-: CONSTANT STRAIN TRIANGLE :-

* Constant strain Triangle Problems of 2-Dimensional :-

We know that, for Two-dimensional problems, the displacements, traction components, and distributed body forces values are functions of (x, y).

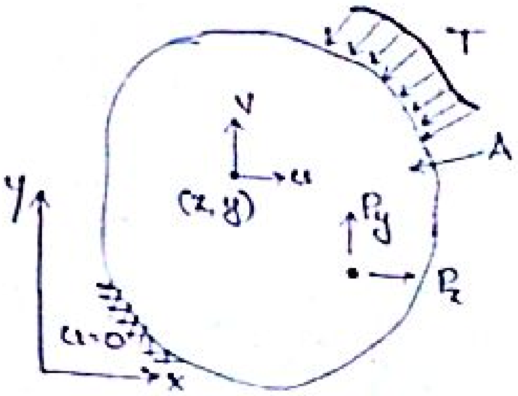
The displacement vector 'u' is given by

$$u = [u, v]^T$$

and the stresses and strains

$$\sigma = [\sigma_x, \sigma_y, \tau_{xy}]^T$$

$$\epsilon = [\epsilon_x, \epsilon_y, \gamma_{xy}]^T$$



$f = [f_x, f_y]^T$, $T = [T_x, T_y]^T$ and $dv = t.dA$.

where

f = Body force (force/unit volume)

T = Traction force (force/unit area)

The strain displacements relations are given by

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \cancel{\frac{\partial y}{\partial y}} \cdot \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

and $\sigma = D\epsilon$

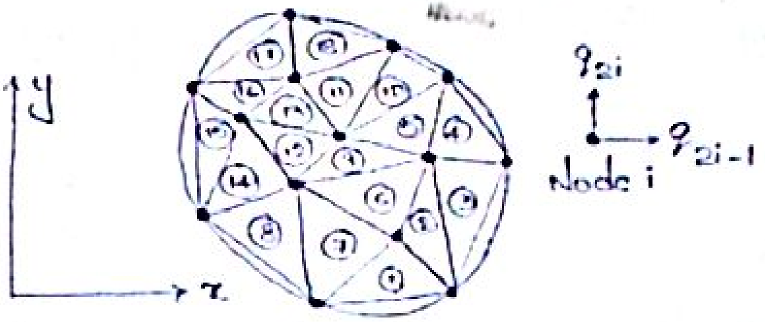
where the value of D is

for Plane stress, $D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

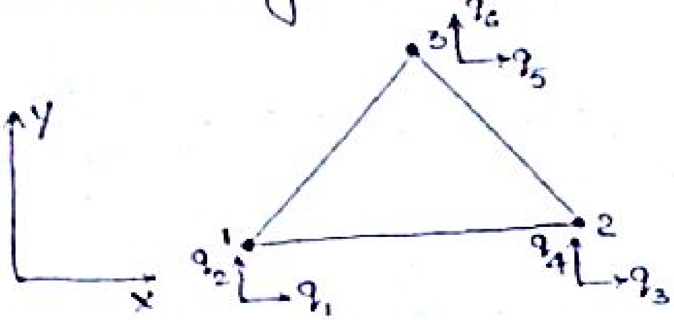
Plane strain, $D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$

* Finite element Modeling :

The two-dimensional region is divided into straight-sided triangles. the points where the corners of the triangle meet are called Nodes, and each triangle formed by three Nodes and three sides is called as elements.



Consider a single element :-



Here. $q_1 \cdot q_3 . q_5$ are the displacements along X dirt.
$q_2 . q_4 . q_6$ are the displacements along Y dirt.

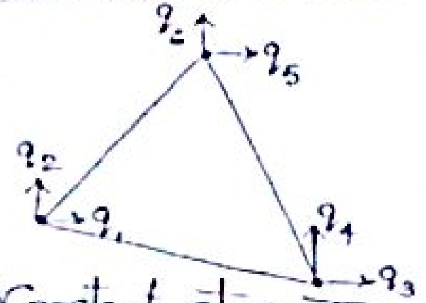
* Applications :-

To find the resultant displacements and stresses in case of Plates under bi-axial loading and bending of plates. For example, the body of an automobile vehicle, ship, train and aircraft etc. can be analysed to find the location of the point in that body where the maximum stress and the maximum displacements occurs.

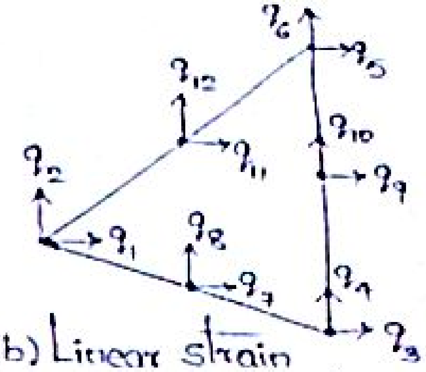
* CST and LST elements :

For the stress analysis of two dimensional objects such as plates and sheets. these objects are idealized into surface elements such as triangular, Rectangular and quadratic elements Among them, the triangular elements are considered as the simplest type of two dimensional elements. Depending upon the no. of nodes selected for the analysis, these Triangular elements are specified as linear elements or non-linear elements.

For linear elements, only three corners nodes are considered for analysis as shown. On the other hand, for the non-linear element, Apart from the corner nodes, some inner nodes are also taken into account as shown.



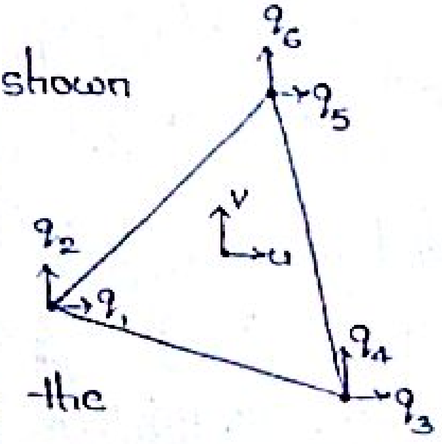
(a) Constant strain Triangle



(b) Linear strain Triangle.

For linear Triangular elements, the displacements are assumed to vary linearly and hence the change of displacement per unit length is constant throughout the element and hence this type of element is called as 'Constant strain triangle' whereas in non-linear triangular element the displacements vary non-linearly in such a way that the strain vary linearly and hence it is called 'Linear strain triangle'. or 'Quadratic triangle'.

* Constant strain Triangle:



Let consider a CST element, as shown

$$N_1 + N_2 + N_3 = 1$$

and let assume

$$N_1 = \xi \quad N_2 = \eta \quad , \& \quad N_3 = 1 - \xi - \eta$$

The displacements are written using the shape functions

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6$$

Now

$$u = q_1 \xi + \eta q_3 + (1-\xi-\eta) q_5$$

$$= (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5$$

$$\therefore u = q_{15}\xi + q_{35}\eta + q_5 \longrightarrow (i)$$

$$\& \quad v = q_2 \xi + q_4 \eta + (1-\xi-\eta) q_6$$

$$= (q_2 - q_6)\xi + (q_4 - q_6)\eta + q_6$$

$$\therefore v = q_{26}\xi + q_{46}\eta + q_6 \longrightarrow (ii)$$

and we $u = Nq$

where $N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$

Similarly we can write

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$= \xi x_1 + \eta x_2 + (1-\xi-\eta) x_3$$

$$= (x_1 - x_3)\xi + (x_2 - x_3)\eta + x_3$$

$$\therefore x = x_{13}\xi + x_{23}\eta + x_3 \longrightarrow (iii)$$

$$\& \quad y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$= \xi y_1 + \eta y_2 + (1-\xi-\eta) y_3$$

$$= (y_1 - y_3)\xi + (y_2 - y_3)\eta + y_3$$

$$\therefore y = y_{13}\xi + y_{23}\eta + y_3 \longrightarrow (iv)$$

We have seen from the above equations that (u,v) and (x, y) are the function $\xi . \eta$ that is $u = u(x(\xi,\eta) . y(\xi,\eta))$ a $v = v(x(\xi,\eta) . y(\xi,\eta))$. By using the chain rule

$$\left.\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi} \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta} \end{aligned}\right\} \longrightarrow (I)$$

Similarly

$$\left.\begin{aligned}\frac{\partial y}{\partial \xi} &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \xi} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \xi}\\ \frac{\partial v}{\partial \eta} &= \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \eta}\end{aligned}\right\} \text{— II}$$

Now considering the eq.'s I, writting in the form of Matrix

$$\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix} = J\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix}$$

Where, J = Jacobian Matrix.

$$J = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} = \begin{bmatrix}x_{13} & y_{13}\\ x_{23} & y_{23}\end{bmatrix}$$

$$\because \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}(x_{13}\xi + x_{23}\eta + x_3) = x_{13}$$

$$\frac{\partial y}{\partial \xi} = \frac{\partial}{\partial \xi}(y_{13}\xi + y_{23}\eta + y_3) = y_{13}$$

Similarly, $\frac{\partial x}{\partial \eta} = x_{23}$ & $\frac{\partial y}{\partial \eta} = y_{23}$

Now
$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = J^{-1}\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix}$$

$$\therefore J^{-1} = \frac{1}{\det J}\begin{bmatrix}y_{23} & -y_{13}\\ -x_{23} & x_{13}\end{bmatrix}$$ and $\det J = x_{13}y_{23} - x_{23}y_{13}$.

$\because A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$ then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$ &

$\det A = ad - bc$.

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = \frac{1}{\det J}\begin{bmatrix}y_{23} & -y_{13}\\ -x_{23} & x_{13}\end{bmatrix}\begin{bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{bmatrix}$$

$$\begin{bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{bmatrix} = \frac{1}{\det J}\begin{bmatrix}y_{23} & -y_{13}\\ -x_{23} & x_{13}\end{bmatrix}\begin{bmatrix}q_{15}\\ q_{35}\end{bmatrix}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{\det J}\left[y_{23}q_{15} - y_{13}q_{35}\right] \quad \& \quad \frac{\partial u}{\partial y} = \frac{1}{\det J}\left[-x_{23}q_{15} + x_{13}q_{35}\right]$$

Similarly, by considering the Equations II

$$\begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix}$$

and we get

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{bmatrix} q_{26} \\ q_{46} \end{bmatrix}$$

$$\therefore \frac{\partial v}{\partial x} = \frac{1}{\det J}\left[y_{23} q_{26} - y_{13} q_{46}\right] \;\&\; \frac{\partial v}{\partial y} = \frac{1}{\det J}\left[-x_{23} q_{26} + x_{13} q_{46}\right]$$

Using the strain-displacement relations

$$\epsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} y_{23} q_{15} - y_{13} q_{35} \\ -x_{23} q_{26} + x_{13} q_{46} \\ -x_{23} q_{15} + x_{13} q_{35} + y_{23} q_{26} - y_{13} q_{46} \end{bmatrix}$$

Let $-y_{13} = y_{31}$ and $-x_{23} = x_{32}$.

$$\epsilon = \frac{1}{\det J} \begin{bmatrix} y_{23}(q_1 - q_5) + y_{131}(q_3 - q_5) \\ x_{32}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ x_{32}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) + y_{31}(q_4 - q_6) \end{bmatrix}$$

$$= \frac{1}{\det J} \begin{bmatrix} y_{23} q_1 + y_{31} q_3 + q_5(-y_{23} - y_{31}) \\ x_{32} q_2 + x_{13} q_4 + q_6(-x_{32} - x_{13}) \\ x_{32} q_1 + y_{23} q_2 + x_{13} q_3 + y_{31} q_4 + q_5(-x_{13} - x_{32}) + q_6 (-y_{23} - y_{31}) \end{bmatrix}$$

Now $-y_{23} - y_{31} = y_{32} + y_{13} = \cancel{y_3} - y_2 + y_1 - \cancel{y_3}$

$= y_{12}$

and $-x_{32} - x_{13} = x_{23} + x_{31} = x_2 - \cancel{x_3} + \cancel{x_3} - x_1 = x_{21}$.

$$\epsilon = \frac{1}{\det J}\begin{bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{bmatrix}$$

$$\epsilon = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\epsilon = Bq$$

$$\therefore B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{21} \end{bmatrix}$$

where B = strain-displacement matrix

Now consider the Potential Energy Approach.

$$\pi = \frac{1}{2}\int_A \epsilon^T D\epsilon\, t.dA - \int_A u^T f\, t.dA - \int_L u^T T.dl - \sum_i u_i^T P_i$$

$$\left[\because T = D\epsilon \quad dv = t.dA\right.$$

consider the strain energy term

$$U_e = \frac{1}{2}\int_A \epsilon^T D\epsilon\, t.dA$$

$$= \frac{1}{2}\int_A B^T q^T D B q.\, t.dA$$

$$= \frac{1}{2}.q^T.B^T D.Bq.t.\int_A dA$$

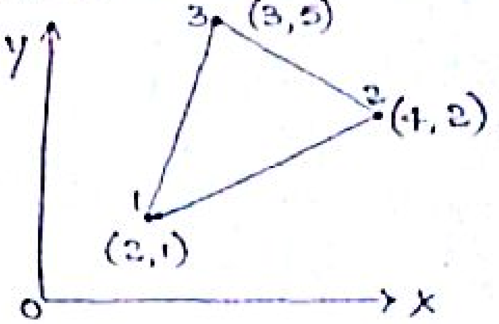
$$= \frac{1}{2}q^T t.A.B^T DB\, q$$

$$U_e = \frac{1}{2}q^T k_e q$$

$\therefore$ element stiffness Matrix, $k_e = q^T.\, t_e A_e B^T DB.$

Problems :

1. For the point 'p' located inside the triangular element as shown if the shape functions N_1 and N_2 are 0.3 and 0.5 resp, find its x and y-coordinates and the left out shape functions.



Sol:- Given

$(x_1, y_1) = (2,1)$, $(x_2, y_2) = (4,2)$, $(x_3, y_3) = (3,5)$.

and $N_1 = 0.3$ and $N_2 = 0.5$

We know that

$N_1 = \xi = 0.3$ & $N_2 = \eta = 0.5$

$x = x_{13}\xi + x_{23}\eta + x_3$

$= (2-3)\xi + (4-3)\eta + x_3$

$= (-1)\xi + \eta + 3.$

$x = -0.3 + 0.5 + 3 = 3.2$

& $y = y_{13}\xi + y_{23}\eta + y_3$

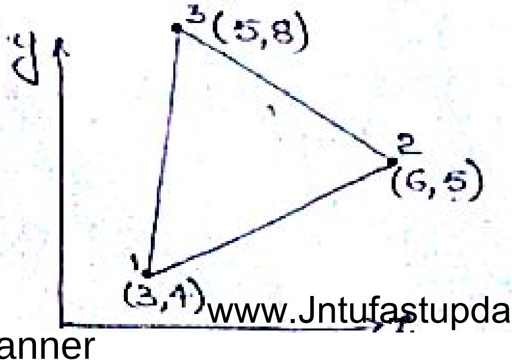
$= (1-5)0.3 + (2-5)0.5 + 5.$

$= 2.3$

we know that, $N_1 + N_2 + N_3 = 1$

$N_3 = 1 - 0.3 - 0.5 = 0.2.$

2. Determine the element stiffness Matrix for the fig shown below. Assume E = 200 GPas and $\mu = 0.3 = \nu$, and thickness = 10mm. Consider plane stress.

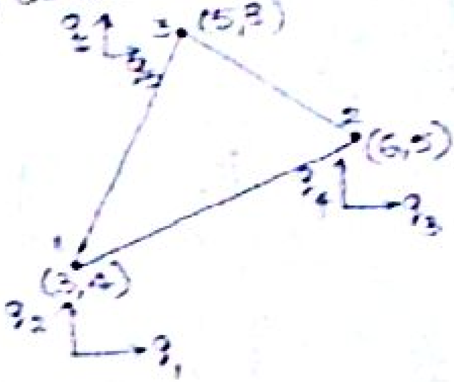


All dimensions are in Millimeters.

Sol:- Given

$(x_1, y_1) = (3,4)$; $(x_2, y_2) = (6,5)$; $(x_3, y_3) = (5,3)$

and G/ $E = 200$ GPa and $\nu = 0.3$, $t = 10$mm.



Area of element, $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$

$= \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 5 & 3 \end{vmatrix}$

$\therefore A = \frac{1}{2}\left[(48-25) - 3(3-5) + 4(5-6)\right]$

$= \frac{1}{2}\left[23 - 9 - 4\right]$

$A = 5\,mm^2$

Since, we need to solve the problem using plane stress condition, we have

$$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{200 \times 10^3}{(1-0.3^2)}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$\because E = 200$ GPa
$= 200 \times 10^9 N/m^2$
$= 200 \times 10^9 \times 10^{-6} N/mm^2$
$= 200 \times 10^3 N/mm^2$

$$\therefore D = 2.197 \times 10^5 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Now, we know, the strain-displacement matrix.

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

Here $J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$

$\det J = x_{13} y_{23} - x_{23} y_{13} = (-2)(-3) - (1)(-4)$

$= 6 + 4$

$\det J = 10$

$$B = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \end{bmatrix}$$

Now

$$B^T D = \frac{1}{10}\begin{bmatrix} -3 & 0 & -1 \\ 0 & -1 & -3 \\ 4 & 0 & -2 \\ 0 & -2 & 4 \\ -1 & 0 & 3 \\ 0 & 3 & -1 \end{bmatrix} 2.197\times10^5 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$B^T D = 2.197\times10^4 \begin{bmatrix} -3 & -0.9 & -0.35 \\ -0.3 & -1 & -1.05 \\ 4 & 1.2 & -0.7 \\ 0.6 & -2 & +1.4 \\ -1 & -0.3 & 1.05 \\ 0.9 & 3 & -0.35 \end{bmatrix}$$

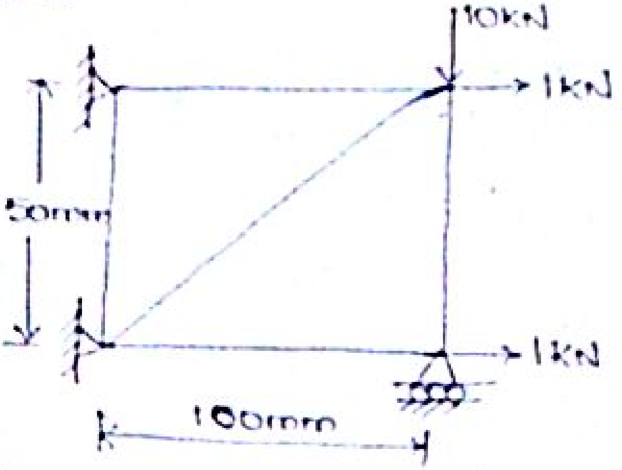
∴ We know,

The element stiffness Matrix $K_e = t_e A_e B^T D B$.

$$K_e = 10\times5\times2.197\times10^3 \begin{bmatrix} -3 & -0.9 & -0.35 \\ -0.3 & -1 & -1.05 \\ 4 & 1.2 & -0.7 \\ 0.6 & -2 & 1.4 \\ -1 & -0.3 & 1.05 \\ 0.9 & 3 & -0.35 \end{bmatrix} \begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4. & 3 & -1 \end{bmatrix}$$

$$\therefore K_e = 10.985\times10^4 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 9.35 & \overset{1.95}{\cancel{2.15}} & -11.3 & -3.2 & 1.95 & 3.05 \\ 1.95 & 4.15 & 0.9 & -2.2 & -2.85 & -1.95 \\ -11.3 & 0.9 & 17.4 & -5.2 & -6.1 & 4.3 \\ -3.2 & -2.2 & -5.2 & 9.6 & 4.8 & -7.4 \\ 1.95 & -2.85 & -6.1 & 4.8 & 4.15 & -1.95 \\ 3.05 & -1.95 & 4.3 & -7.4 & -1.95 & 9.35 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

3. Determine the Nodal displacements and the element stresses for the two dimensional loaded plate as shown in fig. Assume plane stress condition. Take $E = 210 GPa$, $\nu = 0.25$, $t = 10mm$

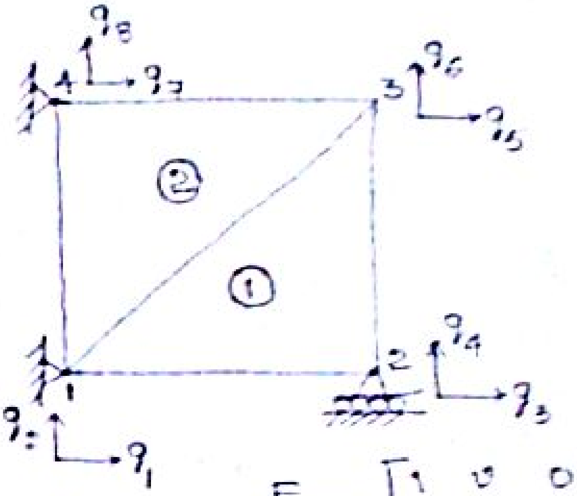


Sol:- Given

$E = 210 GPa = 210 \times 10^3 N/mm^2$

$\nu = 0.25$

$t = 10mm$



elements	Nodes		
①	1	2	3
②	1	4	3

Nodes	co-ordinates x	y
1	0	0
2	100	0
3	100	50
4	0	50

Plane stress $D = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$

$$D = \frac{210 \times 10^3}{(1-(0.25)^2)} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 01 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$D = 56 \times 10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Consider element ① :

$$B' = \frac{1}{\det J} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$(x_1, y_1) = (0,0)$
$(x_2, y_2) = (100,0)$
$(x_3, y_3) = (100,50)$

$$\det J = \left| \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \right| = \left| \begin{bmatrix} -100 & -50 \\ 0 & -50 \end{bmatrix} \right| = 5000.$$

$\det J = 5000.$

$$B' = \frac{1}{5000}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 0 & 0 & 0 & -100 & 0 & 100 \\ 0 & -50 & -100 & 50 & 100 & 0 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix}$$

Let

$$B^T D = \frac{1}{100}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \times 56\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$= 560\begin{bmatrix} -4 & -1 & 0 \\ 0 & 0 & -1.5 \\ 4 & 1 & -3 \\ -2 & -8 & 1.5 \\ 0 & 0 & 3 \\ 2 & 8 & 0 \end{bmatrix}$$

Element stiffness matrix

$$K_e = t_e A_e B^T D B$$

$$K_1 = \frac{\overset{5}{\cancel{10}} \times \frac{1}{\cancel{2}} \times 100 \times 50 \times 560}{\cancel{100}} \begin{bmatrix} -4 & -1 & 0 \\ 0 & 0 & -1.5 \\ 4 & 1 & -3 \\ -2 & -8 & 1.5 \\ 0 & 0 & 3 \\ 2 & 8 & 0 \end{bmatrix} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

$$K_1 = 250\times560 \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 4 & 0 & -4 & 2 & 0 & -2 & 1 \\ 0 & 1.5 & 3 & -1.5 & -3 & 0 & 2 \\ -4 & 3 & 10 & -5 & -6 & 2 & 3 \\ 2 & -1.5 & -5 & 17.5 & 3 & -16 & 4 \\ 0 & -3 & -6 & 3 & 6 & 0 & 5 \\ -2 & 0 & 2 & -16 & 0 & 16 & 6 \end{array}$$

Consider element ②:-

$(x_1, y_1) = (0,0)$: $(x_2, y_2) = (0,50)$: $(x_3, y_3) = (100,50)$

$$B^2 = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$\det J = |x_{13}y_{23} - x_{23}y_{13}| = (-100)(0) - (-100)(-50)$$

$$= 5000$$

$$B^2 = \frac{1}{5000}\begin{bmatrix} 0 & 0 & 50 & 0 & -50 & 0 \\ 0 & 100 & 0 & -100 & 0 & 0 \\ 100 & 0 & -100 & 50 & 0 & -50 \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & -1 \end{bmatrix}$$

$$B^{2^T} D = 560\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$= 560\begin{bmatrix} 0 & 0 & 3 \\ 2 & 8 & 0 \\ 4 & 1 & -3 \\ -2 & -8 & 1.5 \\ -4 & -1 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}$$

Element stiffness Matrix.

$$K_2 = \frac{\overset{5}{\cancel{10}} \times 1/\cancel{2} \times 50 \times \cancel{100} \times 560}{\cancel{100}}\begin{bmatrix} 0 & 0 & 3 \\ 2 & 8 & 0 \\ 4 & 1 & -3 \\ -2 & -8 & 1.5 \\ -4 & -1 & 0 \\ 0 & 0 & -1.5 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & -1 \end{bmatrix}$$

$$= 250 \times 560\begin{bmatrix} 6 & 0 & -6 & 3 & 0 & -3 \\ 0 & 16 & 2 & -16 & -2 & 0 \\ -6 & 2 & 10 & -5 & -4 & 3 \\ 3 & -16 & -5 & 17.5 & 2 & -1.5 \\ 0 & -2 & -4 & 2 & 4 & 0 \\ -3 & 0 & 3 & -1.5 & 0 & 1.5 \end{bmatrix}$$

$$K_2 = 250 \times 560 \times \begin{array}{c} \\ \left[\begin{array}{cccccc} 1 & 2 & 7 & 8 & 5 & 6 \\ 6 & 0 & -6 & 3 & 0 & -3 \\ 0 & 16 & 2 & -16 & -2 & 0 \\ -6 & 2 & 10 & -5 & -4 & 3 \\ 3 & -16 & -5 & 17.5 & 2 & -1.5 \\ 0 & -2 & -4 & 2 & 4 & 0 \\ -3 & 0 & 3 & -1.5 & 0 & 1.5 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 7 \\ 8 \\ 5 \\ 6 \end{array}$$

Global stiffness Matrix

$$K = 250 \times 560 \times \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 10 & 0 & -4 & 2 & 0 & -5 & -6 & 3 \\ 0 & 17.5 & 3 & -1.5 & -5 & 0 & 2 & -16 \\ -4 & 3 & 10 & -5 & -6 & 2 & 0 & 0 \\ 2 & -1.5 & -5 & 17.5 & 3 & -16 & 0 & 0 \\ 0 & -5 & -6 & 3 & 10 & 0 & -4 & 2 \\ -5 & 0 & 2 & -16 & 0 & 17.5 & 3 & -1.5 \\ -6 & 2 & 0 & 0 & -4 & 3 & 10 & -5 \\ 3 & -16 & 0 & 0 & 2 & -1.5 & -5 & 17.5 \end{array}\right] \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

and load vector.

$$F = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} \\ 1000 \\ R_{2y} \\ 1000 \\ -10000 \\ R_{4x} \\ R_{4y} \end{bmatrix}$$

Consider the finite element Eq. $KQ = F$

$$14 \times 10^4 \left[\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 10 & 0 & -4 & 2 & 0 & -5 & -6 & 3 \\ 0 & 17.5 & 3 & -1.5 & -5 & 0 & 2 & -16 \\ -4 & 3 & 10 & -5 & -6 & 2 & 0 & 0 \\ 2 & -1.5 & -5 & 17.5 & 3 & -16 & 0 & 0 \\ 0 & -5 & -6 & 3 & 10 & 0 & -4 & 2 \\ -5 & 0 & 2 & -16 & 0 & 17.5 & 3 & -1.5 \\ -6 & 2 & 0 & 0 & -4 & 3 & 10 & -5 \\ 3 & -16 & 0 & 0 & 2 & -1.5 & -5 & 17.5 \end{array}\right] \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ R_{1y} \\ 1000 \\ R_{2y} \\ 1000 \\ -10000 \\ R_{4x} \\ R_{4y} \end{bmatrix}$$

8.

By using elimination Approach, since $q_1 = q_2 = q_4 = q_7 = q_8 = 0$. So Eliminate 1, 2, 4, 7, 8 rows and Columns, then we get

$$14\times10^4 \begin{bmatrix} 10 & -6 & 2 \\ -6 & 10 & 0 \\ 2 & 0 & 17.5 \end{bmatrix} \begin{bmatrix} q_3 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1000 \\ -10,000 \end{bmatrix}$$

$$140\left[10q_3 - 6q_5 + 2q_6\right] = 1$$

$$140\left[-6q_3 + 10q_5\right] = 1$$

$$14\left[2q_3 + 17.5q_6\right] = -1$$

By solving, we get

$q_3 = 0.0032$, $q_5 = 0.0026$, $q_6 = -0.0044$ mm.

To find element stresses:

element ①

$$\sigma_1 = DBq$$

$$= \frac{56\times10^3}{100}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \\ 0 & -1 & -2 & 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$= 560\begin{bmatrix} -4 & 0 & 4 & -2 & 0 & 2 \\ -1 & 0 & 1 & -8 & 0 & 8 \\ 0 & -1.5 & -3 & 1.5 & 3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0.0032 \\ 0 \\ 0.0026 \\ -0.0044 \end{bmatrix}$$

$$= 560\begin{bmatrix} (4\times0.0032) + 2(-0.0044) \\ (1\times0.0032) + 8(-0.0044) \\ (-3\times0.0032) + 3(0.0026) \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_1 = \begin{Bmatrix} 2.24 \\ -17.92 \\ -1.008 \end{Bmatrix} N/mm^2 = \sigma_1$$

Element ②

$$\sigma_2 = DB\overset{2}{q}$$

$$= 560 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 \\ 2 & 0 & -2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_7 \\ q_8 \\ q_5 \\ q_6 \end{bmatrix}$$

$$= 560 \times \begin{bmatrix} 0 & +2 & 4 & -2 & -4 & 0 \\ 0 & 8 & 1 & -8 & -1 & 0 \\ 3 & 0 & -3 & 1.5 & 0 & -1.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.0026 \\ -0.0044 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_2 = \begin{Bmatrix} 5.824 \\ 1.456 \\ -3.696 \end{Bmatrix} N/mm^2$$

Note:- When any machine components or structure is subjected to two different stresses like normal stresses and shear stress, then at some specific planes inside the element, there may be maximum normal stresses, a minimum normal stress and a Maximum shear stress. The values are.

1. Maximum Normal stress

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x+\sigma_y)+\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

2. Minimum Normal stress

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x+\sigma_y)-\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}\right]$$

3. Maximum shear stress.

$$\tau_m = \frac{1}{2}(\sigma_1-\sigma_2) = \frac{1}{2}\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}$$

4. Principal angle θ_p

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_x-\sigma_y)} \quad \text{or}$$

$$\theta_p = \frac{1}{2}\text{Tan}^{-1}\left(\frac{2\tau_{xy}}{(\sigma_x-\sigma_y)}\right)$$

1. Calculate the element stresses $\sigma_x, \sigma_y, \tau_{xy}$. and Principle stress σ_1 & σ_2 and the principle angle θ_p for the CST element shown in fig.

The Nodal displacements are

$u_1 = 2.0\mu m$ | ∵ 1 Micro meter = 0.001mm or 10^{-6}m
1 Metre = 1000mm.

$v_1 = 1.0\mu m$

$u_2 = 0.5\mu m$ $v_2 = 1.5\mu m$

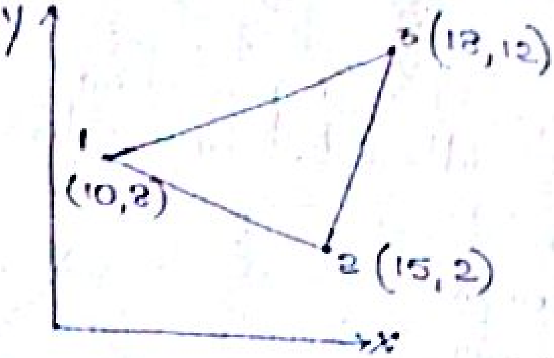
$u_3 = 1.2\mu m$ $v_3 = 2.8\mu m$

Take E = 210 GPa and $\nu = 0.25$. Assume plane strain condition.

$(x_1, y_1) = (10,8)$

$(x_2, y_2) = (15,2)$

$(x_3, y_3) = (18,12)$



Sol:- Given

$u_1 = 2\mu m = 2\times10^{-3} mm = 0.002$

$V_1 = 1.0\mu m = 0.001mm$

$u_2 = 0.5\mu m = 0.0005mm$

$V_2 = 1.5\mu m = 0.0015mm$

$u_3 = 1.2\mu m = 0.0012mm$

$V_3 = 2.8\mu m = 0.0028mm$

and $E = 210GPa = 210\times10^{9}\times10^{-6} N/mm^2 = 210\times10^{3} N/mm^2$

$\nu = 0.25$

and from diagram,

$(x_1, y_1) = (10,8) \quad (x_2, y_2) = (15,2) \quad (x_3, y_3) = (18,12)$

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 10 & 8 \\ 1 & 15 & 2 \\ 1 & 18 & 12 \end{vmatrix}$$

$$= \frac{1}{2}\left[(180-36) - 10(12-2) + 8(18-15)\right]$$

$$= \frac{1}{2}\left[144 - 100 + 24\right]$$

$$= 34.$$

Now for plane strain condition :

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

$$= \frac{210\times10^{3}}{(1+0.25)(1-2(0.25))}\begin{bmatrix} 1-0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0.5-0.25 \end{bmatrix}$$

$$= 0.625\times10^{3}\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

8) $$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -3 & -10 \end{bmatrix}$$

$$\det J = 80 - 12 = 68$$

$$B = \frac{1}{68}\begin{bmatrix} -10 & 0 & 4 & 0 & 6 & 0 \\ 0 & 3 & 0 & -8 & 0 & 5 \\ 3 & -10 & -8 & 4 & 5 & 6 \end{bmatrix}$$

for given CST element, the element stresses is given by

$$\sigma = [D][\epsilon] = DBq$$

8) $\sigma = \{\sigma_x, \sigma_y, \tau_{xy}\}^T$

$$\sigma = 0.625\times10^3\begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \times \frac{1}{68}\begin{bmatrix} -10 & 0 & 4 & 0 & 6 & 0 \\ 0 & 3 & 0 & -8 & 0 & 5 \\ 3 & -10 & -8 & 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$= 9.191\begin{bmatrix} -7.5 & 0.75 & 3 & -2 & 4.5 & 1.25 \\ 2.5 & 2.25 & 1 & -6 & 1.5 & 3.75 \\ 0.75 & -2.5 & -2 & 1 & 1.25 & 1.5 \end{bmatrix}\begin{bmatrix} 0.002 \\ 0.001 \\ 0.0005 \\ 0.0015 \\ 0.0012 \\ 0.0028 \end{bmatrix}$$

$$= 9.191\begin{bmatrix} -0.00685 \\ 0.01105 \\ 0.0052 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} -0.0629 \\ 0.1015 \\ 0.0477 \end{bmatrix} N/mm^2$$

Maximum Normal stress (or Max. principal stress)

$$\sigma_1 = \frac{1}{2}\left[(\sigma_x + \sigma_y) + \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

$$= \frac{1}{2}\left[(-0.0629 + 0.1015) + \sqrt{(-0.0629 - 0.1015)^2 + 4(0.0477)}\right]$$

$$= \frac{1}{2}\left[0.298\right]$$

$$\sigma_1 = 0.1431 \; N/mm^2.$$

- Minimum Normal stress (or Min. Principal stress)

$$\sigma_2 = \frac{1}{2}\left[(\sigma_x + \sigma_y) - \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}\right]$$

$$= \frac{1}{2}\left[(-0.0629 + 0.1015) - \sqrt{(-0.0629 - 0.1015)^2 + 4(0.0477)^2}\right]$$

$$= -0.0757 \; N/mm^2$$

Principal angle $\theta_p = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}\right]$

$$= \frac{1}{2}\tan^{-1}\left[\frac{2(0.0477)}{(-0.0629 - 0.1015)}\right]$$

$$\theta_p = \frac{1}{2}\tan^{-1}(-0.5802)$$

* Temperature Effects :-

At the time of function, if the CST element is at a higher temperature or lower temperature than room temperature, the change in temperature ΔT produces some amount of deformation and the corresponding strain is known as thermal strain, which is considered as initial strain

For plane stress $\{\epsilon_0\} = \begin{Bmatrix} \epsilon_{x0} \\ \epsilon_{y0} \\ \gamma_{xy_0} \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$

For plane strain $\{\epsilon_0\} = (1+\nu)\begin{bmatrix}\alpha\Delta T\\ \alpha\Delta T\\ 0\end{bmatrix}$

α : Coefficient of thermal expansion.

ΔT : Change in temperature.

ν = Poisson's ratio.

element thermal load (i.e, thermal force)

$$\{F_0\} = \{\theta_0\} = [B^T] D\epsilon_0 \times At.$$

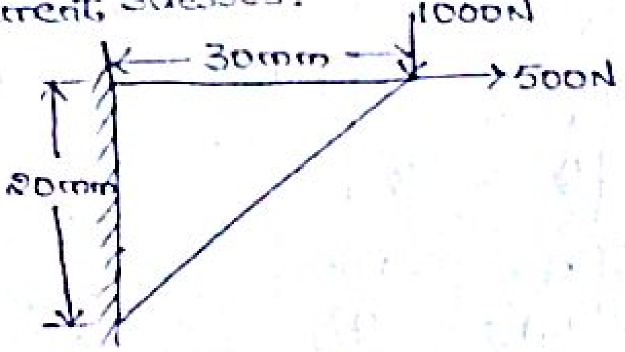
$$= t.A\, B^T D\, \epsilon_0$$

element ~~thermal~~ stress (i.e, due to Mechanical loading & temp)

$$\{\sigma_T\} = D[\epsilon-\epsilon_0] = D[Bq-\epsilon_0]$$

$$= DBq - D\epsilon_0.$$

1. The element shown in the figure is subjected to temperature change of 10c. Find the K and F due to temp. change. Take E = 70GPas, t = 10mm, ν = 0.3 and $\alpha = 7\times10^{-6}/°c$. and also determine the deflection at the point of load application & also the element stresses.



Assume plane stress.

Sol:- Given

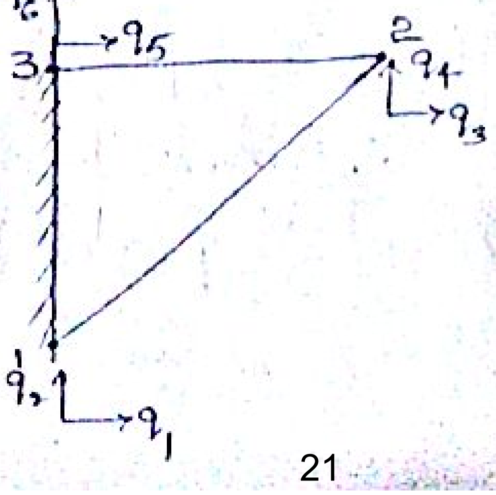
$E = 70\times10^3 N/mm^2$ t=10mm, & $\nu = 0.3$, $\alpha = 7\times10^{-6}/°c$

$\Delta T = 10°c$.

$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (30,20)$

$(x_3, y_3) = (0,20)$



$$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \frac{70\times10^3}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= \frac{7\times10^4}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

then

$$B = \frac{1}{\det J}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \Rightarrow \det J = -(30)(-20) = 600.$$

$$B = \frac{1}{600}\begin{bmatrix} 0 & 0 & 20 & 0 & -20 & 0 \\ 0 & -30 & 0 & 0 & 0 & 30 \\ -30 & 0 & 0 & 20 & 30 & -20 \end{bmatrix} = \frac{1}{60}\begin{bmatrix} 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & 2 & 3 & -2 \end{bmatrix}$$

Let consider

$$B^T D = \frac{7\times10^4}{0.91\times60}\begin{bmatrix} 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 0.128\times10^4\begin{bmatrix} 0 & 0 & -1.05 \\ -0.9 & -3 & 0 \\ 2 & 0.6 & 0 \\ 0 & 0 & 0.7 \\ -2 & -0.6 & 1.05 \\ 0.9 & 3 & -0.7 \end{bmatrix}$$

∴ Element stiffness Matrix

$$K_e = tAB^TDB$$

$$K_e = 10 \times \frac{1}{2} \times 20 \times 30 \times 0.128 \times 10^4 \times \frac{1}{60} \begin{bmatrix} 0 & 0 & -1.05 \\ -0.9 & -3 & 0 \\ 2 & 0.6 & 0 \\ 0 & 0 & 0.7 \\ -2 & -0.6 & 1.05 \\ 0.9 & 3 & -0.7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & -3 & 0 & 0 & 0 & 3 \\ -3 & 0 & 0 & 2 & 3 & -2 \end{bmatrix}$$

$$= 6.4 \times 10^3 \begin{bmatrix} 3.15 & 0 & 0 & -2.1 & -3.15 & 2.1 \\ 0 & 9 & -1.8 & 0 & 1.8 & -9 \\ 0 & -1.8 & 4 & 0 & -4 & 1.8 \\ -2.1 & 0 & 0 & 1.4 & 2.1 & -1.4 \\ -3.15 & 1.8 & -4 & 2.1 & 7.15 & -3.9 \\ 2.1 & -9 & 1.8 & -1.4 & -3.9 & 10.4 \end{bmatrix}$$

Global load vector $F = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix}$

Now we know that

$$\theta = t A B^T D \epsilon_0$$

$$= 10^5 \times \frac{1}{2} \times 20 \times 30 \times \frac{1}{60} \times 0.122 \times 10^4 \begin{bmatrix} 0 & 0 & -1.05 \\ -0.9 & -3 & 0 \\ 2 & 0.6 & 0 \\ 0 & 0 & 0.7 \\ -2 & -0.6 & 1.05 \\ 0.9 & 3 & -0.7 \end{bmatrix} \begin{Bmatrix} \alpha \Delta T \\ \alpha \Delta T \\ 0 \end{Bmatrix}$$

$$= 384 \times 10^4 \begin{bmatrix} 0 & 0 & -1.05 \\ -0.9 & -3 & 0 \\ 2 & 0.6 & 0 \\ 0 & 0 & 0.7 \\ -2 & -0.6 & 1.05 \\ 0.9 & 3 & -0.7 \end{bmatrix}_{6\times3} \begin{bmatrix} 7 \times 10^{-6} \times 10 \\ 7 \times 10^{-6} \times 10 \\ 0 \end{bmatrix}_{3\times1}$$

$$= 384 \times 10^4 \begin{bmatrix} 0 \\ -0.000273 \\ 0.000182 \\ 0 \\ -0.000182 \end{bmatrix}$$

$$
q = \left[\begin{array}{c} 0 \\ -1048.32 \\ 698.88 \\ 0 \\ -698.88 \\ +1048.32 \end{array}\right]\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
$$

∴ Global load vector :

$$
F = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix} = \begin{bmatrix} R_{1x} \\ -1048.32 + R_{1y} \\ 500 + 698.88 \\ -1000 \\ -698.88 + R_{3x} \\ 1048.32 + R_{3y} \end{bmatrix}
$$

Consider the finite element eq.

$$
64 \times 10^3 \begin{bmatrix} 3.15 & 0 & 0 & -2.1 & -3.15 & 2.1 \\ 0 & 9 & -1.8 & 0 & 1.8 & -9 \\ 0 & -1.8 & 4 & 0 & -4 & 1.8 \\ -2.1 & 0 & 0 & 1.4 & 2.1 & -1.4 \\ -3.15 & 1.8 & -4 & 2.1 & 7.15 & -3.9 \\ 2.1 & -9 & 1.8 & -1.4 & -3.9 & 10.4 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} R_{1x} \\ -1048.32 + R \\ 1198.88 \\ -1000 \\ -698.88 + R \\ 1048.32 + R \end{bmatrix}
$$

By using elimination approach, $q_1 = q_2 = q_5 = q_6 = 0$

Eliminate 1, 2, 5, 6 rows & columns

$$
64 \times 10^3 \begin{bmatrix} 4 & 0 \\ 0 & 1.4 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 1198.88 \\ -1000 \end{bmatrix}
$$

$$
64 \times 10^3 \times 4 q_3 = 1198.88 \Rightarrow q_3 = 0.00468 \text{ mm.}
$$

$$
64 \times 10^3 \times 1.4 q_4 = -1000 \Rightarrow q_4 = -0.01116 \text{ mm.}
$$

Element Stresses :

$$\sigma = D\left[\epsilon - \epsilon_0\right]$$

$$= DBq - D\epsilon_0$$

$$= \frac{7\times10^4}{0.91}\times\frac{1}{60}\begin{bmatrix}1 & 0.3 & 0\\ 0.3 & 1 & 0\\ 0 & 0 & 0.35\end{bmatrix}\begin{bmatrix}0 & 0 & 2 & 0 & -2 & 0\\ 0 & -3 & 0 & 0 & 0 & 3\\ -3 & 0 & 0 & 2 & 3 & -2\end{bmatrix}q - \frac{7\times10^4}{0.91}\times\begin{Bmatrix}\alpha\Delta T\\ \alpha\Delta T\\ 0\end{Bmatrix}$$

$$= 1282.051\begin{bmatrix}0 & -0.9 & 2 & 0 & -2 & 0.9\\ 0 & -3 & 0.6 & 0 & -0.6 & 3\\ -1.05 & 0 & 0 & 0.7 & +1.05 & 0.7\end{bmatrix}q - \frac{7\times10^4}{0.91}\times\begin{bmatrix}7\times10^{-5}\\ 7\times10^{-5}\\ 0\end{bmatrix}$$

$$= 1282.051\begin{bmatrix}0 & -0.9 & 2 & 0 & -2 & 0.9\\ 0 & -3 & 0.6 & 0 & -06 & 3\\ -1.05 & 0 & 0 & 0.7 & 1.05 & 0.7\end{bmatrix}\begin{bmatrix}0\\ 0\\ 0.00468\\ -0.0116\\ 0\\ 0\end{bmatrix} - \frac{7\times10^4}{0.91}\begin{bmatrix}7\times10^{-5}\\ 7\times10^{-5}\\ 0\end{bmatrix}$$

$$= 1282.051\begin{bmatrix}2(0.00468)\\ 0.6(0.00468)\\ 0.7(-0.0116)\end{bmatrix} - \frac{7\times10^4}{0.91}\begin{bmatrix}7\times10^{-5}\\ 7\times10^{-5}\\ 0\end{bmatrix}$$

$$= \begin{bmatrix}11.99\\ 3.599\\ -10.015\end{bmatrix} - \begin{bmatrix}5.384\\ 5.384\\ 0\end{bmatrix}$$

$$\sigma = \begin{bmatrix}6.606\\ -1.785\\ -10.015\end{bmatrix}$$

* Axisymmetric Loading :-

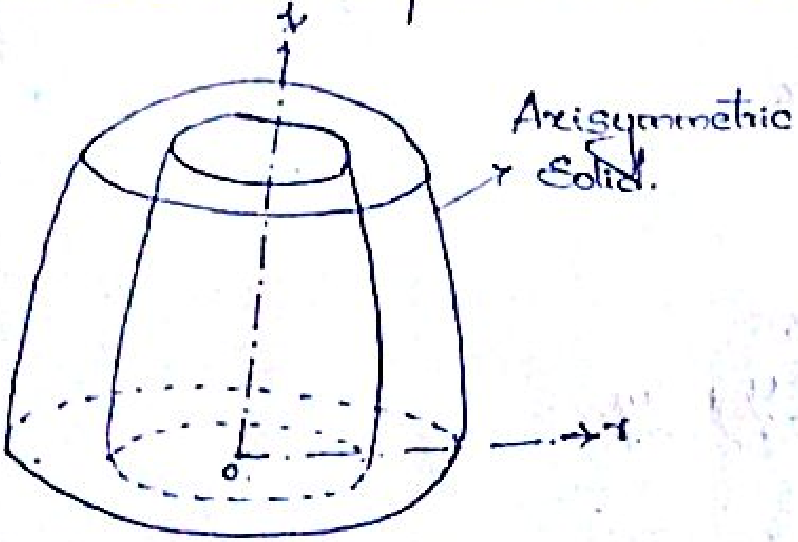
In general, all the structures and machine components are three dimensional members and their properties are analysed in FEM either by 1D, 2D and 3D techniques based on their dimensional ratios.

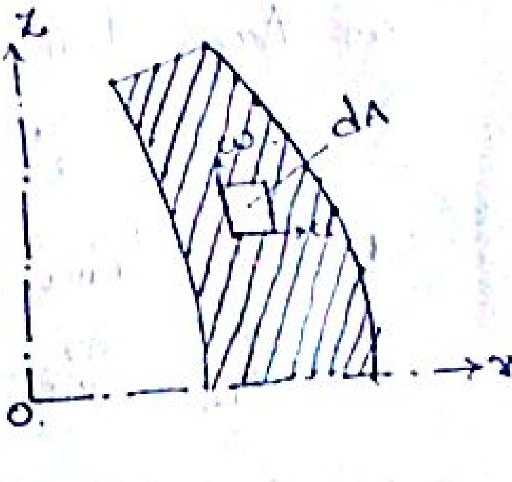
Eg:- 1D - Bars, Beams, Truss.
2D - Plates etc.

At the time same time, for some components known as Axisymmetric solids or Solids of Revolution, eventhough their all the 3 dimensions are comparatively large, they may be analysed using two dimensional techniques due to their axial Symmetry.

Eg:- Pressure Vessels, Cylinders, Fly wheels.

Usually Z-axis may be considered as axis of symmetry and the point in the plane perpendicular to Z-axis is represented by Polar Co-ordinates (r,θ). Because of symmetry all deformations and stresses are independent of co-ordinate 'θ'.



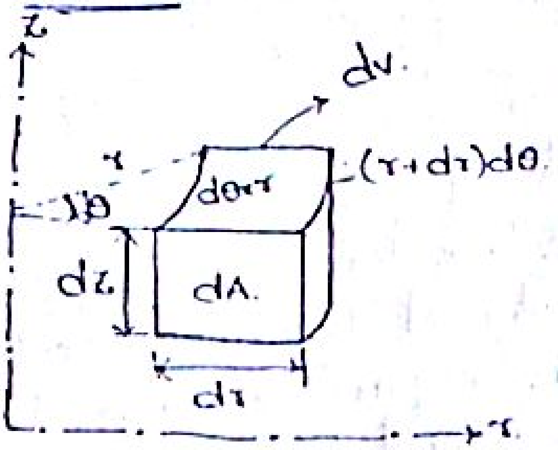


During analysis, body force may be considered wherever necessary. Similarly in the case of Flywheel, centrifugal force may be taken into account.

We know that 2D objects.

$u = [u, w]^T$ $\quad f = [f_r, f_z]^T$, $T = [T_r, T_z]^T$

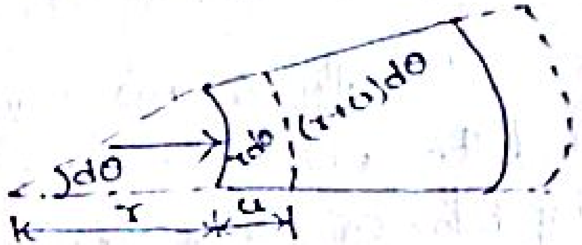
* **Elasticity Relations :-**



(a) Radial strain, ϵ_r.

$$\epsilon_r = \frac{\partial u}{\partial r}$$

(b) Tangential strain ϵ_θ.



$$\epsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta}$$

$$\epsilon_\theta = \frac{u\,\cancel{d\theta}}{r\,\cancel{d\theta}} = \frac{u}{r}$$

(c) Axial strain, ϵ_z

$$\epsilon_z = \frac{\partial w}{\partial z}$$

(d) Shear strain, γ_{rz}

$$\gamma_{rz} = \cancel{\frac{\partial r}{\partial \theta}} \cdot \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}.$$

∴ Strains for Axisymmetric solid, ϵ =

$$\epsilon = \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ u/r \end{Bmatrix}$$

and the stresses

$$\sigma = [\sigma_r , \sigma_z , \tau_{rz} , \sigma_\theta]^T$$

and Stress - strain Relation

$$\sigma = D\epsilon$$

We know that for 3D element.

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \\ \tau_{z\theta} \\ \tau_{r\theta} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \epsilon_\theta \\ \gamma_{rz} \\ \gamma_{z\theta} \\ \gamma_{r\theta} \end{Bmatrix}$$

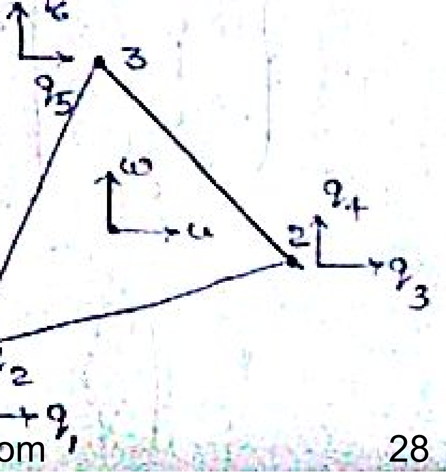
Here $\gamma_{z\theta}$ & $\gamma_{r\theta} = 0$.

then, we get.

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \tau_{rz} \\ \sigma_\theta \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 & \nu \\ \nu & (1-\nu) & 0 & \nu \\ 0 & 0 & \frac{1}{2}-\nu & 0 \\ \nu & \nu & 0 & 1-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{Bmatrix}$$

$$\therefore D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 & \nu \\ \nu & (1-\nu) & 0 & \nu \\ 0 & 0 & \frac{1}{2}-\nu & 0 \\ \nu & \nu & 0 & 1-\nu \end{bmatrix}$$

* **Strain Displacement Matrix :-**



Let $N_1 = \xi$

$N_2 = \eta$

$N_3 = 1 - \xi - \eta$

$N_1 + N_2 + N_3 = 1.$

$u = N_1 q_1 + N_2 q_3 + N_3 q_5$

$v = N_1 q_2 + N_2 q_4 + N_3 q_6.$

Substituting N_1, N_2 & N_3

$u = \xi q_1 + \cancel{\xi}\, \eta q_3 + (1-\xi-\eta) q_5$

$= (q_1 - q_5)\xi + (q_3 - q_5)\eta + q_5.$

$u = q_{15}\xi + q_{35}\eta + q_5$

& $v = q_{26}\xi + q_{46}\eta + q_6.$

Similarly

$r = N_1 r_1 + N_2 r_2 + N_3 r_3.$

$z = N_1 z_1 + N_2 z_2 + N_3 z_3.$

We get

$r = (r_{13})\xi + r_{23}\eta + r_3.$

$z = z_{13}\xi + z_{23}\eta + z_3.$

Using chain Rule.

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial \xi} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial \eta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \eta}.$$

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}}_{J} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{Bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} = \text{Jocatbian Matrix.}$$

$$J = \begin{bmatrix} r_{12} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{Bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & +r_{13} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & +r_{13} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & +r_{13} \end{bmatrix} \begin{Bmatrix} q_{15} \\ q_{35} \end{Bmatrix}$$

Similarly.

$$\begin{Bmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & +r_{13} \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{\det J} \begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & +r_{13} \end{bmatrix} \begin{Bmatrix} q_{26} \\ q_{46} \end{Bmatrix}$$

Now consider the strain.

$$\epsilon = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ u/r \end{Bmatrix} = \begin{bmatrix} \frac{1}{\det J}\left[z_{23} q_{15} - z_{13} q_{35}\right] \\ \frac{1}{\det J}\left[-r_{23} q_{26} + r_{13} q_{46}\right] \\ \frac{1}{\det J}\left[-r_{23} q_{15} + r_{13} q_{35} + z_{23} q_{26} - z_{13} q_{46}\right] \\ \frac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{1}{\det J}\left[z_{23} q_{15} + z_{31} q_{35}\right] \\ \frac{1}{\det J}\left[r_{32} q_{26} + r_{13} q_{46}\right] \\ \frac{1}{\det J}\left[r_{32} q_{15} + r_{13} q_{35} + z_{23} q_{26} + z_{31} q_{46}\right] \\ \frac{N_1}{r} q_1 + \frac{N_2}{r} q_3 + \frac{N_3}{r} q_5 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{1}{\det J}\left[z_{23}q_1 - z_{23}q_5 + z_{31}q_3 - z_{31}q_5\right] \\ \frac{1}{\det J}\left[r_{32}q_2 - r_{32}q_6 + r_{13}q_4 - r_{13}q_6\right] \\ \frac{1}{\det J}\left[r_{32}q_1 - r_{32}q_5 + r_{13}q_3 - r_{13}q_5 + z_{23}q_2 - z_{23}q_6 + z_{31}q_4 - z_{31}q_6\right] \\ \frac{N_1}{r}q_1 + \frac{N_2}{r}q_3 + \frac{N_3}{r}\cdot q_5 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{1}{\det J}\left[z_{23}q_1 + z_{31}q_3 + q_5\left[-z_{23} - z_{31}\right]\right. \\ \frac{1}{\det J}\left\{r_{32}q_2 + r_{13}q_4 + q_6\left[-r_{32} - r_{13}\right]\right. \\ \frac{1}{\det J}\left[r_{32}q_1 + z_{23}q_2 + r_{13}q_3 + z_{31}q_4 + \left[-r_{32} - r_{13}\right]q_5 + \left[-z_{23} - z_{31}\right]q_6\right. \\ \frac{N_1}{r}q_1 + \frac{N_2}{r}q_3 + \frac{N_3}{r}q_5 \end{bmatrix}$$

$$\epsilon = \begin{bmatrix} \frac{z_{23}}{\det J} & 0 & \frac{z_{31}}{\det J} & 0 & \frac{z_{12}}{\det J} & 0 \\ 0 & \frac{r_{32}}{\det J} & 0 & \frac{r_{13}}{\det J} & 0 & \frac{r_{21}}{\det J} \\ \frac{r_{32}}{\det J} & \frac{z_{23}}{\det J} & \frac{r_{13}}{\det J} & \frac{z_{31}}{\det J} & \frac{r_{21}}{\det J} & \frac{z_{12}}{\det J} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix}$$

$$\because \left[-z_{23} - z_{31}\right] = \left[z_{32} + z_{13}\right]$$
$$= \cancel{z_3} - z_2 + z_1 - \cancel{z_3}$$
$$= z_{12}.$$

$$\left[-r_{32} - r_{13}\right] = \left[r_{23} + r_{31}\right]$$
$$= r_2 - \cancel{r_3} + \cancel{r_3} - r_1$$
$$= r_{21}$$

$$B = \begin{bmatrix} \frac{z_{23}}{\det J} & 0 & \frac{z_{31}}{\det J} & 0 & \frac{z_{12}}{\det J} & 0 \\ 0 & \frac{r_{32}}{\det J} & 0 & \frac{r_{13}}{\det J} & 0 & \frac{r_{21}}{\det J} \\ \frac{r_{32}}{\det J} & \frac{z_{23}}{\det J} & \frac{r_{13}}{\det J} & \frac{z_{31}}{\det J} & \frac{r_{21}}{\det J} & \frac{z_{12}}{\det J} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

Here $N_1 = N_2 = N_3 = 1/3$.

$r = \frac{r_1 + r_2 + r_3}{3}$ and $k_e = 2\pi r A_e B^T D B$

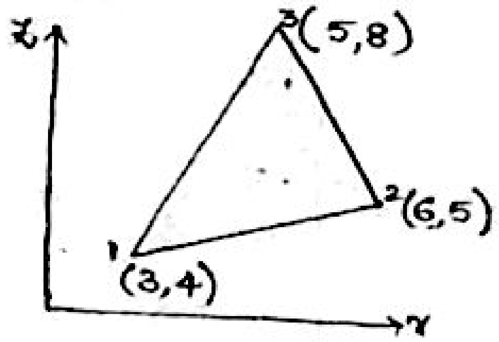
Problems :-

Compute the strain-displacement Matrix for the axisymmetric Triangular element shown in fig. Also determine the element strains. The nodal displacements are found out as.

$u_1 = 0.002$ $w_1 = 0.001$

$u_2 = 0.001$ $w_2 = -0.004$

$u_3 = -0.003$ $w_3 = 0.007$



Sol:- Here

$(r_1, z_1) = (3,4)$; $(r_2, z_2) = (6,5)$;

$(r_3, z_3) = (5,8)$

and Area of $\Delta le = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 5 \\ 1 & 5 & 8 \end{vmatrix}$

$= \frac{1}{2}\left[1(48-25) - 3(8-5) + 4(5-6)\right]$

$A = 5 cm^2$

and $r = \frac{r_1 + r_2 + r_3}{3} = \frac{3+6+5}{3} = \frac{14}{3} = 4.7$

$\therefore N_1 = N_2 = N_3 = 1/3$.

We have

$$B = \frac{1}{\det J}\begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & r_{32} & 0 & r_{13} & 0 & r_{21} \\ r_{32} & z_{23} & r_{13} & z_{31} & r_{21} & z_{12} \\ \frac{N_1(\det J)}{r} & 0 & \frac{N_2(\det J)}{r} & 0 & \frac{N_3(\det J)}{r} & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & -3 \end{bmatrix}$$

$$\det J = 6+4 = 10.$$

or Simply we can write Area $A = \frac{1}{2}[\det J]$

$$B = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \\ \frac{1/3(10)}{4.7} & 0 & \frac{(1/3)(10)}{4.7} & 0 & \frac{1/3(10)}{4.7} & 0 \end{bmatrix}$$

$$B = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \\ 0.709 & 0 & 0.709 & 0 & 0.709 & 0 \end{bmatrix}$$

Now, the element strains

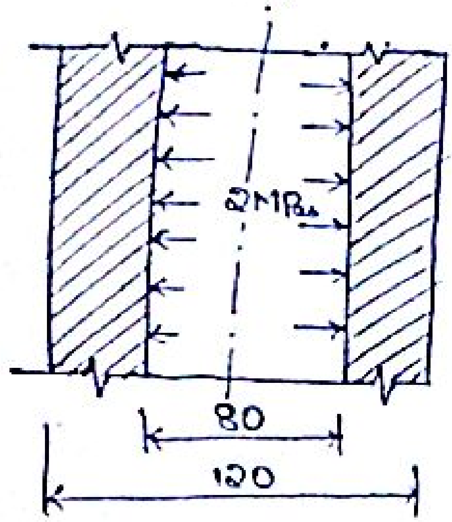
$$\begin{Bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{Bmatrix} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 4 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 3 & -1 \\ 0.709 & 0 & 0.709 & 0 & 0.707 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{bmatrix}$$

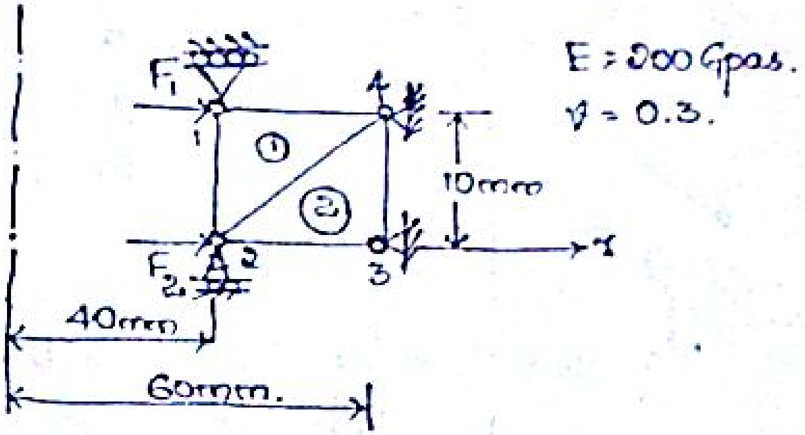
$$\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -3 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 3 \\ -1 & -3 & -2 & 4 & 2 & -1 \\ 0.709 & 0 & 0.709 & 0 & 0.709 & 0 \end{bmatrix}\begin{bmatrix} 0.002 \\ 0.001 \\ 0.001 \\ -0.004 \\ -0.002 \\ 0.007 \end{bmatrix}$$

By solving the above Matrix, we get

$$\begin{bmatrix} \epsilon_r \\ \epsilon_z \\ \gamma_{rz} \\ \epsilon_\theta \end{bmatrix} = \begin{bmatrix} 1\times10^{-4} \\ -28\times10^{-4} \\ 39\times10^{-4} \\ -8\times10^{-6} \end{bmatrix}$$

2. In fig. a cylinder of inside diameter 80mm and outside dia 120 snugly fits in a hole over its full length. The cylinder is then subjected to an internal pressure of 2 MPa. Using 2 elements on the 10mm length shown, find the disp. at the inner Radius.





Sol:

Elements	Nodes		
①	1	2	4
②	2	3	4

Nodes	Co-ordinates	
1	40	10
2	40	0
3	60	0
4	60	10.

$E = 200GPa = 200\times10^{3}$ MPa

$\vartheta = 0.3$

Element ①

$r_1 = 40 \quad , r_2 = 40 , \quad r_3 = 60 .$

$$r_{\textcircled{1}} = \frac{r_1 + r_2 + r_3}{3} = \frac{140}{3} = 46.66 .$$

Element ②

$r_1 = 40 , \quad r_2 = 60 \quad r_3 = 60$

$$r_{\textcircled{2}} = \frac{r_1 + r_2 + r_3}{3} = \frac{160}{3} = 53.34 .$$

Now consider

$$D = \frac{E}{(1+\upsilon)(1-2\upsilon)} \begin{bmatrix} (1-\upsilon) & \upsilon & 0 & \upsilon \\ \upsilon & 1-\upsilon & 0 & \upsilon \\ 0 & 0 & \frac{1}{2}-\upsilon & 0 \\ \upsilon & \upsilon & 0 & 1-\upsilon \end{bmatrix}$$

$$= \frac{200 \times 10^3}{(1+0.3)(1-2(0.3))} \begin{bmatrix} (1-0.3) & 0.3 & 0 & 0.3 \\ 0.3 & 1-0.3 & 0 & 0.3 \\ 0 & 0 & \frac{1}{2}-0.3 & 0 \\ 0.3 & 0.3 & 0 & 1-0.3 \end{bmatrix}$$

$$= 10^5 \begin{bmatrix} 2.68 & 1.15 & 0 & 1.15 \\ 1.15 & 2.68 & 0 & 1.15 \\ 0 & 0 & 0.76 & 0 \\ 1.15 & 1.15 & 0 & 2.68 \end{bmatrix}$$

We, Now consider the element ①

$$J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$$

$$J = \begin{bmatrix} -20 & 0 \\ -20 & -10 \end{bmatrix} \Rightarrow |J| = ad - bc = 200 mm^2$$

Area of element ①

$$A_1 = \frac{1}{2} \times b \times h = \frac{1}{2} \times 10 \times 20 = 100 mm^2$$

$$B = \begin{bmatrix} \frac{z_{23}}{|J|} & 0 & \frac{z_{31}}{|J|} & 0 & \frac{z_{12}}{|J|} & 0 \\ 0 & \frac{r_{32}}{|J|} & 0 & \frac{r_{13}}{|J|} & 0 & \frac{r_{21}}{|J|} \\ \frac{r_{32}}{|J|} & \frac{z_{23}}{|J|} & \frac{r_{13}}{|J|} & \frac{z_{31}}{|J|} & \frac{r_{21}}{|J|} & \frac{z_{12}}{|J|} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -10/200 & 0 & 0 & 0 & 10/200 & 0 \\ 0 & 20/200 & 0 & -20/200 & 0 & 0 \\ 20/200 & -10/200 & -20/200 & 0 & 0 & 10/200 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

Since $r_1 = 46.66$ & $N_1 = N_2 = N_3 = 1/3$.

$$B = \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

Similarly for element ②

$$J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix} = \begin{bmatrix} -20 & -10 \\ 0 & -10 \end{bmatrix} = 200\text{mm}^2$$

$$A_2 = 1/2 \times 10 \times 20 = 100\text{mm}^2$$

$$B = \begin{bmatrix} -10/200 & 0 & 10/200 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20/200 & 0 & 20/200 \\ 0 & -10/200 & -20/200 & 10/200 & 20/200 & 0 \\ 0.0062 & 0 & 0.0062 & 0 & 0.0062 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.0062 & 0 & 0.0062 & 0 & 0.0062 & 0 \end{bmatrix}$$

Now consider Element ①.

$$DB' = 10^5 \begin{bmatrix} 2.68 & 1.15 & 0 & 1.15 \\ 1.15 & 2.68 & 0 & 1.15 \\ 0 & 0 & 0.76 & 0 \\ 1.15 & 1.15 & 0 & 2.68 \end{bmatrix} \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$= 10^5 \begin{bmatrix} -0.125 & 0.115 & 0.0082 & -0.115 & 0.143 & 0 \\ -0.049 & 0.268 & 0.0082 & -0.268 & +0.0657 & 0.01 \\ 0.077 & -0.0385 & -0.077 & 0 & 0 & -0.0385 \\ -0.0384 & 0.115 & 0.0191 & -0.115 & 0.076 & 0 \end{bmatrix}$$

Element Stiffness Matrix $K_1 = 2\pi r A B^T DB$.

$$K_1 = 2\pi(46.66) \times 100 \times \begin{bmatrix} -0.05 & 0 & 0.1 & 0.0071 \\ 0 & 0.1 & -0.05 & 0 \\ 0 & 0 & 0.1 & 0.0071 \\ 0 & -0.1 & 0 & 0 \\ 0.05 & 0 & 0 & 0.0071 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \times$$

$$10^5 \begin{bmatrix} -0.125 & 0.115 & 0.0082 & -0.115 & 0.143 & 0 \\ -0.049 & 0.268 & 0.0082 & -0.268 & +0.0657 & 0.01 \\ 0.077 & -0.0385 & -0.077 & 0 & 0 & -0.0385 \\ -0.0384 & 0.115 & 0.0191 & -0.115 & 0.076 & 0 \end{bmatrix}$$

$$K_1 = 10^7 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 7 & 8 \end{array} \\ \left[\begin{array}{cccccc} 4.03 & -2.58 & -2.34 & 1.45 & -1.932 & 1.13 \\ -2.58 & 8.45 & 1.37 & -7.89 & 1.93 & -0.565 \\ -2.34 & 1.37 & 2.30 & -0.24 & 0.16 & -1.13 \\ 1.45 & -7.89 & -0.24 & 7.89 & -1.93 & 0 \\ -1.932 & 1.93 & 0.16 & -1.93 & 2.25 & 0 \\ 1.13 & -0.565 & -1.13 & 0 & 0 & 0.565 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array}$$

Element ②

$$DB^2 = 10^5 \begin{bmatrix} 2.68 & 1.15 & 0 & 1.15 \\ 1.15 & 2.68 & 0 & 1.15 \\ 0 & 0 & 0.76 & 0 \\ 1.15 & 1.15 & 0 & 2.68 \end{bmatrix} \begin{bmatrix} -0.05 & 0 & 0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0 & 0.1 \\ 0 & -0.05 & -0.1 & 0.05 & 0.1 & 0 \\ 0.0062 & 0 & 0.0062 & 0 & 0.0062 & 0 \end{bmatrix}$$

$$DB^2 = 10^4 \begin{bmatrix} -1.27 & 0 & 1.42 & -1.15 & 0.072 & 1.15 \\ -0.503 & 0 & 0.647 & -2.69 & 0.072 & 2.67 \\ 0 & -0.385 & -0.77 & 0.385 & 0.77 & 0 \\ -0.407 & 0 & 0.743 & -1.15 & 0.168 & 1.15 \end{bmatrix}$$

$$K_2 = 2\pi r A B^T DB$$

$$= 2\pi(53.34)\times 100 \times \begin{bmatrix} -0.05 & 0 & 0 & 0.0062 \\ 0 & 0 & -0.05 & 0 \\ 0.05 & 0 & -0.1 & 0.0062 \\ 0 & -0.1 & 0.05 & 0 \\ 0 & 0 & 0.1 & 0.0062 \\ 0 & 0.1 & 0 & 0 \end{bmatrix} \times 10^4$$

$$\begin{bmatrix} -1.27 & 0 & 1.42 & -1.15 & 0.072 & 1.15 \\ -0.503 & 0 & 0.647 & -2.69 & 0.072 & 2.69 \\ 0 & -0.385 & -0.77 & 0.385 & 0.77 & 0 \\ -0.407 & 0 & 0.743 & -1.15 & 0.168 & 1.15 \end{bmatrix}$$

$$K_2 = 10^7 \begin{array}{c} \begin{array}{cccccc} 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 2.05 & 0 & -2.22 & 1.69 & -0.085 & -1.69 \\ 0 & 0.645 & 1.29 & -0.645 & -1.29 & 0 \\ -2.22 & 1.29 & 5.11 & -3.46 & -2.42 & 2.17 \\ 1.69 & -0.645 & -3.46 & 9.66 & 1.05 & -9.01 \\ -0.085 & -1.29 & -2.42 & 1.05 & 2.62 & 0.241 \\ -1.69 & 0 & 2.17 & -9.01 & 0.241 & 9.01. \end{bmatrix} \end{array} \begin{array}{c} \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Finally we get 8×8 Matrix, Since, the Nodes 3, 4 are completely fixed

$q_5 = q_6 = q_7 = q_8 = 0$, and also, at Node one and 2 there is a Roller support in x-dirt.

$q_2 = 0$ & $q_4 = 0$.

$$10^7 \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}.$$

We have

$$(\text{Pressure})\, P = F/A \left(\text{Force}/\text{Area} \right)$$

$$F = P \times A = 2\pi r l \times P$$

This force F is distributed to two Nodes

$$F_1 = F_2 = \frac{2\pi r l \times P}{2} = \frac{2\pi(40)(10)2}{2} = 2514 N.$$

$$10^7 \begin{bmatrix} 4.03 & -2.34 \\ -2.34 & 4.35 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_3 \end{Bmatrix} = \begin{bmatrix} 2514 \\ 2514 \end{bmatrix}$$

By solving, we get

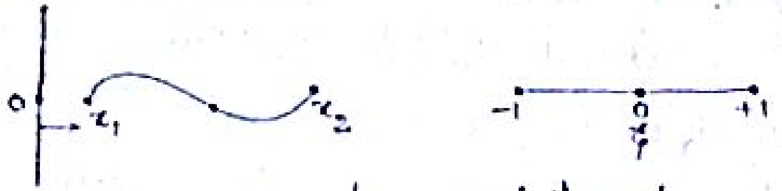
$q_1 = 0.014 \times 10^{-2}$ mm

$q_3 = 0.0133 \times 10^{-2}$ mm.

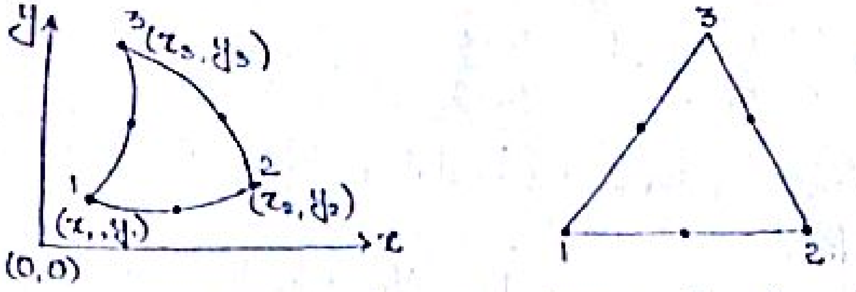
—— × —— × ——.

UNIT - V

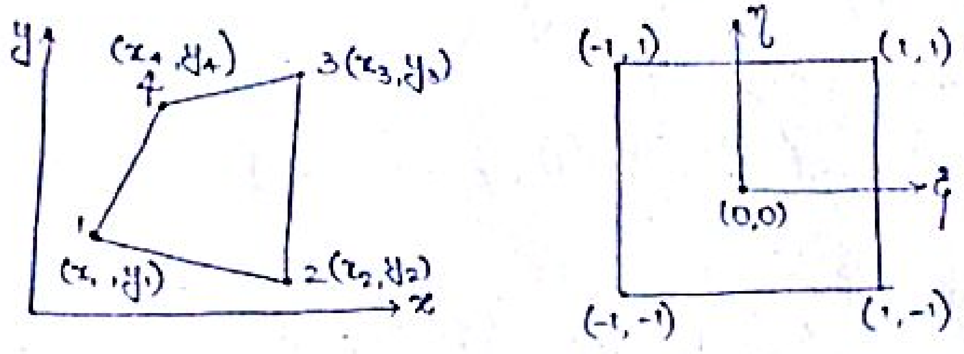
x- Isoparametric element formulation :



(a) One dimensional element.



(b) Two dimensional Triangular element

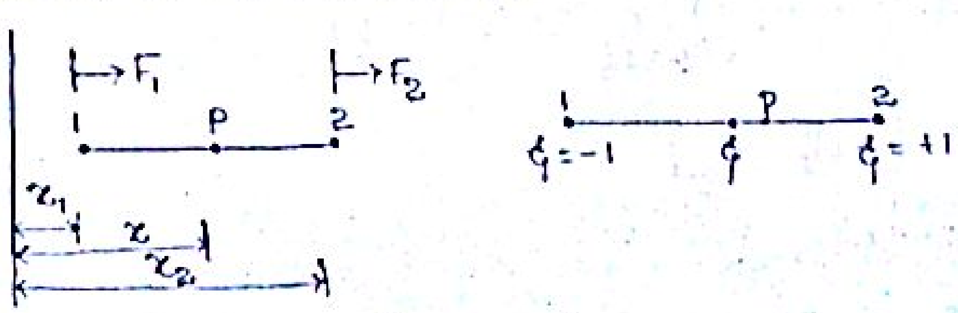


(c) 2D quadrilateral element.

Here x, y = Global co-ordinate system.

ξ, η = Natural co-ordinate system.

x- Consider One dimensional element :—



x_1, x_2 = Global co-ordinates of Nodes 1 & 2.

u_1, u_2 = Disp. at Nodes 1 & 2 due to Axial loads F_1 & F_2

Then, the displacement at P is written as.

$$u = N_1u_1 + N_2u_2$$

for finding the value of N_1 & N_2, we use

$$u = a_1 + a_2x.$$

by solving the above eq. Using the boundary conditions we get.

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \& \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

And also we had the relation b/w Natural coordinate & Global Coordinates

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1. \xrightarrow{} ①$$

i.e, At Node 1 $x = x_1 \Rightarrow \xi = -1$
Node 2 $x = x_2 \Rightarrow \xi = +1.$

Now, we can write :

$$x = \frac{(\xi+1)(x_2 - x_1)}{2} + x_1 \longrightarrow ②$$

Now

$$N_1 = \frac{x_2 - x}{x_2 - x_1} = \frac{1}{x_2 - x_1}\left[x_2 - \left[\frac{(\xi+1)(x_2 - x_1)}{2} + x_1\right]\right]$$

$$= \frac{1}{2(x_2 - x_1)}\left[2x_2 - \xi x_2 + \xi x_1 - x_2 + x_1 - 2x_1\right]$$

$$= \frac{1}{2(x_2 - x_1)}\left[(x_2 - x_1) - \xi(x_2 - x_1)\right]$$

$$= \frac{1}{2\cancel{(x_2 - x_1)}}\cancel{(x_2 - x_1)}\left[1 - \xi\right]$$

$$N_1 = \frac{1 - \xi}{2}$$

Similarly

$N_2 = \frac{x - x_1}{x_2 - x_1}$ by solving this

$$N_2 = \frac{1 + \xi}{2}$$

$$u = \left(\frac{1-\xi}{2}\right)u_1 + \left(\frac{1+\xi}{2}\right)u_2 \longrightarrow (A)$$

Now, from eq ②

$$x = \frac{(\xi+1)(x_2-x_1)}{2} + x_1$$

$$= \frac{(\xi+1)(x_2-x_1)+2x_1}{2}$$

$$= \frac{\xi x_2 - \xi x_1 + x_2 - x_1 + 2x_1}{2}$$

$$= \frac{\xi x_2 - \xi x_1 + x_2 + x_1}{2}$$

$$x = \frac{(1-\xi)}{2}x_1 + \frac{(1+\xi)}{2}x_2 \longrightarrow (B)$$

$$\therefore \; x = N_1x_1 + N_2x_2.$$

Comparing eq's A & B, that the line element displacement and geometry are described by the same shape functions in the same order. this is known as Isoparametric element.

* 2D element :-

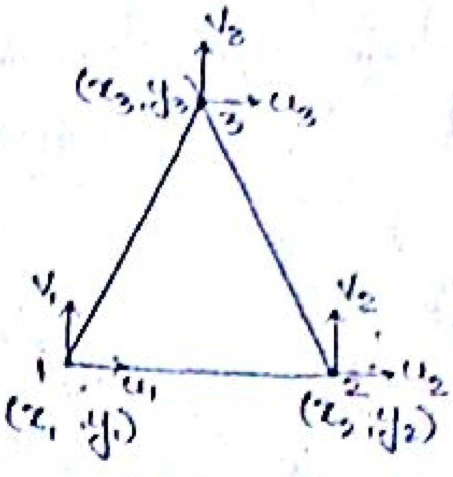
$u = N_1u_1 + N_2u_2 + N_3u_3$

$v = N_1v_1 + N_2v_2 + N_3v_3$.

or

$u = N_1q_1 + N_2q_3 + N_3q_5$

$v = N_1q_2 + N_2q_4 + N_3q_6$.



$N_1 = \xi \quad N_2 = \eta \quad N_3 = 1-\xi-\eta$

$x = N_1x_1 + N_2x_2 + N_3x_3$

$y = N_1y_1 + N_2y_2 + N_3y_3$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$U = [N_i]_d \, q$$

Here $[N_i]_d$ = Shape function for Nodal displacements.

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{bmatrix}$$

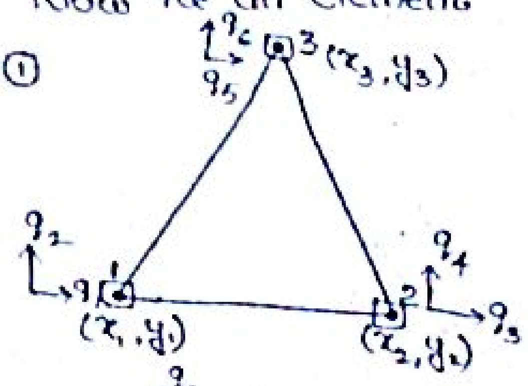
$$= [N_j]_g \, C$$

$[N_j]_g$ = Shape function for Geometry.

and q and c are the Nodal disp. and Nodal co-ordinate vectors.

Now for an element

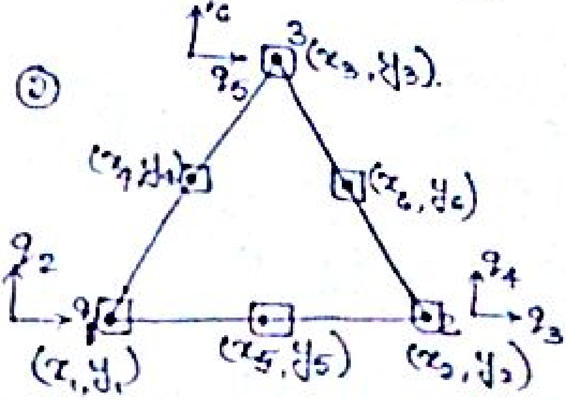
①



$[N_i]_d = [N_j]_g$

It is called Isoparametric element.

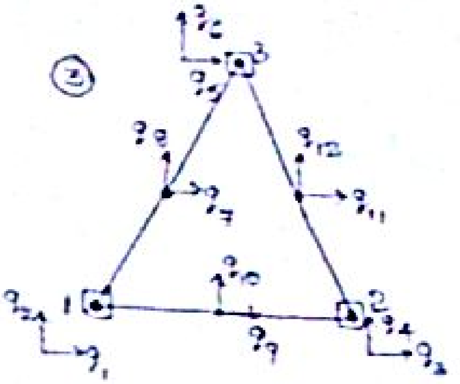
②



$[N_i]_d < [N_j]_g$

It is called Super parametric.

②



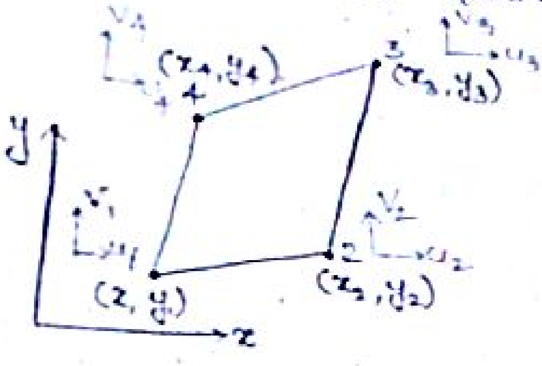
$[N_i]_d > [N_j]_g$

It is called Sub parametric.

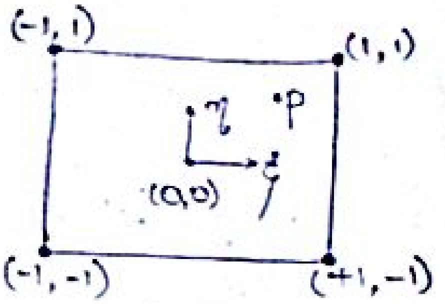
• → Nodes for defining displacement.

□ → Nodes for defining Geometry.

* Consider Four-Node Quadrilateral element :-



(a) General quadrilateral element



(b) Master (Isoparametric) element.

In Isoparametric element, all the disp. & geometry are specified by Natural coordinates ξ & η. Now, the disp. produced at any point P inside the element can be specified by the polynomial expression as.

$$u(\xi,\eta) = a_1 + a_2\xi + a_3\eta + a_4\xi\eta$$

$$\& \; v(\xi,\eta) = a_5 + a_6\xi + a_7\eta + a_8\xi\eta.$$

Let, At Node 1, $\xi = -1$; $\eta = -1$ and $u = u_1$

At Node 2, $\xi = 1$; $\eta = -1$ and $u = u_2$.

At Node 3, $\xi = 1$; $\eta = 1$ and $u = u_3$

At Node 4, $\xi = -1$; $\eta = 1$ and $u = u_4$.

By Substituting the boundary conditions in the above eq and solving the eq's, we get

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta).$$

Similarly, when solving $v(\xi, \eta)$ also we get the same formulas for Shape functions.

For the Four Noded elements, we can write

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$
$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4.$$

$$\& \; x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4.$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

* Strain - Displacement Matrix for the Four-Noded quadrilateral elements :-

$$\epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

Consider the chain Rule

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \xi}$$

$$\& \; \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix}\frac{\partial u}{\partial \xi}\\ \frac{\partial u}{\partial \eta}\end{Bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{Bmatrix}$$

$$= [J]\begin{Bmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{Bmatrix}$$

$[J]$ = Jacobian Matrix

$$[J] = \begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix} = \begin{bmatrix}J_{11} & J_{12}\\ J_{21} & J_{22}\end{bmatrix},$$

where, $J_{11} = \frac{\partial x}{\partial \xi}$, $J_{12} = \frac{\partial y}{\partial \xi}$, $J_{21} = \frac{\partial x}{\partial \eta}$, $J_{22} = \frac{\partial y}{\partial \eta}$.

Now, we know that

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \frac{\partial}{\partial \xi}\left[N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4\right]$$

$$= \frac{\partial N_1}{\partial \xi} x_1 + \frac{\partial N_2}{\partial \xi} x_2 + \frac{\partial N_3}{\partial \xi} x_3 + \frac{\partial N_4}{\partial \xi} \cdot x_4.$$

$$= \frac{1}{4}(-1)(1-\eta)x_1 + \frac{1}{4}(1-\eta)x_2 + \frac{1}{4}(1)(1+\eta)x_3 + \frac{1}{4}(-1)(1+\eta)x_4$$

$$= \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 + (-)(1+\eta)x_4\right]$$

Similarly

$$J_{2} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$J_{21} = \frac{1}{4}\left[-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right]$$

& $J_{22} = \frac{1}{4}\left[-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right]$

Now

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

We know that

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}$$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix}$$

Here we can write

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

Now, to find the values of $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}, \frac{\partial v}{\partial \eta}, \frac{\partial v}{\partial \xi}$.

$$\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left[u_1 N_1 + N_2 u_2 + N_3 u_3 + N_4 u_4\right]$$

$$= \frac{1}{4}\left[-(1-\eta)u_1 + (1-\eta)u_2 + (1+\eta)u_3 - (1+\eta)u_4\right]$$

Similarly

$$\frac{\partial u}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)u_1 - (1+\xi)u_2 + (1+\xi)u_3 + (1-\xi)u_4\right]$$

$$\frac{\partial v}{\partial \xi} = \frac{1}{4}\left[-(1-\eta)v_1 + (1-\eta)v_2 + (1+\eta)v_3 - (1+\eta)v_4\right]$$

$$\frac{\partial v}{\partial \eta} = \frac{1}{4}\left[-(1-\xi)v_1 - (1+\xi)v_2 + (1+\xi)v_3 + (1-\xi)v_4\right]$$

$$\epsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix}$$

Now here

$$\begin{Bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial v}{\partial \xi} \\ \frac{\partial v}{\partial \eta} \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\epsilon = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \times \frac{1}{4} \begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \times \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\epsilon = [B][Q] = [G][H][Q]$$

B = Strain disp. Matrix = [G][H]

where $G = \frac{1}{|J|} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$ &.

$$H = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix}$$

*. Stress-Strain Relationship Matrix :-

$$\sigma = D\epsilon = DBq$$

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad \& \quad \epsilon = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

D = Stress-strain Relationship Matrix.

Plane-Stress

$$D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane-Strain

$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix}$$

*- Element Stiffness Matrix

$$[K] = \int B^T D B . dv$$

For Isoparametric elements

$$[K] = t \iint B^T D B . dA = t \int . \int B^T D B . dx . dy$$

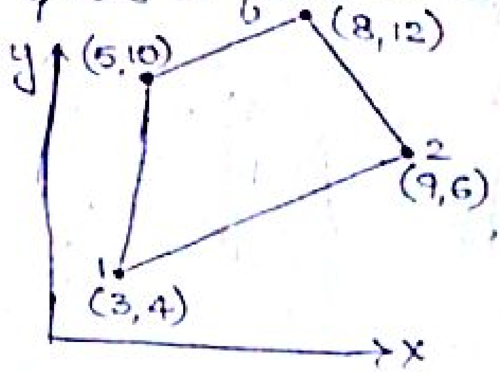
(Global coordinate system)

$$[K] = t \int_{-1}^{1}\int_{-1}^{1} B^T D B \, |J| \, d\xi \, d\eta$$

(For local coordinate system).

Problems:-

1. Determine the Cartesian co-ordinates of the point P which has local co-ordinates $\xi = 0.8$ & $\eta = 0.6$ as shown in fig



Sol: Given

$(x_1, y_1) = (3,4)$; $(x_2, y_2) = (9,6)$; $(x_3, y_3) = (8,12)$; $(x_4, y_4) = (5,10)$

& $\xi = 0.8$, $\eta = 0.6$.

We know that

$N_1 = \frac{1}{4}(1-\xi)(1-\eta) = \frac{1}{4}(1-0.8)(1-0.6) = 0.02.$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta) = \frac{1}{4}(1+0.8)(1-0.6) = 0.18.$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta) = \frac{1}{4}(1+0.8)(1+0.6) = 0.72.$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta) = \frac{1}{4}(1-0.8)(1+0.6) = 0.08.$

$\therefore x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

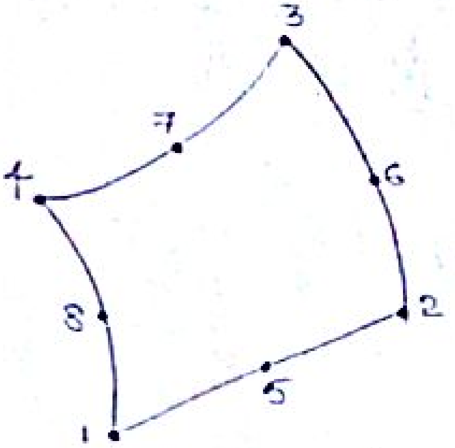
$x = 7.84$ Units.

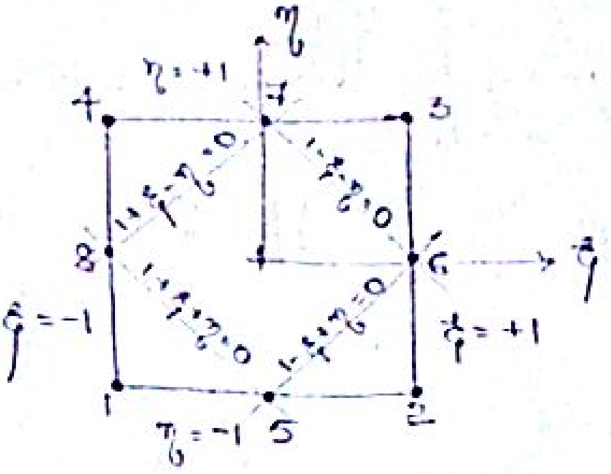
& $y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$= 10.6$ units .

$\therefore$ The cartesian co-ordinates of the Points $P(x, y) = (7.84, 10.6)$

* Eight Noded Quadrilateral Element :-





$$N_1 = -\frac{(1-\xi)(1-\eta)(1+\xi+\eta)}{4}$$

$$N_2 = -\frac{(1+\xi)(1-\eta)(1-\xi+\eta)}{4}$$

$$N_3 = -\frac{(1+\xi)(1+\eta)(1-\xi-\eta)}{4}$$

$$N_4 = -\frac{(1-\xi)(1+\eta)(1+\xi-\eta)}{4}$$

$$N_5 = \frac{(1-\xi^2)(1-\eta)}{2}$$

$$N_6 = \frac{(1+\xi)(1-\eta^2)}{2}$$

$$N_7 = \frac{(1-\xi^2)(1+\eta)}{2}$$

$$N_8 = \frac{(1-\xi)(1-\eta^2)}{2}$$

* Six Noded Element

$N_1 = \xi(2\xi-1)$ $\qquad$ $N_4 = 4\xi\eta$

$N_2 = \eta(2\eta-1)$ $\qquad$ $N_5 = 4\zeta\eta$

$N_3 = \zeta(2\zeta-1)$ $\qquad$ $N_6 = 4\xi\zeta$