STEADY STATE HEAT TRANSFER ANALYSIS

INTRODUCTION:-

We discuss the finite element formation for the solution of steady state heat transfer problem. Heat transfer occurs when there is tempt difference with in a body (or) b/w a body & its surroundings medium.

1-D Heat Transfer:-

To study the steady state heat transfer condition problem is 1-D our object to determine temperature distribution. The heat transfer is in the form of conduction, convenction & thermal radiation. Only conduction & convenction nodes are treated.

Governing Equation:-

Consider heat conduction in a plain wall with uniform heat generation. Let 'A' be the area normal to the direction of heat flow.

Let Q watt/m^3 be the internal heat generated per unit volume. 'A' is the common example for heat generation, the heat produced in a wire carrying current I & having a

Resistance 'R' through a volume 'V':

$$\therefore Q = \frac{I^2 R}{V}$$

$$qA + QA\,dx = \left[q + \frac{dq}{dx}\,dx\right]A$$

$$\cancel{qA} + QA\,dx = \cancel{qA} + \frac{dq}{dx}\,dx\,A$$

$$Q\,\cancel{A}\,\cancel{dx} = \frac{dq}{dx}\,\cancel{dx}\,\cancel{A}$$

$$Q = \frac{dq}{dx}$$

$$\boxed{q = -K\frac{dT}{dx}}$$

where q - internal heat coefficient

K - heat transfer coefficient

dx - length

Boundary Coonditions :-

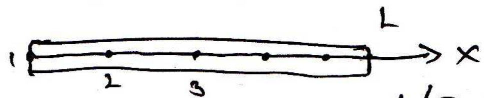
The boundary conditions mainly of three types specified temperature, specified heat flux & convection.

$$T\Big|_{x=0} = T_0$$

$$q\Big|_{x=L} = h(T_L - T_\infty)$$

Element Stiffness matrix 1-D heat transfer coefficient:-

The two node element with a linear shape function is consider subsequently. Now to apply the finite element method the problem is described in x-direction. The temperatures at various nodal displacements are denoted by 'T' as unknowns with in a tipical element e, whose local node numbers are 1 & 2.



$q = h(T_L - T_\infty)$

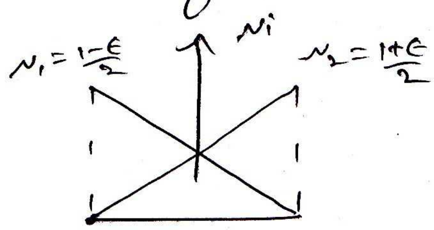
$$T(\epsilon) = N_1 T_1 + N_2 T_2$$

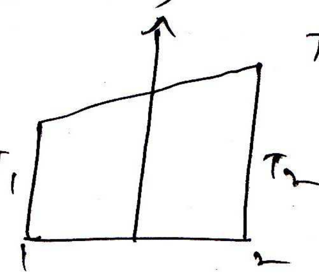
$$= N T_e$$

where $N_1 = \frac{1-\epsilon}{2}$, $N_2 = \frac{1+\epsilon}{2}$

ϵ varies from -1 to $+1$; $N = [N_1 N_2]$

$T = [T_1\ T_2]^T$





$$\epsilon = \frac{2}{x_2 - x_1}(x - x_1) - 1$$

$$\frac{d\epsilon}{dx} = \frac{2}{x_2 - x_1}$$

$$\frac{dT}{dx} = \frac{dT}{dx}\frac{d\epsilon}{dx} = \frac{2}{x_2 - x_1}\frac{dT}{dx}\cdot T_e$$

$$= \frac{\cancel{2}}{x_2 - x_1}\cdot\frac{1}{\cancel{2}}[-1\ \ 1]\cdot T_e$$

$$= \frac{1}{l_e}[-1 \ 1] \, T_e$$

$$\therefore \frac{dT}{dx} = B T_e$$

$$\boxed{B = \frac{1}{l_e}[-1 \ 1]}$$

strain energy equation is :-

$$U_e = \int_0^L K \left(\frac{dT}{dx}\right)^T \left(\frac{dT}{dx}\right) dx$$

$$U_e = K \left(\frac{dT}{dx}\right)^T \left(\frac{dT}{dx}\right) L$$

$$U_e = K \times B_e^T \, T_e^T \, B_e \, T_e \times L$$

$$U_e = K \, T_e^T \frac{1}{l_e} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \frac{1}{\cancel{l_e}} [-1 \ 1] \cancel{l_e}$$

$$U_e = T_e^T \frac{K}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} T_e$$

$$U_e = T_e^T \, K_e \, T_e$$

where $K_e = \frac{K}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

W.K.T $\quad KT = R$

where $R = R_Q + R_q$

$$R_Q = \frac{QAl}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -h_A(T_n - T_\infty) \end{bmatrix}$$

$\because R_Q$ = internal heat generation

n = node

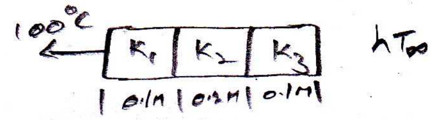
A = area

l = length

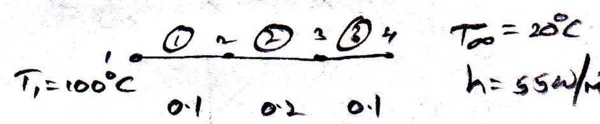
h_A = thermal conductivity

R_q = heat flux.

* Estimate the templerature distribution in 1-D slab as shown in figure.
$K_1 = 25\, w/m\,°K$ $K_2 = 10\, w/m\,°K$ $K_3 = 5\, w/m\,°K$ where
$h = 55\, w/m^2 K$ $T_\infty = 20°c$. find the temperature distribution of the slab.



sl:n



$$K_e = \frac{K}{le}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K_1 = \frac{K}{le}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{25}{0.1}\begin{array}{cc|c} 1 & 2 & \\ \left(\begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix}\right. & \left.\right] & \begin{matrix} 1 \\ 2 \end{matrix} \end{array} = 50\begin{array}{c} 1 \quad 2 \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \end{array}\begin{matrix} 1 \\ 2 \end{matrix}$$

$$K_2 = \frac{10}{0.2}\begin{array}{c} 2 \quad 3 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{matrix} 2 \\ 3 \end{matrix} = 50\begin{array}{c} 2 \quad 3 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{matrix} 2 \\ 3 \end{matrix}$$

$$K_3 = \frac{5}{0.1}\begin{array}{c} 3 \quad 4 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{matrix} 3 \\ 4 \end{matrix} = 50\begin{array}{c} 3 \quad 4 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{matrix} 3 \\ 4 \end{matrix}$$

$$K_e = \begin{array}{c} \begin{matrix} 1 & \quad 2 & \quad 3 & \quad 4 \end{matrix} \\ \begin{bmatrix} 250 & -250 & 0 & 0 \\ -250 & 300 & -50 & 0 \\ 0 & -50 & 100 & -50 \\ 0 & 0 & -50 & 50\,(+h) \end{bmatrix} \end{array}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$KT = R$

$$-250\times100 \qquad 0\times100 \qquad 0\times100$$

$$\begin{bmatrix} \cancel{250} & \cancel{-250} & \cancel{0} & \cancel{0} \\ -250 & 300 & -50 & 0 \\ 0 & -50 & 100 & -50 \\ 0 & 0 & -50 & 105 \end{bmatrix}\begin{bmatrix} \cancel{T_1} \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} \cancel{0} \\ 0-(-25000) \\ 0-(0) \\ 1100-(0) \end{bmatrix}$$

$T_2 = 94.8°$

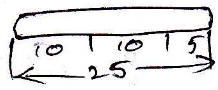
$T_3 = 69.12°$

$T_4 = 43.39°$

5/11

* The heat is generated in a large plate of thermal conductivity 0.8 w/mk. at the rate of 4000 w/m³.. The plate of thickness 25cm the outer surface is exposed to an ambient air with the heat transfer coefficient of 20 w/m² °k at 30°. If the inside surface temp is 500°C. calculate the temp distribution at the distance of 10cm from inner wall. Assume cross sectional area is 62.5 mm²

sol:-

$K = 0.8\ \text{w/m°K}$  $1\,cm = 10\,mm$

$q = 4000\ \text{w/m}^3$ $1m = \frac{1}{10} cm$

$t = 25\,cm$ $= 0.1$

$h = 20\ \text{w/m}^2\ \text{°K}$ $K = \frac{k}{le}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$T_\infty = 30°$

$t_1 = 500°$

$$K_1 = \frac{0.8}{0.1}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{array}{cc|c} 1 & 2 & \\ \hline \end{array}\begin{pmatrix}8 & -8\\ -8 & 8\end{pmatrix}\begin{array}{c}1\\2\end{array} \qquad K_2 = \frac{0.8}{0.1}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}8 & -8\\ -8 & 8\end{pmatrix}\begin{array}{c}2\\3\end{array}$$

$$K_3 = \frac{0.8}{0.05}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}16 & -16\\ -16 & 16\end{pmatrix}\begin{array}{c}3\\4\end{array}$$

$$K = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 8 & -8 & 0 & 0 \\ -8 & 16 & -8 & 0 \\ 0 & -8 & 24 & -16 \\ 0 & 0 & -16 & 16 \end{bmatrix} \end{array} \begin{array}{c}1\\2\\3\\4\end{array}$$

steady state heat transfer coefficient is $KT = R$

where $R = r_Q + r_q$

$$r_{Q_1} = \frac{ql}{2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{4000\times0.1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}200\\200\end{pmatrix}\begin{array}{c}1\\2\end{array}$$

$$r_{Q_2} = \frac{4000\times0.1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}200\\200\end{pmatrix}\begin{array}{c}2\\3\end{array}$$

$$r_{Q_3} = \frac{4000\times0.05}{2}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}100\\100\end{pmatrix}\begin{array}{c}3\\4\end{array}$$

$$\therefore r_Q = \begin{pmatrix}200\\400\\300\\100\end{pmatrix} \qquad r_q = \begin{pmatrix}0\\0\\0\\-h(T_4 - T_\infty)\end{pmatrix} = \begin{pmatrix}0\\0\\0\\+hT_\infty\end{pmatrix} = \begin{pmatrix}0\\0\\0\\60\end{pmatrix}$$

$$\begin{pmatrix}\cancel{8} & \cancel{-8} & \cancel{0} & \cancel{0}\\ \cancel{-8} & 16 & -8 & 0\\ \cancel{0} & -8 & 24 & -16\\ \cancel{0} & 0 & -16 & 16\end{pmatrix}\begin{pmatrix}\cancel{T_1}\\T_2\\T_3\\T_4\end{pmatrix} = \begin{pmatrix}\cancel{200}\\400\\300\\100+600\end{pmatrix}$$

$$\begin{pmatrix}16 & -8 & 0\\ -8 & 24 & -16\\ 0 & -16 & 16\end{pmatrix}\begin{pmatrix}T_2\\T_3\\T_4\end{pmatrix} = \begin{pmatrix}400 - (-8\times500)\\ 300 - (0\times500)\\ 700 - (0\times500\end{pmatrix}$$

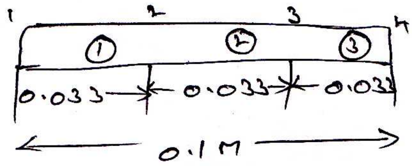
emp is 235°C. Determine temp distribution & the amount of heat transfer from the fin is to the air at 20°C with $h = 9\ W/m^2 °C$. Take width of the fin 1m.

$t = 0.1cm = 1\times10^{-3}\ m$ $\quad K = 360\ W/m°C$

$l = 10cm = 0.1m$ $\quad T_1 = 235°C$

$b = 1m$ $\quad T_\infty = 20°C$

$h = 9\ W/m^2 °C$



W.K.T $\quad KT = R$ $\quad$ where $K = K_T + H_T$

$K_T = \frac{K}{le}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ $\quad H_T = \frac{hle}{3t}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

$KT_1 = \frac{360}{0.033}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = 10909.0\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$ $\quad KT_3 = 10909.0\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$KT_2 = \frac{360}{0.033}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = 10909.0\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}$

$$K_T = 10909\begin{bmatrix}1 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}$$

$H_{t_1} = \frac{9\times0.033}{3\times10^{-3}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$ $\quad H_{t_2} = \frac{9\times0.033}{3\times10^{-3}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

$H_{t_3} = \frac{9\times0.033}{3\times10^{-3}}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$

$$H_T = 99\begin{bmatrix}2 & 1 & 0 & 0\\ 1 & 4 & 1 & 0\\ 0 & 1 & 4 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$K = K_T + H_T = \begin{bmatrix}11107 & -10810 & 0 & 0\\ -10810 & 22214 & -10810 & 0\\ 0 & -10810 & 22214 & -10810\\ 0 & 0 & -10810 & 11107\end{bmatrix}$$

$$h_q = \frac{hT_\infty le}{t}\begin{pmatrix}1\\1\end{pmatrix} = \frac{9\times20\times0.033}{10^{-3}}\begin{pmatrix}1\\2\\2\\1\end{pmatrix} = \begin{pmatrix}5940\\11880\\11880\\5940\end{pmatrix}$$

$KT = R$

$$\begin{pmatrix}\cancel{11107} & \cancel{-10810} & \cancel{0} & \cancel{0}\\ -10810 & 22214 & -10810 & 0\\ 0 & -10810 & 22214 & -10810\\ 0 & 0 & -10810 & 11107+h\end{pmatrix}\begin{pmatrix}T_1\\T_2\\T_3\\T_4\end{pmatrix} = \begin{pmatrix}\cancel{5940}\\11880\\11880\\5940\end{pmatrix} - \begin{bmatrix}(-10810\times234)\\(0\times234)\\(0\times234)\end{bmatrix}$$

$T_2 = 209.17$

$T_3 = 194.74$

$T_4 = 189.92$

DYNAMIC CONSIDERATION

Derivation for constant mass matrix for bar element:-

Since the shape functions for various elements have been discussed in earlier chapters. We know give the element mass matrix by treating the material density ρ to be constant over the element.

we have:

$$m_e = \rho \int N^T N \, dv$$

For 1-D bar element $\quad q = [q_1 \; q_2]^T$

$$N = [N_1 \; N_2]^T$$

where $N_1 = \frac{1-\epsilon}{2} \qquad N_2 = \frac{1+\epsilon}{2}$

$$m_e = \rho \int N^T N \, dv$$

$$= \rho \int N^T N \, A \, dx$$

$$= A\rho \int N^T N \, \frac{l_e}{2} \, d\epsilon$$

$$= \frac{A\rho l_e}{2} \int_{-1}^{1} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} (N_1 \; N_2) \, d\epsilon$$

$$= \frac{A\rho l_e}{2} \int_{-1}^{1} \begin{pmatrix} N_1^2 & N_1 N_2 \\ N_1 N_2 & N_2^2 \end{pmatrix} d\epsilon$$

$$= \frac{A\rho l_e}{2} \int_{-1}^{1} \begin{bmatrix} \frac{(1-\epsilon)^2}{4} & \frac{1-\epsilon^2}{4} \\ \frac{1-\epsilon^2}{4} & \frac{(1+\epsilon)^2}{4} \end{bmatrix} d\epsilon$$

$$= \frac{A l_e \rho}{8} \int_{-1}^{1} \begin{pmatrix} 1+\epsilon^2-2\epsilon & 1-\epsilon^2 \\ 1-\epsilon^2 & 1+\epsilon^2+2\epsilon \end{pmatrix} d\epsilon$$

$$= \frac{A l_e \rho}{8} \begin{pmatrix} \epsilon+\frac{\epsilon^3}{3}-\frac{2\epsilon^2}{2} & \epsilon-\frac{\epsilon^3}{3} \\ \epsilon-\frac{\epsilon^3}{3} & \epsilon+\frac{\epsilon^3}{3}+2\frac{\epsilon^2}{2} \end{pmatrix}_{-1}^{1}$$

$$= \frac{A l_e \rho}{8} \left[\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{7}{3} \end{pmatrix} - \begin{pmatrix} \frac{-7}{3} & \frac{-2}{3} \\ \frac{-2}{3} & \frac{-1}{3} \end{pmatrix} \right]$$

$$= \frac{A l_e \rho}{8} \begin{pmatrix} \frac{8}{3} & \frac{4}{3} \\ \frac{4}{3} & \frac{8}{3} \end{pmatrix}$$

For Truss Element:-

$$u^T = [u\ v]$$

$$q^T = (q_1\ q_2\ q_3\ q_4)$$

where $N = \begin{pmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{pmatrix}$

$$N_1 = \frac{1-\epsilon}{2} \qquad N_2 = \frac{1+\epsilon}{2}$$

$$m_e = \rho \int N^T N \, dV$$

$$= \rho \int N^T N \, A \, dx$$

$$= A\rho \int N^T N \frac{l_e}{2} \, d\epsilon$$

$$= \frac{A l_e \rho}{2} \int_{-1}^{1} \begin{pmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} N_1 & 0 & N_2 & 0 \\ 0 & N_1 & 0 & N_2 \end{pmatrix} d\epsilon$$

$$= \frac{\rho A l_e}{2} \int_{-1}^{1} \begin{bmatrix} N_1^2 & 0 & N_1 N_2 & 0 \\ 0 & N_1^2 & 0 & N_1 N_2 \\ N_1 N_2 & 0 & N_2^2 & 0 \\ 0 & N_1 N_2 & 0 & N_2^2 \end{bmatrix} d\epsilon$$

$$= \frac{\rho A l_e}{2} \begin{bmatrix} \frac{8}{3}\cdot\frac{1}{4} & 0 & \frac{4}{3}\cdot\frac{1}{4} & 0 \\ 0 & \frac{8}{3}\cdot\frac{1}{4} & 0 & \frac{4}{3}\cdot\frac{1}{4} \\ \frac{4}{3}\cdot\frac{1}{4} & 0 & \frac{8}{3}\cdot\frac{1}{4} & 0 \\ 0 & \frac{4}{3}\cdot\frac{1}{4} & 0 & \frac{8}{3}\cdot\frac{1}{4} \end{bmatrix}$$

$$m_e = \frac{\rho A l_e}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

For Beams

$$m_e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -13l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -13l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

Properties of Eigen values and Eigen vectors:-

For a +ve define symmetric stiffness matrix of size 'n' there 'n' real eigen values & corresponding eigen vectors.

$$0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots \lambda_n$$

The eigen values may be arranged in ascending order. If $u_1, u_2 \dots u_n$ then the corresponding values of eigen vector.

$$K u_i = \lambda_i M u_j$$

The eigen vector process the properly of being orthogonal w.r.t both stiffness & mass matrix.

$$u_i^T M u_j = 0 \quad \text{if } i \ne j$$

$$\text{Illy } u_i^T K u_j = 0 \quad \text{if } i \ne j$$

The length of the eigen vectors are generally normalized. so that

$$u_i^T M u_i = 1$$

Illy The eigen vector leads relation

$$u_i^T K u_i = \lambda_i$$

Eigen value & Eigen evaluation:-

This is based on following categories characteristic polynomial

i.e., $(K - \lambda M) U = 0$

is the vector is to be non-retrival

The required condition is $\det(K - \lambda M)U = 0$

Here λ - characteristic of polynomial

m - lompid mass matrix

K - element stiffness matrix.

<u>Formulae's</u>

$$K = \frac{AE}{le}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$m = \frac{\rho Al}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\lambda = \omega^2$$

where $\omega = 2\pi f$

$$\omega = \sqrt{\lambda}$$

$$2\pi f = \sqrt{\lambda}$$

$$f = \frac{\sqrt{\lambda}}{2\pi}$$