CST

INTRODUCTION:-

The 2-D finite formulation in this chapter follows the steps used in the 1-D problem. The displacement, traction components and distributed body force values are functions of the position indicated by (x, y). The displacement vector 'u' is given as:

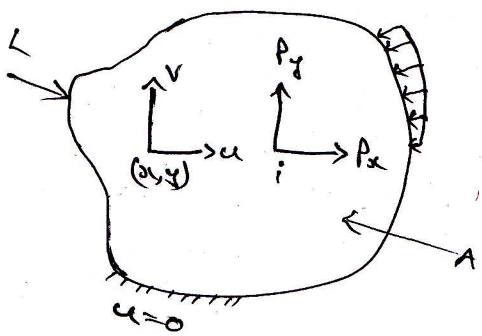
$$u = [u, v]^T$$

where u, v are the x, y components of 'u' respectively.

The stress and strains are given by

$$\sigma = [\sigma_x, \sigma_y, \tau_{xy}]^T$$

$$\epsilon = [\epsilon_x, \epsilon_y, \gamma_{xy}]^T$$



$$f = [f_x, f_y]^T, \quad T = [T_x, T_y]^T$$

The strain displacement relations are given by

$$\epsilon = \left[\frac{du}{dx}, \frac{dv}{dy}, \left(\frac{du}{dy} + \frac{dv}{dx}\right)\right]^T$$

stress and strains are related by

$$\sigma = D\epsilon.$$

ISOPARAMETRIC REPRESENTATION:-

strain-displacement matrix (B) for a 3 noded triangular element. u,v are functions of x,y & x,y are functions of ϵ, η.

i.e., $u = N_1 q_1 + N_2 q_3 + N_3 q_5$

$$V = N_1 q_2 + N_2 q_4 + N_3 q_6$$

shape functions we can write as

$$N_1 = \epsilon \;,\; N_2 = \eta \;,\; N = 1-\epsilon-\eta$$

$$\therefore \quad u = \epsilon q_1 + \eta q_3 + (1-\epsilon-\eta) q_5$$

$$u = \epsilon(q_1 - q_5) + (q_3 - q_5)\eta + q_5$$

$$V = \epsilon q_2 + \eta q_4 + (1-\epsilon-\eta) q_6$$

$$V = (q_2 - q_6)\epsilon + (q_4 - q_6)\eta + q_6$$

The shape function matrix can be written as

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

We have

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

$$x = (x_1 - x_3)\epsilon + (x_2 - x_3)\eta + x_3$$

$$y = (y_1 - y_3)\epsilon + (y_2 - y_3)\eta + y_3$$

Using the notation $x_{ij} = x_i - x_j$, we can write as

$$x = x_{13}\epsilon + x_{23}\eta + x_3$$

$$y = y_{13}\epsilon + y_{23}\eta + y_3$$

we can see that u & v and x & y are the functions of ϵ and η then

$$u = u\left[x(\epsilon,\eta),\ y(\epsilon,\eta)\right]$$

$$v = v\left[x(\epsilon,\eta),\ y(\epsilon,\eta)\right]$$

$$\frac{\partial u}{\partial \epsilon} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \epsilon} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \epsilon}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as.

$$\begin{Bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \epsilon} & \frac{\partial y}{\partial \epsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix}$$

where the (2×2) square matrix is denoted as the Jacobian.

$$J = \begin{bmatrix} \frac{\partial x}{\partial \epsilon} & \frac{\partial y}{\partial \epsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

on taking derivative of x and y.

$$J = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}$$

Also

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

where J^{-1} is the inverse of the Jacobian J, given by.

$$J^{-1} = \frac{1}{\det J}\begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix}$$

$$\det |J| = x_{13} y_{23} - x_{23} y_{13}$$

we have $\quad A = \frac{1}{2} |J|$

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} y_{23} \frac{\partial u}{\partial \epsilon} - y_{13} \frac{\partial u}{\partial \eta} \\ -x_{23} \frac{\partial u}{\partial \epsilon} + x_{13} \frac{\partial u}{\partial \eta} \end{bmatrix}$$

Replacing u by the displacement v,

$$\begin{Bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} y_{23} \frac{\partial v}{\partial \epsilon} - y_{13} \frac{\partial v}{\partial \eta} \\ -x_{23} \frac{\partial v}{\partial \epsilon} + x_{13} \frac{\partial v}{\partial \eta} \end{bmatrix}$$

Using the strain-displacement relations, we get

$$\epsilon = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) \\ + [-y_{13}(q_4 - q_6)] \end{bmatrix}$$

From the definition of x_{ij} & y_{ij} we can write $y_{31} = y_{13}$ & $y_{12} = y_{13} - y_{23}$ & so on.

$$\epsilon = \frac{1}{|J|}\begin{bmatrix} y_{23} q_1 + y_{31} q_3 + y_{12} q_5 \\ x_{32} q_2 + x_{13} q_4 + x_{21} q_6 \\ x_{32} q_1 + y_{23} q_2 + x_{13} q_3 + y_{31} q_4 + x_{21} q_5 + y_{12} q_6 \end{bmatrix}$$

$$\epsilon = Bq$$

where B is a (3×6) strain-displacement matrix.

$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

→ $$D = \frac{E}{1-\gamma^2}\begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$$

→ Element stiffness martix $K = A \times t \times B^T \times D \times B$

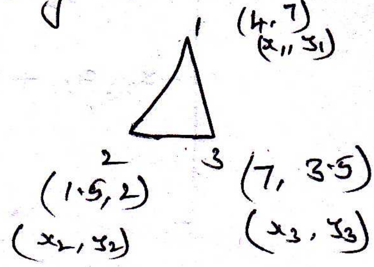
→ stress $(\sigma) = DBq$

D = constant stress matrix

B = strain displacement matrix

10-1)

Evaluate the shape functions N_1, N_2, N_3 at interior point 'p' for the triangular element.



Sol:- P (3.85, 4.8)

W.K.T

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3$$

~~$x = x_1 - x_3$~~

$$N_1 = \epsilon \qquad N_2 = \eta$$

$$N_3 = 1 - \epsilon - \eta$$

$$x = (x_1 - x_3)\epsilon + (x_2 - x_3)\eta + x_3$$

$$y = (y_1 - y_3)\epsilon + (y_2 - y_3)\eta + y_3$$

$$3.85 = -3\epsilon - 5.5\eta + 7$$

$$4.8 = 3.5\epsilon - 1.5\eta + 3.5$$

$$-3\epsilon - 5.5\eta = -3.15 \quad - (1)$$

$$3.5\epsilon - 1.5\eta = 1.3 \quad - (2)$$

$\therefore$ from eqn (1) & (2)

$$\epsilon = 0.5 \qquad \eta = 0.3$$

$$N_1 = 0.5$$

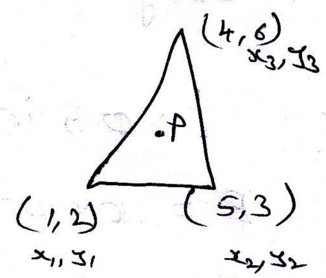
$$N_2 = 0.3$$

$$N_3 = 1 - \epsilon - \eta$$

$$= 1 - 0.5 - 0.3$$

$$= 0.2$$

element are shown in a fig at the interior point at x- coordinate is 3.3 & $N_1 = 0.3$. Determine N_2 & N_3 & coordinate point at P.



Sol: W.K.T $N_1 = \epsilon \quad N_2 = \eta$

$= 0.3$

$x = 3.3$

$x = (x_1 - x_3)\epsilon + (x_2 - x_3)\eta + x_3$

$3.3 = (1-4)\,0.3 + (5-4)\eta + 4$

$3.3 = -0.9 + \eta + 4$

$\eta = 3.3 + 0.9 - 4$

$\boxed{\eta = 0.2}$

$y = (y_1 - y_3)\epsilon + (y_2 - y_3)\eta + y_3$

$= (2-6)\,0.3 + (3-6)\,0.2 + 6$

$= -1.2 - 0.6 + 6$

$= 4.2$

$N_3 = 1 - \epsilon - \eta$

$= 1 - 0.3 - 0.2$

$= 0.5$

$y = N_1 y_1 + N_2 y_2 + N_3 y_3$

$= (0.3 \times 2) + (0.2 \times 3) + (0.5 \times 6)$

$= 0.6 + 0.6 + 3$

$= 4.2$

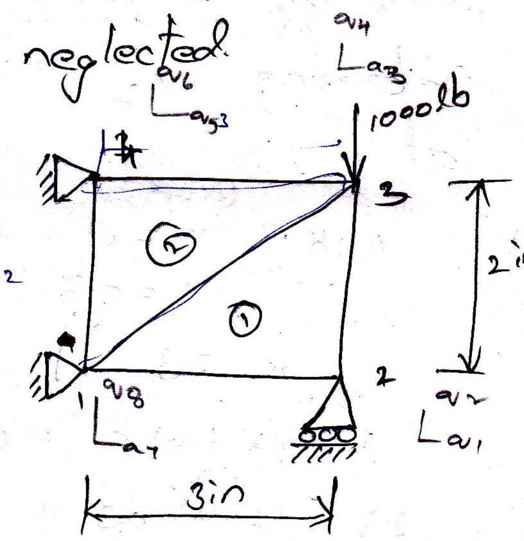
$\therefore N_2 = 0.2 = \eta$

$\therefore N_3 = 0.5$

$\therefore P = (x, y)$

$= (3.3, 4.2)$

* The 2-D loaded plate as shown in the figure. Determine the displacements of nodes 1 & 2 and element stresses using the plane stress conditions the body force may be neglected.



$t = 0.5$ in

$E = 30\times10^6$ psi

$\gamma = 0.25$

S.No:-	nodes	element
1	1-2-4	1
2	2-4-3	2

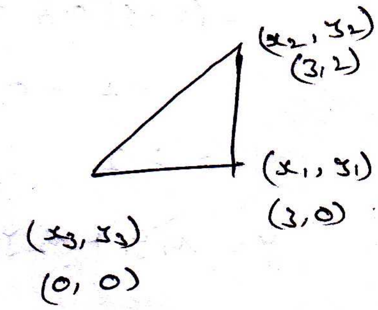
Coordinates

1 - (0,0)

sol:-

$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}_{3\times6}$$

$$B^T = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & x_{32} \\ 0 & x_{32} & y_{23} \\ y_{31} & 0 & x_{13} \\ 0 & x_{13} & y_{31} \\ y_{12} & 0 & x_{21} \\ 0 & x_{21} & y_{12} \end{bmatrix}_{6\times3}$$



$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6\times3}$$

$$D = \frac{30\times10^6}{1-0.25^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$\begin{bmatrix} \cdots & & \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix}_{3\times 3}$$

$$\text{Area} = \frac{1}{2}|J|$$

$$= \frac{1}{2}\left| x_{13}\, y_{23} - y_{13}\, x_{23} \right|$$

$$= \frac{1}{2}\,|3\times 2 - 0|$$

$$= \frac{1}{2}(6)$$

$$= 3$$

$$t = 0.5$$

$$K_1 = t \times A \times B^T \times D \times B$$

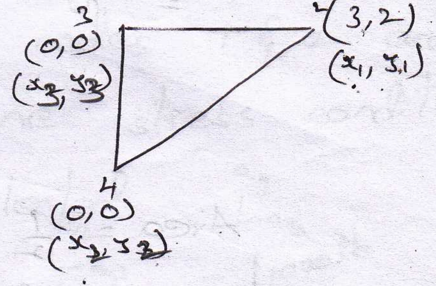
$$= 0.5 \times 3 \times \frac{1}{6}\begin{bmatrix} 2 & 0 & -2 \\ 0 & -2 & 2 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6\times 3} \times 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix}_{3\times 3}$$

$$\times \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

$$= 41666.66 \begin{bmatrix} 64 & 16 & -24.192 \\ -16 & -64 & -24.192 \\ 0 & 0 & 36.288 \\ 24 & 96 & 0 \\ -64 & -16 & 0 \\ 0 & 0 & -24.192 \end{bmatrix}_{6\times 3} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 3 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}_{3\times 6}$$

$$= 41666.66 \begin{bmatrix} 176.384 & -80.384 & -72.576 & 48 & -128 & -48.384 \\ 16.384 & 79.616 & -72.576 & -192 & 48 & 48.384 \\ -72.576 & 72.576 & 108.864 & 0 & 0 & -72.576 \\ 48 & -192 & 0 & 288 & -48 & 0 \\ -128 & 32 & 0 & -48 & 128 & 0 \\ 48.384 & -48.384 & 72.576 & 0 & 0 & 48.384 \end{bmatrix}_{6\times 6}$$

$$B^T = \frac{1}{|J|}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 3 \\ 0 & 3 & -2 \\ 2 & 0 & -3 \\ 0 & -3 & 2 \end{bmatrix}$$



$$D = 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix}_{3\times 3}$$

$x_{13}\,y_{23} - y_{13}\,x_{23}$

$3\times 0 - 2\times 0$

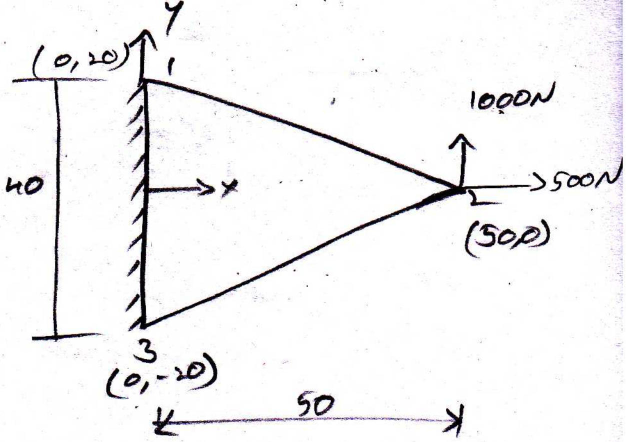
$K_2 =$

$E = 205\times10^3 \ N/mm^2$

$\gamma = 0.3$

$K = t\times A\times B^T D B$

$t = 10$



$$D = \frac{E}{1-\gamma^2}\begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & \frac{1-\gamma}{2} \end{bmatrix}$$

$$= \frac{205\times10^3}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

$$= 225.2\times10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.355 \end{bmatrix}$$

$\det |J|$

$= | x_{13} y_{23} - x_{23} y_{13} |$

$= | 0\times0 - (50\times40) |$

$= 2000$

$$B = \frac{1}{|J|}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{2000}\begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}$$

$K = t\times A\times B^T D B$

$$= 10 \times \frac{1}{2} \times 2000 \times \frac{1}{2000} \begin{bmatrix} 0 & -50 & 20 \\ -40 & 0 & 0 \\ 0 & 0 & -40 \\ 20 & 0 & 50 \\ 0 & 50 & 20 \end{bmatrix}_{6\times 3} 225.2\times 10^{3} \begin{bmatrix} 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}_{3\times 3}$$

$$\times \frac{1}{2000} \begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}_{3\times 6}$$

$$= 563 \begin{bmatrix} 20 & 6 & -17.75 \\ -15 & -50 & 7.0 \\ -40 & -12 & 0 \\ 0 & 0 & -14.0 \\ 20 & 6 & 17.75 \\ 15 & 50 & 7.4 \end{bmatrix}_{6\times 3} \begin{bmatrix} 20 & 0 & -40 & 0 & 20 & 0 \\ 0 & -50 & 0 & 0 & 0 & 50 \\ -50 & 20 & 0 & -40 & 50 & 20 \end{bmatrix}_{3\times 6}$$

$$= 563 \begin{bmatrix} \cancel{1287.5}\ 1275 & -650 & -800 & 700 & \cancel{-487.5}\ -475 & -50 \\ -650 & 2640 & 600 & 280 & 50 & -2360 \\ -800 & 600 & 1600 & 0 & -800 & -600 \\ 700 & -280 & 0 & 560 & -700 & -280 \\ \cancel{-487.5}\ -475 & 50 & -800 & -700 & 1275 & 650 \\ -50 & -2360 & -600 & -280 & 650 & 2640 \end{bmatrix}_{6\times 6}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 500 \\ 1000 \\ 0 \\ 0 \end{bmatrix}$$

563 $\begin{bmatrix} 1600 & 0 \\ 0 & 560 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$

$$\begin{bmatrix} 900800 & 0 \\ 0 & 315280 \end{bmatrix} \begin{bmatrix} q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \end{bmatrix}$$

$$900800\, q_3 = 500$$

$$315280\, q_4 = 1000$$

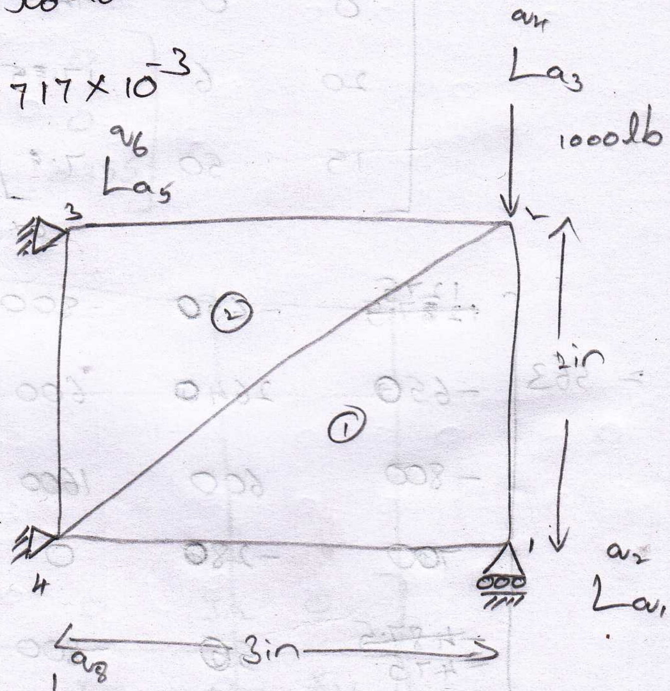
$$q_3 = 5.5506 \times 10^{-4}$$

$$q_4 = 3.1717 \times 10^{-3}$$

$t = 0.5 in$

$\gamma = 0.25$

$E = 30 \times 10^6$ psi



no. of nodes = 4

no. of elements = 2

S.NO	nodes	element
1	1-2-4	1
2	2-4-3	2

$q_1 = 10^{-5} \times 1.913$

$q_3 = 0.875 \times 0^{5}$

$q_4 = -7.436 \times 10^{-5}$

Coordinates

1 (0,0)

2 (0,2)

3 (3,2)

4 (3,0)

Boundary conditions:-

point load at node q_4

no. of fixed nodes

q_2, q_5, q_6, q_7, q_8.

unknown values = q_1, q_3, q_4

$$\begin{bmatrix} 3 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}_{3\times 6}$$

(3,0) (0,0)
$x_3 y_3$ x_1, y_1

$$B^T = \frac{1}{|-3\times 2 - 0|} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6\times 3}$$

$$D = \frac{30\times 10^6}{1-0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

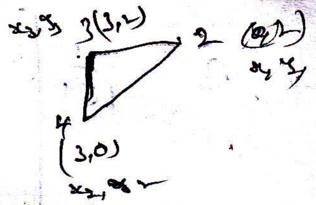
$$= 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix}$$

$$K_1 = t\times A\times B^T \cancel{\times B} \times D\times B$$

$$= 0.5\times \frac{3}{\cancel{6}}\times \frac{1}{6} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}_{6\times 3} \times 10^6 \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}$$

$$= 41666.66 \begin{bmatrix} 64 & 16 & 36.288 \\ 24 & 96 & 24.192 \\ 0 & 0 & -36.288 \\ -24 & -96 & 0 \\ -64 & -16 & 0 \\ 0 & 0 & -24.192 \end{bmatrix}_{6\times 3} \begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & 2 & -3 & 0 & 0 & -2 \end{bmatrix}_{3\times 6}$$

$$0.0416\times 10^6 \; \cancel{= 41666.66} \begin{bmatrix} 236.864 & 120.576 & -108.864 & -48 & -128 & -72.576 \\ 120.576 & 336.384 & -72.576 & -288 & -48 & -48.384 \\ -108.864 & -72.576 & +108.864 & 0 & 0 & 72.576 \\ -48 & -288 & 0 & 288 & 48 & 0 \\ -128 & -48 & 0 & 48 & 128 & 0 \\ -72.576 & -48.384 & 72.576 & 0 & 0 & 48.384 \end{bmatrix}$$



$$K_2 = t \times A \times B^T \times D \times B$$

$$= 0.5 \times \frac{9}{2} \times \frac{1}{6} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 2 \end{bmatrix} 10^6 \times \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12.096 \end{bmatrix} \times \frac{1}{6} \begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$= 10^6 \times 0.0625 \begin{bmatrix} -96 & -24 & 0 \\ 0 & 0 & -36.288 \\ 0 & 0 & -36.288 \\ -24 & -96 & 0 \\ 64 & 16 & 36.288 \\ 24 & 96 & 24.192 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} -6 & -1.5 & 0 \\ 0 & 0 & -2.268 \\ 0 & 0 & -2.268 \\ -1.5 & -6 & 0 \\ 4 & 1 & 2.268 \\ 1.5 & 6 & 1.512 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 & 3 & 2 \end{bmatrix}$$

$$K_2 = 10^6 \begin{array}{c} \begin{array}{cccccc} 3 & 4 & 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 18 & 0 & 0 & 4.5 & -12 & -4.5 \\ 0 & 6.804 & 6.804 & 0 & -6.804 & -4.536 \\ 0 & 6.804 & 6.804 & 0 & -6.804 & -4.536 \\ 4.5 & 0 & 0 & 18 & -3 & -18 \\ -12 & -6.804 & -6.804 & -3 & 14.804 & 7.536 \\ -4.5 & -4.536 & -4.536 & -18 & 7.536 & 21.024 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \\ 5 \\ 6 \end{array}$$

$$K_2 = 10^{6} \begin{array}{c} 2 \\ 3 \\ 4 \\ 7 \\ 8 \end{array} \begin{bmatrix} 5.015 & 13.99 & -3.019 & -11.98 & -1.996 & -2.012 \\ -4.528 & -3.019 & 4.528 & 0 & 0 & 3.019 \\ -1.996 & -11.98 & 0 & 11.98 & 1.996 & 0 \\ -5.324 & -1.996 & 0 & 1.996 & 5.324 & 0 \\ -3.019 & -2.012 & 3.019 & 0 & 0 & 2.012 \end{bmatrix}$$

$$K = K_1 + K_2$$

$$K = 10^{6} \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{c|c|c|c|c|c|c|c|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 9.85 & & -4.528 & -1.996 & & & & \\ 2 & & & & & & & & \\ 3 & -4.528 & & 22.528 & 0 & & & & \\ 4 & -1.996 & & 0 & 18.784 & & & & \\ 5 & & & & & & & & \\ 6 & & & & & & & & \\ 7 & & & & & & & & \\ 8 & & & & & & & & \end{array}$$

$$10^{6} \begin{bmatrix} 9.85 & -4.528 & -1.996 \\ -4.528 & 22.528 & 0 \\ -1.996 & 0 & 18.784 \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -10{,}000 \end{bmatrix}$$

$$a_1 = 10^{-5} \times 1.9$$

$$a_3 = 0.87 \times 10^{-5}$$

$$a_4 = -7.436 \times 10^{-5}$$

AXIS OF SYMMETRY

INTRODUCTION:

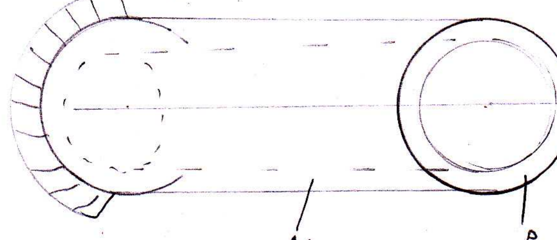
In this chapter evaluing 3-D axis of symmetry solids (or) solids of revolution subjected to axis symmetry loading reduced to simple 2-D problem. Because of total symmetric about z-axis has seen. All the deformations and stresses are independent of rotation angle 'θ'. By considering the element volume the potential approach can be written as

$$\Pi = \frac{1}{2}\int_0^{2\pi}\int_A \sigma^T \epsilon\, r\, dA\, d\theta - \int_0^{2\pi}\int_A u^T f\, r\, dA\, d\theta - \int_0^{2\pi}\int_A u^T T_r\, dl\, d\theta - \Sigma u_i P_i$$

where $u = [u, w]^T$

$f = [f_r, f_z]^T$

$T = [T_r, T_z]^T$



we can write the relation b/w strains & displacement u as $\epsilon = [\epsilon_r\ \epsilon_z\ \gamma_{rz}\ \epsilon_\theta]^T$

$$= \left[\frac{du}{dr}\quad \frac{dw}{dz}\quad \frac{du}{dz}, \frac{dw}{dr}\quad \frac{u}{r}\right]^T$$

The stress-strain relation ship is given by $\sigma = D\epsilon$

$\epsilon = Bq$

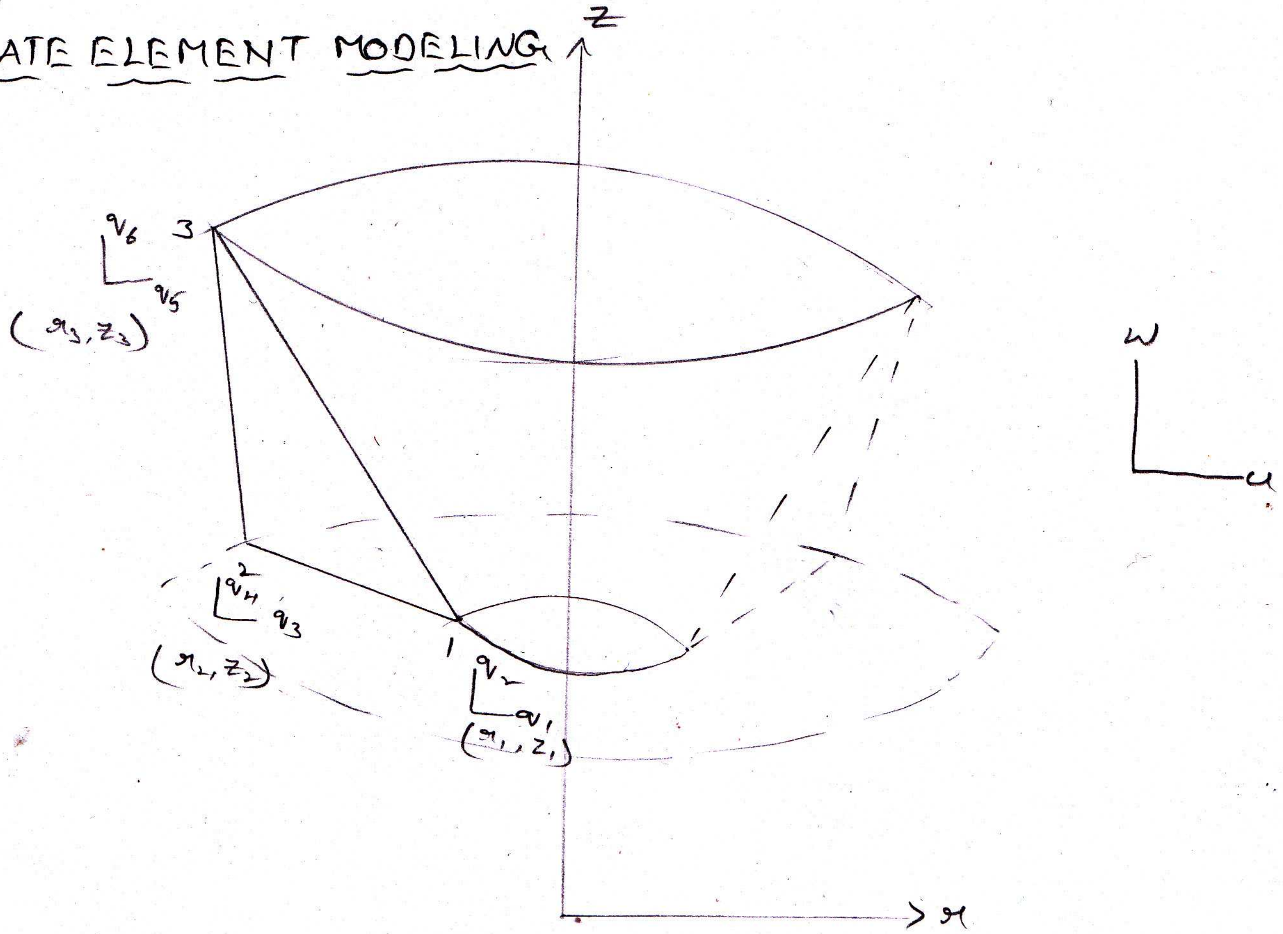
$\sigma = DBq$

$$\because D = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 & \frac{\nu}{1-\nu} \\ \frac{\nu}{1-\nu} & 1 & 0 & \frac{\nu}{1-\nu} \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 1 \end{bmatrix}$$

where D - constant stress matrix

B - strain displacement matrix

FINATE ELEMENT MODELING



The 2-D region defined by revolving area is divided into triangular elements. Though each element is completly represented by the area about r, z coordinates respectively. replace x & y with r & z coordinates. Using these three shape functions N_1, N_2 & N_3 we define.

$$u = Nq$$

$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5$$

$$w = N_1 q_2 + N_2 q_4 + N_3 q_6$$

$$r = N_1 r_1 + N_2 r_2 + N_3 r_3$$

$$z = N_1 z_1 + N_2 z_2 + N_3 z_3$$

$u = \epsilon(q_1 - q_5) + \eta(q_3 - q_5) + q_5$

$w = \epsilon(q_2 - q_6) + \eta(q_4 - q_6) + q_6$

By using isoparametric representation we find.

$r = N_1 r_1 + N_2 r_2 + N_3 r_3$

$z = N_1 z_1 + N_2 z_2 + N_3 z_3$

sub the values of N_1, N_2, N_3 in above eqn's

$r = \epsilon r_1 + \eta r_2 + (1 - \epsilon - \eta) r_3$

$z = \epsilon z_1 + \eta z_2 + (1 - \epsilon - \eta) z_3$

The chain rule of differentiation.

$$\begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{bmatrix}$$

Illy $$\begin{bmatrix} \frac{\partial w}{\partial \epsilon} \\ \frac{\partial w}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \end{bmatrix}$$

where $J = \begin{bmatrix} r_{13} & z_{13} \\ r_{23} & z_{23} \end{bmatrix}$

$$\begin{bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial u}{\partial \epsilon} \\ \frac{\partial u}{\partial \eta} \end{bmatrix}$$

Illy $$\begin{bmatrix} \frac{\partial w}{\partial r} \\ \frac{\partial w}{\partial z} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial w}{\partial \epsilon} \\ \frac{\partial w}{\partial \eta} \end{bmatrix}$$

where $J^{-1} = \frac{1}{|J|}\begin{bmatrix} z_{23} & -z_{13} \\ -r_{23} & r_{13} \end{bmatrix}$

Strain displacement Relations:-

$$\epsilon = \begin{bmatrix} \frac{du}{dr} & \frac{dw}{dz} & \frac{du}{dz} + \frac{dw}{dr} & \frac{u}{r} \end{bmatrix}^T$$

$$\epsilon = \frac{1}{|J|}\begin{bmatrix} z_{23}(q_1 - q_5) - z_{13}(q_3 - q_5) \\ -r_{23}(q_2 - q_6) + r_{13}(q_4 - q_6) \\ -r_{23}(q_1 - q_5) + r_{13}(q_3 - q_6) + z_{23}(q_2 - q_6) - z_{13}(q_4 - q_6) \\ \frac{N_1 q_1 + N_2 q_3 + N_3 q_5}{r} \end{bmatrix}$$

$$\therefore B = \frac{1}{|J|}\begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & r_{32} & 0 & r_{13} & 0 & r_{21} \\ r_{32} & z_{23} & r_{13} & z_{31} & r_{21} & z_{12} \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \end{bmatrix}$$

Element stiffness matrix

$$K = 2\pi r A\, B^T D B.$$

where B = strain displacement matrix

D = constant stress matrix

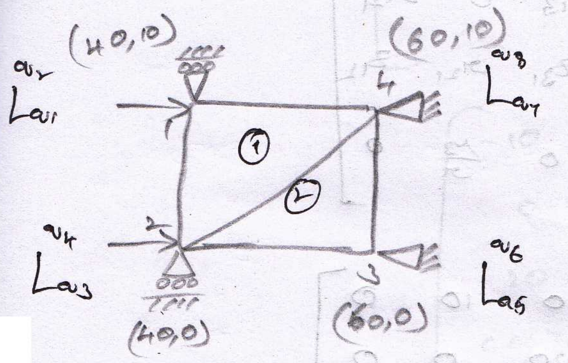
$\bar{r}$ = radius of centroid $= \frac{r_1 + r_2 + r_3}{3}$

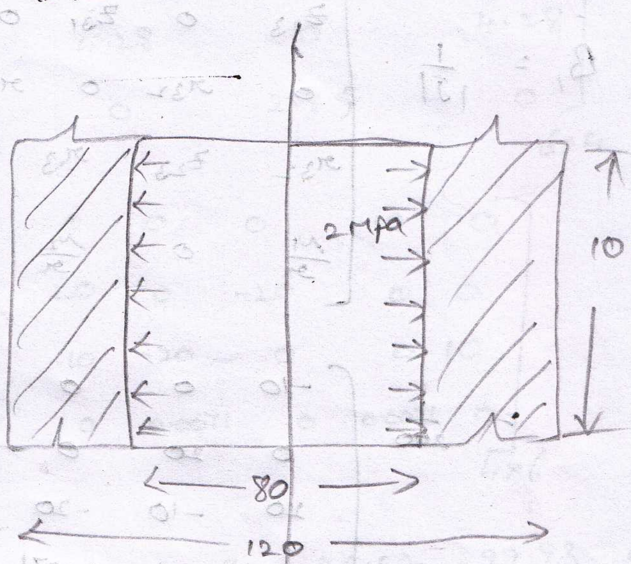
Illy shape functions

$$N_1 = N_2 = N_3 = \frac{1}{3}$$

$$\text{Area } A = \frac{1}{2}|J|$$

outside diameter 120 mm securely fitted into a hole over its full length. The cylinder is then subjected to a internal pressure of 2 MPa using two elements on the 10 mm length. Find the displacements at the inner radius.





E = 200 GPa

$\gamma = 0.3$

s.no	element	nodes
1	1	1-2-4
2	2	2-3-4

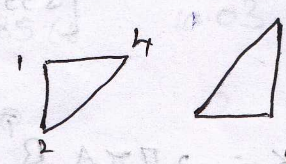
coordinates

1	40, 10
2	40, 0
3	60, 0
4	60, 10

Boundary conditions

no. of fixed nodes. q_5, q_2, q_4, q_5, q_6, q_7, q_8

unknown values $= q_1, q_3$



$$\text{Area} = \frac{1}{2}|J|$$

$$= \frac{1}{2}\left| r_{13}\, z_{23} - r_{23}\, z_{13} \right|$$

$$= \frac{1}{2}\left| -20 \times -10 - 0 \right|$$

$$= 100$$

$$D = \frac{E(1-\gamma)}{(1+\gamma)(1-2\gamma)} \begin{bmatrix} 1 & \frac{\gamma}{1-\gamma} & 0 & \frac{\gamma}{1-\gamma} \\ \frac{\gamma}{1-\gamma} & 1 & 0 & \frac{\gamma}{1-\gamma} \\ 0 & 0 & \frac{1-2\gamma}{2(1-\gamma)} & 0 \\ \frac{\gamma}{1-\gamma} & \frac{\gamma}{1-\gamma} & 0 & 1 \end{bmatrix}$$

$$= \frac{200\times10^3\,(1-0.3)}{(1+0.3)\,(1-2(0.3))} \begin{bmatrix} 1 & \frac{0.3}{0.7} & 0 & \frac{0.3}{0.7} \\ \frac{0.3}{0.7} & 1 & 0 & \frac{0.3}{0.7} \\ 0 & 0 & \frac{1-0.6}{2(1-0.3)} & 0 \\ \frac{0.3}{0.7} & \frac{0.7}{0.7} & 0 & 1 \end{bmatrix}$$

$$= 2.69\times10^{5}\begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.285 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$

$$B_1 = \frac{1}{|J|}\begin{bmatrix} z_{23} & 0 & z_{31} & 0 & z_{12} & 0 \\ 0 & r_{32} & 0 & r_{13} & 0 & r_{21} \\ r_{32} & z_{23} & r_{13} & z_{31} & r_{21} & z_{12} \\ \frac{\alpha_1}{r} & 0 & \frac{\alpha_2}{r} & 0 & \frac{\alpha_3}{r} & 0 \end{bmatrix}$$

$$= \frac{1}{200}\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.05 & 0 & 0 & 0 & 0.05 & 0 \\ 0 & 0.1 & 0 & -0.1 & 0 & 0 \\ 0.1 & -0.05 & -0.1 & 0 & 0 & 0.05 \\ 3.55\times10^{-5} & 0 & 3.55\times10^{-5} & 0 & 3.55\times10^{-5} & 0 \end{bmatrix}$$

$$K_1 = 2\pi r A\, B^T D B.$$

$$= 2\pi\times 46.66\times 100\times\frac{1}{200}\begin{bmatrix} -10 & 0 & 20 & 0.0071 \\ 0 & 20 & -10 & 0 \\ 0 & 0 & -20 & 0.0071 \\ 0 & -20 & 0 & 0 \\ 10 & 0 & 0 & 0.0071 \\ 0 & 0 & 10 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.285 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$

$$\times 2.69\times10^{5}\times\frac{1}{200}\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}$$

$$= 197.159 \times 10^{3} \begin{bmatrix} 8.56 & 20 & -2.85 & 8.56 \\ 3.038\times10^{-3} & 3.038\times10^{-3} & -5.7 & 0.0071 \\ -8.56 & -20 & 0 & -8.56 \\ 10 & 4.28 & 0 & 4.28 \\ 0 & 0 & 2.85 & 0 \end{bmatrix}_{6\times4}$$

$$\begin{bmatrix} -10 & 0 & 0 & 0 & 10 & 0 \\ 0 & 20 & 0 & -20 & 0 & 0 \\ 20 & -10 & -20 & 0 & 0 & 10 \\ 0.0071 & 0 & 0.0071 & 0 & 0.0071 & 0 \end{bmatrix}_{4\times6}$$

$$197.159\times10^{3} \begin{bmatrix} 213.86 & -142.52 & -114.03 & 85.52 & -99.93 & 57 \\ -142.52 & 428.5 & 57 & -400 & 85.66 & -28.5 \\ -114.03 & 57 & -113.99 & -0.06 & 0.03 & -57 \\ 85.52 & -400 & -0.06 & 400 & -85.66 & 0 \\ -99.93 & 85.66 & 0.03 & -85.66 & 10.03 & 0 \\ 57 & -28.5 & -57 & 0 & 0 & 28.5 \end{bmatrix}$$

$$B_2 = \frac{1}{200} \begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \\ 6.25\times10^{-3} & 0 & 6.25\times10^{-3} & 0 & 6.25\times10^{-3} & 0 \end{bmatrix}$$

$$K_2 = 2\pi \times 53.33 \times 100 \times \frac{1}{200} \begin{bmatrix} -10 & 0 & 10 & 6.25\times10^{-3} \\ 0 & 0 & -10 & 0 \\ 10 & 0 & -20 & 6.25\times10^{-3} \\ 0 & -20 & 10 & 0 \\ 0 & 0 & 20 & 6.25\times10^{-3} \\ 0 & 20 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.428 & 0 & 0.428 \\ 0.428 & 1 & 0 & 0.428 \\ 0 & 0 & 0.285 & 0 \\ 0.428 & 0.428 & 0 & 1 \end{bmatrix}$$

HIGHER ORDER ELEMENTS

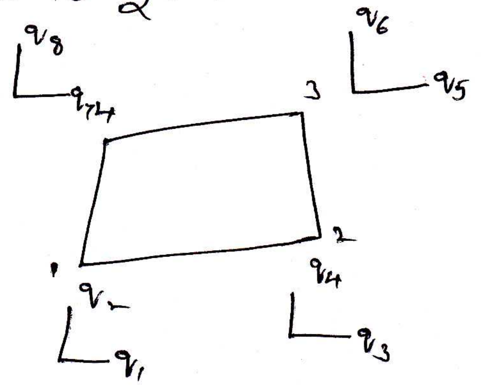
Four noded quadrilateral elements:

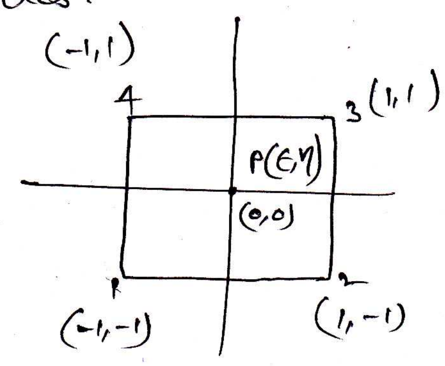
Consider the general quadrilateral element the local nodes are numbered as 1, 2, 3, 4 in counter clock-wise direction. x_i, y_i are the coordinate of node 'i'. The vector $Q = [q_1 q_2 q_3 \cdots q_8]^T$ denotes the element displacement vector. The displacement of an interior point 'p' is located at x, y is represented as 'u'.

$$u = [u(x,y), v(x,y)]^T$$

Shape functions:-

A master is defined in ϵ & η coordinates and is a square shaped the leglanch shape functions where $i = 1, 2, 3, 4$ are defined such that N_i = unity at node 'i' & is equal to zero at other nodes.





In particular consider the defination of N_i ie, $N_1 = 1$ at node 1 & equal to zero at 2, 3 & 4. Now the requirement that $N_1 = 0$ at nodes 2, 3, & 4 is equilent to the requiring that $N_1 = 0$ along the edges. $\epsilon = +1$ & $\eta = +1$. Thus N_1 has been formed

$$N_1 = c\,(1-\epsilon)\,(1-\eta)$$

where c is some constant. The constant is determined from the condition. $N_1 = 1$ at node '1'

$\because \epsilon = -1 \quad \eta = -1$

$\therefore$ sub the values in $N_1 = c\,(1-\epsilon)\,(1-\eta)$

$$1 = c\,(1+1)\,(1+1)$$

$$c = \frac{1}{4}$$

$$\therefore N_1 = \frac{1}{4}(1-\epsilon)(1-\eta)$$

$$|||^{ly} \quad N_2 = \frac{1}{4}(1+\epsilon)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\epsilon)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\epsilon)(1+\eta)$$

$$\therefore \quad u = [u\,v]^T$$

where $u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$ — (A)

$V = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8$ — (B)

which can be written in the form of $u = Nq$

where
$$N = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

For isoperametric formation we use shape functions N_i to also express the coordinates of the points

Thus,
$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$ — (1)

$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$ — (2)

Subsequently we need to express the derivatives of the function in x, y coordinates in terms of its derivatives (ϵ, η) coordinates.

$\therefore$ This is denoted by follows.

$f = f(x, y)$ can be considered to be implecant function of ϵ & η.

$\therefore f = f[x(\epsilon,\eta), y(\epsilon,\eta)]$ by using the chain rule differential eqn

$$\frac{\partial f}{\partial \epsilon} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \epsilon}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \epsilon} & \frac{\partial y}{\partial \epsilon} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$ where J is Jacobian matrix.

sub N_1, N_2 & N_3 & N_4 in above eqn's ① & ②

$$x = \frac{1}{4}(1-\epsilon)(1-\eta)x_1 + \frac{1}{4}(1+\epsilon)(1-\eta)x_2 + \frac{1}{4}(1+\epsilon)(1+\eta)x_3 + \frac{1}{4}(1-\epsilon)(1+\eta)x_4$$

$$\frac{\partial x}{\partial \epsilon} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

Illy $$\frac{\partial y}{\partial \epsilon} = \frac{1}{4}\left[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right]$$

$$\frac{\partial x}{\partial \eta} = \frac{1}{4}\left[-(1-\epsilon)x_1 - (1+\epsilon)x_2 + (1+\epsilon)x_3 + (1-\epsilon)x_4\right]$$

$$\frac{\partial y}{\partial \eta} = \frac{1}{4}\left[-(1-\epsilon)y_1 - (1+\epsilon)y_2 + (1+\epsilon)y_3 + (1-\epsilon)y_4\right]$$

$$\therefore \begin{bmatrix} \frac{\partial f}{\partial \epsilon} \\ \frac{\partial f}{\partial \eta} \end{bmatrix} = J \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial f}{\partial \epsilon} \\ \frac{\partial f}{\partial \eta} \end{bmatrix}$$

$$= \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}\begin{bmatrix} \frac{df}{d\epsilon} \\ \frac{df}{d\eta} \end{bmatrix}$$

$$\therefore \quad \epsilon = \begin{bmatrix} \frac{du}{dx} \\ \frac{dv}{dy} \\ \frac{du}{dy} + \frac{dv}{dx} \end{bmatrix}$$

considering $f=u$

$$\frac{du}{dx} = J^{-1}\begin{bmatrix} \frac{du}{d\epsilon} \\ \frac{du}{d\eta} \end{bmatrix}$$

$$\frac{du}{dy} = J^{-1}\begin{bmatrix} \frac{du}{d\epsilon} \\ \frac{du}{d\eta} \end{bmatrix}$$

lly $\quad \frac{dv}{dx} = J^{-1}\begin{bmatrix} \frac{dv}{d\epsilon} \\ \frac{dv}{d\eta} \end{bmatrix}. \qquad \frac{dv}{dy} = J^{-1}\begin{bmatrix} \frac{dv}{d\epsilon} \\ \frac{dv}{d\eta} \end{bmatrix}$

$$\therefore \quad \epsilon = A\begin{bmatrix} \frac{du}{d\epsilon} \\ \frac{du}{d\eta} \\ \frac{dv}{d\epsilon} \\ \frac{dv}{d\eta} \end{bmatrix}$$

where $A = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$

$$\begin{bmatrix} \frac{du}{d\epsilon} \\ \frac{du}{d\eta} \\ \frac{dv}{d\epsilon} \\ \frac{dv}{d\eta} \end{bmatrix} = Gq$$

sub values of N_1, N_2, N_3 & N_4 in eqn's A & B

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$$

$$u = \frac{1}{4}(1-\epsilon)(1-\eta) q_1 + \frac{1}{4}(1+\epsilon)(1-\eta) q_3 + \frac{1}{4}(1+\epsilon)(1+\eta) q_5 + \frac{1}{4}(1-\epsilon)(1+\eta) q_7$$

$$\frac{du}{d\epsilon} = \frac{1}{4}\left[-(1-\eta) q_1 + (1-\eta) q_3 + (1+\eta) q_5 - (1+\eta) q_7\right]$$

lly $$\frac{du}{d\eta} = \frac{1}{4}\left[-(1-\epsilon) q_1 + (-1)(1+\epsilon) q_3 + (1+\epsilon) q_5 + (1-\epsilon) q_7\right]$$

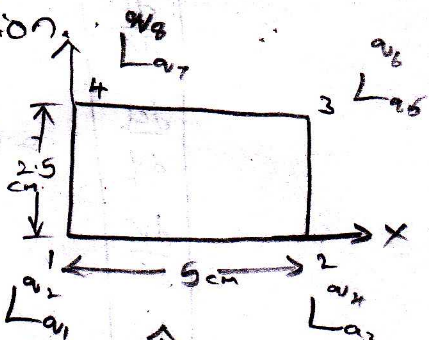
$$\frac{dv}{d\epsilon} = \frac{1}{4}\left[-(1-\eta) q_2 + (1-\eta) q_4 + (1+\eta) q_6 - (1+\eta) q_8\right]$$

$$\frac{dv}{d\eta} = \frac{1}{4}\left[-(1-\epsilon) q_2 - (1+\epsilon) q_4 + (1+\epsilon) q_6 + (1-\epsilon) q_8\right]$$

$\therefore$

$$G = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) \end{bmatrix}$$

← Consider the rectangular element as shown in the figure. Assume plain stress condition.

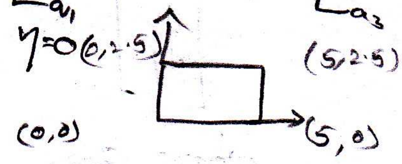


$E = 20.685 \times 10^6 \ N/cm^2$

$\nu = 0.3$

$q = [0, 0, 0.05, 0.075, 0.15, 0.8, 0, 0]^T$

Evaluate Jacobian J, B, & σ $\epsilon = 0$, $\eta = 0$



Sol:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-(0) + 1\times5 + 1\times5 - 0\right] = \frac{10}{4} = 2.5$$

$$J_{12} = \frac{1}{4}\left[-(0) + 1\times \cancel{2.5}\,0 + 1\times2.5 - 1\times2.5\right]$$

$= 0$

$$J_{21} = \frac{1}{4}\left[-1\times0 - 1\times5 + 1\times5 + 0\right] = 0$$

$$J_{22} = \frac{1}{4}\left[-1\times(0) - 1\times0 + 1\times2.5 + 1\times2.5\right] = \frac{5}{4}$$

$= 1.25$

$\det J = |ad - bc| = |2.5\times1.25 - 0|$

$= 3.125$

$$A = \frac{1}{|J|}\begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$

$$= \frac{1}{3.125}\begin{bmatrix} 1.25 & -0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix}_{3\times4}$$

$$G = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\epsilon) & 0 & -(1+\epsilon) & 0 & (1+\epsilon) & 0 & (1-\epsilon) \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 8}$$

$$B = A \times G$$

$$= \frac{1}{3.125}\begin{bmatrix} 1.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5 \\ 0 & 2.5 & 1.25 & 0 \end{bmatrix}_{3\times 4} \times \frac{1}{4}\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}_{4\times 8}$$

$$= 0.08\begin{bmatrix} -1.25 & 0 & 1.25 & 0 & 1.25 & 0 & -1.25 & 0 \\ 0 & -2.5 & 0 & -2.5 & 0 & 2.5 & 0 & 2.5 \\ -2.5 & -1.25 & -2.5 & 1.25 & 2.5 & 1.25 & 2.5 & -1.25 \end{bmatrix}_{3\times 8}$$

$$B = \begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}_{3\times 8}$$

$$\sigma = DBq$$

$$D = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{20.685\times10^6}{1-(0.3)^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$= 22.7307\times10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

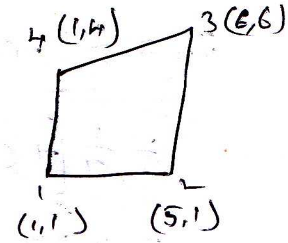
$$\sigma = DBq$$

$$= 22.73\times10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}_{3\times3}\begin{bmatrix} -0.1 & 0 & 0.1 & 0 & 0.1 & 0 & -0.1 & 0 \\ 0 & -0.2 & 0 & -0.2 & 0 & 0.2 & 0 & 0.2 \\ -0.2 & -0.1 & -0.2 & 0.1 & 0.2 & 0.1 & 0.2 & -0.1 \end{bmatrix}_{3\times8}\begin{bmatrix} 0 \\ 0 \\ 0.05 \\ 0.07 \\ 0.15 \\ 0.8 \\ 0 \\ 0 \end{bmatrix}_{8\times1}$$

$$= 22.73\times10^6\begin{bmatrix} -0.1 & -0.06 & 0.1 & -0.06 & 0.1 & 0.06 & -0.1 & 0.06 \\ -0.03 & -0.2 & 0.03 & -0.2 & 0.03 & 0.2 & -0.03 & 0.2 \\ -0.07 & -0.035 & -0.07 & 0.035 & 0.07 & 0.035 & 0.07 & -0.035 \end{bmatrix}_{3\times8}\begin{bmatrix} 0 \\ 0 \\ 0.05 \\ 0.07 \\ 0.15 \\ 0.8 \\ 0 \\ 0 \end{bmatrix}_{8\times1}$$

$$= 22.73\times10^6\begin{bmatrix} 0.0638 \\ 0.152 \\ 0.0374 \end{bmatrix}_{3\times1}$$

node are given in the fig. The element displacement vector is given as $q = [0, 0, 0.20, 0, 0.15, 0.1, 0, 0.05]^T$ the x,y coordinates of the point 'p' whose location in the master element is given by $\epsilon = 0.5$ & $\eta = 0.5$ & also find u, v displacement of point P



$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$u = N_1 q_1 + N_2 q_3 + N_3 q_5 + N_4 q_7$$

$$v = N_1 q_2 + N_2 q_4 + N_3 q_6 + N_4 q_8$$

$$N_1 = \frac{1}{4}(1-\epsilon)(1-\eta) = 0.0625$$

$$N_2 = \frac{1}{4}(1+\epsilon)(1-\eta) = 0.187$$

$$N_3 = \frac{1}{4}(1+\epsilon)(1+\eta) = 0.562$$

$$N_4 = \frac{1}{4}(1-\epsilon)(1+\eta) = 0.1875$$

$$x = \frac{1}{4}(1-\epsilon)(1-\eta)x_1 + \frac{1}{4}(1+\epsilon)(1-\eta)x_2 + \frac{1}{4}(1+\epsilon)(1+\eta)x_3 + \frac{1}{4}(1-\epsilon)(1+\eta)x_4$$

$$= \frac{1}{4}(1-0.5)(1-0.5)1 + \frac{1}{4}1.5 \times 0.5 \times 5 + \frac{1}{4}1.5 \cdot 1.5 \cdot 6 + \frac{1}{4}0.5 \cdot 1.5 \cdot 1$$

$$= 4.56$$

$$y = \frac{1}{4}0.5 \times 0.5 \times 1 + \frac{1}{4}1.5 \times 0.5 \times 1 + \frac{1}{4}1.5 \times 1.5 \times 6 + \frac{1}{4} \times 0.5 \times 1.5 \times 4$$

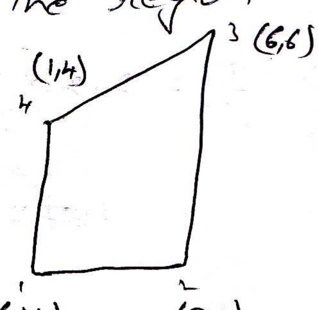
$$= 4.3$$

$$P = (x, y) = (4.56, 4.37)$$

$$u = 0.1218$$

$$v = 0.065$$

k Using a 2×2 rule evaluate the $\iint_A (x^2 + xy^2)\,dx\,dy$ by Gaussian coordinate where 'A' denotes the region shown in the figure. The sampling points are $\pm\frac{1}{\sqrt{3}}$ & weights are unity now



$\cancel{\epsilon = \frac{1}{\sqrt{3}}}$ $\cancel{\eta = \frac{1}{\sqrt{3}}}$

determine the Jacobian for given Gaussian point.

case(i) $\epsilon = \frac{1}{\sqrt{3}}$ $\eta = \frac{1}{\sqrt{3}}$

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$J_{11} = \frac{1}{4}\left[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right]$$

$$= \frac{1}{4}\left[-1\left(1-\frac{1}{\sqrt{3}}\right)1 + (1-0.577)5 + (1+0.577)6 - (1+0.577)1\right]$$

$$= 2.39$$

$$J_{12} = \frac{1}{4}\left(-(1-0.577)1 + (1-0.577)1 + (1+0.577)6 - (1+0.577)4\right]$$

$$= 0.788$$

$$J_{21} = \frac{1}{4}\left[-(1-\epsilon)x_1 - (1+\epsilon)x_2 + (1+\epsilon)x_3 + (1-\epsilon)x_4\right]$$

$$= \cancel{\tfrac{1}{4}}\ 0.394$$

$$J_{22} = 2.288$$

$$\cancel{A}A_1 = \frac{1}{2}\left|(2.39 \times 2.288) - (0.788 \times 0.394)\right|$$

$$= \cancel{5.45}\ \cancel{2.57}\ 2.57$$

case(ii) $\epsilon = \frac{1}{\sqrt{3}}$ $\eta = -\frac{1}{\sqrt{3}}$

$$J_{11} = \frac{1}{4}\left[-1\left(1+\frac{1}{\sqrt{3}}\right)1 + \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1-\frac{1}{\sqrt{3}}\right)6 - \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$$= 2.10$$

$$J_{12} = 0.211$$

$$J_{21} = \frac{1}{4}\left[-\left(1-\frac{1}{\sqrt{3}}\right)1 - \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1+\frac{1}{\sqrt{3}}\right)6 + \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$$= \cancel{4.42}\ 0.39$$

$$J_{22} = 2.288$$

$$|\cancel{J}| = A_2 = \frac{1}{2}|J|$$

$$= \frac{1}{2}\left|(2.10 \times 2.288) - (0.211 \times 0.39)\right|$$

$$J_{11} = \frac{1}{4}\left[-\left(1+\frac{1}{\sqrt{3}}\right)1 + \left(1+\frac{1}{\sqrt{3}}\right)5 + \left(1-\frac{1}{\sqrt{3}}\right)6 - \left(1-\frac{1}{\sqrt{3}}\right)1\right]$$

$$= \cancel{0.8036}\ 2.1$$

$$J_{12} = 0.2115$$

$$J_{21} = \frac{1}{4}\left[-\left(1+\frac{1}{\sqrt{3}}\right)x_1 - \left(1-\frac{1}{\sqrt{3}}\right)x_2 + \left(1-\frac{1}{\sqrt{3}}\right)x_3 + \left(1+\frac{1}{\sqrt{3}}\right)x_4\right]$$

$$= 0.105$$

$$J_{22} = 1.7115$$

$$A_4 = \cancel{0.704}\ 1.78$$

case (iv) $\epsilon = -\frac{1}{\sqrt{3}}$ $\eta = \frac{1}{\sqrt{3}}$

$$J_{11} = \frac{1}{4}\left[-\left(1-\frac{1}{\sqrt{3}}\right)1 + \left(1-\frac{1}{\sqrt{3}}\right)5 + \left(1+\frac{1}{\sqrt{3}}\right)6 - \left(1+\frac{1}{\sqrt{3}}\right)1\right]$$

$$= 2.394$$

$$J_{12} = 0.7885$$

$$J_{21} = 0.105$$

$$J_{22} = 1.7115$$

$$A_3 = 2$$

$$A = \sum_{i=1}^{4} |J| = 2.5 + 2.36 + 2 + 1.78 = 8.64$$

$$\iint_A (x^2 + xy^2)\,dx\,dy = \sum_{j=1}^{n}\sum_{i=1}^{n} w_j w_i\, f(x_i y_i)\cdot dA.$$

$\therefore \cancel{w_1 = w}$

$\therefore w_1 = w_2 = 1$

$$= 8.64\left[w_1^2 f(x_1 y_1) + w_1 w_2 f(x_1 y_2) + w_2 w_1 f(x_2 y_1) + w_2^2 f(x_2 y_2)\right]$$

$(x_1\ 0.57735 \quad y_1\ 0.57735)$

$(x_2\ 0.57735 \quad y_2\ -0.57735)$

$\therefore f = x^2 + xy^2$

$$w_1^2 f(x_1, y_1) = w_1^2 (x_1^2 + x_1 y_1^2)$$

$$= 1\left((0.57735)^2 + (0.57735)(0.57735)^2\right)$$

$$= 0.525$$

$$w_1 w_2 f(x_1, y_2) = w_1 w_2 (x_1^2 + x_1 y_2^2)$$

$$= 1\times 1\left((0.57735)^2 + (0.57735)(-0.57735)^2\right)$$

$$= 0.525$$

$$w_2 w_1 f(x_2 y_1) = w_2 w_1 (x_2^2 + x_2 y_1^2)$$

$$= 1\times 1\left((-0.57735)^2 + (-0.57735)(0.57735)^2\right)$$

$$= 0.14088 = 0.141$$

$$w_2^2 f(x_2 y_2) = w_2 \cdot w_2 (x_2^2 + x_2 y_2^2)$$

$$= 1\times 1\left((-0.57735)^2 + (-0.57735)(-0.57735)^2\right)$$

$$= 0.14088 = 0.141$$

$$\therefore \iint_A (x^2 + xy^2)\, dx\, dy = 8.64\left[0.525 + 0.525 + 0.141 + 0.141\right]$$

$$= 11.50$$