and $x=l$, y is zero.

Putting $x=0$, and $y=0$ in deflection eqn, the constant $C_2=0$.

Putting $x=l$ and $y=0$ " " " the C_1 can be evaluated. Once the constants C_1 & C_2 are known, the slope and the deflection can be determined at any section.

Prob: A beam of length 6m is simply supported at its ends and carries two point loads of 48kN and 40kN at a distance of 1m and 3m respectively from the left support.

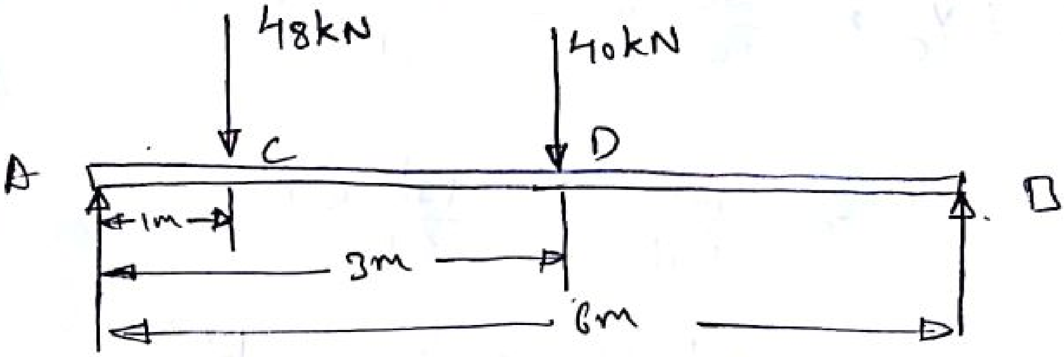
Find:

i) deflection under each load,

ii) Maximum deflection, and.

iii) the point at which maximum deflection occurs

Given $E = 2 \times 10^5$ N/mm^2 and $I = 85 \times 10^6$ mm^4



First calculate R_A & R_B

$$R_A + R_B = 88 \text{ kN}$$

Taking moments at A, we get

$$R_B \times 6 = 48 \times 1 + 40 \times 3$$

$$= \frac{168}{6} = 28 \text{ kN}$$

$R_A = 60\,kN$

Considering the section X in the last part of the beam (ie in length DB) at a distance x from the left support A, the B.M at this section is given by.

$$EI\frac{d^2y}{dx^2} = R_A x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3).$$

$$= 60x \;\vdots\; -48(x-1) \;\vdots\; -40(x-3)$$

Integrating the above equation

$$EI\frac{dy}{dx} = \frac{60x^2}{2} + C_1 \;\vdots\; -\frac{48(x-1)^2}{2} \;\vdots\; -\frac{40(x-3)^2}{2}.$$

$$EI\frac{dy}{dx} = 30x^2 + C_1 \;\vdots\; -24(x-1)^2 \;\vdots\; -20(x-3)^2 \quad \text{---- (i)}$$

Integrating the above equation again, we get.

$$EIy = \frac{30x^3}{3} + C_1 x + C_2 \;\vdots\; -\frac{24(x-1)^3}{3} \;\vdots\; -\frac{20(x-3)^3}{3}.$$

$$EIy = 10x^3 + C_1 x + C_2 \;\vdots\; -8(x-1)^3 \;\vdots\; -\frac{20}{3}(x-3)^3 \quad \text{--- (ii)}$$

To find the values of C_1 and C_2, use two boundary conditions.

i) at $x=0$, $y=0$

ii) at $x=6m$, $y=0$

Substituting the (i) boundary conditions in eq (2)

$$0 = 0 + 0 + C_2$$

$$C_2 = 0$$

Substituting the second boundary condition ie $x = 6$ and $y = 0$ in equation (ii) and consider the complete equation.

$$0 = 10 \times 6^3 + C_1 \times 6 + 0 - 8(6-1)^3 - \frac{20}{3}(6-3)^3$$

$$= 10 \times 36 \times 6 + 6C_1 - 8(5)^3 - \frac{20}{3}3^3$$

$$= 980 + 6C_1$$

$$C_1 = \frac{-980}{6}$$

$$C_1 = -163.33.$$

Now substituting the values of c_1 and c_2 in equation (ii), we get

Deflection eq

$$EIy = 10x^3 - 163.33x \; \vdots \; - 8(x-1)^3 \; \vdots \; - \frac{20}{3}(x-3)^3 \qquad \text{— (iii)}$$

Now substituting the value of c_1 in equation (i) we get

Slope equation

$$EI\frac{dy}{dx} = 30x^2 - 163.33 - 24(x-1)^2 - 20(x-3)^2 \qquad \text{— (iv)}$$

i) a) Deflection at point C, $x = 1$ substitute in eq iii

$$EI\,Y_c = 10\times1^3 - 163.33\times1$$
$$= 10 - 163.33 - 153.33\ kNm^3$$
$$= -153.33\times10^{12}\ Nmm^2.$$
$$Y_c = \frac{-153.33\times10^{2}}{2\times10^5\times85\times10^6}$$
$$= -9.019\ mm.$$

negative sign indicates deflection in downwards

b) Deflection under second load at D, $x = 3m$ substitute in equation (iii)

$$EI\,Y_D = 10\times3^3 - 163.33\times3 - 8(3-1)^3$$
$$= \frac{-283.99\times10^{12}}{2\times10^5\times85\times10^6} = -16.7\ mm$$

ii) <u>Maximum Deflection</u>.: The deflection is maximum at a section between C and D, at maximum deflection $\frac{dy}{dx} = 0$

Hence equating (i)

$$30x^2 + C_1 - 24(x-1)^2 = 0 \quad (C_1 = -163.33)$$
$$6x^2 + 48x - 187.33 = 0$$

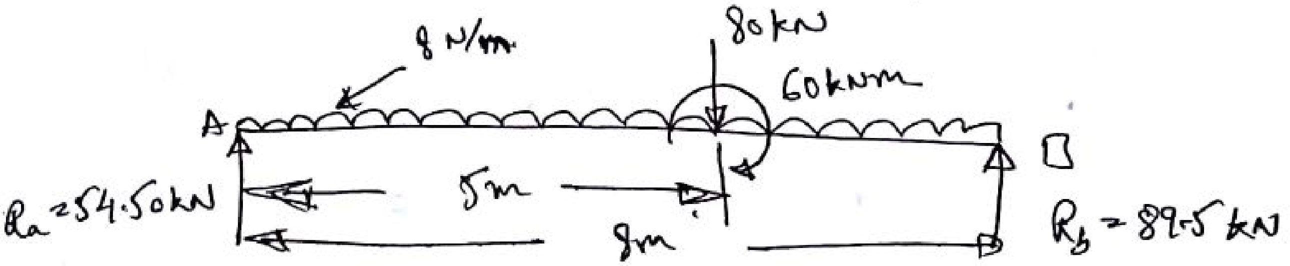
by solving quadratic equation

$$x = \frac{-48 \pm \sqrt{48^2 + 4\times6\times187.33}}{2\times6} = 2.87m$$

Now substituting $x = 2.87m$ in equation (iii) upto 2nd dotted line

$$EI\,Y_{max} = 10\times2.87^3 - 163.33\times2.87 - 8(2.87-1)^3$$
$$= \frac{-284.67\times10^{12}\ Nmm^3}{2\times10^5\times85\times10^6}$$
$$Y_{max} = -16.745\ mm.$$

Prob: A steel beam is simply supported at the ends on a span of 8 metres, and carries a uniformly distributed load of 8 kN/m on the whole span. In addition, a connection made to the beam at 5 metres from the left end exerts a downward load of 80 kN together with a clockwise couple of 60 kNm acting in the plane of bending of the beam. Determine the location and magnitude of the maximum deflection. I_{xx} for the beam section $= 4.79 \times 10^8$ mm^4 and $E = 200$ kN/mm^2



First calculate R_a & R_b

Taking moments about the right end B,

$$R_a \times 8 + 60 = 8 \times 8 \times 4 + 80 \times 3$$

$$R_a = 54.50 \text{ kN}$$

Taking moments about the left end A.

$$R_b \times 8 = 80 \times 5 + 8 \times 8 \times 4 + 60$$

$$R_b = 89.5 \text{ kN}$$

The equation to the deflected shape of the beam is given by

Anti clock + | Anti clock −

$$EI\frac{d^2y}{dx^2} = B.M.$$

$$EI\frac{d^2y}{dx^2} = 54.50x - 4x^2 \;\vdots\; + 60(x-5)^0 - 80(x-5)$$

Integrating, we get,

$$EI\frac{dy}{dx} = 27.25x^2 - \frac{4}{3}x^3 + C_1 \;\vdots\; + 60(x-5) - 40(x-5)^2$$

Integrating again, we get,

$$EIy = \frac{27.25}{3}x^3 - \frac{x^4}{3} + C_1x + C_2 \;\vdots\; + 30(x-5)^2 - \frac{40}{3}(x-5)^3$$

boundary condition At $x = 0, y = 0$ $\therefore C_2 = 0$

At $x = 8m$, $y = 0$

$$\therefore 0 = \frac{27.25 \times 8^3}{3} - \frac{8^4}{3} + 8C_1 + 270 - 360$$

$$C_1 = -399.417$$

Assuming the deflection to be maximum in the range AC and equating the slope to zero, we get,

$$EI\cdot\frac{dy}{dx} = 27.25x^2 - \frac{4}{3}x^3 + C_1$$

$$\frac{dy}{dx} = 0; \quad C_1 = -399.41$$

$$x = 4.31m.$$

The value of x obtained is less than 5m and is therefore in the range AC. Hence the position of maximum deflection determined is correct.

Put $x = 4.31\,m$ in the deflection equation.

$$EIy_{max} = \frac{27.25}{3}(4.31)^3 - \frac{4.31^4}{3} - 399.417 \times 4.31$$

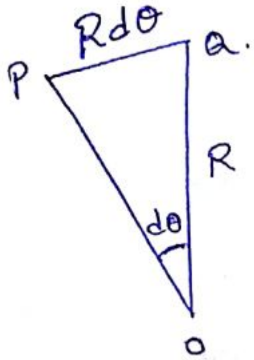
$$= -1189.271$$

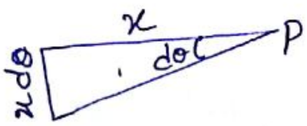
$$\therefore y_{max} = -\frac{1189.271 \times 10^9}{200 \times 4.79 \times 10^8} = \underline{\underline{-11.58\,mm}}$$

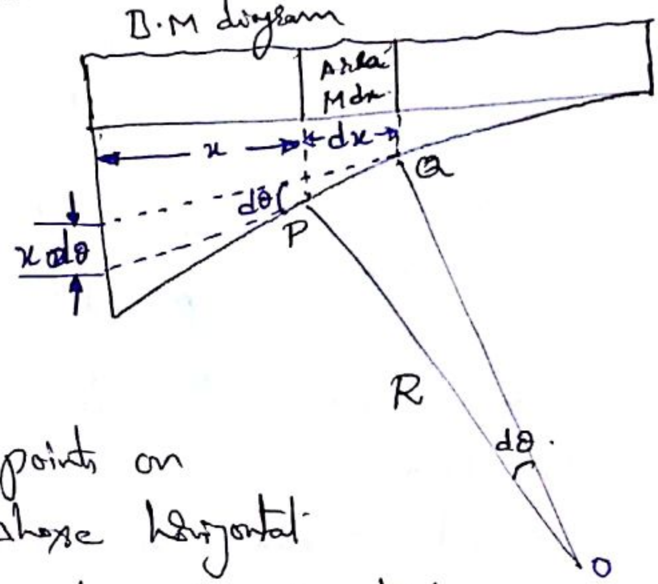
Moment Area Method – Mohr's Theorems

Let AB represent part of the deflected form of a beam of uniform section.

Let A be a point of zero slope and zero deflection.







Let P and Q be two points on the deflected curve whose horizontal distances from B are x and $x + dx$ respectively.

Let the angle between the tangents at P and Q be $d\theta$. Obviously the angle between the normals at P and Q will also be equal to $d\theta$.

Let R be the radius of curvature of the elemental part PQ.

$$d\theta = \frac{PQ}{R}$$

$$PQ = R\,d\theta = dx \qquad d\theta = \frac{dx}{R}$$

But
$$\frac{M}{I} = \frac{E}{R} \qquad \text{——— (1)}$$

~~But~~ where M is the bending moment at any section between P and Q.

$$d\theta = dx \times \frac{1}{R} \qquad \frac{1}{R} = \frac{M}{EI}.$$

$$= dx \cdot \frac{M}{EI}.$$

Since A is point of zero slope, the total slope at B is given by

$$\Theta = \frac{1}{EI} \sum_{x=0}^{x=BA} M dx.$$

$$= \frac{1}{EI} \text{ (area of the B.M diagram between A and B).}$$

In case, the slope at A is not zero, we have,
Total change in slope b/w B and A equals the area of B.M diagram between B and A divided by the flexural rigidity 'EI'

Deflection, due to the bending of the portion PQ

$$dy = x\, d\theta$$

From the equation $\frac{M}{I} = \frac{E}{R}$

$$d\theta = \frac{M}{EI} dx. \text{ substituting in } dy = x\,d\theta$$

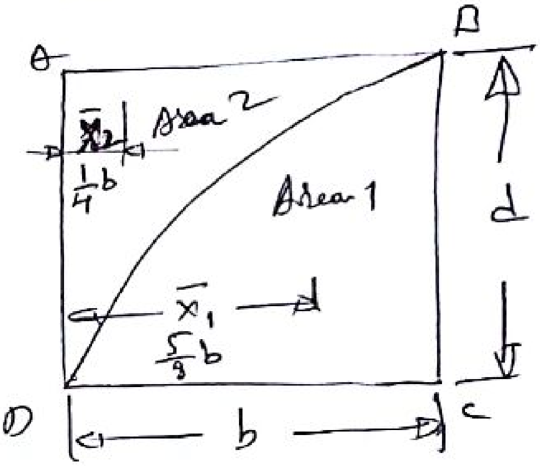
$$dy = \frac{M}{EI} x dx \quad \text{—— (ii)}$$

$\therefore$ Total deflection at B due to bending of all elemental portions like PQ.

$$Y = \frac{1}{E.I} \sum_{x=0}^{x=BA} M_x \cdot dx.$$

$$Y = \frac{1}{EI}$$ (the first moment of the area of the B.M diagram).

For a simply supported beam with UDL. the BMD. being parabolic



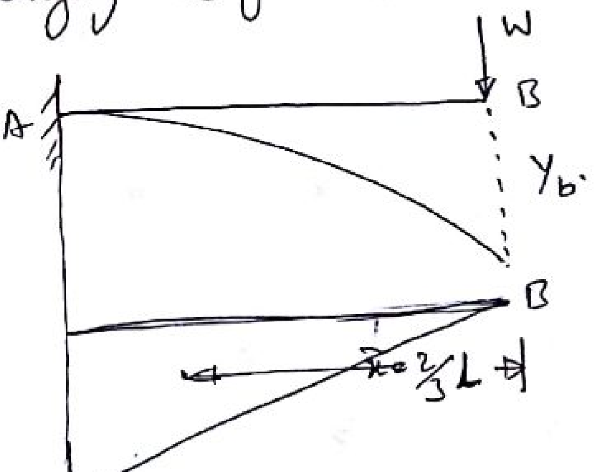
$$\text{Area DBC} = A_1 = \frac{2}{3}bd$$

$$\bar{x}_1 = \frac{5}{8}b$$

$$\text{Area ABDA} = \frac{1}{3}bd$$

$$\bar{x}_2 = \frac{1}{4}b$$

i) Cantilever carrying a point load at the free end.



$$\theta_b = \frac{\text{Area of B.M. diagram between A and B}}{EI}$$

$$\text{Area of B.M diagram} = \frac{1}{2} l \cdot Wl = \frac{Wl^2}{2}.$$

$$\theta_b = \frac{wl^2}{2EI}$$

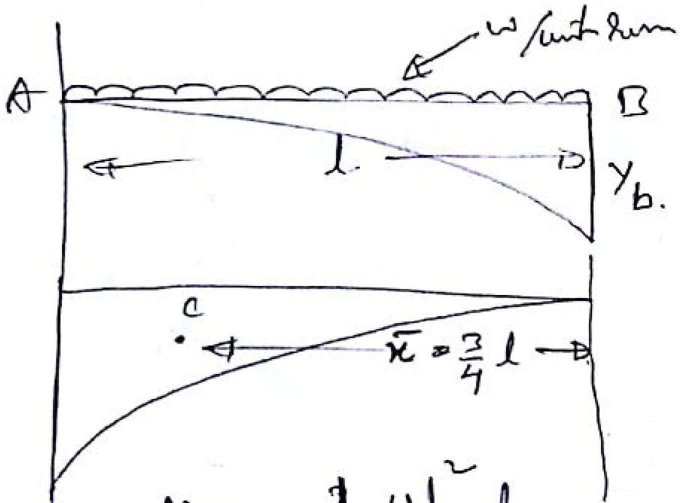
$$Y_b = \frac{A\bar{x}}{EI}.$$

$$\bar{x} = \frac{2}{3}l.$$

$$Y_b = \frac{wl^2}{2EI}.\frac{2}{3}l$$

$$= \frac{wl^3}{3EI}.$$

Cantilever beam with a UDL



$$\text{Area} = \frac{1}{3}\frac{wl^2}{2}\times l$$

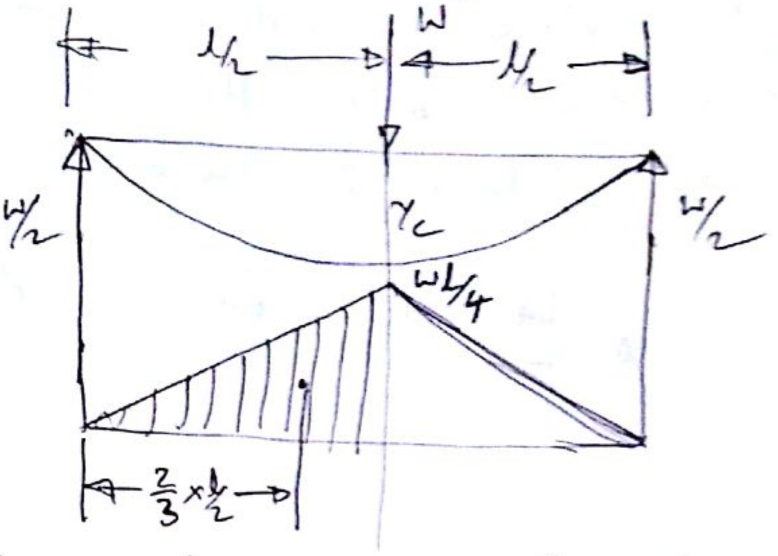
$$= \frac{wl^3}{6}$$

$$\theta_b = A/EI = \frac{wl^3}{6EI}$$

$$Y_b = \frac{A\times\bar{x}}{EI}. = \frac{\frac{wl^3}{6}\times\frac{3}{4}\times l}{EI}$$

$$= \frac{wl^4}{8EI}.$$

Simply supported beam carrying a point load at mid span.



$$A = \frac{1}{2}\times\frac{wl}{4}\times\frac{l}{2} = \frac{wl^2}{16}$$

$$\theta = \frac{\text{Area}}{EI} = \frac{wl^2}{16EI}$$

$$Y_c = \frac{\text{Area}\times\bar{x}}{EI} \qquad \left[\begin{array}{l}\bar{x} = \frac{2}{3}\times\frac{l}{2} \\ \bar{x} = \frac{l}{3}\end{array}\right]$$

$$= \frac{\frac{wl^2}{16}\times\frac{l}{3}}{EI} = \frac{wl^3}{48EI}$$

Simply supported beam carrying a uniformly distributed load

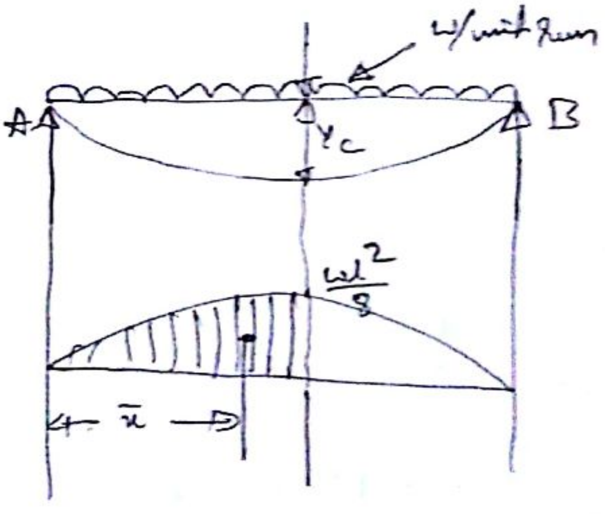
$$\text{Area} = \frac{2}{3}\frac{wl^2}{8}\times\frac{l}{2}$$

$$= \frac{wl^3}{24}$$

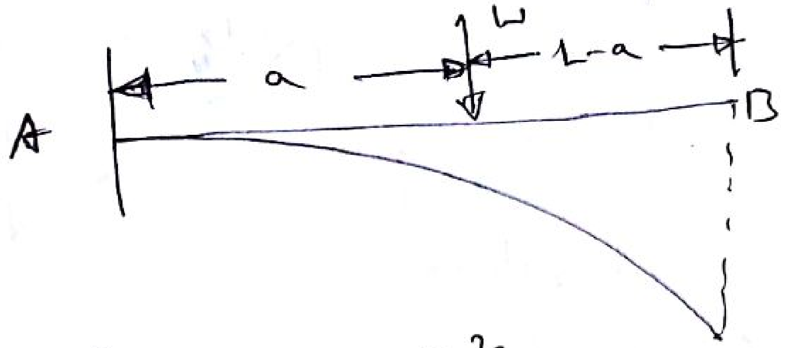
$$\bar{x} = \frac{5}{8}\times\frac{l}{2} = \frac{5l}{16}$$

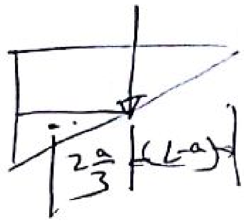
$$\theta = \frac{wl^3}{24EI}$$

$$Y_c = \frac{\frac{wl^3}{24}\times\frac{5l}{16}}{EI} = \frac{5wl^4}{384EI}$$



Prob A cantilever of length l carries a point load W at a distance a from the fixed end. Find the slope and deflection at the free end.





$$A = \frac{1}{2} \cdot a \cdot Wa = \frac{Wa^2}{2}.$$

$$\frac{3l - 3a + 2a}{3}$$

$$\text{Slope at } B = \frac{A}{EI} = \frac{Wa^2}{2EI}$$

$$\frac{3l - a}{3}$$

Deflection at B.

$$= \frac{\text{Moment of the area of BM diagram b/w A\&B about B}}{EI}$$

$$= \frac{A\bar{x}}{EI} = \frac{\frac{Wa^2}{2}\left[(l-a) + \frac{2a}{3}\right]}{EI}$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L-a)$$

Relation between Maximum Bending stress and Maximum deflection

Case i) Simply supported beam carrying a point load at midpoint

Maximum bending moment $= M = \frac{Wl}{4}$

Let the section be symmetrical about the neutral axis.

Section modulus $= Z = \frac{I}{d/2} = \frac{2I}{d}$

Maximum bending stress $= \sigma$ or $f = \frac{M}{Z}$

$$= \frac{Wl}{4} \times \frac{d}{2I}$$

$$f = \frac{Wld}{8I} \quad \text{— (1)}$$

Maximum deflection $= \delta = \frac{Wl^3}{48EI}$ — (2)

$$\frac{f}{\delta} = \frac{1}{2} = \frac{\cancel{Wl}d}{\cancel{8I}} \times \frac{\overset{6}{\cancel{48EI}}}{\cancel{Wl^3}^{2}}$$

$$\frac{f}{\delta} = \frac{6Ed}{l^2}$$

$$\boxed{\delta = \frac{fl^2}{6Ed}}$$

$$\frac{\delta}{l}\frac{d}{l} = \frac{1}{6}\frac{f}{E}$$

Case 2: simply supported beam of span l carrying a uniformly distributed load w per unit run over the whole span.

$$\text{Maximum bending moment} = \frac{wl^2}{8}$$

$$\text{section modulus} = Z = \frac{2I}{d}$$

$$\text{Maximum bending stress} = f = \frac{M}{Z}$$

$$= \frac{wl^2}{8} \times \frac{d}{2I}$$

$$f = \frac{wl^2}{16}\frac{d}{I} \quad \text{--- (1)}$$

$$\text{Maximum deflection } \delta = \frac{5wl^4}{384EI} \quad \text{--- (2)}$$

$$\frac{f}{\delta} = \frac{\cancel{w}l^2 d}{16\cancel{I}} \times \frac{384E\cancel{I}}{5\cancel{w}l^{4}}$$

$$= \frac{24dE}{5l^2}$$

$$\delta = \frac{5fl^2}{24dE}$$

$$\frac{\delta}{l}\frac{d}{l} = \frac{5}{24}\frac{f}{E}$$

UNIT - VI

TORSION & COLUMNS

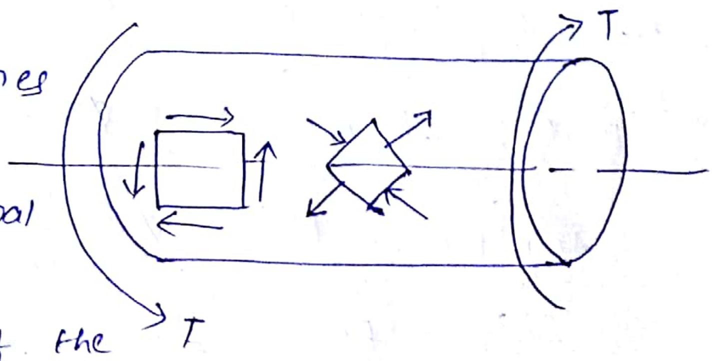
A shaft of circular ~~cro~~ section is said to be in pure torsion when it is subjected to equal and opposite end couples whose axes coincide with the axis of the shaft. Since all sections of the shaft are identical and are subjected to the same torque we say the shaft is in pure torsion.

While a beam bends as an effect of a bending moment, a shaft twists as an effect of torsion.

The angle of twist changes along the longitudinal axis of the shaft. For a shaft of uniform radius and subjected to the same torque, the angle of twist will vary linearly from one end to the other end.

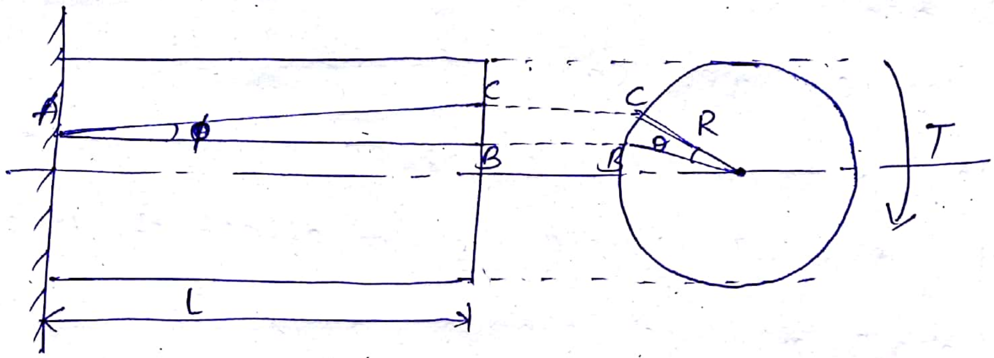
Diagonal Tensile and compressive stresses:

shear stresses acting on the cross-sectional planes and on the longitudinal planes give rise to diagonal tensile and diagonal compressive stresses of the same magnitude as the shear stresses. These tensile and compressive stresses are oriented at 45° to the longitudinal axis and are obviously maximum at the surface of the shaft in torsion.



Theory of pure torsion (or) Torsion equation:

Fig. shows a solid cylindrical shaft of radius R and length l subjected to a couple (or) a twisting moment T at the one end, while the other end is held (or) fixed by the balancing couple of the same magnitude.



The torsion:

Let AB be a line on the surface of the shaft and parallel to the axis of the shaft before deformation. After the application of torque, the deformation takes place as AC, thus cross-section will be twisted through angle θ and surface by angle ϕ.

Here, shear strain $\phi = \frac{BC}{L}$

We know that $\phi = \frac{\tau}{G}$

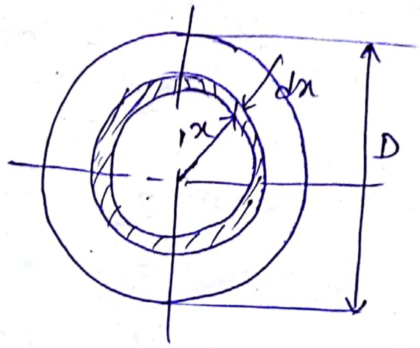
$$\therefore \frac{BC}{L} = \frac{\tau}{G} \qquad [\because BC = R\theta]$$

$$\frac{R\theta}{L} = \frac{\tau}{G}$$

$$\therefore \frac{\tau}{R} = \frac{G\theta}{L} \longrightarrow (1)$$

For solid circular shaft

Now, consider an elementary ring of thickness dx at radius x and let the shear stress at this radius be τ_x



Turning force on the elementary ring $= \tau_x \times 2\pi x\, dx$

Turning moment due to this turning force $(dT) = \tau_x \times 2\pi x\, dx \times x$

Total turning moment

$$\int dT = \int_0^R \tau_x \times 2\pi x^2 dx .$$

$$\left[\because \frac{\tau}{R} = \frac{\tau_x}{x} \qquad \tau_x = \frac{\tau}{R} \times x \right]$$

$$= \int_0^R \frac{\tau x}{R} \times 2\pi x^2 dx$$

$$T = 2\pi \times \frac{\tau}{R} \times \int_0^R x^3 dx$$

$$T = \frac{2\pi \tau}{R} \times \left(\frac{x^4}{4} \right)_0^R$$

$$\left[\frac{\cancel{2}\pi \tau}{\cancel{R}} \times \frac{\cancel{R^4}}{\cancel{4}2} = \frac{\pi R^3}{2} \times \tau \right]$$

$$T = \frac{\cancel{2}\pi \tau}{R} \times \frac{R^4}{\cancel{4}2}$$

For solid shaft.

$$J = \frac{\pi R^4}{2}$$

$$T = \frac{\tau}{R} \times \frac{\pi R^4}{2}$$

$$T = \frac{\tau}{R} \times J$$

$$\frac{T}{J} = \frac{\tau}{R} \longrightarrow (2)$$

From (1) & (2) $\boxed{\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}} \longrightarrow$ torsion equation.

Where T - Maximum twisting torque

R - Radius of the shaft

J - polar moment of inertia

τ - shear stress.

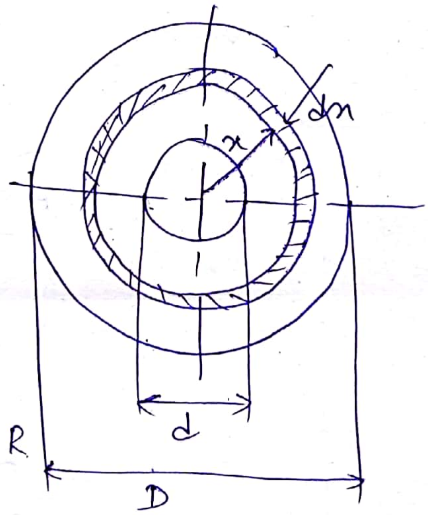
G - Modulus of rigidity

θ - angle of twist (radians)

l - length of the shaft.

<u>For hollow circular shaft</u>

consider a hollow circular shaft subjected to a torque T.



let R - outer radius of the shaft

r - Inner radius of the shaft

τ - shear stress at radius R

We have, $dT = \tau_x \times 2\pi x\, dx \times x$.

Integrating on both sides

$$\tau_x = \frac{\tau}{R} \times x.$$

$$\int dT = \int_r^R \tau_x \times 2\pi x \times dx \times x$$

$$= \int_r^R \frac{\tau}{R} \times x \times 2\pi x^2 dx$$

$$= \frac{2\pi\tau}{R} \int_r^R x^3 dx$$

$$T = \frac{2\pi\tau}{R} \left(\frac{x^4}{4}\right)_r^R$$

$$T = \frac{\cancel{2}\pi\tau}{R} \times \frac{R^4 - r^4}{\cancel{4}2}$$

$$T = \frac{\pi \tau}{R} \times \left(\frac{R^4 - r^4}{2}\right) \qquad J = \frac{\pi (R^4 - r^4)}{2}$$

$$= \frac{\tau}{R} \times J$$

$$\frac{T}{J} = \frac{\tau}{R} \longrightarrow (3)$$

From (1) & (3)

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}.$$

Assumptions

The torsion equation is based on the following assumptions :

1. The material of the shaft is uniform throughout
2. The shaft circular in section remains circular after loading.
3. A plane section of shaft normal to its axis before loading remains plane after the torques have been applied.
4. The twist along the length of shaft is uniform throughout.
5. The distance b/w any two normal cross-sections remains the same after the application of torque.
6. Maximum shear stress induced in the shaft due to application of torque does not exceed its elastic limit value.

polar modulus

From $\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L}$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$T = \tau \times \frac{J}{R}$$

$$T = \tau \times Z_p$$

Z_p - polar modulus.

polar modulus of the section is thus measure of strength of shaft in torsion.

(i) For solid shaft

$$J = \frac{\pi R^4}{2}$$

$$Z_p = \frac{J}{R}$$

$$= \frac{\frac{\pi R^4}{2}}{R} = \frac{\pi R^3}{2}$$

(ii) For hollow shaft

$$J = \frac{\pi (R^4 - r^4)}{2}$$

$$Z_p = \frac{\pi (R^4 - r^4)}{2R}$$

Torsional rigidity:

From $\frac{T}{J} = \frac{G\theta}{L}$.

$$\theta = \frac{TL}{GJ}.$$

Torsional rigidity - GJ

the quantity GJ stands for the torque required to produce a twist of 1 radian per unit length of the shaft.

$\frac{GJ}{L}$ - Torsional stiffness.

It is the torque required to produce a twist of 1 radian over the length of the shaft.

Torsional flexibility is the reciprocal of torsional stiffness.

Torsional flexibility = $\frac{L}{GJ}$. It is the angle of rotation produced by a unit torque.

Modulus of rupture :

The maximum fictitious shear stress calculated by the torsion formula by using the experimentally found maximum torque required to rupture a shaft.

$$\tau_r = \frac{T_u R}{J}$$

τ_r - Modulus of rupture (or computed ultimate twisting strength

T_u - ultimate torque at failure

R - outer radius of the shaft.

power transmitted by a shaft :

let a shaft turning at N rpm transmit P kW. let the mean torque to which the shaft is subjected to be T N-m.

∴ Power transmitted (P) = Mean torque × Angle turned per second.

$$P = T \times \frac{N}{60} \times 2\pi \text{ watts}$$

$$= T \times \frac{N}{60} \times \frac{2\pi}{1000} \text{ kW}$$

$$\boxed{\therefore P = \frac{2\pi NT}{60000} \text{ kW}}$$

Sometimes angular speed is expressed as frequency f of rotation.

1 Hz = 1 revolution per sec.

$$\boxed{P = \frac{2\pi f T}{1000}}$$

Pblm In a tensile test a test piece of 25 mm diameter, 200 mm gauge length, stretched 0.0975 mm under a pull of 50 kN. In a torsion test, the same rod twisted 0.025 radian over a length of 200 mm when a torque of 0.4 kN-m was applied. Evaluate poisson's ratio and the three elastic modulii for the material.

Sol

$$\delta l = \frac{WL}{AE}$$

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} \times \left(\frac{25}{1000}\right)^2$$

$$= 4.91 \times 10^{-4} \text{ m}^2$$

$$\delta l = \frac{WL}{AE}$$

$$0.0975 \times 10^{-3} = \frac{50 \times 10^3 \times 0.2}{4.91 \times 10^{-4} \times E}$$

$$E = 208 \text{ GN/m}^2$$

$$J = \frac{\pi D^4}{32}$$

$$= \frac{\pi}{32} \times \left(\frac{25}{1000}\right)^4 = 3.835 \times 10^{-8} \ m^4$$

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$G = \frac{TL}{\theta J} = \frac{0.4 \times 10^3 \times 0.2}{0.025 \times 3.835 \times 10^{-8}} \times 10^{-9}$$

$$G = 83.44 \ GN/m^2.$$

W.K.T

$$E = 2G(1+\mu)$$

$$208 \times 10^9 = 2 \times 83.44 \times 10^9 \times (1+\mu)$$

$$\mu = 0.246.$$

$$E = 3K(1-2\mu)$$

$$208 \times 10^9 = 3 \times K(1 - 2 \times 0.246)$$

$$K = 136.4 \ GN/m^2.$$

pblm A solid circular shaft transmits 75 kw power at 200 rpm. Calculate the shaft diameter, if the twist in the shaft is not to exceed 1° in 2m length of shaft, and shear stress is limited to 50 MN/m². Take G = 100 GN/m².

Sol Given

$P = 75$ kw, $N = 200$ rpm

$\theta = 1° = 1 \times \frac{\pi}{180}$, $L = 2m$

$\tau = 50 \ MN/m^2$, $G = 100 \ GN/m^2$.

$$P = \frac{2\pi NT}{60 \times 1000}$$

$$75 = \frac{2\pi \times 200 \times T}{60 \times 1000}$$

$$T = \frac{75 \times 60 \times 1000}{2\pi \times 200} = 3581 \text{ N-m}.$$

case-1 Allowable shear stress (50 MN/m²)

$$T = \tau \times \frac{\pi}{16} \times D^3$$

$$3581 = 50 \times 10^6 \times \frac{\pi}{16} \times D^3.$$

$$D^3 = \frac{3581 \times 16}{50 \times 10^6 \times \pi}$$

$$D = 0.0714 m \text{ (or) } 71.4 mm.$$

case-2 Angle of twist (1°)

$$\frac{T}{J} = \frac{G\theta}{L}$$

$$\frac{3581}{\frac{\pi}{32} \times D^4} = \frac{100 \times 10^9 \times 1 \times \frac{\pi}{180}}{2}$$

$$D^4 = \frac{3581 \times 2 \times 180 \times 32}{\pi \times \pi \times 100 \times 10^9}$$

$$D = 0.0804 m \text{ (or) } 80.4 mm$$

From the above two cases, we find that suitable diameter for the shaft is 80.4 mm (Ans)

Shafts in series and parallel :

(i) Shafts in series :

When two shafts are connected so as to remain continuous lengthwise, they are said to be in series. In such cases, each shaft transmits the same torque. The angle of twist is the sum of the angle of twist of the two shafts connected in series.

Consider the shafts AB and BC connected in series.

let l_1 & l_2 - Lengths of AB & BC

d_1 & d_2 - diameters of AB & BC.

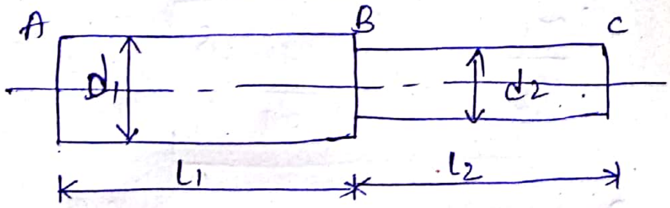
T - Torque on the shaft.

τ_1 & τ_2 - extreme shear stresses of two shafts.

θ_1 & θ_2 - angle of twists

J_1 & J_2 - polar MOI.

From



$$\frac{T}{J} = \frac{G\tau}{R} = \frac{G\theta}{L}.$$

$$\frac{T}{J_1} = \frac{\tau_1}{d_1/2} = \frac{G_1\theta_1}{l_1} \longrightarrow (1)$$

$$\frac{T}{J_2} = \frac{\tau_2}{d_2/2} = \frac{G_2\theta_2}{l_2}. \longrightarrow (2)$$

δ

From ① & ②

$$\frac{T}{J_1} = \frac{\tau_1}{d_1/2}$$

$$\frac{T}{J_1} = \frac{\tau_1}{d_1/2}$$

$$T = \frac{2\tau_1 \times J_1}{d_1} \longrightarrow ⓐ$$

$$T = \frac{2\tau_2 \times J_2}{d_2} \longrightarrow ⓑ$$

$$\frac{ⓐ}{ⓑ} \quad \frac{\frac{\cancel{2}\tau_1 J_1}{d_1}}{\frac{\cancel{2}\tau_2 J_2}{d_2}} = 1$$

$$\frac{\tau_1 J_1}{d_1} \times \frac{d_2}{\tau_2 J_2} = 1$$

$$\frac{\tau_1}{\tau_2} = \frac{J_2}{J_1} \times \frac{d_1}{d_2}.$$

$$= \frac{\frac{\cancel{\pi} d_2^4}{\cancel{32}}}{\frac{\cancel{\pi} d_1^4}{\cancel{32}}} \times \frac{d_1}{d_2}.$$

$$\boxed{\frac{\tau_1}{\tau_2} = \left(\frac{d_2}{d_1}\right)^3}.$$

$$\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1} \times \frac{l_1}{l_2} \times \frac{G_2}{G_1}$$

$$= \left(\frac{d_2}{d_1}\right)^4 \times \frac{l_1}{l_2} \times \frac{G_2}{G_1}$$

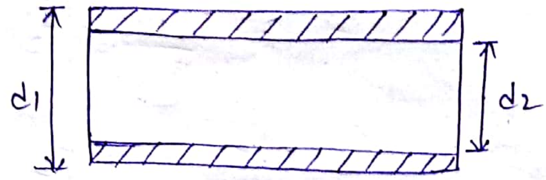
Total angle of twist $= \theta_1 + \theta_2$

$$= \frac{T l_1}{G_1 J_1} + \frac{T l_2}{G_2 J_2}.$$

$$\theta = T\left(\frac{l_1}{G_1 J_1} + \frac{l_2}{G_2 J_2}\right)$$

(ii) shafts in parallel

Fig. shows two shafts connected in parallel.



let T - torque applied on the composite shaft.

the torque gets distributed to the two shafts.

let the torques on the two shafts be T_1 & T_2.

$$\therefore T = T_1 + T_2$$

Assuming no slip b/w the two shafts, the twist will be same for each shaft.

$$\theta_1 = \theta_2 = \theta.$$

$$\frac{T_1}{J_1} = \frac{G_1\theta_1}{l_1}, \quad \frac{T_2}{J_2} = \frac{G_2\theta_2}{l_2}$$

$$T_1 = \frac{G_1\theta_1 J_1}{l_1}, \quad T_2 = \frac{G_2\theta_2 J_2}{l_2}$$

$$T = T_1 + T_2$$

$$T = \frac{G_1\theta_1 J_1}{l_1} + \frac{G_2\theta_2 J_2}{l_2}.$$

$$\left[\begin{array}{l}\theta_1 = \theta_2 = \theta \\ l_1 = l_2 = l\end{array}\right]$$

$$T = \theta\left(\frac{G_1 J_1}{l_1} + \frac{G_2 J_2}{l_2}\right)$$

$$T = \frac{\theta}{l}\left[G_1 J_1 + G_2 J_2\right]$$

$$\boxed{\theta = \frac{TL}{G_1 J_1 + G_2 J_2}}$$

$$\frac{T_1}{T_2} = \frac{G_1 \cancel{\theta_1} J_1}{\cancel{l_1}} \times \frac{\cancel{l_2}}{G_2 \cancel{\theta_2} J_2}$$

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$$

$$T = T_1 + T_2.$$

$$T_1 = \frac{G_1 J_1}{G_2 J_2} \times T_2$$

$$T = \frac{G_1 J_1}{G_2 J_2} T_2 + T_2.$$

$$T = \left(\frac{G_1 J_1 + G_2 J_2}{G_2 J_2}\right) T_2$$

$$T_2 = \left[\frac{G_2 J_2}{G_1 J_1 + G_2 J_2}\right] T$$

i.e. $$T = T_1 + \left(\frac{G_2 J_2}{G_1 J_1 + G_2 J_2}\right) T$$

$$T_1 = T - \left(\frac{G_2 J_2}{G_1 J_1 + G_2 J_2}\right) T$$

$$T_1 = \left(\frac{G_1 J_1}{G_1 J_1 + G_2 J_2}\right) T$$

Relation b/w the maximum shear stresses in the shafts.

$$\frac{\tau_1}{R_1} = \frac{T_1}{J_1}, \qquad \frac{\tau_2}{R_2} = \frac{T_2}{J_2}$$

$$\tau_1 = \frac{T_1}{J_1} \times R_1, \qquad \tau_2 = \frac{T_2}{J_2} \times R_2.$$

$$\frac{\tau_1}{\tau_2} = \frac{T_1}{T_2} \times \frac{J_2}{J_1} \times \frac{R_1}{R_2}.$$

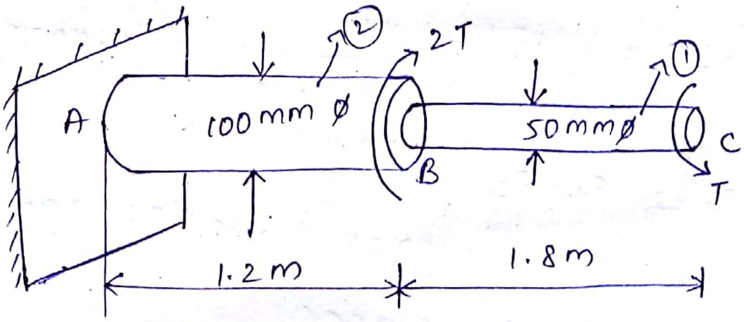
But $$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2}$$

$$\frac{\tau_1}{\tau_2} = \cancel{\frac{T_1}{T_2}} \frac{G_1 \cancel{J_1}}{G_2 \cancel{J_2}} \times \frac{\cancel{J_2}}{\cancel{J_1}} \times \frac{R_1}{R_2}$$

$$\boxed{\frac{\tau_1}{\tau_2} = \frac{G_1}{G_2} \times \frac{R_1}{R_2}}$$

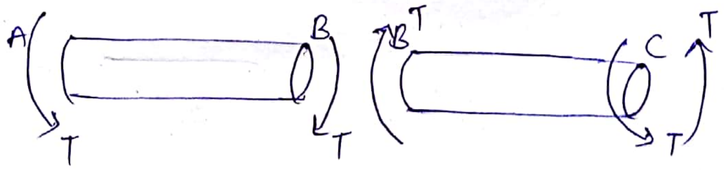
pblm ⑧

The stepped steel shaft shown in fig. is subjected to a torque T at the free end and a torque of 2T in the opposite direction at the junction of two sizes. What is the total angle of twist at the free end, if the maximum shear stress in the shaft is limited to ~~70 MN/m²~~ 70 MN/m^2. Assume the $G = 84\ GN/m^2$.



Sol

For BC



$T_1 = T$, $l_1 = 1.8\,m$

$J_1 = \frac{\pi}{32} \times (0.05)^4 = 6.136 \times 10^{-7}\ m^4$

θ_1 - Angle of twist in BC

$$\theta_1 = \frac{T_1 l_1}{G_1 J_1}$$

$$= \frac{T \times 1.8}{84 \times 10^9 \times 6.136 \times 10^{-7}}$$

For AB

$T_2 = T$, $l_2 = 1.2\,m$

$J_2 = \frac{\pi}{32} \times (0.1)^4 = 9.817 \times 10^{-6}\ m^4$.

$$\theta_2 = \frac{T_2 l_2}{G_2 J_2} = \frac{T_2 \times 1.2}{84 \times 10^9 \times 9.817 \times 10^{-6}}$$

θ_1 & θ_2 are in opposite directions, Hence θ_c is the total angle of twist at c.

$$\theta_c = \theta_1 - \theta_2$$

$$\frac{T}{J} = \frac{\tau}{R}$$

$$T = \frac{\tau \times J}{R} = \frac{70 \times 10^6 \times 6.136 \times 10^{-7}}{0.025} = 1718.1 \text{ N-m.}$$

$$\theta_1 = \frac{1718.1 \times 1.8}{84 \times 10^9 \times 6.136 \times 10^{-7}} = 0.06 \text{ rad.}$$

$$\theta_2 = \frac{1718.1 \times 1.2}{84 \times 10^9 \times 9.817 \times 10^{-6}} = 0.0025 \text{ rad.}$$

$$\theta_c = \theta_1 - \theta_2$$

$$= 0.06 - 0.0025 = 0.0575 \text{ rad.}$$

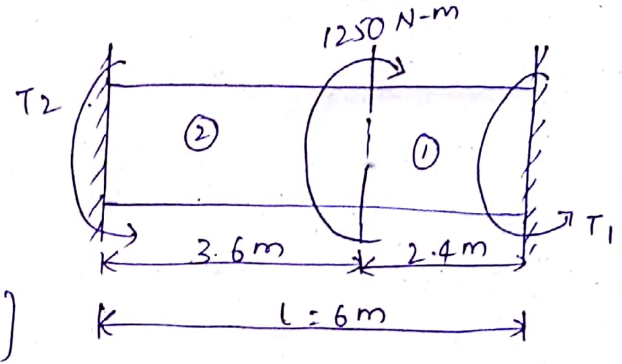
$$= 3.29 \text{ degrees.}$$

———— x ————

(9)

Pblm A solid steel shaft 6m long is securely fixed at each end. A torque of 1250 N-m is applied to the shaft at a section 2.4m from one end. What are the fixing torques set up at the ends of the shaft? If the diameter of the shaft is 40mm. what are the maximum shear stresses in the two portions? calculate also the angle of twist for the section where the torque is applied, G = 84.G N/m^2.

Sol



$$\theta_1 = \theta_2$$

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2}$$

$$[G_1 = G_2, \quad J_1 = J_2]$$

$$T_1 L_1 = T_2 L_2$$

$$T_1 + T_2 = 1250$$

$$T_2 = \frac{T_1 L_1}{L_2} = \frac{T_1 \times 2.4}{3.6}$$

$$T_1 + \frac{T_1 \times 2.4}{3.6} = 1250$$

$$T_1 (1 + 0.667) = 1250$$

$$T_1 = 749.8 \text{ N-m}$$

$$T_2 = 1250 - 749.8 = 500.2 \text{ N-m}$$

$$\theta = \theta_1 = \theta_2 = \frac{749.8 \times 2.4}{84 \times 10^9 \times \frac{\pi}{32} \times (0.04)^4} = 0.0852 \text{ rad.}$$

$$\theta = 4.88 \text{ degrees}$$

$$\tau_1 = \frac{16 T_1}{\pi D^3} = \frac{16 \times 749.8}{\pi \times (0.04)^3} \times 10^{-6} = 59.66 \text{ MN/m}^2$$

$$\tau_2 = \frac{16 T_2}{\pi D^3} = \frac{16 \times 500.2}{\pi \times (0.04)^3} \times 10^{-6} = 39.8 \text{ MN/m}^2$$

TORSION OF SHAFTS

Shaft is a machine element which is used to transmit power in machines.

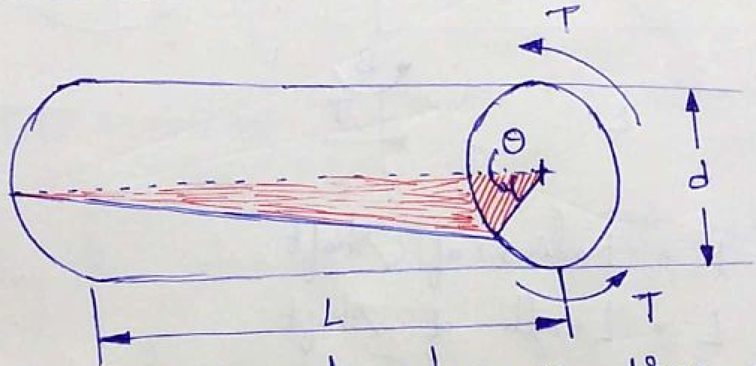
Types of shafts

1) Solid
2) Hollow
3) Tapered

Cylindrical
Square
etc....

1) Transmission shafts (transmit power)
2) Machine shafts (Integral part of the machine itself)

What is Torsion



Torsion is experienced when a force on a rod is applied to twist & turn the rod.

Effects of Torsion

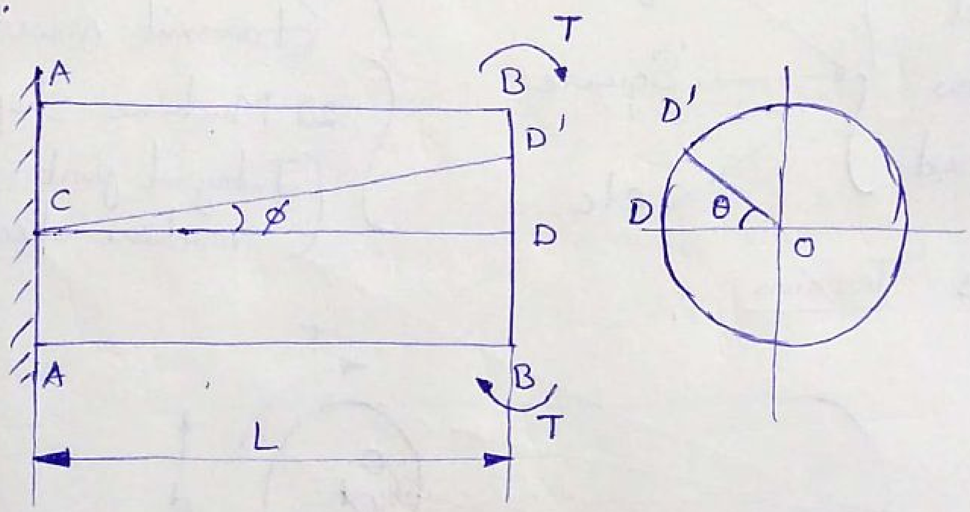
→ Torsion creates distortion.
→ It also causes loss of energy.

Assumptions made in the derivation of shear stress produced in a circular shaft subjected to torsion

1) Material of the shaft is uniform throughout.
2) The twist along the shaft is uniform.
3) The shaft is of uniform circular section throughout
4) Cross-section of the shaft is plain after twist
5) All radii which are straight before twist remain straight after twist.

Derivation of Shear stress produced in a circular shaft subjected to torsion

When a circular shaft is subjected to torsion, shear stresses are setup in the material of the shaft.



Let R = Radius of shaft

L = Length of shaft

T = Torque applied at the end BB

τ = Shear stress induced at the surface of the shaft due to torque T

C = Modulus of rigidity of the material of the shaft

ϕ = $\angle DCD'$ also equal to shear strain

θ = $\angle DOD'$ and is also called angle of twist

Now distortion at the outer surface due to torque T = DD'

∴ Shear strain at outer surface

= Distortion per unit length

$$= \frac{\text{Distortion at the outer surface}}{\text{Length of shaft}} = \frac{DD'}{L}$$

$$= \frac{DD'}{CD} = \tan\phi \quad \left(\text{if } \phi \text{ is very small then } \tan\phi = \phi\right)$$

$$= \phi$$

∴ Shear strain at outer surface,

$$\phi = \frac{DD'}{L} \text{ ——— (i)}$$

(∵ OD = R (Radius of shaft))

$$\text{Arc } DD' = OD \times \theta$$
$$= R\theta$$

Substituting the value of DD' in equation (i), we get

shear strain at outer surface

$$\phi = \frac{R \times \theta}{L} \text{ ——— (ii)}$$

Now the modulus of rigidity (C) of the material of the shaft is given as

$$C = \frac{\text{Shear stress induced (outer surface)}}{\text{Shear strain Produced (outer surface)}}$$

$$= \frac{\tau}{\frac{R\theta}{L}} = \frac{\tau \times L}{R\theta}.$$

$$C = \frac{\tau \times L}{R\theta}$$

$$\frac{C\theta}{L} = \frac{\tau}{R} \text{ ——— (iii)}$$

$$\therefore \quad \tau = \frac{R \times C \times \theta}{L}$$

Now for a given shaft subjected to a given torque (T), the values of C, θ and L are constant. Hence shear stress produced is proportional to the radius R.

$$\tau \propto R \text{ or } \frac{\tau}{R} = \text{constant} \quad \text{—— (iv)}$$

If q is the shear stress induced at a radius 'r' from the centre of the shaft then

$$\frac{\tau}{R} = \frac{q}{r} \quad \text{—— (V)}$$

But $\quad \dfrac{\tau}{R} = \dfrac{C\theta}{L}$

$$\therefore \quad \frac{\tau}{R} = \frac{C\theta}{L} = \frac{q}{r} \quad \text{—— (VI)}$$

From the equation 'IV' it is clear that shear stress at any point in the shaft is proportional to the distance of the point from the axis of the shaft.

Hence the shear stress is maximum at the outer surface and shear stress is zero at the axis of the shaft.

Maximum Torque Transmitted by a Circular Solid Shaft (Strength of a Solid Circular shaft)

The maximum torque transmitted by a circular solid shaft, is obtained from the maximum shear stress induced at the outer surface of the solid shaft.

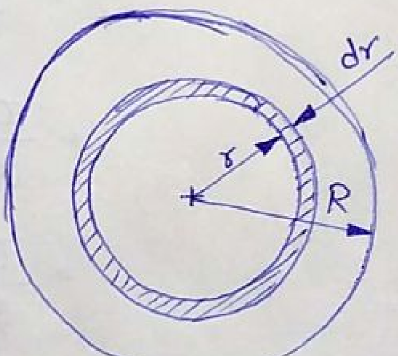
Consider a shaft subjected to a torque T.

Let τ = maximum shear stress induced at the outer surface

R = Radius of the shaft.

q = Shear stress at a radius 'r' from the centre.

Consider an elementary circular ring of thickness 'dr' at a distance 'r' from the centre as shown in the figure.



Then the area of the ring

$$dA = 2\pi r\,dr$$

From the equation $\frac{\tau}{R} = \frac{q}{r}$

$\therefore$ Shear stress at the radius r,

$$q = \frac{\tau}{R} r = \tau \frac{r}{R}$$

$\therefore$ Turning force on the elementary circular ring

= Shear stress acting on the ring × Area of ring

$$= \tau \times \frac{r}{R} \times 2\pi r\,dr$$

$$= \frac{\tau}{R} \times 2\pi r^2 dr$$

Now turning moment due to the turning force on the elementary ring,

dT = Turning force on the ring × Distance of the ring from the axis.

$$= \frac{\tau}{R} \times 2\pi r^2 dr \times r$$

$$= \frac{\tau}{R} \times 2\pi r^3 dr \quad \text{——— (i)}$$

∴ The total turning moment (or total torque) is obtained by integrating the above equation between the limits 0 and R.

$$T = \int_0^R dT = \int_0^R \frac{\tau}{R} \times 2\pi r^3 dr$$

$$= \frac{\tau}{R} \times 2\pi \int_0^R r^3 dr$$

$$= \frac{\tau}{R} \times 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{\tau}{R} \times 2\pi \times \frac{R^4}{4}$$

$$= \tau \times \frac{\pi}{2} \times R^3 \qquad \left[R = \frac{D}{2}\right]$$

$$= \tau \times \frac{\pi}{2} \times \left[\frac{D}{2}\right]^3$$

$$= \tau \times \frac{\pi}{2} \times \frac{D^3}{8} = \tau \times \frac{\pi D^3}{16}$$

$$\boxed{T = \frac{\pi}{16} \tau D^3} \quad \text{——— (2)}$$

Prob:- The shearing stress of a solid shaft is not to exceed 40 N/mm² when the torque transmitted is 20000N-m, Determine the minimum diameter of the shaft.

Given:

Maximum shear stress $\tau = 40\ N/mm^2$

Torque transmitted $T = 20000 N\text{-}m$

$= 20000 \times 10^3\ N\text{-}mm$

Let D = Minimum diameter of the shaft in mm.

$$T = \frac{\pi}{16} \tau D^3$$

$$D = \left[\frac{16T}{\tau \pi}\right]^{1/3}$$

$$= \left[\frac{16 \times 20000 \times 10^3}{\pi \times 40}\right]^{1/3}$$

$$= \underline{136.2}\ mm.$$

Torque Transmitted by a Hollow Circular Shaft

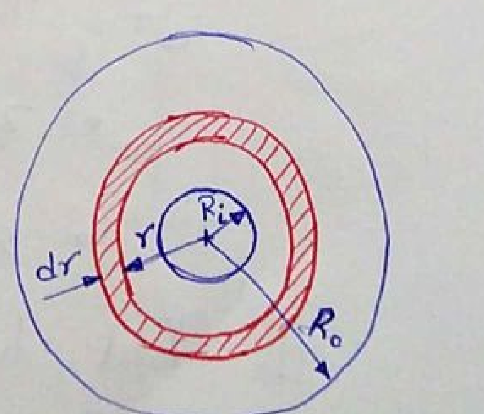
Consider a hollow shaft. Let it is subjected to a torque 'T'. Take an elementary circular ring of thickness 'dr' at a distance r from the centre

Let R_o = outer radius of the shaft.

R_i = Inner radius of the shaft

r = Radius of elementary circular ring

dr = Thickness of the ring



τ = Maximum shear stress induced at outer surface of the shaft.

q = Shear stress induced on the elementary ring

dA = Area of the elementary circular ring

$= 2\pi r \times dr$

Shear stress at the elementary ring is obtained from the equation

$$\frac{\tau}{R_o} = \frac{q}{r}$$

$$q = \frac{\tau}{R_o} \times r$$

$\therefore$ Turning force on the ring = stress $\times$ Area

$$= q \times dA$$

$$= \frac{\tau}{R_o} \times r \times 2\pi r dr$$

$$= \frac{\tau}{R_o} \times 2\pi \times r^2 dr$$

Turning moment (dT) on the ring

dT = Turning force $\times$ Distance of the ring from centre

$$= 2\pi \frac{\tau}{R_o} r^2 dr \times r$$

$$= 2\pi \frac{\tau}{R_o} r^3 dr$$

The total turning moment is obtained by integrating the above equation between the limits R_i and R_o

$$T = \int_{R_i}^{R_o} dT$$

$$= \int_{R_i}^{R_o} 2\pi \frac{\tau}{R_o} r^3 \, dr$$

$$= 2\pi \frac{\tau}{R_o} \int_{R_i}^{R_o} r^3 \, dr$$

$$= 2\pi \frac{\tau}{R_o} \left[\frac{r^4}{4} \right]_{R_i}^{R_o} = 2\pi \frac{\tau}{R_o} \left[\frac{R_o^4 - R_i^4}{4} \right]$$

$$\boxed{T = \cancel{2} \frac{\pi}{2} \tau \left[\frac{R_o^4 - R_i^4}{R_o} \right]} \quad \text{--- } (1)$$

Let D_o = outer diameter of the shaft

D_i = Inner diameter of the shaft.

Then $R_o = \frac{D_o}{2}$ and $R_i = \frac{D_i}{2}$

$$T = \frac{\pi}{2} \tau \left[\frac{\left(\frac{D_o}{2}\right)^4 - \left(\frac{D_i}{2}\right)^4}{\left(\frac{D_o}{2}\right)} \right]$$

$$= \frac{\pi}{2} \tau \left[\frac{D_o^4 - D_i^4}{16} \times \frac{2}{D_o} \right]$$

$$\boxed{T = \frac{\pi}{16} \tau \left[\frac{D_o^4 - D_i^4}{D_o} \right]} \quad \text{--- } (2)$$

Power Transmitted by Shafts

Power transmitted by the shafts can be determined by

$$\text{Power} = \frac{2\pi N T^*}{60} \text{ watts}$$

N = r.p.m of the shafts

T = Mean torque transmitted in N-m

ω = Angular speed of shaft

Then $\text{Power} = \frac{2\pi N T^*}{60} = \underline{\underline{\omega \times T}}$ $\left[\because \frac{2\pi N}{60} = \omega\right]$

Prob: Two shafts of the same material and of same lengths are subjected to the same torque, if the first shaft is of a solid circular section and the second shaft is of hollow circular section, whose internal diameter is 2/3 of the outside diameter and the maximum shear stress developed in each shaft is the same, Compare the weights of the shafts.

Given: Two shafts of the same material and same lengths transmit the same torque and develops the same maximum stress.

Let T = Torque transmitted by each shaft

τ = Max-shear stress developed in each shaft

D = outer diameter of the solid shaft.

D_o = outer diameter of the hollow shaft

D_i = Internal diameter of the hollow shaft = $\frac{2}{3}D_o$

W_s = Weight of the solid shaft

W_h = Weight of the hollow shaft.

L = Length of each shaft.

w = weight density of the material of each shaft

Torque transmitted by the solid shaft is given by

$$T = \frac{\pi}{16}\tau D^3 \quad \text{——— (1)}$$

Torque transmitted by the hollow shaft is given by

$$T = \frac{\pi}{16}\tau\left[\frac{D_o^4 - D_i^4}{D_o}\right] = \frac{\pi}{16}\tau\left[\frac{D_o^4 - \left(\frac{2}{3}D_o\right)^4}{D_o}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_o^4 - \frac{16}{81}D_o^4}{D_o}\right] = \frac{\pi}{16}\tau \times \frac{65}{81}D_o^3$$

$$= \frac{\pi}{16}\tau \times \frac{65D_o^3}{81} \quad \text{——— (2)}$$

As torque transmitted by solid and hollow shafts are equal, hence equating equations 1 & 2

$$\cancel{\frac{\pi}{16}}\cancel{\tau}D^3 = \cancel{\frac{\pi}{16}}\cancel{\tau} \times \frac{65}{81}D_o^3$$

$$D^3 = \frac{65}{81}D_o^3$$

$$D = 0.929D_o \quad \text{——— (3)}$$

Now weight of solid shaft,

$$W_s = \text{Weight density} \times \text{volume of solid shaft}$$

$$= \omega \times \text{Area of cross-section} \times \text{Length}$$

$$= \omega \times \frac{\pi}{4} D^2 \times L \quad \text{———} \quad (4)$$

Weight of hollow Shaft,

$$W_h = \omega \times \text{Area of cross section of hollow shaft} \times \text{Length}$$

$$= \omega \times \frac{\pi}{4} [D_o^2 - D_i^2] \times L$$

$$= \omega \times \frac{\pi}{4} \left[D_o^2 - \left(\frac{2}{3}\right)^2 D_o^2 \right] \times L$$

$$= \omega \times \frac{\pi}{4} \left[D_o^2 - \frac{4}{9} D_o^2 \right] \times L$$

$$= w \times \frac{\pi}{4} \times \frac{5}{9} D_o^2 \times L \quad \text{———} \quad (5)$$

Dividing equation '4' by equation '5'

$$\frac{W_s}{W_h} = \frac{w \times \frac{\pi}{4} D^2 \times L}{w \times \frac{\pi}{4} \times \frac{5}{9} D_o^2 \times L}$$

$$= \frac{9}{5} \frac{D^2}{D_o^2}$$

$$= \frac{9}{5} \times \frac{(0.929 D_o)^2}{D_o^2}$$

$$= \frac{9}{5} \times 0.929^2$$

$$\frac{\text{weight of solid shaft}}{\text{weight of hollow shaft}} = \frac{1.55}{1}$$

Prob: Find the maximum shear stress induced in a solid circular shaft of diameter 15cm when the shaft transmits 150kW power at 180 rpm.

Given:

Diameter of shaft $D = 150$ mm.

Power transmitted $P = 150$ kW $= 150 \times 10^3$ W

Speed of shaft $N = 180$ rpm

Let τ = maximum shear stress induced in the shaft

Power transmitted is given by

$$P = \frac{2\pi N T}{60}$$

$$150 \times 10^3 = \frac{2 \times \pi \times 180 \times T}{60}$$

$$T = 7957700 \text{ Nmm.}$$

for solid shaft

$$T = \frac{\pi}{16} \tau D^3$$

$$7957700 = \frac{\pi}{16} \times \tau \times 150^3$$

$$\tau = \frac{16 \times 7957700}{\pi \times 150^3}$$

$$\tau = 12 \text{ N/mm}^2.$$

Prob:- A hollow shaft, having an inside diameter 60% of its outer diameter, is to replace a solid shaft transmitting the same power at the same speed. Calculate the percentage saving in material, if the material to be used is also the same.

Given:

Let D_0 = Outer diameter of the hollow shaft

D_i = Inside diameter of the hollow shaft = 60% of D_0.

$$= \frac{60}{100} D_0 = 0.6 D_0$$

D = Diameter of the solid shaft

P = Power transmitted by hollow & solid shafts

N = Speed of each shaft

τ = Maximum shear stress induced in each shaft. Since the material of both shafts is same and hence shear stress will be same.

Power of solid shaft & hollow shaft is given by

$$P = \frac{2\pi N T}{60}$$

$$T = \frac{P \times 60}{2\pi N}$$ [P & N are same for Solid & hollow shaft]

Then torque transmitted by solid shaft and hollow shaft is same.

T for solid shaft $T = \frac{\pi}{16} \tau D^3$ ——— (1)

T for hollow shaft

$$T = \frac{\pi}{16}\tau\left[\frac{D_o^4 - D_i^4}{D_o}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_o^4 - (0.6\,D_o)^4}{D_o}\right]$$

$$= \frac{\pi}{16}\tau\left[\frac{D_o^4 - 0.1296 D_o^4}{D_o}\right]$$

$$= \frac{\pi}{16}\tau \times 0.8704\, D_o^3 \quad \text{———} \quad (2)$$

equating 1 & 2

$$\frac{\pi}{16}\tau D^3 = \frac{\pi}{16}\tau \times 0.8704\, D_o^3$$

$$D = (0.8704)^{1/3} D_o$$

$$D = 0.9548\, D_o$$

$\therefore$ Area of solid shaft $= \frac{\pi}{4} D^2 = \frac{\pi}{4}(0.9548\, D_o)^2$

$$= 0.716\, D_o^2.$$

$\therefore$ Area of hollow shaft $= \frac{\pi}{4}(D_o^2 - D_i^2)$

$$= \frac{\pi}{4}(D_o^2 - (0.6\,D_o)^2)$$

$$= \frac{\pi}{4}(D_o^2 - 0.36\,D_o^2)$$

$$= \frac{\pi}{4}(0.64\,D_o^2)$$

$$= 0.502\, D_o^2$$

For the shafts of the same material, the weight of the shafts is proportional to the areas.

$\therefore$ Saving in material = Saving in area

$$= \frac{\text{Area of solid shaft} - \text{Area of hollow shaft}}{\text{Area of solid shaft}}$$

$$= \frac{0.716 D_o^2 - 0.502 D_o^2}{0.716 D_o^2}$$

$$= 0.2988$$

$\therefore$ Percentage of saving in material $= 0.2988 \times 100$

$$= \underline{\underline{29.88}}$$

<u>Expression for torque in terms of polar moment of inertia</u>

Polar moment of inertia of a plane area is defined as the moment of inertia of the area about an axis perpendicular to the plane of the figure and passing through the C.G of the area. It is denoted by symbol J

The moment (dT) on the circular ring is given by

$$dT = \frac{\tau}{R} 2\pi r^3 dr$$

$$= \frac{\tau}{R} r^2 dA \qquad \left(\because dA = 2\pi r\, dr\right)$$

$$\text{Total torque} = T = \int_0^R dT$$

$$= \int_0^R \frac{\tau}{R} r^2 dA = \frac{\tau}{R}\int_0^R r^2 dA \longrightarrow (1)$$

But $r^2 dA$ = Moment of inertia of the elementary ring about an axis perpendicular to the plane and passing through the centre of the circle.

$\therefore \int_0^R r^2 dA$ = Moment of inertia of the circle about an axis perpendicular to the plane of the circle and passing through the centre of the circle

$$= \text{Polar moment of inertia } (J) = \frac{\pi}{32} D^4$$

Hence eq (1) becomes as

$$T = \frac{\tau}{R} \times J$$

$$\frac{T}{J} = \frac{\tau}{R}. \qquad \text{———} \quad (2)$$

$\therefore$ But from the equation

$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\boxed{\frac{T}{J} = \frac{\tau}{R} = \frac{C\theta}{L}}$$

Polar Modulus

Polar modulus is defined as the ratio of the polar moment of inertia to the radius of the shaft. It is also called torsional section modulus. It is denoted Z_p.

$$Z_p = \frac{J}{R}.$$

a) For a solid shaft, $J = \frac{\pi}{32} D^4$

$$Z_p = \frac{\frac{\pi}{32} D^4}{R} = \frac{\frac{\pi}{32} D^4}{D/2}$$

$$= \frac{\pi}{16} D^3 \quad \text{——— (1)}$$

b) For a hollow shaft

$$J = \frac{\pi}{32} (D_o^4 - D_i^4)$$

$$Z_p = \frac{\frac{\pi}{32} (D_o^4 - D_i^4)}{R}.$$

$$= \frac{\frac{\pi}{32}\left[D_o^4 - D_i^4\right]}{D_o/2}$$

$$= \frac{\pi}{16 D_o} \times \left[D_o^4 - D_i^4\right] \quad \text{——— (2)}$$

Strength of a shaft and Tortional Rigidity

* The strength of a shaft means the maximum torque or maximum power the shaft can transmit
* Torsional rigidity or stiffness of the shaft is defined as the product of modulus of rigidity (C) and polar moment of inertia of the shaft (J)

$$\text{Torsional rigidity} = C \times J$$

* Torsional rigidity is also defined as the torque required to produce a twist of one radian per unit length of the shaft.

Let a twisting moment T produce a twist of θ radians in a shaft of length L

$$\frac{T}{J} = \frac{C\theta}{L} \quad ; \quad C \times J = \frac{T \times L}{\theta}$$

$$\therefore \text{Torsional rigidity} = \frac{T \times L}{\theta} \quad \left[\begin{array}{l} L = \text{one metre} \\ \theta = \text{one radian} \end{array}\right]$$

Then torsional rigidity = Torque.

Prob: Determine the diameter of a solid steel shaft which will transmit 90kW at 160 r.p.m. Also determine the length of the shaft if the twist must not exceed 1° over the entire length. The maximum shear stress is limited to 60 N/mm². Take the value of modulus of rigidity = 8×10⁴ N/mm²

Given:

Power $P = 90kW = 90 \times 10^3 W$ $\left[\because 1° = \frac{\pi}{180} \text{radian}\right]$

Speed $N = 160$ r.p.m

Angle of twist $\theta = 1°$ or $\frac{\pi}{180}$ radians

Max. shear stress, $\tau = 60 \ N/mm^2$

Modulus of rigidity, $C = 8 \times 10^4 \ N/mm^2$

Let D = Diameter of the shaft and

L = Length of the shaft.

i) Diameter of the shaft.

$$T = \frac{\pi}{16} \tau D^3 \quad \text{——— (1)}$$

To calculate T, $P = \frac{2\pi NT}{60}$

$$90 \times 10^3 = \frac{2\pi \times 160 \times T}{60}$$

$$T = 5371.48 \times 10^3 \ N\text{-}mm$$

Substituting the values in eq ①

$$5371.48 \times 10^3 = \frac{\pi}{16} \times 60 \times D^3$$

$$D^3 = \frac{5371.48 \times 10^3 \times 16}{\pi \times 60} = 455945$$

$$D = (455945)^{1/3}$$

$$\boxed{D = 76.8 \text{ mm.}}$$

ii) Length of the shaft.

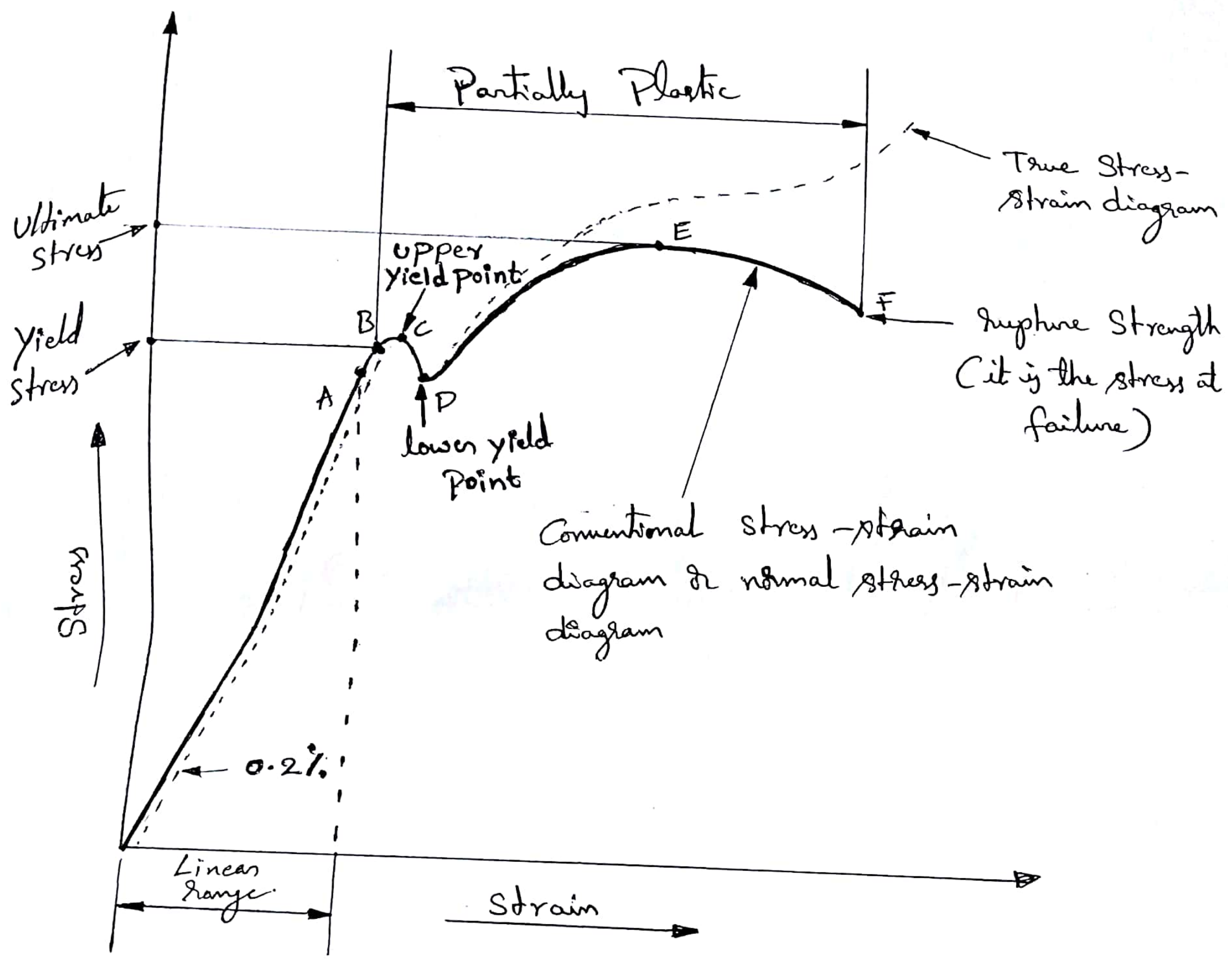
$$\frac{\tau}{R} = \frac{C\theta}{L}$$

$$\frac{60}{\frac{76.8}{2}} = \frac{8 \times 10^4 \times \pi}{180 \times L}$$

$$L = \frac{8 \times 10^4 \times \pi \times 76.8}{60 \times 180 \times 2}$$

$$\boxed{L = 893.6 \text{ mm}}$$

Tensile Test Curve for Mild Steel. (Stress-Strain Curve) 13



→ So it is evident from the graph that the strain is proportional to strain 'or' elongation is proportional to the load giving a straight line relationship. This law of proportionality is valid upto a point A (proportionality limit). 'A' (upto).

→ For a short period of time beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as Elastic limit.

→ Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation when load is removed. Point 'C' & 'D' are upper yield point and lower yield points respectively. The stress at the yield point is called yield strength.

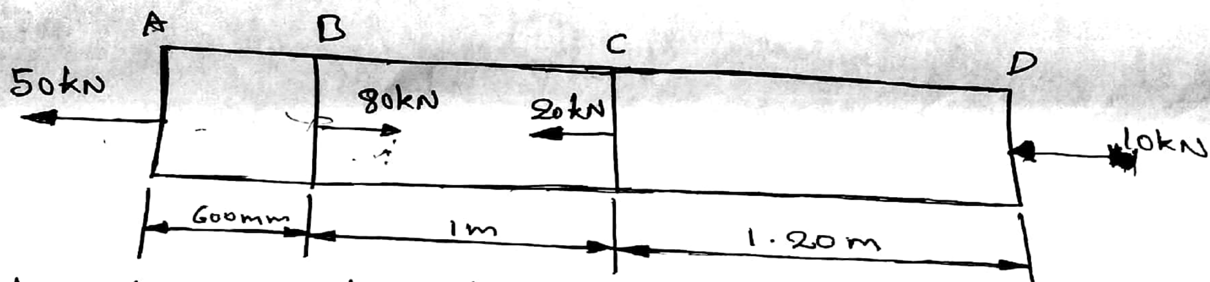
→ A Further increase in the load will cause deformation in the whole volume of the metal. The maximum load which the specimen can withstand without failure is called the load at the ultimate strength (E)

→ After the specimen has reached the ultimate stress a neck is formed which decreases the cross sectional area of the specimen. The stress is reduced until the specimen breaks away at point 'F' & is called Breaking Stress

Principle of Superposition :-

When a number of loads are acting on a body the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads.

Prob No: 8 A brass bar, having cross-sectional area of 1000 mm², is subjected to axial force as shown in the figure.



Find the total elongation of the bar.
Take $E = 1.05 \times 10^5$ N/mm²

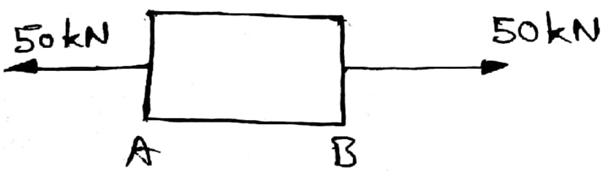
Given Data:

Area 'A' = 1000 mm²

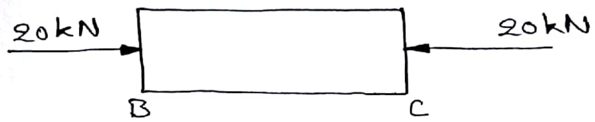
Value of 'E' = 1.05×10^5 N/mm²

dL = Total elongation of the bar.

The force of 80kN acting at B is split up into three forces of 50kN, 20kN, and 10kN. Then



3.



Part AB This part is subjected to a tensile load of 50 kN. Hence there will be increase in length of this part.

Increase in the length of AB

$$= \frac{P_1}{AE} \times L_1$$

$$= \frac{50 \times 1000}{1000 \times 1.05 \times 10^5} \times 600$$

$$= 0.2857 \text{ mm}$$

Part BC This part is subjected to a compressive load of 20kN or 20,000N. Hence there will be decrease in length of this part.

$\therefore$ Decrease in the length of BC.

$$= \frac{P_2}{AE} \times L_2 = \frac{20000}{1000 \times 1.05 \times 10^5} \times 1000$$

$$= 0.1904 \text{ mm.}$$

Part BD this part is subjected to a compressive load of 10 kN, Hence there will be decrease in length of this part.

Decrease in the length of BD

$$= \frac{P_3}{AE} \times L_3 = \frac{10000}{1000 \times 1.05 \times 10^5} \times 2200$$

$$= 0.2095 \text{ mm.}$$

$\therefore$ Total elongation $= 0.2857 - 0.1904 - 0.2095$

$$= \underline{\underline{-0.1142 \text{ mm}}}$$

Negative sign shows, that there will be decrease in length of the bar.

<u>Prob: 9</u> A tensile load of 40 kN is acting on a rod of diameter 40mm and of length 4m. A bore of diameter 20mm is made centrally on the rod. To what length the rod should be bored so that the ~~length~~ total extension will increase 30% under the same tensile load. Take $E = 2 \times 10^5$ N/mm²

Given data

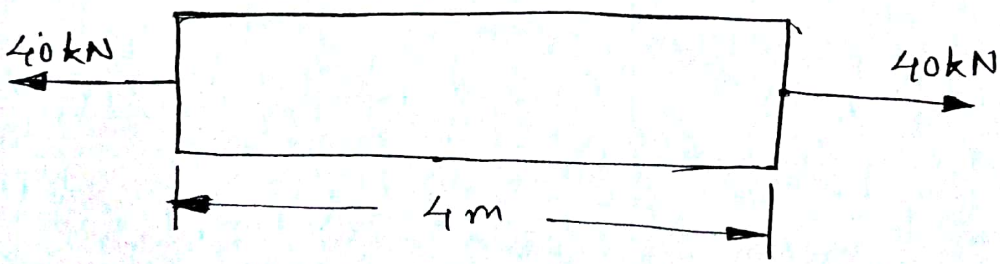
P (tensile load) = 40kN

D (diameter of the rod) = 40mm.

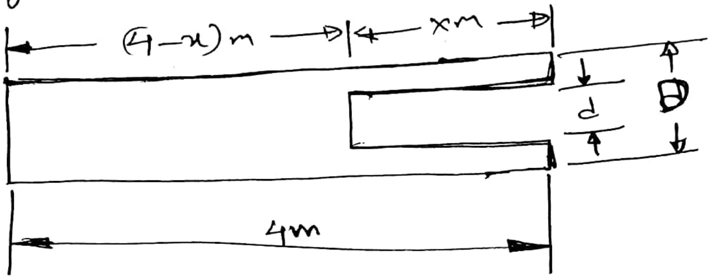
L (length of the rod) = 4m.

$E = 2 \times 10^5$ N/mm²

d (bore dia) = 20mm



Area of rod, $A = \frac{\pi}{4}(40^2) = 400\pi \text{ mm}^2$



Area of bore, $a = \frac{\pi}{4} \times 20^2 = 100\pi \text{ mm}^2$

Total extension after bore = 1·3 × Extension before bore

change in length of the ~~bar~~ without bore

$$\delta l = \frac{P}{AE} \times L$$

$$= \frac{40000 \times 4000}{400\pi \times 2 \times 10^5}$$

$$= \frac{2}{\pi} \text{ mm}$$

Extension after the bar is made

$$= 1{\cdot}3 \times \delta l.$$

$$= 1{\cdot}3 \times \frac{2}{\pi}$$

$$= \frac{2{\cdot}6}{\pi} \text{ mm}$$

Stress in unbored portion

$$= \frac{\text{Load}}{\text{Area}} = \frac{P}{A} = \frac{\overset{100}{\cancel{40000}}}{\cancel{400}\pi} = \frac{100}{\pi} \text{ N/mm}^2$$

Stress in bored portion

$$= \frac{P}{A-a} = \frac{40000}{(400\pi - 100\pi)} = \frac{40000}{300\pi}$$

Extension of unbored portion

$$\delta l = \frac{\text{Stress}}{E} \times \text{length of unbored portion}$$

$$= \frac{100}{\pi \times 2 \times 10^5} \times (4-x) \times 1000$$

$$= \frac{(4-x)}{2\pi} \text{ mm.}$$

Extension of bored portion

$$= \frac{\text{Stress}}{E} \times \text{length of bored portion}$$

$$= \frac{4000}{300\pi \times 2 \times 10^5} \times 1000x = \frac{4x}{6\pi} \text{ mm.}$$

Total extension after the bore is made

$$= \frac{4-x}{2\pi} + \frac{4x}{6\pi}$$

$$\frac{2.6}{\pi} = \frac{4-x}{2\pi} + \frac{4x}{6\pi}$$

$$\frac{2.6}{\cancel{\pi}} = \frac{3(4-x)+4x}{6\cancel{\pi}}$$

$$6 \times 2.6 = 12 - 3x + 4x$$

$$x = 15.6 - 12$$

$$x = 3.6 \text{ meters}$$

Rod should be bored upto a length of 3.6m

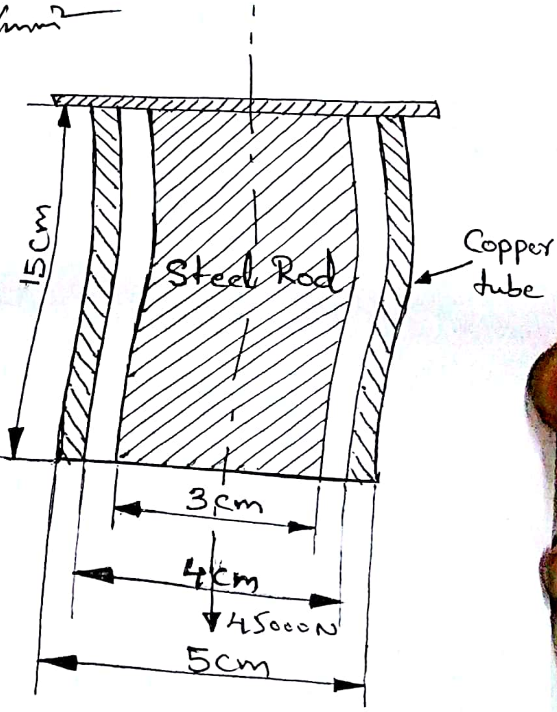
Prob: 10, A steel rod of 3cm diameter is enclosed centrally in a hallow copper tube of external diameter 5cm and internal diameter 4cm. The composite bar is then subjected to an axial pull of 45000N. If the length of each bar is equal to 15cm, determine: i) The stresses in the rod and tube, and ii) Load carried by each bar.

Take E for steel = 2.1×10^5 N/mm²

Copper = 1.1×10^5 N/mm²

Sol: Diameter of steel rod = 3cm

= 30mm.



Area of steel rod $A_s = \frac{\pi}{4}(30)^2$

$A_s = 706.86$ mm²

External diameter of Copper tube = 5cm

= 50mm.

Internal diameter of Copper tube = 4cm

= 40mm

Area of copper tube,

$$A_c = \frac{\pi}{4}[50^2 - 40^2] \text{ mm}^2$$

$$= 706.86 \text{ mm}^2$$

Axial pull on Composite bar, P = 4500 N

Length of each bar, L = 15cm.

Young's modulus for steel, $E_s = 2.1 \times 10^5$ N/mm²

" Copper $E_c = 1.1 \times 10^5$ N/mm²

i) Stress in the rod and tube.

$$\frac{\sigma_s}{E_s} = \frac{\sigma_c}{E_c}$$

$$\sigma_s = \frac{E_s}{E_c} \times \sigma_c$$

$$= \frac{2.1 \times 10^5}{1.1 \times 10^5} \times \sigma_c$$

$$\sigma_s = 1.909\, \sigma_c$$

load on steel + load on copper = Total load.

$$P_s + P_c = P$$

$$\sigma_s A_s + \sigma_c A_c = 45000\,N.$$

$$1.909\, \sigma_c \times 706.86 + \sigma_c \times 706.86 = 45000\,N$$

$$2056.25\, \sigma_c = 45000\,N$$

$$\sigma_c = \frac{45000}{2056.25}$$

$$\boxed{\sigma_c = 21.88\ N/mm^2}$$

$\sigma_s = 1.909\, \sigma_c$; $\sigma_s = 1.909 \times 21.88$

$$\boxed{\sigma_s = 41.77\ N/mm^2}$$

ii) Load carried by each bar.

Load = stress × Area

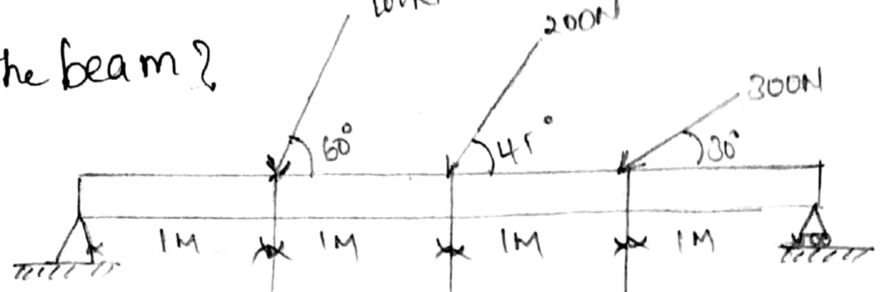
Load carried by steel bar = $P_s = \sigma_s \times A_s$

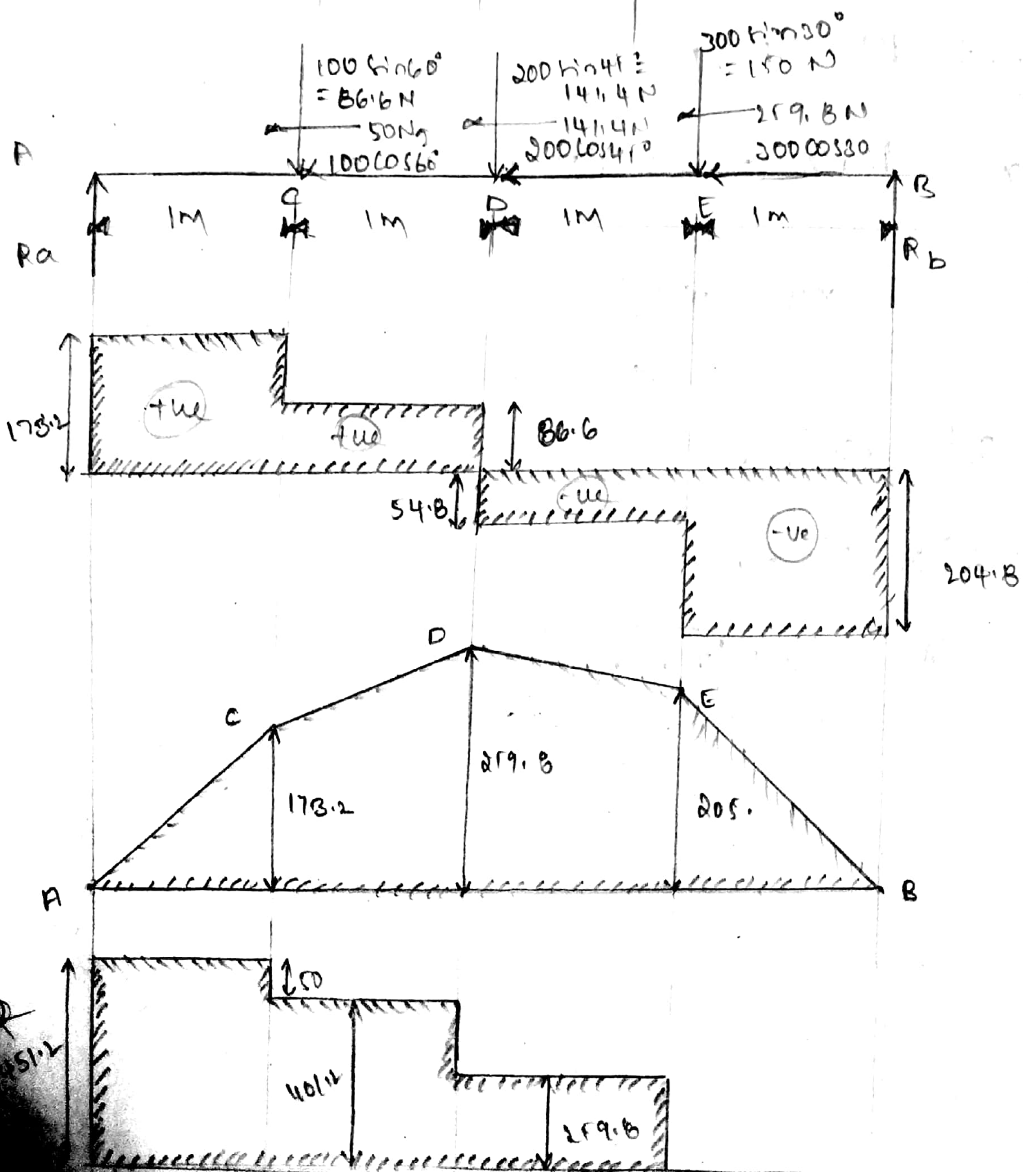
$P_s = 41.77 \times 706.86$ $\boxed{P_s = 29525.5\,N}$

Load carried by copper tube = $P_c = 45000 - P_s$

$$\boxed{P_c = 15474.5\ N}$$

A Horizantal Beam AB of length 4m is hinged at A and supported on rollers at B. The beam carrie the inclined loads of 100N, 200N, 300N inclined at 60°, 45°, & 30° to the horizantal as shown in figure. Draw the shearforce, Bending moment & thrust diagrams for the beam?





$Ra + Rb = 86.6 + 141.4 + 150 = 378N$

Taking moment at A then,

$86.6 \times 1 + 141.4 \times 2 + 150 \times 3 = R_b \times 4$

$\boxed{R_b = 204.8N}$, $\boxed{Ra = 173.2N}$

Thrust force acting on A

$50 + 141.4 + 259.8$

$= 451.2N.$

SFD: $F_A = Ra = 173.2N$

$F_C = Ra - 86.6 = 173.2 - 86.6 = 86.6N$

$F_D = Ra - 86.6 - 141.4 = -54.8N$

$F_E = Ra - 86.6 - 141.4 - 150 = -204.8N$

$\therefore \boxed{F_E = F_B = -204.8N}$

BMD: Btn A & C

$M_x = Ra\,x = 173.2 \times x$

At A; $x = 0$ $\therefore M_A = 0$

At C; $x = 1m$ $\therefore \boxed{M_C = 173.2\,N\text{-}m}$

Btn C & D

$M_x = Ra\,x - 86.6(x-1)$

At D; $x = 2m$

$M_D = 173.2 \times 2 - 86.6(1)$

$= 346.4 - 86.6 = 259.8\,N\text{-}m.$

Btn D & E. At E; $x = 3$

$M_x = Ra\,x - 86.6(x-1) - 141.4(x-2)$

Btn E & B

$M_x = R_a x - 86.6(x-1) - 141.4(x-2) - 150(x-3)$

At B; $x = 4$

$$\boxed{M_B = 0}$$

P.14 A Beam 10m long and simply supported at each

... load of 1000 N/m exten