Unit - VI

: STEADY - STATE HEAT TRANSFER :

* Heat Transfer problems :

In problems, all machineries and structures cannot be operated in the same room-temperature. Due to the surrounding variation of heat or change of seasons, the operating temperatures of the above items may vary. In some cases, some equipments are purposely heated. For example, Boiler. Because of these heat variations, thermal stresses will be induced in the structures and machines which will affect their mechanical properties like strength.

The problems dealing with temperature variation, thermal stress distribution and their controlling techniques are specified as Heat Transfer problems.

→ **Applications :**

In mechanical engineering field, attention is focused to know about the mechanism of heat-transfer involved in the operation of power-plant equipments such as Boilers, Condensers, air pre-heaters, economizers and so on in order to improve their performance. Similarly the refrigeration and air-conditioning systems also involve heat exchanging devices, which need careful design, etc.

* Modes of Heat Transfer :

Heat transfer is nothing but the transmission of heat energy between material bodies due to temperature difference. Usually heat transfer takes place from the region of high temp. to the region of low temperature. There are 3 modes of Heat Transfer.

→ **Conduction :** In this mode of heat transfer, heat energy is transmitted through the same material from the high temp. region to low temp. region without the actual movement of the molecules. Pure conduction occurs only in the solid material.

→ **Convection :** Here the heat energy is transmitted from a solid material to the fluid material surroundings it due to the temperature difference. It occurs in boiling and condensation process.

→ **Radiation :** Heat energy is transmitted from one body to another body without any transmitting medium.

Laws of Heat Transfer :

The amount of heat energy transferred per unit time in conduction, convection and radiation may be expressed as some kind of rate equations. This rate equation is specified by different laws for different modes of heat transfer.

For heat conduction, the rate equation is known as Fourier's law which is expressed for one dimension as

$$q = -k \frac{dT}{dx}$$

Where, q = heat flux (W/m^2), k = Thermal conductivity. (W/mK).

$\frac{dT}{dx}$ = Temperature gradient (K/m)

This equation is analogous to the 1D stress/strain law for the stress analysis problem which is

$$\sigma = E \frac{du}{dx}.$$

For Convection heat transfer, the rate equation is given by Newton's law of cooling

$$q = h(T - T_{\infty})$$

q = Convection heat flux (W/m^2)

h = Convection heat transfer coefficient (W/m^2K)

T = Temp. of Solid Surface (K)

T_{∞} = Temp. of fluid surrounding (K).

For heat transfer by radiation, the maximum heat flux that can be emitted by radiation from a black surface is given by Stefan-Boltzmann law.

$$q = \sigma T_s^4$$

q = Radiation heat flux (W/m^2).

σ = Stefan-Boltzmann constant $(5.669 \times 10^{-8})\ W/m^2K^4$.

T_s = Surface temperature.

Heat flux emitted by a real surface is less than that of black surface and is given by

$$q = \epsilon \sigma T_s^4. \quad , \epsilon = \text{Emissivity}.$$

From the 1st of thermodynamics, we know that the increase of energy in a system is equal to the difference between the energy transfer by heat to the system and the energy transfer by the work done on the surroundings by the system.

That is

$$dE = dQ - dW$$

dE = Increase of energy

Q = Total heat entering the system.

W = Workdone on the surroundings.

* Formulation of Heat transfer problems :

Based on the heat transfer mechanisms, the heat transfer problems are classified into two major types such as

Scalar Variable problems.

Vector Variable problems.

In Scalar Variable problems, only the magnitude of the variable can be identified in the specific locations.

Example : Temp. at different points of the heated rods, plates etc.

In Vector Variable problems, apart from the quantity (magnitude), the direction of heat flow, the deformations occuring at different directions due to thermal stresses etc can be evaluated.

Depending upon the size of the components through which heat-transfer takes place.

1. One dimensional problems.
2. Two dimensional problems.
3. Three dimensional problems.

* Scalar Variable problems :

Steady State Heat conduction Problems : $\frac{dT}{dt} = 0$.

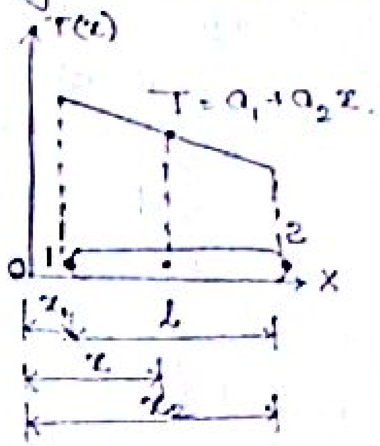
In steady state condition, the temperature does not depend on time

$$\frac{dT}{dt} = 0.$$

* Determination of shape function for one dimensional element by assuming linear variation of temperature :

Consider a polynomial form with global co-ordinates for the temperature

$$T = a_1 + a_2 x$$



Let, At Node 1

$$T = T_1 \quad \& \quad x = x_1$$

At Node 2

$$T = T_2 \quad \& \quad x = x_2$$

We get

$$T_1 = \cancel{a_1} + a_2 x_1$$

$$\frac{T_2 = \cancel{a_1} + a_2 x_2}{T_1 - T_2 = a_2(x_1 - x_2)}$$

$$a_2 = \frac{T_1 - T_2}{x_1 - x_2}$$

Now,

$$T_2 = a_1 + \frac{T_1 - T_2}{x_1 - x_2} \cdot x_2$$

$$T_2(x_1 - x_2) = a_1(x_1 - x_2) + T_1 x_2 - T_2 x_2$$

$$T_2 x_1 - \cancel{T_2 x_2} - T_1 x_2 + \cancel{T_2 x_2} = a_1(x_1 - x_2)$$

$$a_1 = \frac{T_2 x_1 - T_1 x_2}{x_1 - x_2}$$

$\therefore$ Substituting in the eq.

$$T = \frac{T_2 x_1 - T_1 x_2}{x_1 - x_2} + \frac{T_1 - T_2}{x_1 - x_2} \cdot x$$

$$= \frac{T_2 x_1 - T_1 x_2 + T_1 x - T_2 x}{x_1 - x_2}$$

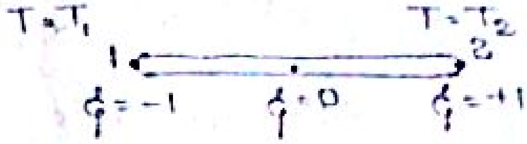
$$T = T_1 \frac{(x - x_2)}{x_1 - x_2} + T_2 \frac{(x_1 - x)}{x_1 - x_2} \quad \text{or}$$

$$T = \frac{x_2 - x}{x_2 - x_1} T_1 + \frac{x - x_1}{x_2 - x_1} \cdot T_2.$$

$$\therefore N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \text{and} \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

Similar consider a polynomial form with Natural co-ordinate system

$$T = a_1 + a_2 \xi$$



At Node 1, $T = T_1$ & $\xi = -1$

$$T_1 = a_1 + a_2(-1)$$

$$T_1 = a_1 - a_2 \longrightarrow (i)$$

At Node 2, $T = T_2$ & $\xi = 1$

$$T_2 = a_1 + a_2 \longrightarrow (ii)$$

Solving

$$\begin{array}{l} T_1 = a_1 - \cancel{a_2} \\ T_2 = a_1 + \cancel{a_2} \\ \hline T_1 + T_2 = 2a_1 \Rightarrow a_1 = \dfrac{T_1 + T_2}{2} \end{array}$$

Consider eq (i)

$$T_1 = \frac{T_1 + T_2}{2} - a_2$$

$$2T_1 = T_1 + T_2 - 2a_2$$

$$\frac{T_2 - T_1}{2} = a_2$$

$$\therefore T = \frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{2}\xi$$

$$= \frac{T_1 + T_2 + (T_2 - T_1)\xi}{2}$$

$$T = \left(\frac{1-\xi}{2}\right)T_1 + \left(\frac{1+\xi}{2}\right)T_2$$

$$\therefore N_1 = \frac{1-\xi}{2} \quad \& \quad N_2 = \frac{1+\xi}{2}$$

Now for finding the value of ξ

$$\xi = a_1 + a_2 x$$

At Node 1, $\xi = -1$ & $x = x_1$

Node 2, $\xi = 1$ & $x = x_2$

$$\begin{array}{l} -1 = \cancel{a_1} + a_2 x_1 \\ 1 = \cancel{a_1} + a_2 x_2 \\ (-) \quad (-) \quad (-) \\ \hline -2 = a_2(x_1 - x_2) \end{array}$$

$$a_2 = \frac{2}{x_2 - x_1}$$

Consider

$$-1 = a_1 + \frac{2}{x_2 - x_1} \cdot x_1$$

$$-(x_2 - x_1) = a_1(x_2 - x_1) + 2x_1$$

$$-x_2 + x_1 - 2x_1 = a_1(x_2 - x_1)$$

$$a_1 = \frac{-(x_1 + x_2)}{x_2 - x_1}$$

$$\therefore \xi = -\frac{(x_1 + x_2)}{x_2 - x_1} + \frac{2}{x_2 - x_1} \cdot x$$

$$= \frac{-(x_1 + x_2) + 2x}{x_2 - x_1}$$

$$= \frac{2x - 2x_1 - x_2 + x_1}{x_2 - x_1}$$

$$= \frac{2(x - x_1)}{x_2 - x_1} - \frac{(x_2 - x_1)}{x_2 - x_1}$$

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

Finding Strain-displacement Matrix.

We have

$$\frac{dT}{dx} = \frac{dT}{d\xi} \cdot \frac{d\xi}{dx}$$

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

$$\frac{d\xi}{dx} = \frac{2}{x_2 - x_1} \Rightarrow d\xi = \frac{2}{x_2 - x_1} dx.$$

$$\& \frac{dT}{d\xi} = \frac{d}{d\xi}(N_1 T_1 + N_2 T_2) = \frac{d}{d\xi}\left[\frac{1-\xi}{2} T_1 + \frac{1+\xi}{2} T_2\right]$$

$$\frac{dT}{d\xi} = -\frac{T_1}{2} + \frac{T_2}{2}$$

Now, $\frac{dT}{d\xi} = \frac{1}{2}[-T_1 + T_2] \times \frac{2}{x_2 - x_1}$

$$\frac{dT}{dx} = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$\therefore B = \frac{1}{x_2 - x_1}[-1 \quad 1]$ or $\frac{1}{l_e}[-1 \quad 1]$

Now, to find the element thermal stiffness matrix. Let. Now to find the heat flux equation for one dimensional element. consider the Heat flux equation

$$q = -k\frac{dT}{dx}$$

$$Q = -kA.\frac{dT}{dx} \quad \Big| \because q = \frac{Q}{A}, \quad Q = \text{Heat Flow}, \; q = \text{Heat flux}$$

Consider the heat flow at Node 1

$$Q_1 = kA.\frac{1}{l_e}[T_1 - T_2]$$

The heat flow at Node 2

$$Q_2 = kA\frac{1}{l_e}[T_2 - T_1]$$

$$\therefore Q_2 = \frac{-kA}{l_e}[T_1 - T_2]$$

Consider the values Q_1 & Q_2 and write in the matrix form

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{kA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$\{Q\} = K\{T\}$$

$$\therefore [K] = \frac{kA}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Here $[K]$ = Thermal stiffness.

k = Thermal conductivity.

$\{T\}$: Temp. vector

$\{Q\}$ = Nodal heat flow vector.

Now the finite-element equation for 1D element

$$\{Q\} = KT$$

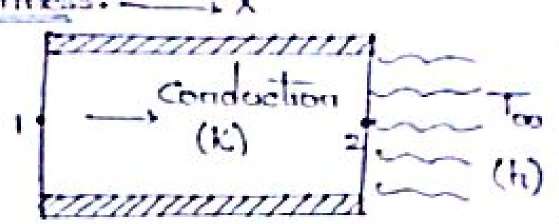
Since, the Nodal heat flow Q_1 & Q_2 are considered as nodal thermal forces for analysis, $\{F\} = [K]T$.

* Derivation of thermal stiffness Matrix for 1D element subjected to conduction and convection :-

In practice, a few equipments are insulated thoroughly in order to transfer heat through conduction only. But most of the equipments are operated in open air or with cooling fluids. In such cases, heat can be transferred by conduction as well as by convection. Usually, convection heat transferring will occur through end surface and/or lateral surface depending upon the equipment set-up

* Heat conduction along with free end heat convection :

(i) thermal stiffness :-



Consider 1D solid rod element of length l, area of cross-section A, with nodes 1 and 2 and also with insulation at lateral surface as shown in the above figure

Since the above rod element is subjected to heat conduction along with free end convection, the required thermal stiffness matrix must have conduction part and convection part.

The conduction part of thermal stiffness matrix can be derived from the equation such as :

$$\{K\}_k = \int_v B^T D B \, dv.$$

We know that

$$B = \frac{1}{l_e}[-1 \quad 1] \quad \& \quad dv = dA . l.$$

$$\therefore \{K\}_k = B^T D B \int l_x \, dA = \frac{1}{l_e^2}\begin{bmatrix}-1\\1\end{bmatrix}[-1 \quad 1].D.\cancel{l}_x A$$

$$= \frac{1}{l_e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} D A.$$

Here $[D] = K$ (thermal conductivity) for 1D.

$$\{K\}_k = \frac{KA}{l_e}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Similarly, the convection part of thermal stiffness matrix can be derived from another relation

$$[k]_h = \int_A h\, N^T N \, dA$$

h = Convection heat transfer coefficient

$$N = [N_1 \ N_2] = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

Since, the convection heat transfer takes place at $x = l$, the shape functions at Node 2 are

$$\left. N_1 = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2} \right]_{\xi = +1}$$

$$N_1 = 0 \ \ \& \ \ N_2 = 1$$

$$N = [N_1 \ \ N_2] = [0 \ \ 1] \ \ \& \ \ N^T = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[k]_h = h N^T N \int dA$$

$$= h \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ \ 1] A.$$

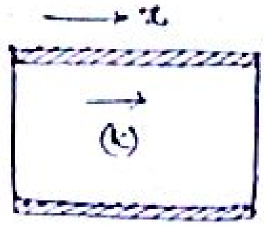
$$[k]_h = hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ The combined thermal stress is given by

$$[k] = [k]_k + [k]_h = \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

<u>Note</u> :- If the body/rod is completely insulated, then the thermal stiffness is given by

$$[k] = [k]_k = \frac{kA}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



(ii) Convection force from the free end :

$$\{F_h\}_{end} = \int_A h . T_\infty [N]^T . dA$$

$$\{F_h\}_{end} = h A T_\infty \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(iii) Finite element equation:

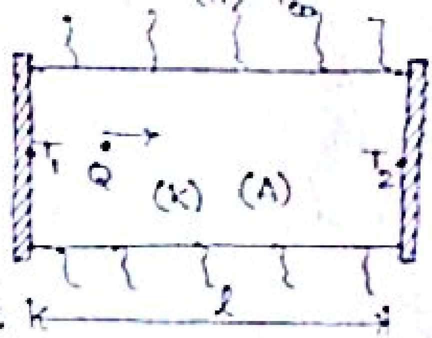
$$\{k\} [T] = \{F\}$$

$$\left\{\frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}\begin{bmatrix}T_1\\ T_2\end{bmatrix} = hAT_\infty\begin{Bmatrix}0\\ 1\end{Bmatrix}$$

Heat conduction along with lateral surface heat-convection:

(i) Thermal stiffness for 1D heat conduction with lateral surface & with internal heat generation:-

T_∞ = Surrounding Temperature.
A = Cross-sectional area.
Q = Internally generated heat.
K = Thermal conductivity.
h = heat convection coefficient



we know that, the conduction thermal stiffness

$$[K]_k = \frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Now, the convection part of stiffness matrix is.

$$[K]_h = \int_s h\,[N]^T[N]\,ds.$$

$$= \int_0^l h\,[N]^T[N]\,P.dx \quad \Big|\ \because ds = P.dx$$

P = Perimeter.

$$= hP\int_{-1}^{+1}[N]^T[N].\frac{x_2-x_1}{2}.d\xi$$

$$= hP\int_{-1}^{+1}\begin{bmatrix}\frac{1-\xi}{2}\\ \frac{1+\xi}{2}\end{bmatrix}\begin{bmatrix}\frac{1-\xi}{2} & \frac{1+\xi}{2}\end{bmatrix}\times\frac{l_e}{2}d\xi.$$

$$= \frac{hP}{4}\int_{-1}^{+1}\begin{bmatrix}\left(\frac{1-\xi}{2}\right)^2 & \frac{(1-\xi)(1+\xi)}{4}\\ \frac{(1+\xi)(1-\xi)}{4} & \left(\frac{1+\xi}{2}\right)^2\end{bmatrix}\frac{l_e}{2}.d\xi.$$

$$= \frac{hPl}{4\times 2}\int_{-1}^{+1}\begin{bmatrix}(1+\xi^2-2\xi) & (1+\xi-\xi-\xi^2)\\ 1-\xi+\xi-\xi^2 & 1+\xi^2+2\xi\end{bmatrix}d\xi$$

$$= \frac{hPl}{8}\int_{-1}^{+1}\begin{bmatrix}1+\xi^2-2\xi & 1-\xi^2\\ 1-\xi^2 & 1+\xi^2+2\xi\end{bmatrix}d\xi.$$

$$= \frac{hPl}{8}\begin{bmatrix} \xi + \frac{\xi^3}{3} - 2\frac{\xi^2}{2} & \xi - \frac{\xi^3}{3} \\ \xi - \frac{\xi^3}{3} & \xi + \frac{\xi^3}{3} + 2\frac{\xi^2}{2} \end{bmatrix}_{-1}^{+1}$$

$$= \frac{hPl}{8}\begin{bmatrix} (1+1/3-1)-(-1-1/3-2/2) & (1-1/3)-(-1+1/3) \\ (1-1/3)-(-1+1/3) & (1+1/3+1)-(-1-1/3+1) \end{bmatrix}$$

$$= \frac{hPl}{8}\begin{bmatrix} 1+1/3-1+1+1/3+1 & 1-1/3+1-1/3 \\ 1-1/3+1-1/3 & 1+1/3+1+1+1/3-1 \end{bmatrix}$$

$$= \frac{hPl}{8}\begin{bmatrix} 8/3 & 4/3 \\ 4/3 & 8/3 \end{bmatrix}$$

$$= \frac{hPl \times 2}{8 \cdot 4}\begin{bmatrix} 4/3 & 2/3 \\ 2/3 & 4/3 \end{bmatrix}$$

$$= hPl\begin{bmatrix} 4/12 & 2/12 \\ 2/12 & 4/12 \end{bmatrix}$$

$$[K]_h = \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

∴ Combined stiffness Matrix is given by

$$[K] = [K]_K + [K]_h = \frac{KA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPl}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(ii) Thermal force due to lateral heat convection & internal heat generation: force due to lateral surface

$$[F_h] = \int_S h T_\infty [N]^T ds = \int_{-1}^{+1} h T_\infty [N]^T P \cdot \frac{l_e}{2} \cdot d\xi$$

$$= hPT_\infty \frac{l_e}{2}\int_{-1}^{+1}\begin{bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{bmatrix} d\xi$$

$$= \frac{hPT_\infty l_e}{2 \times 2}\begin{bmatrix} \xi - \frac{\xi^2}{2} \\ \xi + \frac{\xi^2}{2} \end{bmatrix}_{-1}^{+1}$$

$$= \frac{hPT_\infty l_e}{4}\begin{bmatrix}(1-\frac{1}{2})-(-1-1/2)\\(1+1/2)-(-1+1/2)\end{bmatrix}$$

$$= \frac{hPT_\infty l_e}{4}\begin{bmatrix}1-1/2+1+1/2\\1+1/2+1-1/2\end{bmatrix}$$

$$= \frac{hPT_\infty l_e}{4}\begin{bmatrix}2\\2\end{bmatrix}$$

$$\{F_h\} = \frac{hPT_\infty l_e}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Force due to internal heat generation.

$$\{F_Q\} = \int_v [N]^T Q\,dv = \int_0^l [N]^T Q \times A\,dx.$$

$$= \int_{-1}^{+1}\begin{bmatrix}\frac{1-\xi}{2}\\\frac{1+\xi}{2}\end{bmatrix} Q \times A \times \frac{l_e}{2}.d\xi.$$

$$= \frac{QAl}{2\times 2}\int_{-1}^{+1}\begin{bmatrix}1-\xi\\1+\xi\end{bmatrix}.d\xi.$$

$$= \frac{QAl}{4}\begin{bmatrix}\xi-\xi^2/2\\\xi+\xi^2/2\end{bmatrix}_{-1}^{+1}$$

$$= \frac{QAl}{4}\begin{bmatrix}2\\2\end{bmatrix}$$

$$\{F_Q\} = \frac{QAl}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Total Force, $\{F\} = \{F_h\} + \{F_Q\}$.

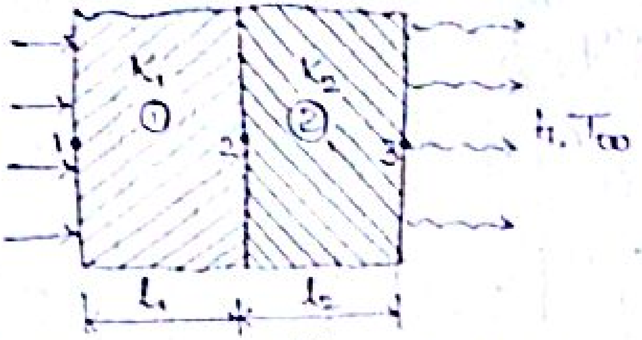
$$= \frac{hPlT_\infty}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + \frac{QAl}{2}\begin{Bmatrix}1\\1\end{Bmatrix}.$$

$$\{F\} = \frac{hPlT_\infty + QAl}{2}\begin{Bmatrix}1\\1\end{Bmatrix}$$

(iii) Finite element Equation :- $\{K\}\{T\} = \{F\}$.

$$\left\{\frac{KA}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \frac{hPl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right\}\begin{Bmatrix}T_1\\ T_2\end{Bmatrix} = \frac{hPlT_\infty + QAl}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix}$$

* Heat Transfer Analysis for Composite Wall :



Where q = Heat inflow at the innerside of element ①

K_1, K_2 = Thermal conductivities of element (1) & (2)

h = Heat convection coefficient.

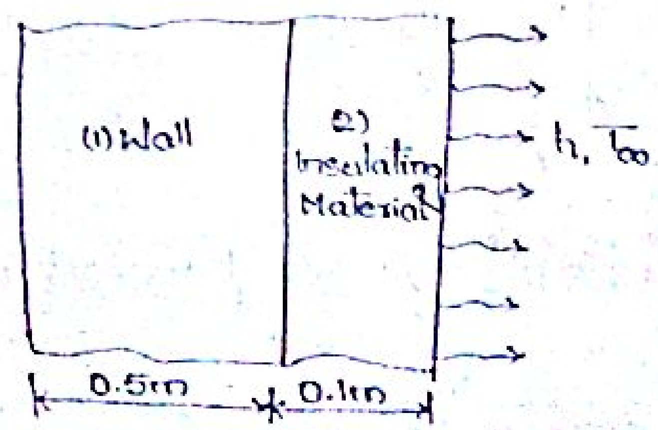
T_∞ = Atmospheric temperature.

l_1, l_2 = Wall thickness of element (1) & (2)

A = Area of cross-section.

Problems :-

1. A Wall of 0.5m thickness having thermal conductivity of 6 W/mK. The wall is to insulated with a material of thickness 0.1m having an average thermal conductivity of 0.3 W/mK. The inner surface temperature is 1200°c and the outside of the insulation is exposed to atmospheric air at 30°c with heat transfer coefficient of 40 W/m²K. Calculate the Nodal temperatures.



Sol:- Element (1) → Wall   element (2) → Insulating Material.

$l_1 = 0.5m$ $l_2 = 0.1m$

$K_1 = 6W/mk$ $K_2 = 0.3W/mk$.

T_1, T_2, T_3 are the temperatures at Nodes 1,2,3 resp. & T_∞ is the atmospheric air temperature.

T_1 = Innerwall Temp. = $1200°c = 1200+273 = 1473°K$.

T_∞ = Atm. Temp. = $30°c + 273 = 303K$.

h = Heat Transfer coefficient = $40 W/m^2K$.

Consider element (1):

$$[K]_1 = [K]_1 = \frac{K_1 A_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{6}{0.5}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \quad \Big| \because \text{Assume } A = 1m^2$$

$$= \begin{array}{cc} & \begin{array}{cc}1 & 2\end{array} \\ & \begin{bmatrix}12 & -12\\ -12 & 12\end{bmatrix} \end{array}\begin{array}{c}1\\2\end{array}$$

element (2):

Here both conduction & the convection at the outer wall takes place

$$[k_k] = \frac{K_2 A_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{0.3}{0.1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix}$$

$$[k_h]_{end} = hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = 40\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 40\end{bmatrix}$$

$$[K]_2 = [k_k] + [k_h]_{end} = \begin{bmatrix}3 & -3\\ -3 & 3\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 40\end{bmatrix} = \begin{array}{cc} & \begin{array}{cc}2 & 3\end{array} \\ & \begin{bmatrix}3 & -3\\ -3 & 43\end{bmatrix} \end{array}\begin{array}{c}2\\3\end{array}$$

Global Stiffness Matrix.

$$[k] = \begin{bmatrix}12 & -12 & 0\\ -12 & 15 & -3\\ 0 & -3 & 43\end{bmatrix}$$

The finite element Equation $k\{T\} = \{Q\}$ or $\{F\}$.

We know that

$$Q_1 = 0 \quad Q_2 = 0.$$

$$\therefore Q_3 = hAT_\infty \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = hAT_\infty = 40 \times 303 = 12120W.$$

Now.

$$\begin{bmatrix} 12 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 43 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12120 \end{bmatrix}$$

$$T_1 = 1473°K.$$

$$\begin{bmatrix} 12 & -12 & 0 \\ -12 & 15 & -3 \\ 0 & -3 & 43 \end{bmatrix} \begin{bmatrix} 1473 \\ 0T_2 \\ 0T_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12120 \end{bmatrix}$$

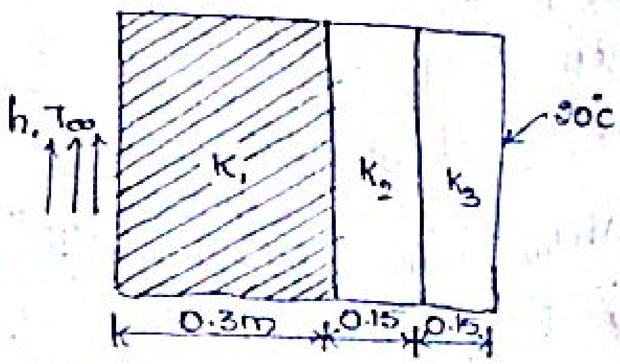
$$17676 - 12T_2 = 0$$

$$-17676 + 15T_2 - 3T_3 = 0$$

$$-3T_2 + 43T_3 = 12120$$

By Solving, $T_2 = 1252.22°K$, $T_3 = 369.23°K$.

2. A Composite Wall consists of 3 Materials, as shown in fig. The outer temperature is $T_0 = 20°c$ convection heat transfer takes place on the inner surface of the Wall with $T_\infty = 800°c$ and $h = 25 W/m^2°c$. Determine the temp. distribution.



$K_1 = 20 W/m°c$

$K_2 = 30 W/m°c$

$K_3 = 50 W/m°c$

Sol:

Element	Nodes	
1	1	2
2	2	3
3	3	4

Element ①

$$K_1 = [K_k] + [K_h]_{end}$$

$$= \frac{K_1 A_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + Ah \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix}0 & 0\\ 0 & 25\end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 1 \end{array}$$

$$= \frac{20}{0.3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix}0 & 0\\ 0 & 25\end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 1 \end{array}$$

$$= \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}66.67 & -66.67\\ -66.67 & 66.67\end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} + \begin{array}{c} \begin{array}{cc} 2 & 1 \end{array} \\ \begin{bmatrix}0 & 0\\ 0 & 25\end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 1 \end{array}$$

$$K_1 = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}91.67 & -66.67\\ -66.67 & 66.67\end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array}$$

Element ②

$$K_2 = \frac{30}{0.15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}200 & -200\\ -200 & 200\end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

Element ③

$$K_3 = \frac{50}{0.15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix}333.34 & -333.34\\ -333.34 & 333.34\end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \end{array}$$

Global stiffness Matrix.

$$K = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix}91.67 & -66.67 & 0 & 0\\ -66.67 & 266.67 & -200 & 0\\ 0 & -200 & 533.34 & -333.34\\ 0 & 0 & -333.34 & 333.34\end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

and F at $Q = AhT_\infty \begin{Bmatrix}0\\1\end{Bmatrix}\begin{array}{c}2\\1\end{array} = 25\times 800$ added at Node 1

$$KT_Q = F$$

$$\begin{bmatrix}91.67 & -66.67 & 0 & 0\\ -66.67 & 266.67 & -200 & 0\\ 0 & -200 & 533.34 & -333.34\\ 0 & 0 & -333.34 & 333.34\end{bmatrix}\begin{bmatrix}T_1\\ T_2\\ T_3\\ T_4\end{bmatrix} = \begin{bmatrix}25\times 800\\ 0\\ 0\\ 0\end{bmatrix}$$

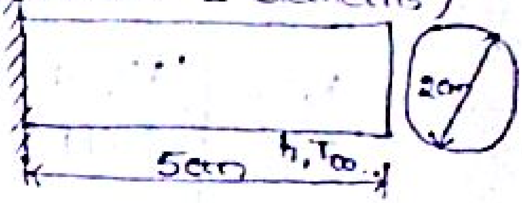
Given $T_4 = 20°c$, assuming this as a boundary condition, by using elimination approach eliminate 4th Row and column.

$$\begin{bmatrix} 91.67 & -66.67 & 0 & 0 \\ -66.67 & 266.67 & -200 & 0 \\ 0 & -200 & 533.34 & -333.34 \\ \cancel{0} & \cancel{0} & \cancel{-333.34} & \cancel{333.34} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \cancel{20°c} \end{bmatrix} = \begin{bmatrix} 25 \times 200 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 91.67 & -66.67 & 0 \\ -66.67 & 266.67 & -200 \\ 0 & -200 & 533.34 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 25 \times 200 \\ 0 \\ 0 \end{bmatrix}$$

By solving, we gets T_1, T_2, & T_3 values.

3. Find the temperature distribution in a straight fin with the physical properties as shown in fig. thermal conductivity $K = 70\ W/cm°c$ convection heat transfer coefficient $h = 10\ W/cm^2°c$ Temp. at root of the fin $T_0 = 140°c$, Surrounding temp. $T_\infty = 40°c$
   1. Assume free end of the fin is Insulated.
   2. Heat from the free end also. (Consider 2 elements)



Sol:- Given $T_0 = T_1 = 140°c$

$T_\infty = 40°c$.

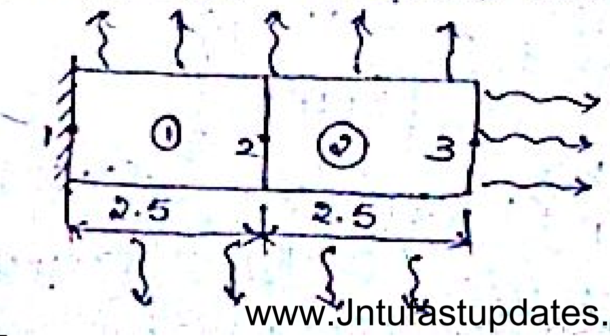
$K_1 = K_2 = 70\ W/cm°c$.

$h_1 = h_2 = 10\ W/cm^2°C$.

$A_1 = A_2 = \pi r^2 = \pi(1)^2 = \pi$

$L_1 = L_2 = 2.5cm$

Case II:- ~~Free End Insulation~~ Convection from free End also



Consider. element ①

$$[K_k] = \frac{A_1 k_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix}87.96 & -87.96\\ -87.96 & 87.96\end{bmatrix} & \begin{array}{c}1\\2\end{array}\end{array}$$

$$[K_h]_l = \frac{hPl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}52.35 & 26.17\\ 26.17 & 52.35\end{bmatrix}$$

$$[K_1] = [K_k] + [K_h]_l = \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix}140.31 & -61.79\\ -61.79 & 140.31\end{bmatrix} & \begin{array}{c}1\\2\end{array}\end{array}$$

element ②

$$[K_k] = \frac{A_2 k_2}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix}87.96 & -87.96\\ -87.96 & 87.96\end{bmatrix} & \begin{array}{c}2\\3\end{array}\end{array}$$

$$[K_h]_l = \frac{hPl}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix}52.35 & 26.17\\ 26.17 & 52.35\end{bmatrix} & \begin{array}{c}2\\3\end{array}\end{array}$$

$$[K_h]_{end} = hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix}0 & 0\\ 0 & 31.415\end{bmatrix} & \begin{array}{c}2\\3\end{array}\end{array}$$

$$[K_2] = [K_k] + [K_h]_l + [K_h]_{end} = \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix}140.31 & \cancel{26.17}\ -61.79\\ -61.79 & 171.725\end{bmatrix} & \begin{array}{c}2\\3\end{array}\end{array}$$

Global stiffness Matrix

$$K = \begin{bmatrix}140.31 & -61.79 & 0\\ -61.79 & 280.62 & -61.79\\ 0 & -61.79 & 171.725\end{bmatrix}$$

Consider the Heat flow

$$Q_{e\text{-}1} = \frac{hPlT_\infty}{2}\begin{Bmatrix}1\\1\end{Bmatrix} = \frac{10 \times 2\pi \times 2.5 \times 40}{2}\begin{Bmatrix}1\\1\end{Bmatrix}\begin{array}{c}1\\2\end{array}$$

$$= \begin{Bmatrix}3141.59\\ 3141.59\end{Bmatrix}\begin{array}{c}1\\2\end{array}$$

$$Q_{e\text{-}2} = \frac{hPlT_\infty}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + hAT_\infty\begin{Bmatrix}0\\1\end{Bmatrix} = \begin{Bmatrix}3141.59\\ 3141.59 + 1256.63\end{Bmatrix}\begin{array}{c}2\\3\end{array}$$

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 3141.59 \\ 6283.18 \\ 4398.22 \end{Bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

Consider the Finite element equation.

$$KT = Q$$

$$\begin{bmatrix} 140.31 & -61.79 & 0 \\ -61.79 & 280.62 & -61.79 \\ 0 & -61.79 & 171.725 \end{bmatrix} \begin{bmatrix} 140 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 3141.59 \\ 6283.18 \\ 4398.22 \end{bmatrix}$$

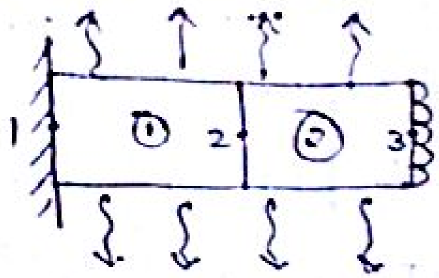
$$140.31(140) - 61.79T_2 = 3141.59$$

$$-61.79(140) + 280.62T_2 - 61.79T_3 = 6283.18$$

$$-61.79T_2 + 171.725T_3 = 4398.22$$

$T_2 = 267.06°C$ & $T_3 = 121.705°C$.

Case-1 :- Free End Insulation.



element ①

$$[K_k] = \begin{bmatrix} 87.96 & -87.96 \\ -87.96 & 87.96 \end{bmatrix}$$

$$[K_h]_e = \begin{bmatrix} 52.35 & 26.17 \\ 26.17 & 52.35 \end{bmatrix}$$

$$[K_1] = \begin{array}{cc} 1 & 2 \\ \end{array}$$
$$\begin{bmatrix} 140.31 & -61.79 \\ -61.79 & 140.31 \end{bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

Similarly.

$$\begin{array}{cc} 2 & 3 \end{array}$$
$$[K_2] = \begin{bmatrix} 140.31 & -61.79 \\ -61.79 & 140.31 \end{bmatrix}\begin{matrix} 2 \\ 3 \end{matrix}$$

Global Stiffness Matrix :-

$$K = \begin{bmatrix} 140.31 & -61.79 & 0 \\ -61.79 & 280.62 & -61.79 \\ 0 & -61.79 & 140.31 \end{bmatrix}$$

Consider the Heat Flow :

$$Q_{e-1} = \frac{hPl T_\infty}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 3141.59 \\ 3141.59 \end{Bmatrix}.$$

$$Q_{e-2} = \begin{Bmatrix} 3141.59 \\ 3141.59 \end{Bmatrix}.$$

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 3141.59 \\ 6283.18 \\ 3141.59 \end{Bmatrix}$$

Finite Element eq.

$$K.T = Q$$

$$\begin{bmatrix} 140.31 & -61.79 & 0 \\ -61.79 & 280.62 & -61.79 \\ 0 & -61.79 & 140.31 \end{bmatrix}\begin{bmatrix} 140 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 3141.59 \\ 6283.18 \\ 3141.59 \end{bmatrix}$$

$$(140.31)(140) - 61.79T_2 = 3141.59$$

$$T_2 = 267.06°C.$$

$$-61.79T_2 + 140.31T_3 = 3141.59.$$

$$T_3 = 139.99°C.$$

* Dynamic Analysis :-

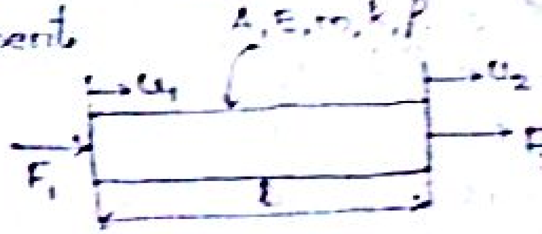
In this chapter we are going to learn about the dynamic analysis of the structures using FEM. For static analysis, the loads are applied very slowly and the displacements are produced gradually and the movement of mass is negligible & invisible. On the other hand, when the load is applied suddenly, the mass of the body is largely disturbed and it will move to a faster rate from its initial position. The analysis of such type of fast moving mass and the resultant displacement is termed as dynamic analysis. In dynamic problems, the displacements, velocities, strains, & stresses, loads are all time-dependent. That is, their magnitudes vary with respect to time.

→ Applications :-

The concept of dynamic analysis is applied to all kinds of structural components like bar, trusses, beams, frames and also machine components like piston rod, connecting rod, spindle etc., when they are undergoing sudden loads.

* Different Formulae depending on the shape / type :-

Model 1 :- 1 Dimensional Bar Element



$$k = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$m = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

m = Mass Matrix.

Finite element Eq. of dynamic eq: $\left[[K] - [M]\lambda\right]\{u\} = 0.$

& $\lambda = \omega^2 = \sqrt{k/m}$.

ω = Natural frequency.

λ = Eigen values.

or

$$\left[[K] - [M]\omega^2\right]\{u\} = 0.$$

Model: 2 :- Truss element

$$k_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$m_e = \frac{\rho A_e l_e}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

Model: 3 : Beam Element

$$k_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

$$m_e = \frac{\rho A_e l_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

Note :-

*** Lumped Mass Matrix :-

In lumped mass techniques, the total element mass in each direction is distributed equally to the nodes of element, and the masses are associated with translational degrees of freedom only.

For Truss Element

$$m_e = \frac{\rho A_e l_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For Beam Element

$$m_e = \frac{\rho A_e l_e}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Evaluation of Eigenvalues & Eigenvectors :-

Evaluating an eigen value $\lambda (=\omega^2)$ which is a measure of the frequency of vibration together with the corresponding eigen vector U indicating the mode shape

$$KU = \lambda MU.$$

or

$$\{K - \lambda M\} U = 0$$

Condition $|K - \lambda M| = 0$

where K = Global stiffness Matrix.

M = Global Mass Matrix.

λ = Eigen Value $= \omega^2$

$$\omega = \sqrt{K/m}$$

ω = Natural Circular frequency.

The time interval required for the mass to complete one full cycle of Motion is called the Period of the vibration τ (sec)

$$\tau = \frac{2\pi}{\omega}$$

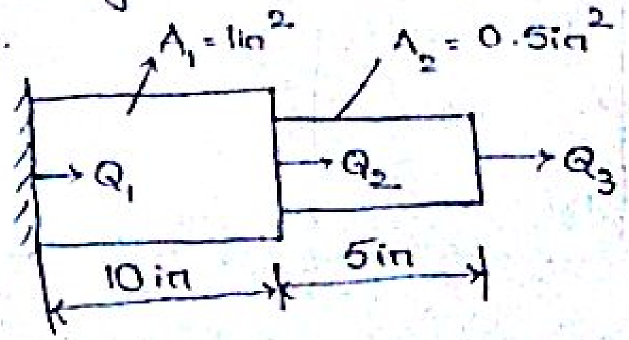
** Frequency, f (rad/sec) $= \frac{1}{\tau} = \frac{\omega}{2\pi}$

$$\omega = \frac{2\pi}{\tau} = 2\pi f.$$

$$\lambda = (2\pi f)^2. \Rightarrow f = \sqrt{\lambda}/2\pi$$

Problem :-

1. Determine the eigenvalue and eigen vector for the stepped bar shown in fig.

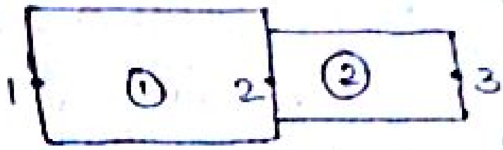


$E = 30 \times 10^6$ psi

Specific wt $f = 0.283$ lb/in

Sol:- Elements Nodes

①	1	2
②	2	3



Element ①

$$K_1 = \frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{30\times10^6}{10}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^6 \begin{array}{cc} & \begin{array}{cc}1 & 2\end{array}\\ & \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$

$$m_1 = \frac{\rho A_1 l_1}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

Density $\rho = \frac{f}{g} = \frac{0.283}{32.2\times12} = 7.324\times10^{-4}\ lbs^2/in^4$

$$m_1 = \frac{7.324\times10^{-4}\times10}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$g = 32.2\ ft/s^2$
$= 32.2\times12\ in/s^2$

$$m_1 = 10^{-4}\begin{array}{c}\begin{array}{cc}1 & 2\end{array}\\ \begin{bmatrix}24.41 & 12.206\\12.206 & 24.41\end{bmatrix}\end{array}\begin{array}{c}1\\2\end{array}$$

Element ②

$$K_2 = \frac{30\times10^6\times0.5}{5}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^6\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}$$

$$m_2 = \frac{\rho A_2 l_2}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix} = \frac{7.324\times10^{-4}\times10.5\times5}{6}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$= 10^{-4}\begin{array}{c}\begin{array}{cc}2 & 3\end{array}\\ \begin{bmatrix}6.10 & 3.05\\3.05 & 6.10\end{bmatrix}\end{array}\begin{array}{c}2\\3\end{array}$$

Global stiffness. Matrix :-

$$K = 10^6\begin{bmatrix}3 & -3 & 0\\-3 & 6 & -3\\0 & -3 & 3\end{bmatrix}$$

& Global Mass Matrix

$$M = 10^{-4}\begin{bmatrix} 24.41 & 12.206 & 0 \\ 12.206 & 30.51 & 3.05 \\ 0 & 3.05 & 6.10 \end{bmatrix}$$

(or)

$$M = \begin{bmatrix} 2.44\times10^{-3} & 1.22\times10^{-3} & 0 \\ 1.22\times10^{-3} & 3.05\times10^{-3} & 3.05\times10^{-4} \\ 0 & 3.05\times10^{-4} & 6.10\times10^{-4} \end{bmatrix}$$

Consider the eq.

$$[K - \lambda M]\,U = 0.$$

To find the value of λ, Let

$$|K - \lambda M| = 0.$$

$$\left|\begin{bmatrix} 3\times10^{6} & -3\times10^{6} & 0 \\ -3\times10^{6} & 6\times10^{6} & -3\times10^{6} \\ 0 & -3\times10^{6} & 3\times10^{6} \end{bmatrix} - \lambda \begin{bmatrix} 2.44\times10^{-3} & 1.22\times10^{-3} & 0 \\ 1.22\times10^{-3} & 3.05\times10^{-3} & 3.05\times10^{-4} \\ 0 & 3.05\times10^{-4} & 6.10\times10^{-4} \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} (3\times10^{6}) - \lambda(2.44\times10^{-3}) & (-3\times10^{6}) - \lambda(1.22\times10^{-3}) & 0 \\ (-3\times10^{6}) - \lambda(1.22\times10^{-3}) & (6\times10^{6}) - \lambda(3.05\times10^{-3}) & (-3\times10^{6}) - \lambda(3.05\times10^{-4}) \\ 0 & (-3\times10^{6}) - \lambda(3.05\times10^{-4}) & (3\times10^{6}) - \lambda(6.10\times10^{-4}) \end{vmatrix} = 0.$$

Since Node 1 is fixed, so eliminate 1st Row and column.

$$\begin{vmatrix} 6\times10^{6} - \lambda(3.05\times10^{-3}) & -(3\times10^{6} + \lambda(3.05\times10^{-4})) \\ -(3\times10^{6} + \lambda(3.05\times10^{-4})) & 3\times10^{6} - \lambda(6.10\times10^{-4}) \end{vmatrix} = 0.$$

$$\Rightarrow \left[6\times10^{6} - \lambda\left(3.05\times10^{-3}\right)\right]\left[3\times10^{6} - \lambda\left(6.10\times10^{-4}\right)\right] -$$

$$\left[3\times10^{6} + \lambda\left(3.05\times10^{-4}\right)\right]\left[3\times10^{6} + \lambda\left(3.05\times10^{-4}\right)\right] = 0$$

$$\Rightarrow \left[18\times10^{12} - 36.6\times10^{2}\lambda - 9.15\times10^{3}\lambda + \lambda^{2}\left(1.86\times10^{-6}\right)\right] -$$

$$\left[9\times10^{12} + 9.15\times10^{2}\lambda + 9.15\times10^{2}\lambda + 9.302\times10^{-8}\lambda^{2}\right] = 0$$

$$\Rightarrow 1.766\times10^{-6}\lambda^{2} - 14640\lambda + 9\times10^{12} = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^{2}-4ac}}{2a}$$

3 Q_2 }

$$\lambda = \frac{14640 \pm \sqrt{(14640)^{2} - 4(1.766\times10^{-6})(9\times10^{12})}}{2(1.766\times10^{-6})}$$

$$\lambda = \frac{14640 \pm 12278.175}{3.532\times10^{-6}}$$

$$\therefore \lambda_1 = 7.62\times10^{9} \quad \text{and} \quad \lambda_2 = 7.1\times10^{8}$$

Now To find the frequency

We have

$$\omega = 2\pi f$$

$$\lambda = \omega^{2} = (2\pi f)^{2}$$

$$f = \sqrt{\lambda}/2\pi$$

$$f_1 = 13893.05\,Hz \;\&\; f_2 = 4240.814\,Hz.$$

Now, to find the eigen vector for $\lambda_1 = 7.69 \times 10^{9}$

$$\{K - \lambda M\} U_1 = 0$$

$$\begin{bmatrix} -17.24 \times 10^6 & -5.32 \times 10^6 \\ -5.32 \times 10^6 & -1.64 \times 10^6 \end{bmatrix} \begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = 0.$$

$$17.24 Q_2 + 5.32 \times Q_3' = 0 \longrightarrow (1)$$

$$+5.32 Q_2 + 1.64 Q_3 = 0 \longrightarrow (2).$$

Since the determinant is zero

$$17.24 Q_2 = -5.32 Q_3. \longrightarrow \text{from } (1)$$

$$5.32 Q_2 = -1.64 Q_3 \longrightarrow \text{from } (2)$$

Let $U_1^T = \{17.24 Q_2 ,\ 5.32 Q_2\}$ or $\{Q_2 ,\ 0.308 Q_2\}$

or

$$U_1^T = \{+5.32 Q_3 ,\ -1.64 Q_3\}.$$