FINITE ELEMENT METHODS.

* **Finite Element Methods:**

The Finite Element Method is described as a Numerical Method to obtain approximate solutions to a wide Variety of mostly Complex problems arised in different engineering fields.

The process of Finite element Analysis / Methods can be specified by three stages of Activities such as:

1. **Pre-processing:**

   It involves the data.
   - → Nodal Coordinates
   - → Connectivity
   - → Boundary Conditions.
   - → Loading and Material information.

2. **Processing:**

   It involves the data.
   - → Stiffness Generation.
   - → Stiffness Modification.
   - → Solution of Variables.

3. **Post Processing:** The Strain and Stress distribution, temperature etc are computed at this stage.

* **Basic Steps in Finite Element Method:**

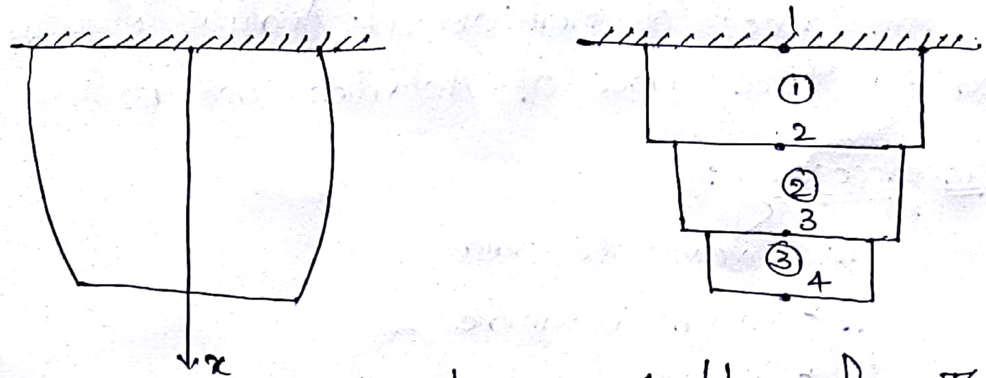
To find the Solution for a Complex problem in the Finite Element method, some basic steps are followed. These steps are almost similar except slight modification in any field.

Eg: Structural Analysis, heat transfer, Fluid flow etc

The General steps involved in Structural Engineering problems are discussed as follows:

1. Discretization of Domain:

The first step in the Finite Element Method is to divide (i.e, discretize) the whole complex structure called domain into small parts i.e, Sub-domains, called Finite Elements or Lines or Surfaces. The main function of discretization of the domain is due to the following reasons.



Finite element Modeling of a Bar.

→ In Most Engineering problems, the domains has irregular shapes. Hence the field variables used for their design such as displacements due to loading, temperature distribution due to H.T, Mass Transfer due to fluid flow etc., cannot be predicted accurately in all places by using analytical methods.

→ The irregular shapes of domain has infinite no. of degrees of freedom which results in highly complicated and time consuming computational Methods.

That is why, the irregular shaped domain is splitted into convenient no. of regular shaped sub-domains or finite elements. and thus the problem is simplified.

During discretization, the shapes, sizes, number and Configuration of the elements have to be choosen carefully.

2. Selection of displacement function:

It involves choosing a displacement function within each element. The function is defined with in the elements using the nodal values of the element.

Linear, quadratic & Cubic polynomials are frequently used functions because they are simple to work.

3. <u>Formation of Element Stiffness Matrix & Load Vector.</u>

After discretizing the domain with desired element shapes, the element stiffness matrix and load vector are formulated. These can be done by using either equilibrium conditions or a suitable variational principle.

4. <u>Formation of Global Stiffness Matrix & Load Vector.</u>

The Global Stiffness Matrix and the Global load vector are formulated from the element stiffness matrix and Element load vector.

The Final Global finite element equation for the complete structure can be written in the Matrix form as

$$[K][U] = \{F\}$$

Where, $[K]$ = Global Stiffness Matrix.

$[U]$ = Nodal displacement vector.

$\{F\}$ = Global Load Vector.

5. <u>Incorporation/Applying Boundary Conditions:</u>

After forming Global Finite Element Equation, the boundary conditions are imposed in that equation. The size of the stiffness matrix may be reduced and the equations required for the solution of the problem are developed. 1. Elimination Method 2. Penalty Method.

6. <u>Solution of Simultaneous Equations:</u>

The developed Equation (from step:5) are solved to get nodal displacements commonly by Gauss Elimination method or Gauss Seidal or Jocobi Iteration Method.

7. Calculating Strains & Stresses:

From the Calculated Nodal displacements, the element strains & stresses can be Computed by using the necessary equations.

8. Interpretation of the Results Obtained:

The final goal of FEA is to interpret and analyse the results obtained in the design and production process. Determining the Location in the structure where large deformation and large stresses occur is generally important & essential in making design & analysis decisions.

* Coordinates and Shape Functions:

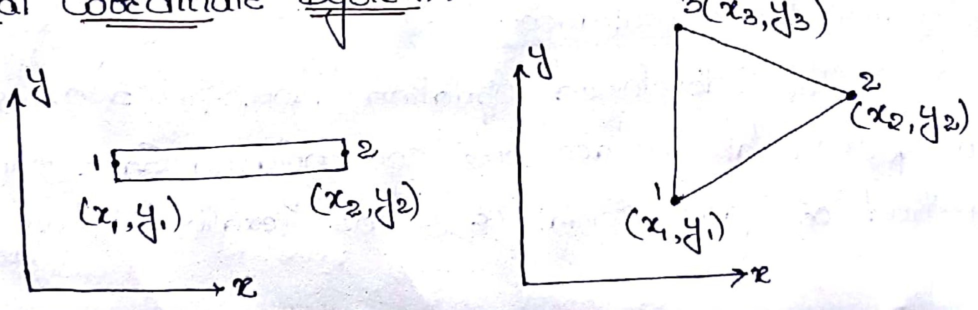
→ Co-ordinate System:

In Finite Element Method, the location of various nodes of the element must be expressed with respect to some fixed axes, for easy identification of the element and further process. These axes are called as coordinates, which may be placed away from the elements or on the elements themselves depending upon Complexity of the problem.

In practice two types of Co-ordinate Systems are adopted, they are:

1. Global Coordinate System.
2. Local Coordinate System.

* Global Coordinate System:

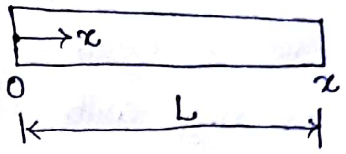


In this system, the axis is placed completely outside the element. So, the coordinate points are represented with respect to the coordinate Axis.

* Local Coordinate System:

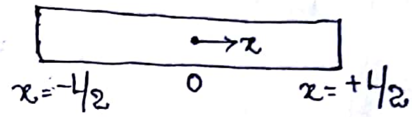
(a) Axis at the Extreme End.

Here, the axis is placed at the extreme end as shown so that the value of x changes from 0 to L.



(b) Axis at the Mid-point:

Here, the axis is placed at the centre of the element so that the value of x changes from $-L/2$ to $L/2$.



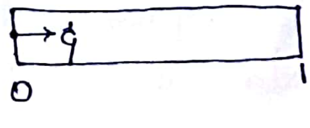
* Natural Coordinate System:

A Natural Coordinate system is another type of Local coordinate system that permits the specification of a point within the element by a set of dimensionless numbers whose magnitude never exceeds Unity.

There are two types of Natural Coordinate system.

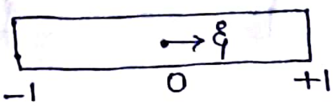
1. Axis at Extreme End

Here, the magnitude of 'ξ' varies from 0 to 1



2. Axis at Mid-point

Here, the Magnitude of 'ξ' varies from -1 to $+1$



Note: This Natural or Intrinsic coordinate system is denoted by $\xi(x_i y_i)$

→ **Shape Functions:**

For achieving the displacements at the Nodal points and inside the elements, we have to make use of two mathematical expressions namely.

1. Finite Element Equations
2. Shape Functions.

* **Finite Element Equation:** This equations relates the applied force with Nodal displacements, and hence by using this Equations the displacements at the primary nodes (i.e., at the extreme ends of the elements) can be evaluated.

$$F = KQ.$$

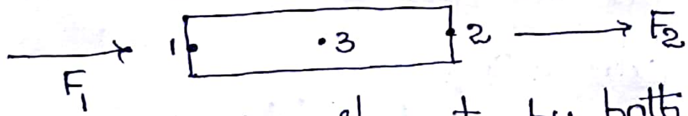
* **Shape Functions:** Shape functions are employed to find the displacements at the interior points of the element using the values of Nodal displacements. It is denoted by N.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + \ldots\ldots$$

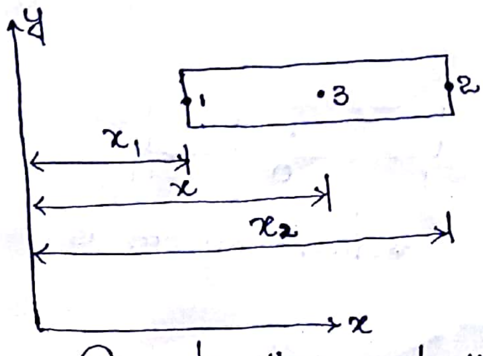
Where $u_1, u_2, u_3 \longrightarrow$ Nodal displacements

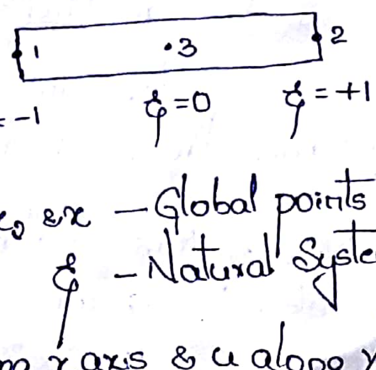
$u \longrightarrow$ Interior displacement.

Consider a Bar (1D) element as shown.



Let represent the Element by both the Global and Natural coordinate system.

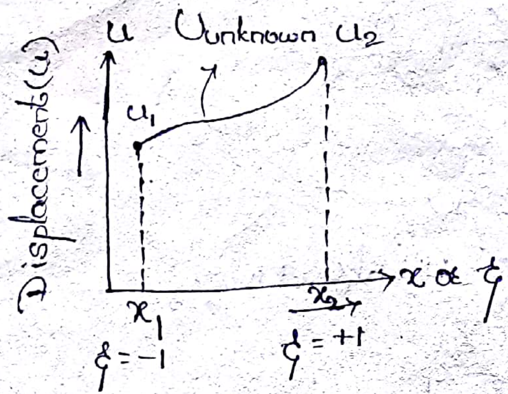


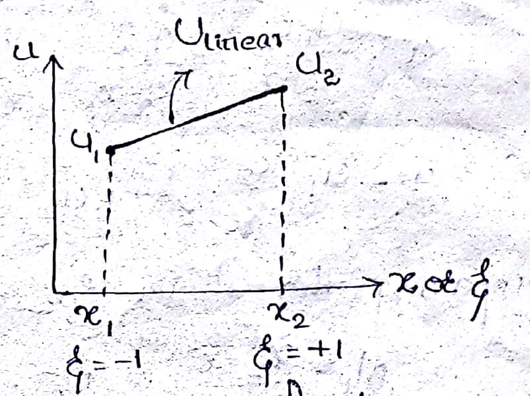


Here, x_1, x_2 & x — Global points

ξ — Natural System.

Consider the graph, the x & ξ along x axis & u along y axis.





Interpolation function.

Now, the u can be written as a Linear polynomial function.

$u = a_1 + a_2 x$ for Absolute or Global system.

$u = a_1 + a_2 \xi$ for Natural or Intrinsic system.

(i) Consider the Absolute co-ordinate system.

$$u = a_1 + a_2 x \quad \text{—— (A)}$$

At Node 1   At Node 2

$x = x_1$, $u = u_1$ $x = x_2$, $u = u_2$

$u_1 = a_1 + a_2 x_1 \rightarrow (1)$ $u_2 = a_1 + a_2 x_2$ —— (2)

By solving (1) and (2)

$$u_1 = \cancel{a_1} + a_2 x_1$$
$$u_2 = \cancel{a_1} + a_2 x_2$$
$$(-) \quad (-) \quad (-)$$

$$u_1 - u_2 = a_2 (x_1 - x_2)$$

$$a_2 = \frac{u_2 - u_1}{x_2 - x_1}$$

& $a_1 = u_1 - a_2 x_1$

$$= u_1 - \frac{u_2 - u_1}{x_2 - x_1} \cdot x_1 = \frac{u_1 x_2 - \cancel{u_1 x_1} - u_2 x_1 + \cancel{u_1 x_1}}{(x_2 - x_1)}$$

$$a_1 = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1}$$

Substitute in Eq. (A)

$$u = \frac{u_1 x_2 - u_2 x_1}{x_2 - x_1} + \frac{u_2 - u_1}{x_2 - x_1} \cdot x$$

$$u = \frac{u_1 x_2}{x_2 - x_1} - \frac{u_1 x}{x_2 - x_1} - \frac{u_2 x_1}{x_2 - x_1} + \frac{u_2 x}{x_2 - x_1}$$

$$u = u_1\left(\frac{x_2 - x}{x_2 - x_1}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$

Comparing the above Eq. with $u = N_1 u_1 + N_2 u_2$

$$N_1 = \frac{x_2 - x}{x_2 - x_1} \quad \& \quad N_2 = \frac{x - x_1}{x_2 - x_1}$$

(ii) * Consider the Natural co-ordinate system

$$u = a_1 + a_2 \xi \quad \text{—(B)}$$

At Node 1   At Node 2

$u = u_1$ & $\xi = -1$ $u = u_2$ & $\xi = +1$

$u_1 = a_1 - a_2 \rightarrow (3)$ $u_2 = a_1 + a_2$ —(4)

By solving (3) & (4)

$$u_1 = a_1 - \cancel{a_2}$$
$$u_2 = a_1 + \cancel{a_2}$$
$$u_1 + u_2 = 2a_1$$

$$\therefore a_1 = \frac{u_1 + u_2}{2}$$

$$\& \quad a_2 = a_1 - u_1$$
$$= \frac{u_1 + u_2}{2} - u_1 = \frac{u_1 + u_2 - 2u_1}{2}$$

$$\therefore a_2 = \frac{-u_1 + u_2}{2}$$

∴ Substituting in Eq. (B)

$$u = \frac{u_1 + u_2}{2} + \frac{-u_1 + u_2}{2}\xi$$

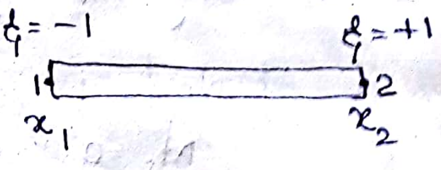
$$= \frac{u_1}{2} - \frac{u_1}{2}\xi + \frac{u_2}{2} + \frac{u_2}{2}\xi$$

$$\therefore u = u_1\left(\frac{1-\xi}{2}\right) + u_2\left(\frac{1+\xi}{2}\right)$$

Again, by comparing with $u = N_1 u_1 + N_2 u_2$

$$N_1 = \frac{1-\xi}{2} \quad N_2 = \frac{1+\xi}{2}$$

→ To find the value of ξ :-



Consider

$$\xi = a_1 + a_2 x$$

At Node 1

$\xi = -1$ & $x = x_1$

$$-1 = a_1 + a_2 x_1$$

At Node 2

$\xi = +1$ & $x = x_2$

$$1 = a_1 + a_2 x_2$$

Solving,

$$\cancel{a_1} + a_2 x_1 = -1$$
$$\cancel{a_1} + a_2 x_2 = +1$$
$$(-) \quad (-) \qquad (-)$$

$$a_2(x_1 - x_2) = -2$$

$$\therefore a_2 = \frac{2}{x_2 - x_1}$$

& $a_1 = -1 - a_2 x_1$

$$\therefore a_1 = -1 - \frac{2}{x_2 - x_1} \cdot x_1$$

$$\therefore \xi = -1 - \frac{2}{x_2 - x_1} x_1 + \frac{2}{x_2 - x_1} \cdot x$$

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

We see that $\xi = -1$ at Node 1 & $\xi = 1$ at Node 2. The length of the element covered when ξ changes from −1 to +1.

*** Properties of Shape function :**

(1) Kronecker delta property :

The shape function has the value of 1 at that node and zero at other nodes.

Check :- Let $N_1 = \frac{x_2 - x}{x_2 - x_1}$

At Node 1 ⇒ $x = x_1$

$$N_1 = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

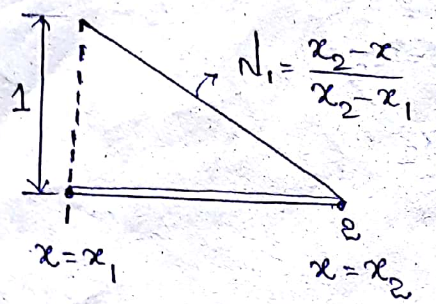
At Node 2 ⇒ $x = x_2$

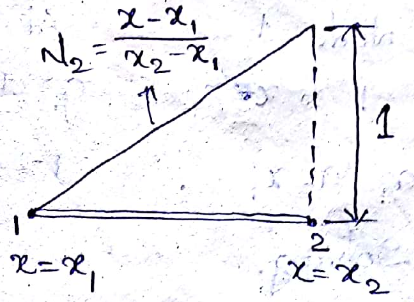
$$N_1 = \frac{x_2 - x_2}{x_2 - x_1} = 0.$$

Similarly $N_2 = \frac{x - x_1}{x_2 - x_1}$

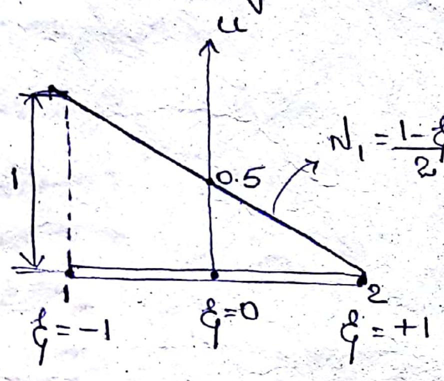
At Node 1 $\Rightarrow x = x_1$ $\quad$ At Node 2 $\Rightarrow x = x_2$

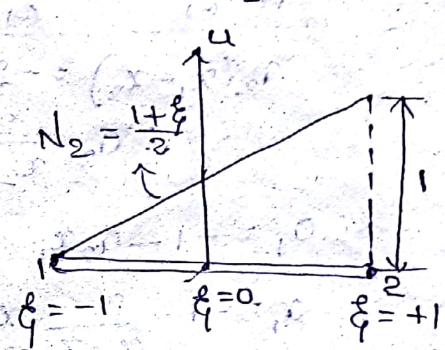
$N_2 = 0$ $\quad$ $N_2 = 1$





Similarly, when $N_1 = \frac{1-\xi}{2}$ $\quad$ $N_2 = \frac{1+\xi}{2}$





(2). Completeness :

1. Always the sum of the ~~(Nodal)~~ shape functions is equal to unity.

$$N_1 + N_2 = 1$$

Check :

$$\frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = \frac{x_2 - \cancel{x} + \cancel{x} - x_1}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1.$$

2. $N_1 u_1 + N_2 u_2 = u$

& $N_1 x_1 + N_2 x_2 = x$ for all 'x'

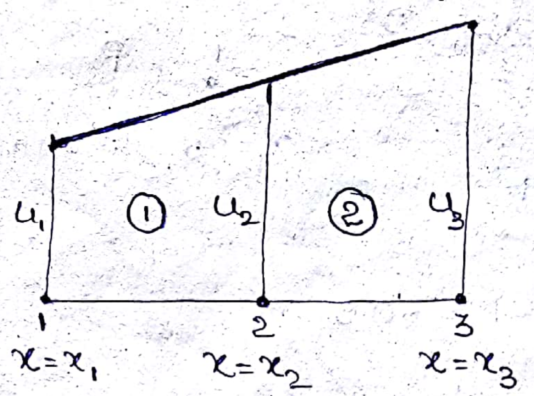
Check

$$\frac{x_2 - x}{x_2 - x_1} \cdot x_1 + \frac{x - x_1}{x_2 - x_1} \cdot x_2 = \frac{\cancel{x_2 x_1} - x x_1 + x x_2 - \cancel{x_1 x_2}}{x_2 - x_1}$$

$$N_1 x_1 + N_2 x_2 = \frac{x\cancel{(-x_1 + x_2)}}{\cancel{x_2 - x_1}} = x.$$

(3) Compatibility:

The shape function should be a continuous function across the element boundary.



$$u^1 = \frac{x_2 - x}{x_2 - x_1} \cdot u_1 + \frac{x - x_1}{x_2 - x_1} \cdot u_2 \quad \& \quad u^2 = \frac{x_3 - x}{x_3 - x_2} u_2 + \frac{x - x_2}{x_3 - x_2} \cdot u_3$$

Check:-

Let $x = x_2$

$$u^1 = \frac{x_2 - x_2}{x_2 - x_1} \cdot u_1 + \cancel{\frac{x_2 - x_1}{x_2 - x_1}} u_2 = u_2$$

$$u^2 = \cancel{\frac{x_3 - x_2}{x_3 - x_2}} u_2 + \frac{x_2 - x_2}{x_3 - x_2} u_3 = u_2$$

(4) If the Nodal displacement is a Linear variation then the element displacement must be a linear variation.

(5) The sum of the derivatives of shape function must be zero.

$$\frac{\partial N_1}{\partial \xi} + \frac{\partial N_2}{\partial \xi} = 0 \quad \text{or}$$

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial x} = 0$$

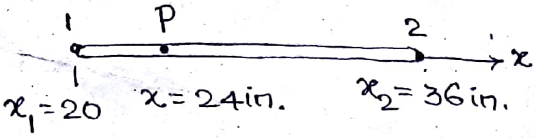
Check:-

$$\frac{\partial}{\partial \xi}\left(\frac{1-\xi}{2}\right) + \frac{\partial}{\partial \xi}\left(\frac{1+\xi}{2}\right) = -\frac{\partial \xi}{\partial \xi} + \frac{\partial \xi}{\partial \xi} = -1 + 1 = 0.$$

Problem :-

contd:

1. Consider the following fig: (a) Evaluate ξ, N_1, & N_2 at point p. & (b) If $q_1 = 0.003$ in & $q_2 = -0.005$in, determine the value of the displacement q at point 'P'



sol:- W.K.T

$$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

$$\xi = \frac{2}{16}(24 - 20) - 1 = -0.5$$

$$\& \; N_1 = \frac{1-\xi}{2} = 0.75 \quad \& \quad N_2 = \frac{1+\xi}{2} = 0.25.$$

(b) W.K.T

$$u = u_1 N_1 + u_2 N_2$$

$$= 0.75(0.003) + 0.25(-0.005)$$

$$= 0.001 \text{in.}$$

* Strain Displacement Matrix :

$$\epsilon = \frac{du}{dx}$$

$$\epsilon = \frac{du}{d\xi} \times \frac{d\xi}{dx}$$

W.K.T $$\xi = \frac{2(x - x_1)}{x_2 - x_1} - 1$$

$$\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}$$

We know the Isoparametric formulation $u = q_1 N_1 + q_2 N_2$

$$u = \frac{1-\xi}{2} q_1 + \frac{1+\xi}{2} q_2$$

$$\frac{du}{d\xi} = \frac{-q_1}{2} + \frac{q_2}{2} = \frac{-q_1 + q_2}{2}$$

$$\epsilon = \frac{du}{d\xi} \times \frac{d\xi}{dx} = \frac{-q_1 + q_2}{2} \times \frac{2}{(x_2 - x_1)}$$

$$= \frac{1}{(x_2 - x_1)}(-q_1 + q_2)$$

$$e = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}q_1 \\ q_2\end{Bmatrix}$$

$$\epsilon = Bq$$

$$\therefore B = \frac{1}{x_2 - x_1}\begin{bmatrix}-1 & 1\end{bmatrix}$$

where B = Strain-displacement Matrix.

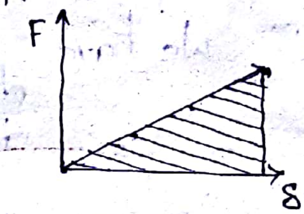
& stress $= E\epsilon$.

$$\sigma = EBq.$$

Potential Energy Approach :

We know that

π = Strain Energy (U) ± Work potential (W_p).



We have $U = \frac{1}{2} F \times \delta$

$= \frac{1}{2} \times \sigma \times A \times \epsilon l.$

$= \frac{1}{2}\sigma . \epsilon . V. \quad \left| \because \sigma = \frac{F}{A}, \epsilon = \frac{\delta}{l}, V = A \times l. \right.$

$U = \frac{1}{2} \times$ stress $\times$ strain $\times$ volume.

it can be written as

$$U = \frac{1}{2}\sigma^T \epsilon . V.$$

In Integration form.

$$U = \int_V \frac{1}{2}\sigma^T \epsilon dv.$$

Let work done = Force × Displacement.

W_p = Body force + Traction force + Point load force.

$$= -\int_V u^T f . dv - \int_S u^T T . ds - \sum_{i=1}^{n} P_i u_i$$

$$\therefore \pi = \int_V \frac{1}{2}\sigma^T \epsilon . dv - \int_V u^T f . dv - \int_S u^T T . ds - \sum_{i=1}^{n} P_i u_i^T$$

Since, the Domain or Continuum has been discretized into Finite Elements, the Expression of π becomes.

$$\Pi = \sum_e \frac{1}{2}\int_e \sigma^T \epsilon A.dx - \sum_e \int u^T f A.dx - \sum_e \int_e u^T T.dx - \sum_i Q_i^T P_i.$$

→ Element Stiffness Matrix :

Consider the Strain Energy

$$U_e = \frac{1}{2}\int_e \sigma^T \epsilon A.dx.$$

$$\sigma = EBq \quad , \epsilon = Bq$$

$$U_e = \frac{1}{2}\int_e q^T.B^T E Bq A.dx.$$

$$U_e = \frac{1}{2} q^T \int_e [B^T.EBA.dx]\, q.$$

We have

$$\frac{dx}{d\xi} = \frac{x_2 - x_1}{2} \quad \text{or} \quad \frac{d\xi}{dx} = \frac{2}{x_2 - x_1}$$

$$dx = \frac{x_2 - x_1}{2} \times d\xi.$$

$$dx = \frac{l_e}{2}.d\xi \quad \text{where } -1 \le \xi \le 1 \text{, \& } l_e = x_2 - x_1$$

$$\therefore U_e = \frac{1}{2} q^T B.AB^T E_e \int_e \left[\frac{x_2 - x_1}{2}\, d\xi\right] q.$$

$$= \frac{1}{2} q^T \left[A_e \frac{l_e}{2} E_e B^T.B \int_{-1}^{+1} d\xi\right] q$$

We have, $\int_{-1}^{+1} d\xi = \xi\Big]_{-1}^{+1} = 1 + 1 = 2$, & We know that

$$B = \frac{1}{l_e}[-1 \quad 1] \; ; \; \& \; B^T = \frac{1}{l_e}\begin{bmatrix}-1\\1\end{bmatrix}$$

$$= \frac{1}{\cancel{2}} q^T A_e \frac{\cancel{l_e}}{2} E_e \frac{1}{l_e^{\cancel{2}}}\begin{bmatrix}-1\\1\end{bmatrix}[-1 \;\; +1] \times \cancel{2} \times q.$$

$$= \frac{1}{2} q^T \frac{A_e E_e}{l_e}\begin{bmatrix}+1 & -1\\ -1 & +1\end{bmatrix} \times q.$$

$$U_e = \frac{1}{2} q^T K_e q$$

Where K_e = Element Stiffness Matrix

$$= \frac{A_e E_e}{l_e} \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ Force Terms :

Consider the body force terms. $\int_e u^T f . A . dx$

$$\int_e u^T f . A_e dx = \int_e f_e A_e (N_1 q_1 + N_2 q_2)^T . dx$$

$$= A_e f_e \int_e \left[(N_1 \;\; N_2) \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \right]^T . dx .$$

$$= A_e f_e q^T \begin{bmatrix} \int N_1 dx \\ \int N_2 dx \end{bmatrix}$$

Let $dx = \frac{l_e}{2} d\xi$, $-1 \le \xi \le +1$, We have $N_1 = \frac{1-\xi}{2}$ & $N_2 = \frac{1+\xi}{2}$

$$\int N_1 dx = \frac{l_e}{2} \int_{-1}^{+1} N_1 d\xi = \frac{l_e}{2} \int_{-1}^{+1} \left(\frac{1-\xi}{2}\right) d\xi = \frac{l_e}{2}$$

$$\int N_2 dx = \frac{l_e}{2} \int_{-1}^{+1} N_2 d\xi = \frac{l_e}{2} \int_{-1}^{+1} \left(\frac{1+\xi}{2}\right) d\xi = \frac{l_e}{2}$$

$$\therefore \int_e u^T f . A_e dx = q^T . A_e f \frac{l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= q^T . f^e$$

Where f^e = Element Body force vector

$$= \frac{A_e f l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

→ Traction force :- Here $ds = dx$. Perimeter.

Consider the element traction force term

$$\int u^T . T . dx = \int_e (N_1 q_1 + N_2 q_2)^T . T . dx .$$

$$= T q^T \begin{Bmatrix} \int N_1 dx \\ \int N_2 dx \end{Bmatrix}$$

Where, $\int_e N_1 dx = \int_e N_1 dx = l_e/2$.

$$\int_e u^T T\, dx = q^T T^e$$

T^e = Element Traction Force.

$$= \frac{T l_e}{2}\begin{Bmatrix}1\\1\end{Bmatrix}.$$

We finded the element Matrices k^e, f^e, T^e.

The total Potential energy can be written as.

$$\Pi = \frac{1}{2}Q^T K Q - Q^T F.$$

Where K = Global stiffness Matrix

Q = Global displacement Vector.

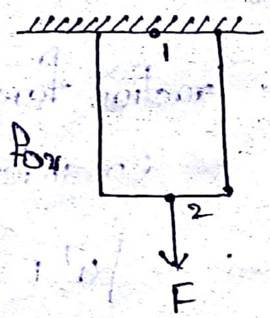
F = Global Load vector.

* Treatment of Boundary Conditions :

For any system under analysis, the information about the Nature of Loading in different locations, Nature of nodal displacements and other fields variables such as the values of temperatures at different Nodes etc are considered as boundary conditions. The boundary conditions are of two types.

1. Geometric or Essential boundary Conditions.
2. Natural or Non-Essential boundary Conditions.

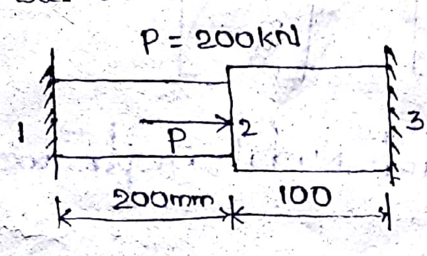
* In the figure, the fixed Node at 1 is the Essential boundary Condition.
* The load at the free end is an Example for Non-Essential boundary Condition.



Two Approaches are discussed for handling specified ⑨
displacement boundary conditions. They are
1. Elimination Approach.
2. Penalty Approach.

* Problems on Elimination Approach :

1. Determine the Nodal displacements, stresses & strains induced in the bar as shown.



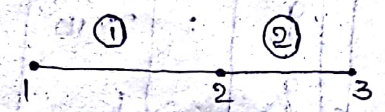
$A_1 = 1000mm^2$ $\quad A_2 = 2000mm^2$

$E_1 = 200\,GPa$ $\quad E_2 = 83\,GPas$

$= 200\times10^9 N/m^2$ $\quad = 0.83\times10^5 N/mm^2$

$= 2\times10^5 N/mm^2$

Sol :- (i) Discrilization of structure :



Element	Nodes	
①	1	2
②	2	3

(ii) & (iii) Assuming an Appropriate solution, & obtaining element stiffness Matrix, & load vector.

$$K^e = \frac{A_e E_e}{l_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$K^1 = \frac{A_1 E_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{1000\times2\times10^5}{200}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\therefore K^1 = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}10^6 & -10^6\\ -10^6 & 10^6\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$K^2 = \frac{2000\times0.83\times10^5}{100}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}1.6\times10^6 & -1.6\times10^6\\ -1.6\times10^6 & 1.6\times10^6\end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

And From the figure $F_2 = 200kN = 200\times10^3 N$. &

R_1 & R_3 are the Reaction forces.

(iv) Global stiffness Matrix & global force vectors:

$$K = K^1 + K^2 = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left[\begin{array}{ccc} 10^6 & -10^6 & 0 \\ -10^6 & 10^6+1.6\times10^6 & -1.6\times10^6 \\ 0 & -1.6\times10^6 & +1.6\times10^6 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$F = \begin{bmatrix} F_1 + T_1 + P_1 \\ F_2 + T_2 + P_2 \\ F_3 + T_3 + P_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 2\times10^5 \\ R_3 \end{bmatrix}$$

(v) Solving the Unknowns : (a) To find Nodal Displacements.

$$KQ = F$$

$$\begin{bmatrix} 10^6 & -10^6 & 0 \\ -10^6 & 10^6+1.6\times10^6 & -1.6\times10^6 \\ 0 & -1.6\times10^6 & 1.6\times10^6 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_1 \\ 2\times10^5 \\ R_3 \end{bmatrix} \longrightarrow (A)$$

Here, the Nodes at 1 and 3 are fixed, therefore the Nodal displacements at 1 and 3 are zero.

$\therefore q_1 = q_3 = 0.$

By using the principle of Elimination Approach; Eliminate 1row 1column, & 2row, 2nd column in the Matrix.

$$\begin{bmatrix} \cancel{10^6} & \cancel{-10^6} & \cancel{0} \\ \cancel{-10^6} & 10^6+1.6\times10^6 & \cancel{-1.6\times10^6} \\ \cancel{0} & \cancel{-1.6\times10^6} & \cancel{1.6\times10^6} \end{bmatrix} \begin{bmatrix} \cancel{q_1}^{\,0} \\ q_2 \\ \cancel{q_3}^{\,0} \end{bmatrix} = \begin{bmatrix} \cancel{R_1} \\ 2\times10^5 \\ \cancel{R_3} \end{bmatrix}$$

$$\therefore (10^6 + 1.6\times10^6) q_2 = 2\times10^5$$

$$(2.6\times10^6) q_2 = 2\times10^5$$

$$q_2 = 0.076$$

$$q_1 = 0$$

$$q_3 = 0.$$

(b) To find Reaction forces :

The Point load applied at Node 2 is resisted by the reaction forces developed at Nodes 1 and 3 (i.e at fixed ends) in order to Maintain the Equilibrium Condition of the system.

From, the global finite element Matrix (A)

We can write the Equations for the reaction forces at Nodes 1 and 3, such as

$$-10^6 q_2 = R_1 \Rightarrow -10^6(0.076) = F_1 \quad \Big| \because q_2 = 0.076.$$

$$\therefore R_1 = -76 \times 10^3 N. = -76 KN.$$

& $-1.6\times10^6 q_2 = R_3$

$$R_3 = -121.6\times10^3 N. = -121.6 KN.$$

The Negative sign to F_1 and F_3 is due to their directions of action which are opposite to the Applied force F_2.

(vi) Stresses & Strains :

ϵ_1 for Element 1 $= \dfrac{u_2 - u_1}{l_1} = \dfrac{0.076}{200} = 3.84\times10^{-4}$

ϵ_2 for Element 2 $= \dfrac{q_3 - q_2}{l_2} = \dfrac{0-0.076}{100} = -7.69\times10^{-4}$

σ_1 for Element 1 $= E_1\epsilon_1 = 2\times10^5 \times 3.84\times10^{-4}$

$= 76.8 \ N/mm^2$

σ_2 for Element 2 $= E_2\epsilon_2 = 0.83\times10^5 \times -7.69\times10^{-4}$

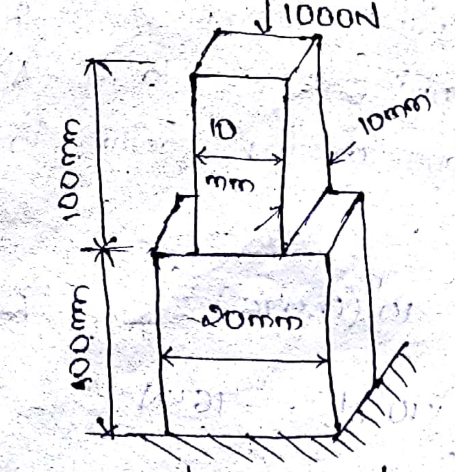
$= -63.8 \ N/mm^2$

(or)

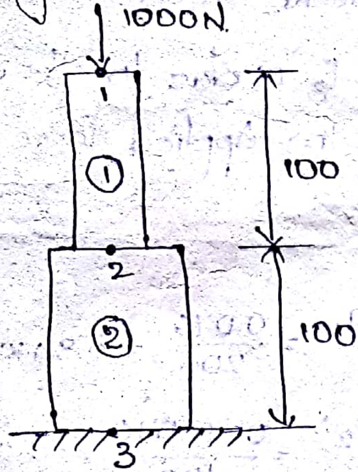
The strains & stresses can be also calculated by using the formula.

$$\epsilon = Bq, \ \sigma = EBq, \ \& \ B = \frac{1}{l_e}[-1 \quad 1]$$

2. For the column made of Mild steel as shown in fig. Find (a) Nodal displacement, (b) Reaction force at the supports (c) stresses & strains in Elements. Assume $E = 2\times10^5\ N/mm^2$



Sol :- The given column can be redrawn as spring element model consisting of 2 elements & 3 Nodes.



from figure

$A_1 = 10\times10 = 100\ mm^2$

$A_2 = 20\times10 = 200\ mm^2$

$E_1 = E_2 = 2\times10^5\ N/mm^2$

$l_1 = l_2 = 100\ mm.$

(a) Nodal displacements :

Consider the Element Stiffness Matrix

$$K^1 = \frac{A_1E}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{100\times2\times10^5}{100}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 2\times10^5 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$K^2 = \frac{A_2E}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{200\times2\times10^5}{100}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 2\times10^5 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3. \end{array}$$

Global stiffness Matrix $K = K^1 + K^2$

$$K = 2\times10^5 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

∴ The finite element eq $[K]Q = \{F\}$

$$2\times10^5 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ F_3 \end{bmatrix}$$

From the boundary conditions.

$q_3 = 0$, $F_1 = 1000$, $F_2 = 0$, so eliminate 3^{rd} row & Column.

$$2\times10^5 \begin{bmatrix} 1 & -1 & \cancel{0} \\ -1 & 3 & \cancel{-2} \\ \cancel{0} & \cancel{-2} & \cancel{2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \\ \cancel{F_3} \end{bmatrix}$$

$$2\times10^5 \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$$

$$2\times10^5 (q_1 - q_2) = 1000.$$

$$2\times10^5 (-q_1 + 3q_2) = 0.$$

$$\Rightarrow -q_1 + 3q_2 = 0.$$

By solving the Above Equations, We get

$$q_1 = 75\times10^{-4}\,mm \;\&\; q_2 = 25\times10^{-4}\,mm.$$

(b) Reaction forces :-

The Reaction force is at Node F_3.

$$2\times10^5 (-2q_2) = F_3$$

$$F_3 = -1000N \text{ (Acting Upwards)}.$$

(c) Stress & Strains :-

For element 1

$$\epsilon_1 = Bq = \frac{1}{l_1}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}q_1 \\ q_2\end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}75\times10^{-4} \\ 25\times10^{-4}\end{Bmatrix}$$

$$= -50\times10^{-6}$$

$$\sigma_1 = EBq = 2\times10^{5} \times -50\times10^{-6} = -10\ N/mm^2$$

For element 2.

$$\epsilon_2 = Bq = \frac{1}{l_2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}q_2 \\ q_3\end{Bmatrix}$$

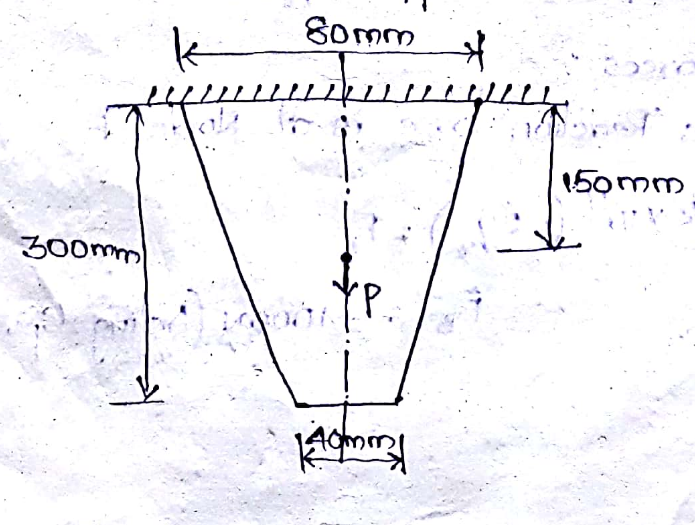
$$= \frac{1}{100}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}25\times10^{-4} \\ 0\end{Bmatrix}$$

$$= -25\times10^{-6}$$

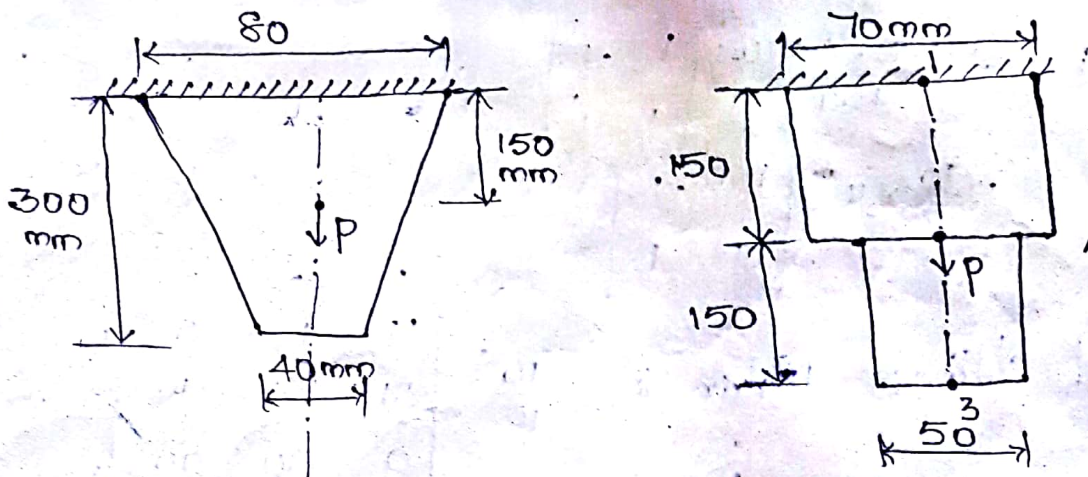
$$\sigma_2 = EBq = 2\times10^{5}\left(-25\times10^{-6}\right) = -5\ N/mm^2$$

The Negative sign indicates Compressive stresses.

3. For a tapered bar of uniform thickness $t = 10mm$ as shown Find the displacements at the nodes by deforming into two elements. The bar has a Mass density $\rho = 7800\ kg/m^3$ Young's Modulus $E = 2\times10^5\ MN/m^2$. In addition to self weight, the bar is subjected to a point load $P = 1KN$ at its Centre. Also determine the reaction force at the Support.



Sol.



$P = 1000\,N$

$\rho = 7800\,kg/m^3$

$E = 2\times10^5\,MN/m^2$

$t = 10\,mm$

Now for the tapered bar

The area of Node 1 = Width x thickness.

$= 80\times10 = 800\,mm^2$

The area at Node 3 = Width & Thickness

$= 40\times10 = 400\,mm^2$

The Area at Node 2 $= \dfrac{\text{Area at Node 1} + \text{Area at Node 3}}{2}$

$= \dfrac{800+400}{2}$

$= 600\,mm^2$

For the stepped bar

$A_1 = \dfrac{(\text{Area at Node 1} + \text{Area at Node 2})}{2}$

$= \dfrac{800+600}{2} = 700\,mm^2$ = 70mm width x 10mm thickness

$A_2 = \dfrac{\text{Area at Node 2} + \text{Area at Node 3}}{2}$

$= \dfrac{600+400}{2} = 500\,mm^2$ = 50mm Width x 10mm thickness

(a) To find Nodal Displacements :-

For element 1

$$K^1 = \frac{A_1E_1}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{700\times2\times10^5}{150}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= \frac{14}{15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$k^2 = \frac{500 \times 2 \times 10^5}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{10}{15} \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Global stiffness Matrix

$$k = \frac{10^6}{15}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix} \end{array}\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$

To find the force vectors, Since the bar is having a self weight. it is assumed that the body force due to self weight of the element is equally distributed in its two nodes.

We know that, for Element 1

$$f_1 = \frac{A_1 l_1 f_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \cancel{700 \times 2 \times 10^5 \times 150 \times 7800 \times 10^{-9} \times 9.81}$$

$g = 9.81\ m/sec^2$

$= 9.81 \times 10^3\ mm/sec^2$

$$= \frac{700 \times 150 \times 7800 \times 10^{-9} \times 9.81 \times 10^3}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\begin{matrix} 1 \\ 2 \end{matrix}$$

$\because f = \rho g v$

f is at unit volume

$f = \rho g$

$$f_1 = 4.017\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4.017 \\ 4.017 \end{Bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} \times 10^3$$

For Element 2.

$$f_2 = \frac{A_2 l_2 f_2}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= \frac{500 \times 150 \times 7800 \times 10^{-9} \times 9.81 \times 10^3}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}\begin{matrix} 2 \\ 3 \end{matrix}$$

$$f_2 = 2.869\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2.869 \\ 2.869 \end{Bmatrix} \times 10^3$$

$\therefore$ Global force vector

$$f\ \text{or}\ F = \begin{Bmatrix} f_1 + T_1 + P_1 \\ f_2 + T_2 + P_2 \\ f_3 + T_3 + P_3 \end{Bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} = \begin{Bmatrix} 4.017 \times 10^3 \\ (4.017 + 2.869 + \cancel{1}) 1000. \\ 2.869 \times 10^3 \end{Bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$= \begin{Bmatrix} 4.017 \\ 7886.\cancel{88} \\ 2.869 \end{Bmatrix}$$

$\therefore \{K\} Q = \{F\}$.

$$\frac{10^6}{15}\begin{bmatrix} 14 & -14 & 0 \\ -14 & 24 & -10 \\ 0 & -10 & 10 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

Here, At Node 1 $q_1 = 0$ & F_1 is the reaction force and also having a body force

$\therefore F_1 = f_1 + R_1$

Since $q_1 = 0$, Eliminate 1st row & 1st column.

$$\frac{10^6}{15}\begin{bmatrix} \cancel{14} & \cancel{-14} & \cancel{0} \\ \cancel{-14} & 24 & -10 \\ \cancel{0} & -10 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 4.017 + R_1 \\ 7886.80 \\ 2.869 \end{bmatrix}$$

$$\frac{10^6}{15}\begin{bmatrix} 24 & -10 \\ -10 & 10 \end{bmatrix}\begin{bmatrix} q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 7886.80 \\ 2.869 \end{bmatrix}$$

$$\frac{10^6}{15}\left[24q_2 - 10q_3\right] = 7886.80$$

$$\frac{10^6}{15}\left[-10q_2 + 10q_3\right] = 2.869$$

By solving, We get

$q_2 =$, $q_3 =$

To find the Reaction forces

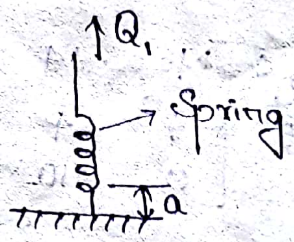
$$\frac{10^6}{15}\left[-14q_2\right] = 4.017 \times 10^3 + R_1$$

$$R_1 = \frac{10^6}{15}\left[-14q_2\right] - 4.017 \times 10^3$$

$\therefore R_1 =$

Penalty Approach :

$$* \begin{bmatrix} K_{11}+C & K_{12} & K_{13} & \cdots & K_{1N} \\ K_{21} & K_{22} & K_{23} & \cdots & K_{2N} \\ \vdots & & & & \\ K_{N1} & K_{N2} & \cdots & \cdots & K_{NN} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_3 \end{bmatrix} = \begin{bmatrix} F_1 + Ca_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix}$$

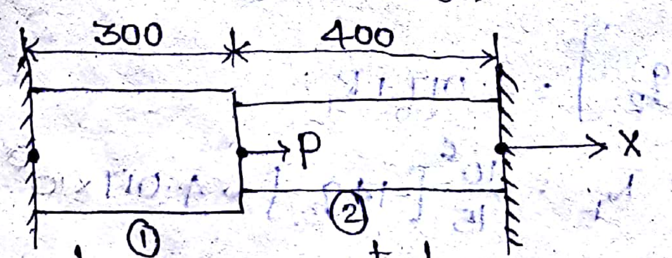


Where $C = \text{Max}|K_{ij}| \times 10^4$
$1 \le i \le N$
$1 \le j \le N$

Here, the value of C is added at the fixed Node places.

Problem :-

* Consider the bar shown as Fig. An axial load $P = 200 \times 10^3 N$ is applied as shown. Using the penalty Approach for handling boundary conditions, do the following.
  (a) Determine the Nodal displacements.
  (b) Determine the stress in each Material.
  (c) Determine the ~~stress~~ Reation forces.



$A_1 = 2400 mm^2$ $\qquad$ $A_2 = 600 mm^2$
$E_1 = 70 \times 10^9 N/m^2$ $\qquad$ $E_2 = 200 \times 10^9 N/m^2$

Sol:- (a) Element stiffness Matrix

$$k^1 = \frac{A_1 E_1}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{2400 \times 70 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.56 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^6 \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 0.56 & -0.56 \\ -0.56 & 0.56 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$k^2 = \frac{600 \times 200 \times 10^3}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.3 \times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= 10^6 \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 0.3 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

Global stiffness Matrix 'k' $= k^1 + k^2$

$$k = 10^6 \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 0.3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

Global load vector

$$F = [0, 200 \times 10^3, 0]^T$$

Here, as 1 and 3 are fixed, when using the penalty approach therefore, a large no. c is added to the first and third diagonal element of 'k' and the value "Ca" is added to the 1 & 3 values of 'F'

$$\therefore q_1 = 0 \Rightarrow a_1 = 0, \quad q_2 = 0 \Rightarrow a_2 = 0.$$

$$\& \quad C = \text{Max}|k_{ij}| \times 10^4 = [0.86 \times 10^6] \times 10^4.$$

$$C = 8600 \times 10^6.$$

$$k = 10^6 \begin{array}{cc} \begin{array}{ccc} 1 & 2 & 3 \end{array} & \\ \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 8600.3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \end{array}$$

& Modified Load Vector $F = \begin{bmatrix} F_1 + Cq_1 \\ F_2 \\ F_3 + Cq_3 \end{bmatrix}$

$$= \begin{bmatrix} 0 \\ 200\times10^3 \\ 0 \end{bmatrix}$$

$$\Rightarrow 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.3 \\ 0 & -0.3 & 8600.3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 200\times10^3 \\ 0 \end{bmatrix}$$

By solving the above Matrix, We get

$q_1 = 15.14\times10^{-6}$ $q_2 = 0.232$ & $q_3 = 8.11\times10^{-6}$.

(b) The element Stresses

$$\sigma_1 = 70\times10^3 \times \frac{1}{300}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}15.14\times10^{-6} \\ 0.232\end{Bmatrix}$$

$= 54.27$ MPas | 1 MPas $= 1\times10^6 N/m^2 = 1N/mm^2$

$$\sigma_2 = 200\times10^3 \times \frac{1}{400}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0.232 \\ 8.11\times10^{-6}\end{Bmatrix}$$

$= -116.29\ N/mm^2$ or MPa

(c) Reaction forces

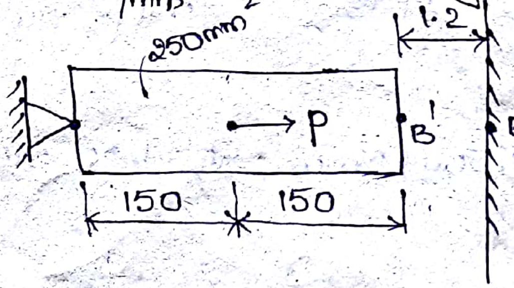
$$R_1 = -CQ_1 = -\left[0.86\times10^6\times10^4\right]\times 15.1432\times10^{-6}$$

$$= -130.23\times10^3 N.$$

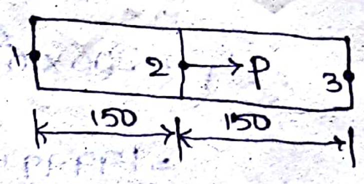
$$R_2 = -CQ_3 = -\left[0.86\times10^{10}\right]\times 8.11\times10^{-6}$$

$$= -69.77\times10^3 N.$$

2. In fig, a load $P = 60 \times 10^3 N$ is applied as shown. Determine the displacement field, stress & Support Reactions in the body. Take $E = 20 \times 10^3 N/mm^2$. Solve using Penalty Approach.



Sol: (a) Element Stiffness Matrix



$$K^1 = \frac{250 \times 20 \times 10^3}{150}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{10^5}{3}\begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \end{array}$$

$$K^2 = \frac{250 \times 20 \times 10^3}{150}\begin{bmatrix} -1 & -1 \\ +1 & 1 \end{bmatrix}$$

$$= \frac{10^5}{3}\begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 3 \end{array}$$

∴ Global Stiffness Matrix

$$K = \frac{10^5}{3}\begin{bmatrix} -1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

& Global load vector

$$F = [0,\ 60 \times 10^3,\ 0]$$

Again, Using the penalty Approach, the value of c is to be added to the 1^{st} & 3^{rd} diagonal Elements.

$$C = \frac{2}{3} \times 10^5 \times 10^4 = 20000 \times \frac{10^5}{3}$$

& $a_1 = 0$ & $a_3 = 1.2$. & $Q_1 = 0$.

And the value $ca_3 = C \times 1.2$ is added to the 3^{rd} component F.

$$\frac{10^5}{3}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60\times 10^3 \\ 0+\left(\frac{2}{3}\times 10^9\times 1.2\right) \end{Bmatrix}$$

By solving

$Q_1 = 7.49\times 10^{-5}$, $Q_2 = 1.50045\,mm$, $Q_3 = 1.200015\,mm$.

(b) Stresses

$$\sigma_1 = 200\times 10^3 \times \frac{1}{150}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 7.499\times 10^{-5} \\ 1.50045 \end{Bmatrix}$$

$$= 199.99\ N/mm^2$$

$$\sigma_2 = 200\times 10^3 \times \frac{1}{150}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 1.50045 \\ 1.200015 \end{Bmatrix}$$

$$= -40.004\ N/mm^2.$$

(c) Reation forces.

$$R_1 = -C\times 7.499\times 10^{-5} = -\frac{2}{3}\times 10^9 \times 7.499\times 10^{-5}$$

$$= -49.99\times 10^3 N.$$

$$R_2 = -C\times (Q_3 - a_3) = -\frac{2}{3}\times 10^9 (1.200015 - 1.2)$$

$$= -9999.99\,N.$$

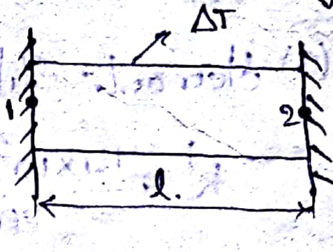
* Temperature Effects :

When a Machine member is loaded at room temperature its dimensions change and the corresponding stresses will be induced in the Machine member. Sometimes, without the application of external loading the member will expand or contract due to change of temperature. If the member is allowed to expand or contract freely, even for the change of temperature & corresponding change in dimensions, no stresses will be induced in the member.

If the ends of the element are restricted, during the rise or fall of temperature, a stress called 'Thermal stress' is induced in the member which may be a Tensile or Compressive in Nature.

The strain due to change of temperature is known as Thermal Strain.

Let consider a bar whose ends are fixed & named as nodes 1 & 2 as shown in figure, undergo a change of temperature ΔT.



ΔT = Rise in Temp. (°C)

α = Coefficient of Thermal expansion (mm/mm °C).

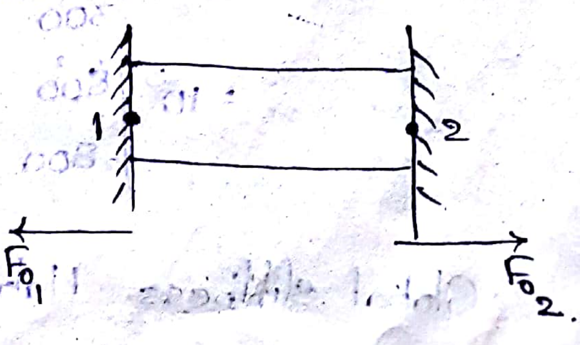
then,

Thermal strain, $\epsilon_0 = \alpha \Delta T$.

Thermal stress, $\sigma_0 = E\alpha\Delta T$.

Thermal Force, $F_0 = AE\alpha\Delta T$.

$$\therefore [F_0] = \begin{Bmatrix} F_{01} \\ F_{02} \end{Bmatrix} = EA\alpha\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$



If the ~~Meber~~ Member, is subjected to both external loading and change in temperature, the resultant stress induced in the bar

$$\sigma_x = \sigma - \sigma_0 = E\epsilon - E\epsilon_0$$

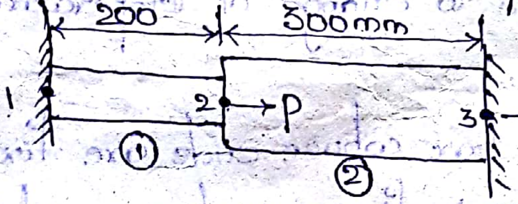
$$= EBq - E\alpha\Delta T$$

$$\sigma_x = \frac{E}{x_2 - x_1}[-1 \;\; 1]\,q - E\alpha\Delta T$$

$$F = \sum_e (f^e + T^e + \Theta^e) + P$$

Problems :-

1. An Axial load $P = 300\times10^3 N$ is applied at $20°c$ to the rod as shown in fig. The temperature is then raised to $60°c$
   (a) Assemble k & F Matrices.
   (b) Determine the Nodal Disp. & element stresses.



Aluminium ①	Steel ②
$E_1 = 70\times10^9 N/m^2$	$E_2 = 200\times10^9 N/m^2$
$A_1 = 900 mm^2$	$A_2 = 1200 mm^2$
$\alpha_1 = 23\times10^{-6}/°C$	$\alpha_2 = 11.7\times10^{-6}/°C$

Sol:-

(a) element stiffness Matrices.

$$k^1 = \frac{70\times10^3\times900}{200}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 10^3 \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix}315 & -315\\ -315 & 315\end{bmatrix} \end{array}\begin{array}{c}1\\2\end{array}$$

$$k^2 = \frac{200\times10^3\times1200}{300}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$= 10^3 \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix}800 & -800\\ -800 & 800\end{bmatrix} \end{array}\begin{array}{c}2\\3\end{array}$$

$\therefore$ Global stiffness Matrix $K = 10^3 \begin{bmatrix}315 & -315 & 0\\ -315 & 1115 & -800\\ 0 & -800 & 800\end{bmatrix} N/mm.$

(17)

Now, to find the Global load vector 'F', both temperature and point load effects have been considered.

$$\theta^1 = EA\alpha\Delta T\begin{bmatrix}-1\\1\end{bmatrix} \quad \& \quad \Delta T = 60-20 = 40°C$$

$$= 70\times10^3 \times 900 \times 23\times10^{-6}\times 40 \begin{Bmatrix}-1\\1\end{Bmatrix}\begin{matrix}1\\2\end{matrix} \ N.$$

$$\theta^1 = 10^3\begin{Bmatrix}-57.96\\57.96\end{Bmatrix}\begin{matrix}1\\2\end{matrix}$$

$$\theta^2 = 200\times10^3 \times 1200\times 11.7\times10^{-6}\times 40 \begin{Bmatrix}-1\\1\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

$$= 10^3\begin{Bmatrix}-112.32\\112.32\end{Bmatrix}\begin{matrix}2\\3\end{matrix}$$

$$\therefore F = 10^3\begin{Bmatrix}-57.96\\+57.96-112.32+300\\112.32\end{Bmatrix}\begin{matrix}1\\2\\3\end{matrix}$$

$$F = 10^3\begin{Bmatrix}-57.96\\245.64\\112.32\end{Bmatrix} N$$

(b)

Consider $KQ = F$.

$$10^3\begin{bmatrix}315 & -315 & 0\\-315 & 1115 & -800\\0 & -800 & 800\end{bmatrix}\begin{bmatrix}q_1\\q_2\\q_3\end{bmatrix} = \begin{Bmatrix}F_1\\F_2\\F_3\end{Bmatrix} = \begin{Bmatrix}-57.96\times10^3\\245.64\times10^3\\112.32\times10^3\end{Bmatrix}$$

By Using the Elimination Approach, $q_1=0$ & $q_3=0$, so Eliminate 1st & 3rd rows & Columns.

$$\cancel{10^3}[1115]q_2 = \cancel{10^3}\times 245.64$$

$$q_2 = 0.22mm$$

$$\therefore Q = [0, 0.22, 0]^T mm.$$

<u>Stresses</u>: $\sigma_1 = \frac{E}{l_e}[-1 \ \ 1]\begin{Bmatrix}q_1\\q_2\end{Bmatrix} - E\alpha\Delta T$

$$= \frac{70\times10^3}{200}[-1 \ \ 1]\begin{Bmatrix}0\\0.22\end{Bmatrix} - 70\times10^3\times 23\times10^{-6}\times 40.$$

$= 12.60\,MPa.$

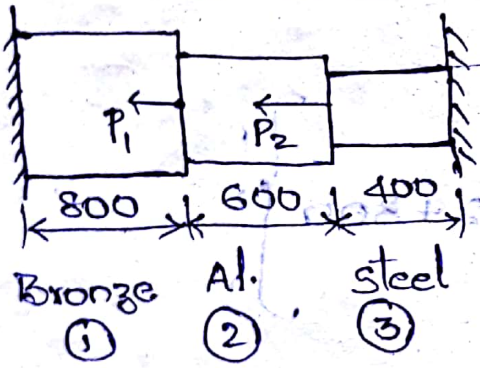
8)

$$\sigma_2 = \frac{200\times10^3}{300}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0.22\\0\end{Bmatrix} - 200\times10^3\times11.7\times10^{-6}\times40.$$

$$= -240.27\,MPa.$$

2. Assignment :

The structure is subjected to an increase in temp, $\Delta T = 80$. Determine the displacements, stresses, & support reactions. Solve this problem using elimination method for handling B.C.



$P_1 = 60\,kN$

$P_2 = 75\,kN$

$\Delta T = 80^\circ C$

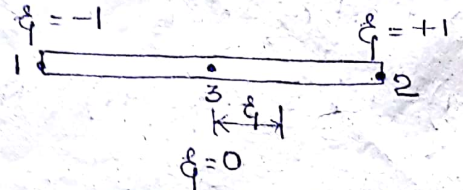
$(1\,GPa = 10^9\,N/m^2)$.

	Bronze	Al.	steel
A	$2400\,mm^2$	$1200\,mm^2$	$600\,mm^2$
E	83 GPa	70 GPa	200 GPa
α	$18.9\times10^{-6}/^\circ C$	$23\times10^{-6}/^\circ C$	$11.7\times10^{-6}/^\circ C$

(18)

Quadratic shape functions :

Consider typical three nodes element as shown.



Consider the polynomial equation

$$u = a_1 + a_2 \xi + a_3 \xi^2$$

At Node 1: $u = u_1$ & $\xi = -1$

$$u_1 = a_1 - a_2 + a_3 \longrightarrow \text{(i)}$$

At Node 2: $u = u_2$ & $\xi = +1$

$$u_2 = a_1 + a_2 + a_3 \longrightarrow \text{(ii)}$$

At Node 3: $u = u_3$ & $\xi = 0$

$$u_3 = a_1 \longrightarrow \text{(iii)}$$

Solving (i) & (ii)

$$u_1 = \cancel{a_1} - a_2 + \cancel{a_3}$$
$$\underset{(-)}{u_2} = \underset{(-)}{\cancel{a_1}} + \underset{(-)}{a_2} + \underset{(-)}{\cancel{a_3}}$$

$$u_1 - u_2 = -2a_2$$

$$a_2 = \frac{u_2 - u_1}{2}$$

$$\& \; a_1 = u_3$$

$$\therefore a_3 = u_1 - a_1 + a_2$$

$$= u_1 - u_3 + \frac{u_2 - u_1}{2}$$

$$= \frac{2u_1 - 2u_3 + u_2 - u_1}{2}$$

$$a_3 = \frac{u_1 + u_2 - 2u_3}{2}$$

$$u = u_3 + \frac{u_2 - u_1}{2}\xi + \frac{u_1 + u_2 - 2u_3}{2}\xi^2$$

$$= -\frac{u_1}{2}\xi + \frac{u_1}{2}\xi^2 + \frac{u_2}{2}\xi + \frac{u_2}{2}\xi^2 + u_3 - u_3\xi^2$$

$$= \frac{u_1}{2}\left(\xi^2 - \xi\right) + \frac{u_2}{2}\left(\xi^2 + \xi\right) + u_3\left(1 - \xi^2\right)$$

$$u = \frac{1}{2}u_1 \xi(\xi-1) + \frac{1}{2}\xi(\xi+1)u_2 + (1-\xi^2)u_3.$$

Comparing with $u = N_1u_1 + N_2u_2 + N_3u_3$

$$N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = \frac{1}{2}\xi(\xi+1), \quad N_3 = (1-\xi^2)$$
$$= (1+\xi)(1-\xi)$$

Another procedure :-

For example since $N_1 = 0$ at $\xi = 0$ and $N_1 = 0$ at $\xi = 1$ and $N_1 = 1$ at $\xi = -1$,

$$N_1 = C_1 \xi(\xi-1)$$
$$1 = C_1(-1)(-2)$$
$$C_1 = 1/2$$
$$\therefore N_1 = \frac{1}{2}\xi(\xi-1).$$

$$\& \quad N_2 = C_2(1+\xi)(\xi)$$
$$N_2 = 1 \;\&\; \xi = 0.+1$$
$$1 = C_2(2)(1)$$
$$C_2 = 1/2$$
$$\Rightarrow N_2 = \frac{1}{2}\xi(1+\xi).$$

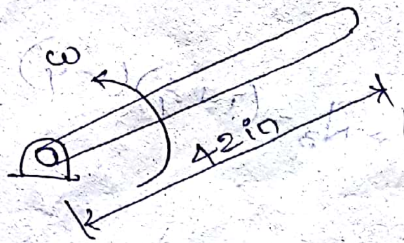
Similarly, $N_3 = C_3(\xi+1)(\xi-1)$

$$N_3 = 1 \text{ and } \xi = 0.$$
$$1 = C_3(1)(-1)$$
$$C_3 = -1$$
$$N_3 = -1(\xi+1)(\xi-1)$$
$$= -1(\xi^2-1)$$
$$\therefore N_3 = (1-\xi^2).$$

$$\therefore N_1 = \frac{1}{2}\xi(\xi-1), \quad N_2 = \frac{1}{2}\xi(1+\xi), \quad N_3 = (1-\xi^2)$$

Problem:-

1. Consider the Rod as shown in fig, which is rotating at constant angular velocity $\omega = 30\,rad/sec$. Determine the axial stress distribution in the rod, using 2 quadratic elements. Consider only the Centrifugal force. Ignore bending of the Rod.

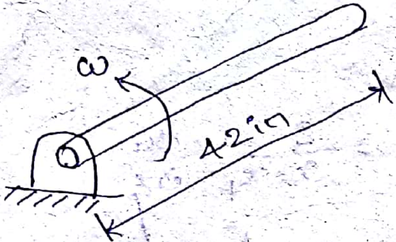


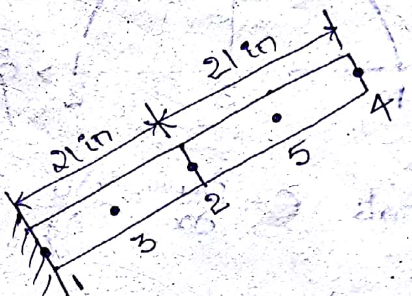
$A = 0.6\,in^2$

$E = 10^7\,Psi$

Weight Density, $\rho = 0.2836\,lb/in^3$

Sol:-





Element	Nodes		
1	1	2	3
2	2	4	5

We know, that for quadratic elements

$$K^1 = \frac{E_e A_e}{3 l_e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} = \frac{10^7 \times 0.6}{3 \times 21}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

$$K^2 = \frac{10^7 \times 0.6}{3 \times 21}\begin{array}{c} \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 4 \\ 5 \end{array}$$

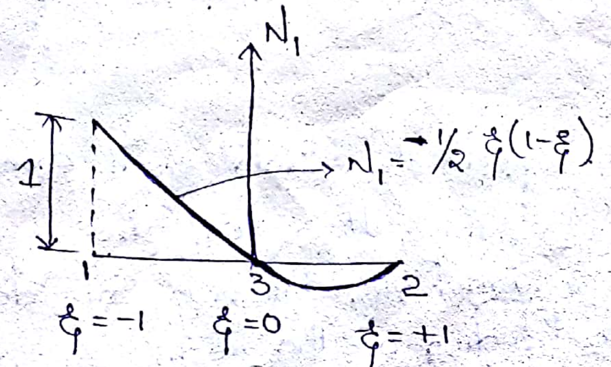
$$K = \frac{10^7 \times 0.6}{3 \times 21}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \end{array} + \frac{10^7 \times 0.6}{3 \times 21}\begin{array}{c} \begin{array}{ccc} 2 & 4 & 5 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 \\ 7 & 1 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 2 \\ 4 \\ 5 \end{array}$$

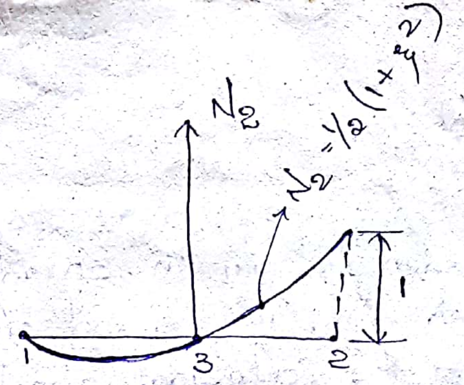
$$K = \frac{10^7 \times 0.6}{3 \times 21}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 7 & 1 & -8 & 0 & 0 \\ 1 & 14 & -8 & 1 & -8 \\ -8 & -8 & 16 & 0 & 0 \\ 0 & 1 & 0 & 7 & -8 \\ 0 & -8 & 0 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$

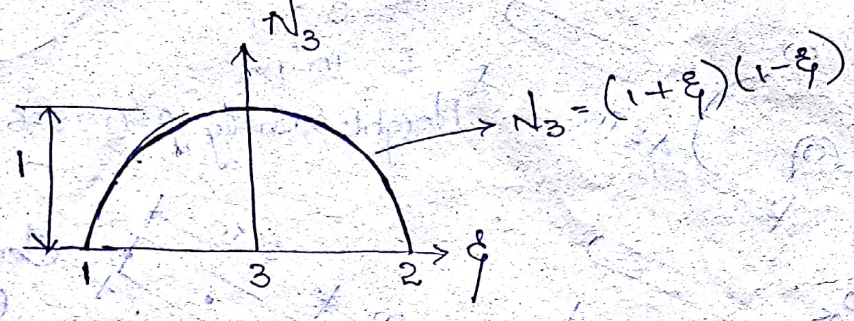
We know that, the centrifugal force

$$f = \frac{\rho r \omega^2}{g}\,lb/in^3$$

$\rho = 0.2836\,lb/in^3$, $g = 32.2\,ft/s^2$







We know that $\epsilon = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} = \frac{2}{x_2-x_1}\cdot\frac{\partial u}{\partial \xi}$

$$= \frac{2}{x_2-x_1}\left[\frac{\partial}{\partial \xi}\left[N_1q_1+N_2q_2+N_3q_3\right]\right]$$

$$\epsilon = \frac{2}{(x_2-x_1)}\left[\frac{\partial N_1}{\partial \xi} \quad \frac{\partial N_2}{\partial \xi} \quad \frac{\partial N_3}{\partial \xi}\right]\begin{bmatrix}q_1\\ q_2\\ q_3\end{bmatrix}$$

Here $N_1 = \frac{1}{2}(\xi^2-1)$ or $-\frac{1}{2}(1-\xi^2)$

$N_2 = \frac{1}{2}(1+\xi^2)$

$N_3 = (1+\xi)(1-\xi) = (1-\xi+\xi-\xi^2)$

$$\epsilon = \frac{2}{(x_2-x_1)}\left[\frac{-(1-2\xi)}{2} \quad \frac{(1+2\xi)}{2}, \quad -2\xi\right]q$$

$$\therefore B = \frac{2}{x_2-x_1}\left[\frac{-(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]$$

Consider the Potential Energy ξ :-

$$\Pi = \Sigma \frac{1}{2}\int \sigma^T \epsilon A\,dx - \sum_e \int_e u^T f A\,dx - \Sigma\int_e u^T T\,dx - \Sigma Q_i P_i$$

By solving, we get

$$K_e = \frac{E_eA_e}{3l_e}\begin{bmatrix}7 & 1 & -8\\ 1 & 7 & -8\\ -8 & -8 & 16\end{bmatrix}, \quad f^e = \frac{A_el_ef}{2}\int_{-1}^{+1} N^T d\xi = A_el_ef\begin{Bmatrix}1/6\\ 1/6\\ 2/3\end{Bmatrix}$$

$$\& \ T^e = l_eT\begin{Bmatrix}1/6\\ 1/6\\ 2/3\end{Bmatrix}.$$

Element ①

$$f_1 = \frac{0.2836 \times 10.5 \times 30^2}{32.2 \times 12} = 6.94 \quad \Big| \because r_1 = \frac{21}{2} = 10.5$$

Element ②

$$f_2 = \frac{0.2836 \times 31.5 \times 30^2}{32.2 \times 12} = 20.81 \qquad r_2 = 21 + \frac{21}{2} = 21 + 10.5 = 31.5$$

we know that the element body force

$$f^1 = ALf_1 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} = 0.6 \times 21 \times 6.94 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} = \begin{Bmatrix} 14.57 \\ 14.57 \\ 58.29 \end{Bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$f^2 = 0.6 \times 21 \times 20.81 \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix} \begin{matrix} 2 \\ 5 \\ 4 \end{matrix} = \begin{Bmatrix} 43.7 \\ 43.7 \\ 174.80 \end{Bmatrix} \begin{matrix} 2 \\ 5 \\ 4 \end{matrix}$$

$$\therefore \text{Total force}, \ F = \begin{Bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{Bmatrix}$$

Consider the finite element equation

$$KQ = F$$

$$\frac{10^7 \times 0.6}{3 \times 21} \begin{bmatrix} 7 & 1 & -8 & 0 & 0 \\ 1 & 14 & -8 & 1 & -8 \\ -8 & -8 & 16 & 0 & 0 \\ 0 & 1 & 0 & 1 & -8 \\ 0 & -8 & 0 & -8 & 16 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{bmatrix}$$

Since Node 1 is fixed, by using the elimination approach, eliminate 1^{st} row & 1^{st} column then we get

$$\frac{10^7 \times 0.6}{3 \times 21} \begin{bmatrix} 14 & -8 & 1 & -8 \\ -8 & 16 & 0 & 0 \\ 1 & 0 & 1 & -8 \\ -8 & 0 & -8 & 16 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} = \begin{bmatrix} 14.57 \\ 58.27 \\ 58.29 \\ 174.80 \\ 43.7 \end{bmatrix}$$

By solving we get

q_2, q_3, q_4 & q_5 Values.

and we have stress

$$\sigma = EBq$$

where $B = \frac{2}{l_e}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right] q$

Element ①

$$\sigma_1 = 10^7 \times \frac{2}{21}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Not stress at Node 1, $\xi = -1$

$$\sigma_{1/1} = 10^7 \times \frac{2}{21}\left[-1.5 \quad 0.5 \quad 2\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Stress at Node 2, $\xi = +1$

$$\sigma_{1/2} = 10^7 \times \frac{2}{21}\left[0 \quad 3/2 \quad -2\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

& ~~By~~ Stress at Node 3, $\xi = 0$.

$$\sigma_{1/3} = 10^7 \times \frac{2}{21}\left[-0.5 \quad 0.5 \quad 0\right]\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

By substituting the values of q_1, q_2, q_3 we get the stress at Node 1, 2, & 3.

Element ②

$$\sigma_2 = 10^7 \times \frac{2}{21}\left[-\frac{(1-2\xi)}{2} \quad \frac{1+2\xi}{2} \quad -2\xi\right]\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 2, $\xi = -1$

$$\sigma_{2/2} = 10^7 \times \frac{2}{21}\left[-1.5 \quad 0.5 \quad 2\right]\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 4, $\xi = 1$

$$\sigma_{2/4} = 10^7 \times \frac{2}{21}\left[0 \quad 3/2 \quad -2\right]\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

At Node 5, $\xi = 0$

$$\sigma_{2/5} = 10^7 \times \frac{2}{21}\left[-0.5 \quad 0.5 \quad 0\right]\begin{bmatrix} q_2 \\ q_4 \\ q_5 \end{bmatrix}$$

22-12-2020

UNIT-III

1. Analysis of Trusses

Truss: Is a structural member Constructed by no. of bars and 'L' angles and are Connected eachother firmly at their ends by means of bolts or rivets. The truss elements such bars or angles transmit force only and they Cannot move individually or relative to each other. The truss is mainly used to sustain the load of building roof and railway bridges etc. A two-dimensional truss is shown in fig.

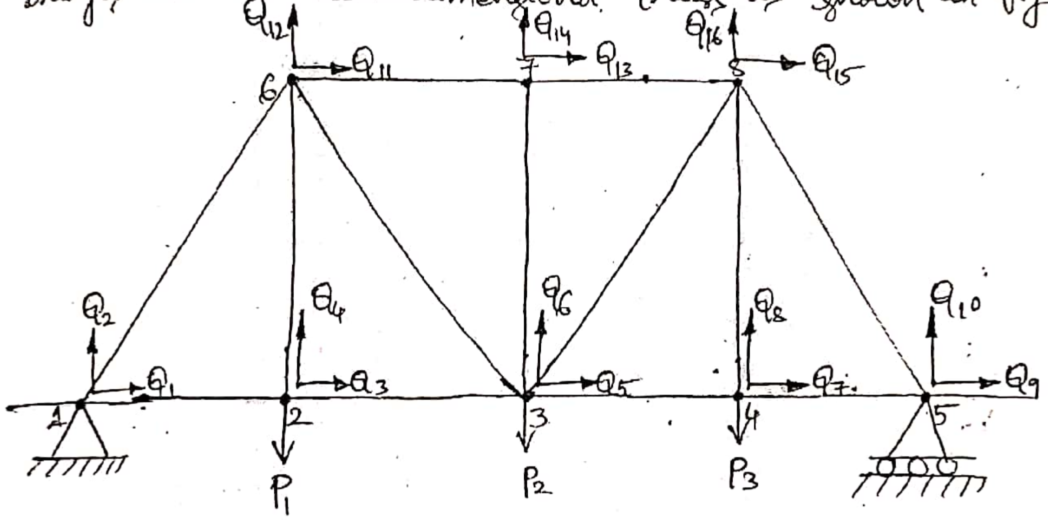


Fig. Two-dimensional truss

A Truss structure Consists only of two force members i.e, every truss element is in direct tension or Compression (axial forces) and Can deform only in the axial direction. It will not be able to Carry transverse loads (lateral loads) or bending moments. In a truss it is required that all loads and reactions are applied only at the joints and that all members are Connected together at their ends by frictionless pin joints.

The following assumptions are made while finding the forces in a truss.

1. The truss is loaded only at the joints.
2. All the members are pin jointed.
3. Self weight of the members are neglected unless stated.

In a 2 dimensional truss analysis, each of the two nodes can have a components of displacement parallel to X and Y axis.
In a 3-Dimensional truss analysis each node can have displacement component in X, Y and Z axis. A two force member is represented as

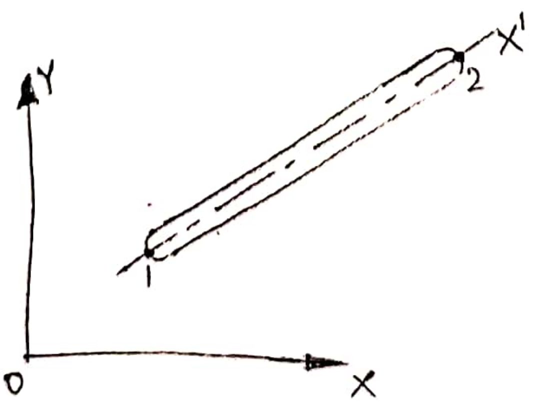
Two-force member

The two-dimensional bar joint is constructed by no. of bars jointed in different directions as shown in fig.

Local and Global Coordinate System:

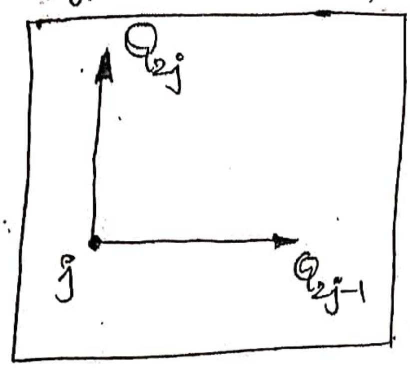
The main difference between one-dimensional structures and trusses is that the elements of a truss have various orientations. To account for this different orientations local and Global Coordinate systems are introduced as follows.

A typical plane truss is shown in local and global coordinate systems in the fig below.



In local numbering scheme the two nodes of the element are numbered 1 & 2. The local coordinate system consists of the X' axis which runs along the element from node-1 towards node-2. All quantities in local Co-ordinate system will be denoted by a prime (').

The Global X and Y Coordinate system is fixed and does not depend on orientation of the element. In Global Coordinate system every node has 2 DOF. A systematic ~~no of~~ numbering scheme is adapted. A node whose global node number is (j) has associated with it degrees of freedom $2j-1$ and $2j$ for that global displacements associated with node-j are Q_{2j-1} & Q_{2j} as shown in the fig.



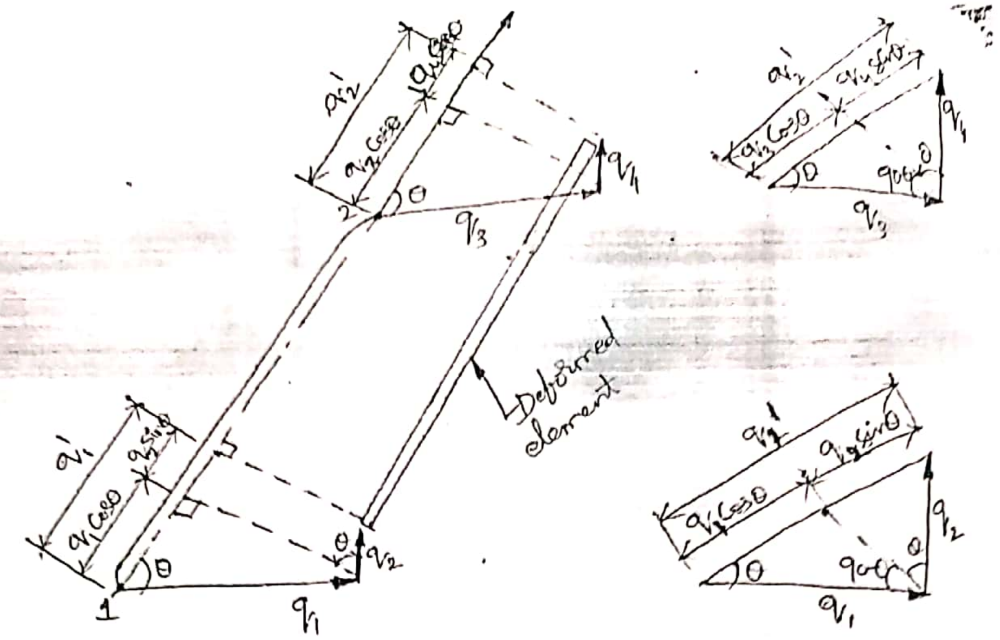
Let q_1' & q_2' be the displacements at nodes 1 & 2 in the local coordinate system. Thus the element displacement vector in the local coordinate system is denoted by

$\{q'\} = \{q_1' \ q_2'\}^T$.

The element displacement vector in the Global Coordinate system is a (4×1) vector (or) matrix, denoted by

$$\{q\} \text{ or } \{\delta\} = \{Q_1 \ Q_2 \ Q_3 \ Q_4\}^T \text{ or } \{q_1 \ q_2 \ q_3 \ q_4\}^T$$

- The relationship between $\{q'\}$ & $\{q\}$ or $\{\delta\}$ is developed as follows.



In fig. q_1' equals the sum of the projections of q_1 & q_2 onto the x'-axis. Thus $q_1' = q_1 \cos\theta + q_2 \sin\theta$. Similarly $q_2' = q_3 \cos\theta + q_4 \sin\theta$. At this stage the direction cosines l & m are introduced as $l = \cos\theta$ and $m = \cos\phi$ i.e $\sin\theta$. These direction cosines are the cosines of the angles that the local x'-axis makes with the global x and Y axis respectively.

The above equations can be written as $q_1' = q_1 l + q_2 m$

$q_2' = q_3 l + q_4 m$

The above equations can be written in matrix form as

$$\{q'\} = [L]\{q\} \quad (\text{or}) \quad \{q'\} = [L]\{\delta\}$$

where, q' = The element displacement vector as per local Coordinate system. $= \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$

$\{q\}$ or $\{\delta\}$ = element displacement vector as per Global Coordinate System.

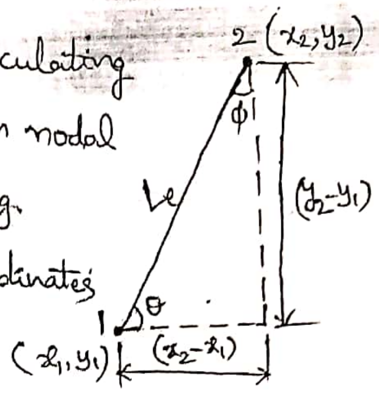
$$= \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}.$$

$[L]$ = Transformation matrix (or) rotation matrix.

$$[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

Formulas for Calculating direction Cosine l & m.

Simple formulas are given for Calculating the direction Cosines l & m from nodal Coordinate data. Refering to fig. Let (x_1, y_1) & (x_2, y_2) be the Coordinates of nodes 1 & 2 respectively.



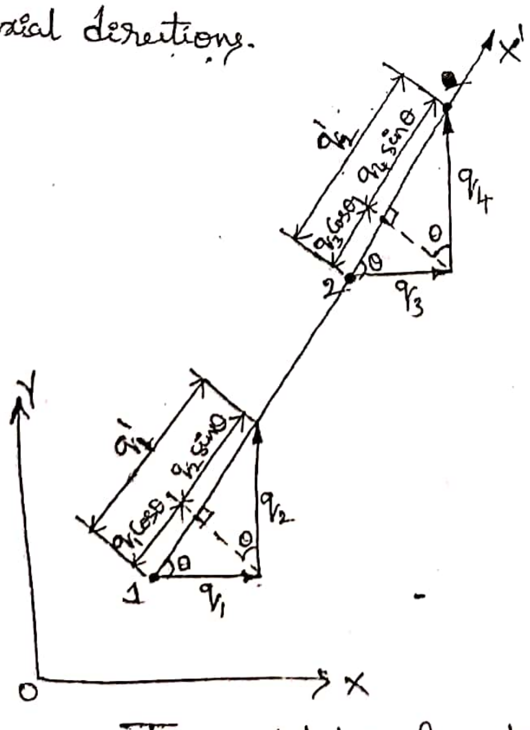
We have, $l = \cos\theta = \dfrac{x_2 - x_1}{L_e}$

$m = \sin\theta = \dfrac{y_2 - y_1}{L_e}$.

where, $L_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Stiffness matrix [K] for truss element

Consider a two noded bar element as shown in fig. for the analysis of trusses. This element is subjected to only axial forces. So the displacements are only in axial directions.



Two noded bar element

Let q_1' & q_2' = displacements of the nodes 1 & 2 in local Coordinate system (along the element axis).

q_1, q_2 & q_3, q_4 = Components of displacements q_1' & q_2' denoted in Global coordinate system (along X & Y axis)

From fig. We have $q_1' = q_1 \cos\theta + q_2 \sin\theta$

$$q_2' = q_3 \cos\theta + q_4 \sin\theta$$

Consider l & m are the direction Cosines, So $l = \cos\theta$ & $m = \sin\theta$. then the above equation becomes,

$$q_1' = q_1 l + q_2 m$$
$$q_2' = q_3 l + q_4 m$$

The above equations Can be written in the matrix form as

$$\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

i.e., $\{q'\} = [L]\{q\}$ (or) $\{q'\} = [L]\{\delta\}$

[Note:- The problems related to truss is one-dimensional when viewed (Considered) in local Coordinate systems and two dimensional when Considered in Global Coordinate System]

Since, the truss is treated as one-dimensional element in the local Coordinate system. The element stiffness matrix for the truss on the basis of One-dimensional Can be given by

$$[K']_e = \frac{A_e E_e}{L_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{——— (1)}$$

where, A_e = Element Cross-sectional area

E_e = Young's modulus

L_e = Length of the truss element

To develop element stiffness matrix in global coordinate system, the strain energy concept may be adapted. This is obtainable by considering the strain energy in local coordinate system and is given by.

$$U_e = \frac{1}{2}\{q'\}^T [K'] \{q'\}$$

Substituting for $\{q'\} = [L]\{q\}$

$$U_e = \frac{1}{2}([L]\{q\})^T [K'] ([L]\{q\}) \quad \left(\because (AB)^T = B^T A^T\right)$$

$$U_e = \frac{1}{2} \{q\}^T [L]^T [K'] ([L]\{q\})$$

The strain energy in global coordinate can be written as

$$U_e = \frac{1}{2}\{q\}^T [K]_e \{q\}.$$

where, $[K]_e$ is the element stiffness matrix in global coordinate system and is given as $[K]_e = [L]^T [K'] [L]$.

substituting for $[L]$ & $[K']$

$$\therefore [K]_e = \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \cdot \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[K]_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix}_{4\times 2} \begin{bmatrix} l & m & -l & -m \\ -l & -m & l & m \end{bmatrix}_{2\times 4}$$

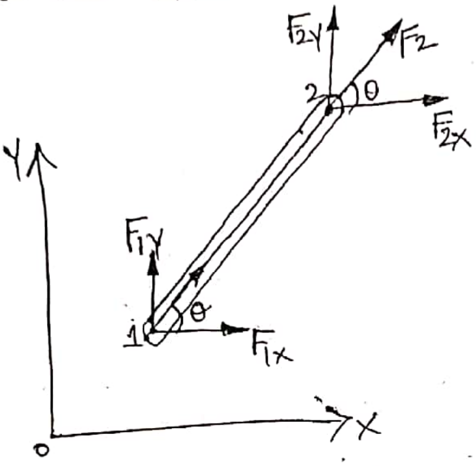
$$[K]_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -ml \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}_{4\times 4}$$

It may be noted that stiffness matrix properties are satisfied.

i.e., $[K]_e$ is symmetric and the sum of elements in any Column is equal to Zero.

Formulation of Finite element equation

Similar to nodal displacements, nodal forces can also be resolved into their X and Y Components



If F_1 & F_2 are the nodal forces acting at the nodes 1&2 along the element as shown in fig. Then we can write as F_{1X} & F_{1Y} as X & Y Components of F_1.

And F_{2X} & F_{2Y} as X & Y Components of F_2.

Combining the element stiffness, nodal displacement and nodal forces, the finite element equation for the truss element can be written in matrix form as $\{F\}=[K]\{\delta\}$

i.e.,

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \end{Bmatrix} = \frac{A_e E_e}{L_e} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

For the given structure Consisting of two & more no. of elements, the element stiffness matrices for all the elements are derived individually and then by Combining them in suitable format, the Global stiffness matrix is formulated and the nodal displacements and forces can be determined from the suitable global finite element equation.

<u>Stress Calculation</u>

Expressions for element stresses can be obtain by noting the truss element in local coordinate is a simple two force member. Thus the stress (σ) in a truss element is given by $\sigma = E_e \epsilon$. Since, the strain (ϵ) is the change in length per unit original length.

$$\epsilon = \frac{\text{change in length}}{\text{original length}}$$

$$\epsilon = \frac{q_2' - q_1'}{L_e}$$

$$\therefore \text{stress } (\sigma) = E_e \left(\frac{q_2' - q_1'}{L_e} \right)$$

$$\sigma = \frac{E_e}{L_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix}$$

$$\sigma = \frac{E_e}{L_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \{q'\}$$

This equation can be written in terms of Global displacements $\{q\}$ using transformation, $\{q'\} = [L]\{q\}$

$$\text{i.e, } \{q'\} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

$$\sigma = \frac{E_e}{L_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \{q\}$$

$$\sigma = \frac{E_e}{L_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \{q\}$$

Using this equation stresses are calculated in all the elements or members if forces are required the stresses may be multiplied by cross sectional area, positive value indicates tension, negative value indicates compression.

Analysis procedure for Truss-element

1. Divide the given truss assemble into suitable no. of truss elements and name them as ①, ② & ③., etc. in the counterclockwise direction from left side (usually from left bottom of the assembly).

2. Name the nodal points (or) nodes as 1, 2, 3.. etc. from left bottom end of truss assembly in counter-clockwise direction.

3. For any element locate its angle of inclination 'θ' at the first node & measure the magnitude of angle counter clockwise from global +ve X-axis.

4. For analysing the selected element write the X & Y coordinates of axial displacements q_1' & q_2' as q_1, q_2, q_3 & q_4.

5. Similarly mention the X & Y coordinates of nodal forces F_1, F_2 as F_{1X}, F_{1Y} & F_{2X}, F_{2Y} etc.

6. Find the element stiffness matrix for the i^{th} element in the global coordinate system such as

$$[K]_i = \frac{A_i E_i}{L_i}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

where, $l = \cos\theta$, $m = \sin\theta$

Sometimes $\cos\theta$ & $\sin\theta$ are directly calculated using the trigonometric relationship between elements & its locational distance in X & Y axis.

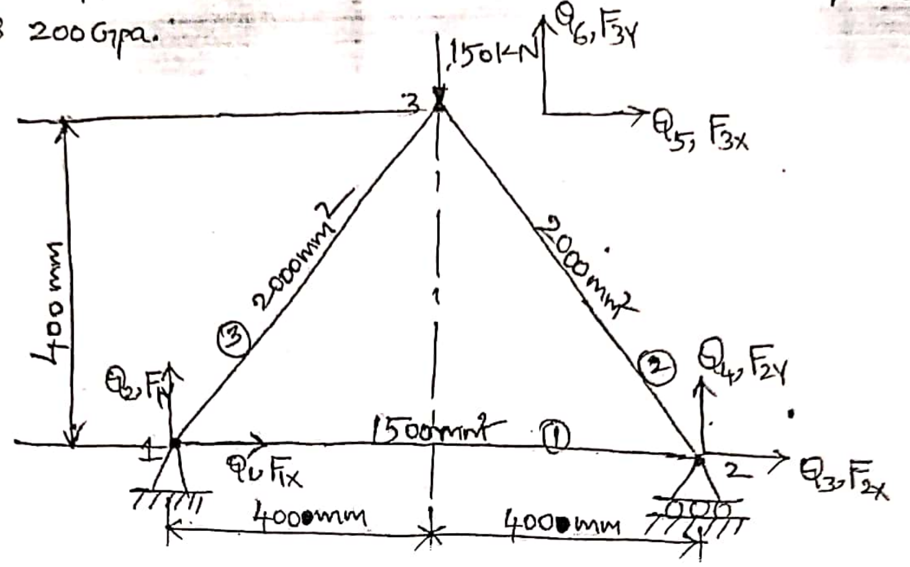
7. using stiffness matrices of various elements of truss form the Global stiffness matrix [K]

8. Write the Global finite element equation for the entire truss element assembly as $\{F\} = [K]\{\delta\}$

9. Apply the boundary conditions, in global finite element equation for example, $U_i = 0$, $Q_i = 0$, neglect i^{th} row & i^{th} column of the global stiffness matrix.

10. From the remaining items of the Finite element equation determine the nodal displacements, element stresses, support reactions using proper formulas and guidelines. If temperature is included consider thermal effect during analysis.

1. For the 3 bar truss shown in fig. Determine the nodal displacements & stress in each member. Find the support reactions. Also take modulus of Elasticity as 200 GPa.



A) The given truss is a 3 bar structure element members, node numbers & global displacements are as shown in fig.

I. Taking node-1 as origin, the coordinates of various nodes (nodal coordinates data) are as follows.

Nodal Coordinates table :-

Nodes	Nodal Coordinates	
	x	y
1	0	0
2	800	0
3	400	400

Element Connectivity table :- (Nodal Connectivity details are given in table)

Element No.	Element Node 1	Element Node 2
1	1	2
2	2	3
3	1	3

Length of element ①, $L_1 = 800mm$

" " ②, $L_2 = 400\sqrt{2} = 565.565mm$.

" " ③, $L_3 = \sqrt{(400-0)^2 + (400-0)^2} = 565.565mm$

direction Cosine table :- $l = \cos\theta$, $m = \sin\theta$

Element No.	L_e (mm)	θ	l	m
1	800	0	1	0
2	565.565	135	0.707	0.707
3.	565.565	45	0.707	0.707

given, $A_1 = 1500\,mm^2$; $A_2 = 2000mm^2$; $A_3 = 2000mm^2$

$E_1 = E_2 = E_3 = E = 2\times10^5\ N/mm^2$

The element stiffness matrix is given by

$$[K]_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element stiffness matrix for element ① Can be written as

$$[K]_1 = \frac{1500\times200\times10^3}{800}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{Global DOF (or) nodes.} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

$$[K]_1 = 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{nodes} \\ \begin{bmatrix} 375 & 0 & -375 & 0 \\ 0 & 0 & 0 & 0 \\ -375 & 0 & 375 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Element stiffness matrix for element ② Can be written as:

$$[K]_2 = \frac{2000 \times 200 \times 10^3}{565.565} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \leftarrow \text{nodes} \\ \left[\begin{array}{cccc} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{array}\right] \end{array} \begin{array}{c} \downarrow \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$\therefore [K]_2 = 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \leftarrow \text{nodes} \\ \left[\begin{array}{cccc} 353.62 & -353.62 & -353.62 & 353.62 \\ -353.62 & 353.62 & 353.62 & -353.62 \\ -353.62 & 353.62 & 353.62 & -353.62 \\ 353.62 & -353.62 & -353.62 & 353.62 \end{array}\right] \end{array} \begin{array}{c} \downarrow \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Element stiffness matrix for element 3

$$[K]_3 = \frac{2000 \times 200 \times 10^3}{565.565} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$\therefore [K]_3 = 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \leftarrow \text{nodes} \\ \left[\begin{array}{cccc} 353.62 & 353.62 & -353.62 & -353.62 \\ 353.62 & 353.62 & -353.62 & -353.62 \\ -353.62 & -353.62 & 353.62 & 353.62 \\ -353.62 & -353.62 & 353.62 & 353.62 \end{array}\right] \end{array} \begin{array}{c} \downarrow \\ 1 \\ 2 \\ 5 \\ 6 \end{array}$$

The Global stiffness matrix $[K]$ is assembled from the element stiffness matrices $[K]_1$, $[K]_2$ & $[K]_3$. By adding the element stiffness contributions by noting the element connectivity, we get,

$$[K] = 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} (375+353.62) & 353.62 & -375 & 0 & -353.62 & -353.62 \\ 353.62 & 353.62 & 0 & 0 & -353.62 & -353.62 \\ -375 & 0 & (375+353.62) & 353.62 & -353.62 & 353.62 \\ 0 & 0 & -353.62 & 353.62 & 353.62 & -353.62 \\ -353.62 & -353.62 & -353.62 & 353.62 & (353.62+353.62) & (-353.62+353.62) \\ -353.62 & -353.62 & 353.62 & -353.62 & 0 & (353.62+353.62) \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$[K] = 10^3 \begin{bmatrix} 728.62 & 353.62 & -375 & 0 & -353.62 & -353.62 \\ 353.62 & 353.62 & 0 & 0 & -353.62 & -353.62 \\ -375 & 0 & 728.62 & 353.62 & -353.62 & 353.62 \\ 0 & 0 & -353.62 & 353.62 & 353.62 & -353.62 \\ -353.62 & -353.62 & -353.62 & 353.62 & 707.24 & 0 \\ -353.62 & -353.62 & 353.62 & -353.62 & 0 & 707.24 \end{bmatrix}$$

Global displacement vector $\{\delta\}$ or $\{Q\} = \{Q_1 \ Q_2 \ Q_3 \ Q_4 \ Q_5 \ Q_6\}^T$

Global load vector $\{F\} = \{F_{1X} \ F_{1Y} \ F_{2X} \ F_{2Y} \ F_{3X} \ F_{3Y}\}^T$

The Global finite element equation or equilibrium equation in matrix form is $\{F\} = [K]\{\delta\}$.

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix} = 10^3 \begin{bmatrix} 728.62 & 353.62 & -375 & 0 & -353.62 & -353.62 \\ 353.62 & 353.62 & 0 & 0 & -353.62 & -353.62 \\ -375 & 0 & 728.62 & 353.62 & -353.62 & 353.62 \\ 0 & 0 & -353.62 & 353.62 & 353.62 & -353.62 \\ -353.62 & -353.62 & -353.62 & 353.62 & 707.24 & 0 \\ -353.62 & -353.62 & 353.62 & -353.62 & 0 & 707.24 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

The given boundary conditions are $Q_1 = 0$, $Q_2 = 0$ & $Q_4 = 0$ and $F_{2X} = 0$, $F_{3X} = 0$; $F_{3Y} = -150 \times 10^3 N$. Applying the boundary conditions, then the global finite element equation becomes

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ 0 \\ F_{2Y} \\ 0 \\ -150\times10^3 \end{Bmatrix} = 10^3 \begin{bmatrix} 728.62 & 353.62 & -375 & 0 & -353.62 & -353.62 \\ 353.62 & 353.62 & 0 & 0 & -353.62 & -353.62 \\ -375 & 0 & 728.62 & 353.62 & -353.62 & 353.62 \\ 0 & 0 & -353.62 & 353.62 & 353.62 & -353.62 \\ -353.62 & -353.62 & -353.62 & 353.62 & 707.24 & 0 \\ -353.62 & -353.62 & 353.62 & -353.62 & 0 & 707.24 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ Q_3 \\ 0 \\ Q_5 \\ Q_6 \end{Bmatrix} \quad (1)$$

The rows & columns corresponding to DOF 1, 2 & 4. which corresponds to fixed supports are neglected. The reduced finite element equations are

$$\begin{Bmatrix} 0 \\ 0 \\ -150\times10^3 \end{Bmatrix} = 10^3 \begin{bmatrix} 728.62 & -353.62 & 353.62 \\ -353.62 & 707.24 & 0 \\ 353.62 & 0 & 707.24 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

$$0 = 10^3 [728.62Q_3 - 353.62Q_5 + 353.62Q_6] \quad \text{--- (2)}$$

$$0 = 10^3 [-353.62Q_3 + 707.24Q_5] \quad \text{--- (3)}$$

$$-150\times10^3 = 10^3 [353.62Q_3 + 707.24Q_6] \quad \text{--- (4)}$$

$$-353.62Q_3 + 707.24Q_5 = 0$$

$$353.62\,Q_3 = 707.24\,Q_5$$

$$Q_3 = 2Q_5 \quad \text{--- (5)}$$

$$\cancel{-150\times10^3 = 10^3 [353.62(2) + 707.24Q_6]}$$

$$-150 = 353.62(2Q_5) + 707.24Q_6 \quad \text{--- (6)}$$

$$-150 = 707.24\,Q_5 + 707.24\,Q_6 \quad \text{--- (6)}$$

$$0 = 728.62(2Q_5) - 353.62Q_5 + 353.62Q_6$$

$$0 = 1457.24Q_5 - 353.62Q_5 + 353.62Q_6$$

$$0 = 1103.62\,Q_5 + 353.62Q_6 \quad \text{--- (7)}$$

from (6) & (7) $\Rightarrow$ $Q_5 = 0.1\,mm$

$Q_6 = -0.312\,mm$

$Q_3 = 0.2\,mm$

ii) Stress in each element can be obtained from the connectivity of element (1) is 1–2 consequently the nodal displacement vector for element (1) is given by.

$$\{q_1\} = \{Q_1 \;\; Q_2 \;\; Q_3 \;\; Q_4\}^T$$

$\therefore$ The stress in element ①, $\sigma_1 = \frac{E_1}{L_1}[-l \ -m \ l \ m]\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix}$

$$\sigma_1 = \frac{200\times10^3}{800}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.2 \\ 0 \end{Bmatrix}$$

$\sigma_1 = 250\,(0+0+0.2+0)$

$\sigma_1 = 50\ N/mm^2$ (Tensile) $\qquad (\because \sigma = \frac{P}{A})$

$P_1 = \sigma_1 \times A_1 = 50 \times 1500 = 75\ KN$ (Tensile)

Stress in element ②, $\sigma_2 = \frac{E_2}{L_2}[-l \ -m \ l \ m]\{q_2\}$

$\{q_2\} = \{Q_3 \ Q_4 \ Q_5 \ Q_6\}^T$

$$\sigma_2 = \frac{200\times10^3}{565.565}\begin{bmatrix} 0.707 & -0.707 & -0.707 & 0.707 \end{bmatrix}\begin{Bmatrix} 0.2 \\ 0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$\sigma_2 = 353.62\,[0.707(0.2) - 0.707(0) - 0.707(0.1) - 0.707(0.312)]$

$\sigma_2 = -53\ N/mm^2$ (or) $53\ N/mm^2$ (Compressive)

$\therefore P_2 = \sigma_2 \times A_2 = -53 \times 2000 = -106\ KN$

$P_2 = 106\ KN$ (Compressive)

Connectivity of element ③ is 1–3 Consequently the nodal displacement vector for element ③ is given by

$\{q_3\} = \{Q_1 \ Q_2 \ Q_5 \ Q_6\}^T$

The stress in element ③ is, $\sigma_3 = \frac{E_3}{L_3}[-l \ -m \ l \ m]\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_5 \\ Q_6 \end{Bmatrix}$

$$\sigma_3 = \frac{200\times10^3}{565.565}\begin{bmatrix} -0.707 & +0.707 & 0.707 & 0.707 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.1 \\ -0.312 \end{Bmatrix}$$

$\sigma_3 = 353.62\,[0 - 0 + (0.707\times0.1) + (0.707)(-0.312)]$

$\sigma_3 = -53\ N/mm^2$ (or) $53\ N/mm^2$ (Compressive)

$\therefore P_3 = \sigma_3 A_3 = -53 \times 2000 = -106\ KN$

$P_3 = 106\ KN$ (Compressive)

iii) To find support reactions

We need to find reaction forces along the DOF 1, 2 & 4 which correspond to fixed supports. From the Global finite element eq ① we have

$$F_{1X} = 10^3(0+0-375Q_3+0-353.62Q_5-353.62Q_6)$$

$$F_{1Y} = 10^3(0+0+0+0-353.62Q_5-353.62Q_6)$$

$$F_{2Y} = 10^3(0+0-353.62Q_3+0+353.62Q_5-353.62Q_6)$$

$$F_{1X} = 10^3[(-375\times 0.2)-353.62(0.1)+353.62(0.312)]$$

$$F_{1X} = -32.56N$$

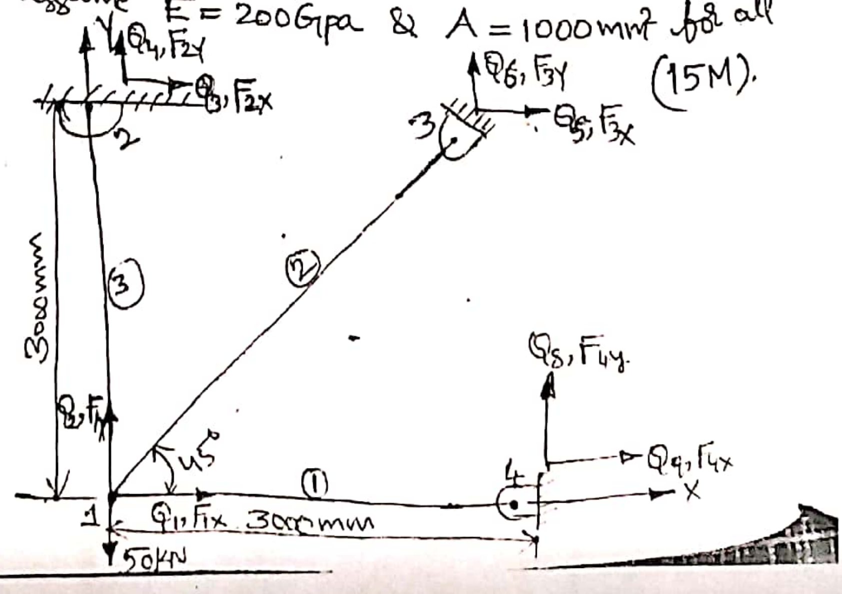
$$F_{1Y} = 10^3[-353.62(0.1)+353.62(0.312)]$$

$$F_{1Y} = 74.96 \simeq 75KN$$

$$F_{2Y} = 10^3[-353.62(0.2)+353.62(0.1)+353.62(0.312)]$$

$$F_{2Y} = 74.96 \simeq 75KN$$

2. For the plane truss composed of 3 elements shown in fig subjected to a downward force of 50KN applied at node 1. Determine the X and Y displacement at node 1 and stresses in each element assume E = 200Gpa & A = 1000mm² for all elements. (15M).



The given truss is a 3 element structure, node numbers and global displacements are shown in Fig.

Taking node 1 as the origin, the Coordinates of various nodes i.e. nodal Coordinate data are as follows.

Nodal Coordinates table

Nodes	Nodal Coordinates	
	x	y
1	0	0
2	0	3000
3	3000	3000
4	3000	0

Element Connectivity table i.e, nodal Connectivity Coordinates areas follows

Element Connectivity table

Element no	Element node 1	Element node 2
1	1	4
2	1	3
3	1	2

Length of element ①, $L_1 = 3000\,mm$

" " " ②, $L_2 = \sqrt{(3000-0)^2 + (3000-0)^2}$

$L_2 = 4242.64\,mm$

" " " ③, $L_3 = 3000\,mm$

Direction Cosines table:

$l = \cos\theta$, $m = \sin\theta$

Element no.	L_e (mm)	θ	l	m
1	3000	0	1	0
2	4242.64	45°	0.707	0.707
3	3000	90°	0	1

$A = 1000 mm^2$ & $E_1 = E_2 = E_3 = 200 GPa = 200 \times 10^3 N/mm^2$

The element stiffness matrix is given by

$$[K]_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element stiffness matrix for element ①

$$[K]_1 = \frac{1000 \times 200 \times 10^3}{3000} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \leftarrow \text{nodes} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

$$[K]_1 = 10^3 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 7 & 8 \end{array} \\ \begin{bmatrix} 66.67 & 0 & -66.67 & 0 \\ 0 & 0 & 0 & 0 \\ -66.67 & 0 & 66.67 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 7 \\ 8 \end{array}$$

Element stiffness matrix for element ②

$$[K]_2 = \frac{1000 \times 200 \times 10^3}{4242.64} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \leftarrow \text{nodes} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \leftarrow \text{nodes} \\
[K]_2 = 10^3 & \begin{bmatrix} 23.57 & 23.57 & -23.57 & -23.57 \\ 23.57 & 23.57 & -23.57 & -23.57 \\ -23.57 & -23.57 & 23.57 & 23.57 \\ -23.57 & -23.57 & 23.57 & 23.57 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}
\end{array}
$$

Element stiffness matrix for element 3,

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{nodes} \\
[K]_3 = \dfrac{1000 \times 200 \times 10^3}{3000} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
= 10^3 & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 66.67 & 0 & -66.67 \\ 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array}
$$

The Global stiffness matrix $[K]$ is assembled from element stiffness matrices. By adding the element stiffness matrices, noting the element Connectivity, we get

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
[K] = 10^3 & \begin{bmatrix}
90.24 & 23.57 & 0 & 0 & -23.57 & -23.57 & -66.67 & 0 \\
23.57 & 90.24 & 0 & -66.67 & -23.57 & -23.57 & \cancel{66.67}\ 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -66.67 & 0 & 66.67 & 0 & 0 & 0 & 0 \\
-23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\
-23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\
-66.67 & 0 & 0 & 0 & 0 & 0 & 66.67 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\end{array}
$$

Global displacement vector $\{\delta\}$ or $\{Q\} = \{Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8\}^T$

Global load vector $\{F\} = \{F_{1X}\ F_{1Y}\ F_{2X}\ F_{2Y}\ F_{3X}\ F_{3Y}\ F_{4X}\ F_{4Y}\}^T$

The Global finite element equation or equilibrium equation in matrix form is $\{F\} = [K]\{\delta\}$

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \\ F_{4X} \\ F_{4Y} \end{Bmatrix} = 10^3 \begin{bmatrix} 90.24 & 23.57 & 0 & 0 & -23.57 & -23.57 & -66.67 & 0 \\ 23.57 & 90.24 & 0 & -66.67 & -23.57 & -23.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 & 0 & 0 & 0 & 0 \\ -23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\ -23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\ -66.67 & 0 & 0 & 0 & 0 & 0 & 66.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix}$$

The given boundary conditions are $Q_3=0$, $Q_4=0$, $Q_5=0$, $Q_6=0$ & $Q_7=0$, $Q_8=0$ and $F_{1X}=0$, $F_{1Y}=-50\,kN$; Applying the boundary conditions, then the global finite element equation becomes

$$\begin{Bmatrix} 0 \\ -50\times10^3 \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \\ F_{4X} \\ F_{4Y} \end{Bmatrix} = 10^3 \begin{bmatrix} 90.24 & 23.57 & 0 & 0 & -23.57 & -23.57 & -66.67 & 0 \\ 23.57 & 90.24 & 0 & -66.67 & -23.57 & -23.57 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -66.67 & 0 & 66.67 & 0 & 0 & 0 & 0 \\ -23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\ -23.57 & -23.57 & 0 & 0 & 23.57 & 23.57 & 0 & 0 \\ -66.67 & 0 & 0 & 0 & 0 & 0 & 66.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The rows & columns corresponding to DOF 3, 4, 5, 6, 7 & 8 which corresponds to fixed supports are neglected. The reduced finite element equation becomes

$$\begin{Bmatrix} 0 \\ -50\times10^3 \end{Bmatrix} = 10^3 \begin{bmatrix} 90.24 & 23.57 \\ 23.57 & 90.24 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}.$$

$$0 = 10^3 [90.24 Q_1 + 23.57 Q_2] \quad \text{—— (2)}$$

$$-50\times10^3 = 10^3 [23.57 Q_1 + 90.24 Q_2] \quad \text{—— (3)}$$

$$Q_1 = 0.155 \text{ mm}$$

$$Q_2 = -0.594 \text{ mm}$$

ii) Stress in each element can be obtained from the connectivity of element (1) is 1–4. Consequently the nodal displacement vector for element (1) is given by. $\sigma_1 = \frac{E_1}{L}[-l\ -m\ l\ m]\{q_1\}$

$$\{q_1\} = \{Q_1\ Q_2\ Q_7\ Q_8\}^T$$

$\therefore$ The stress in element (1), $\sigma_1 = \frac{E_1}{L_1}[-l\ -m\ l\ m]\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_7 \\ Q_8 \end{Bmatrix}$

$$\sigma_1 = \frac{200\times10^3}{3000}[-1\ \ 0\ \ 1\ \ 0]\begin{Bmatrix} 0.155 \\ -0.594 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma_1 = 66.666(-0.155)$$

$\sigma_1 = -10.33\ N/mm^2$ (or) $10.33\ N/mm^2$ (Compressive)

The stress in element (2), $\sigma_2 = \frac{E_2}{L_2}[-l\ -m\ l\ m]\{q_2\}$

$$\{q_2\} = \{Q_1\ Q_2\ Q_5\ Q_6\}^T$$

$$\sigma_2 = \frac{200\times10^3}{4242.64}[-0.707\ \ -0.707\ \ 0.707\ \ 0.707]\begin{Bmatrix} 0.155 \\ -0.594 \\ 0 \\ 0 \end{Bmatrix}$$

$$\sigma_2 = 47.140[(-0.707\times0.155)+(0.707\times0.594)]$$

$$\sigma_2 = 47.140(-0.5295)$$

$$\sigma_2 = -14.63$$

$\sigma_2 = 14.63\ N/mm^2$ (Tensile)

The stress in element ③, $\sigma_3 = \frac{E_3}{L_3}\begin{bmatrix}-l & -m & l & m\end{bmatrix}\{q_3\}$

$\{q_3\} = \{Q_1 \ Q_2 \ Q_3 \ Q_4\}$

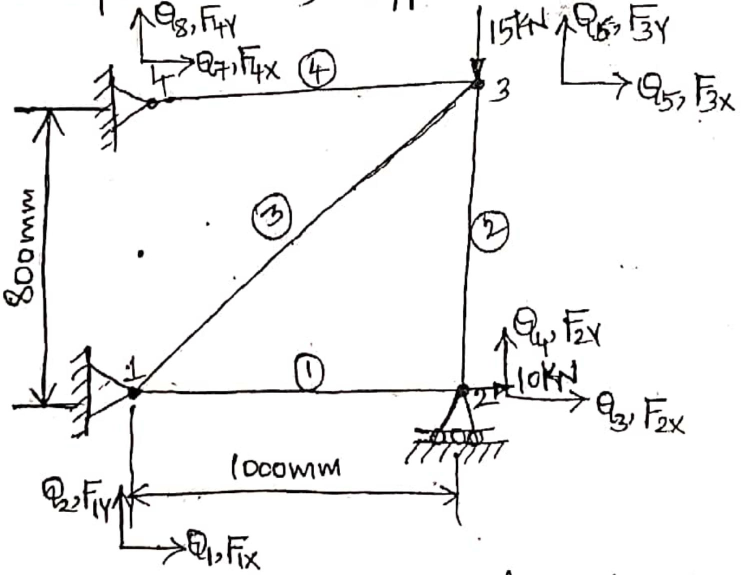
$$\sigma_3 = \frac{200\times10^3}{3000}\begin{bmatrix}0 & -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}0.155\\-0.594\\0\\0\end{Bmatrix}$$

$$\sigma_3 = 66.666\,(0.594)$$

$$\sigma_3 = 39.59 \simeq 39.6\ N/mm^2\ (\text{Tensile})$$

26-12-2020

3. Consider a 4 bar truss as shown in fig. it is given that $E = 200\,GPa$, $A = 500\,mm^2$ for all the elements. Determine i) Nodal displacements ii) Support reactions iii) element stresses



A) The given truss structure is divided into 4 elements. Element numbers, Node numbers and global displacements are as shown in fig.

Let Q_1, Q_2 displacements along global axis X and Y at node-1. Q_3, Q_4 displacements along global axis X and Y at node-2. Q_5, Q_6 displacements along global axis X and Y at node-3. Q_7, Q_8 displacements along global axis X and Y at node-4.

F_{1X}, F_{1Y} nodal forces along global axis X and Y at node-1.
F_{2X}, F_{2Y} nodal forces along global axis X, Y at node-2.

F_{3X}, F_{3Y} nodal forces along global axis X, Y at node-3.

F_{4X}, F_{4Y} nodal forces along global axis X, Y at node-4.

Taking node-1 is the origin the coordinates of various nodes are follows.

Nodal Coordinates Table :-

Node No.	Nodal Coordinates (mm)	
	x	y
1	0	0
2	1000	0
3	1000	800
4	0	800

Element Connectivity table :-

Element	Element Node 1	Element Node-2
1	1	2
2	2	3
3	1	3
4	4	3

Global numbers.

Length of element ①, $L_1 = 1000\,mm$

" " ②, $L_2 = 800\,mm$

" " ③, $L_3 = \sqrt{(1000-0)^2 + (800-0)^2} = 1280.624\,mm$

" " ④, $L_4 = 1000\,mm$.

Directions Cosine table, where, $l = \cos\theta$, $m = \sin\theta$

Element No.	Le (mm)	θ	l	m
1	1000	0	1	0
2	800	90	0	1
3	1280.624	38.65	0.780	0.624
4	1000	0	1	0

$\tan\theta = \frac{800}{1000}$

$\theta = 38.65°$

ii) Nodal displacements

$A_1 = A_2 = A_3 = A_4 = A = 500\,mm^2$

$E_1 = E_2 = E_3 = E_4 = E = 200\,Gpa = 200\times10^3\,N/mm^2$

Element stiffness matrix, for element is given by

$$[K]_e = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element stiffness matrix for element ①

$$[K]_1 = \frac{500\times200\times10^3}{1000}\begin{array}{c|c} \begin{matrix} 1 & 2 & 3 & 4 \leftarrow \text{nodes} \end{matrix} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

$$= 10^5 \begin{array}{c|c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} & \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \end{array}$$

Element stiffness matrix for element ②

$$[K]_2 = \frac{500\times200\times10^3}{800}\begin{array}{c|c} \begin{matrix} 3 & 4 & 5 & 6 \leftarrow \text{nodes} \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$= 10^5 \begin{array}{c|c} \begin{matrix} 3 & 4 & 5 & 6 \leftarrow \text{nodes} \end{matrix} & \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -1.25 \\ 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 1.25 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{array}$$

Element stiffness matrix for element ③

$$[K]_3 = \frac{500\times200\times10^3}{1280.624}\begin{array}{c|c} \begin{matrix} 1 & 2 & 5 & 6 \end{matrix} & \\ \begin{bmatrix} 0.6084 & 0.48 & -0.6084 & -0.48 \\ 0.48 & 0.389 & -0.48 & -0.389 \\ -0.6084 & -0.48 & 0.6084 & 0.48 \\ -0.48 & -0.389 & 0.48 & 0.389 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix} \end{array}$$

$$[K]_3 = 10^5 \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc|c} 1 & 2 & 5 & 6 & \\ \hline 0.475 & 0.374 & -0.475 & -0.374 & 1 \\ 0.374 & 0.303 & -0.374 & -0.303 & 2 \\ -0.475 & -0.374 & 0.475 & 0.374 & 5 \\ -0.374 & -0.303 & 0.374 & 0.303 & 6 \end{array}$$

Element stiffness matrix for element (4).

$$[K]_4 = \frac{500 \times 200 \times 10^3}{1000} \begin{array}{cccc|c} 5 & 6 & 7 & 8 & \\ \hline 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \\ -1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 8 \end{array}$$

$$= 10^5 \begin{array}{cccc|c} 5 & 6 & 7 & 8 & \\ \hline 1 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \\ -1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 8 \end{array}$$

The Global stiffness matrix [K] is assembled from the element stiffness matrices By adding the element stiffness matrices, noting the element Connectivity, we get

$$[K] = 10^5 \begin{array}{cccccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline 1.475 & 0.374 & -1 & 0 & -0.475 & -0.374 & 0 & 0 & 1 \\ 0.374 & 0.303 & 0 & 0 & -0.374 & -0.303 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1.25 & 0 & -1.25 & 0 & 0 & 4 \\ -0.475 & -0.374 & 0 & 0 & 1.475 & 0.374 & -1 & 0 & 5 \\ -0.374 & -0.303 & 0 & -1.25 & 0.374 & \cancel{0.303}\ 1.553 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{array}$$

Global displacement vector $\{Q\}$ or $\{\delta\} = \{Q_1, Q_2\ Q_3\ Q_4\ Q_5\ Q_6\ Q_7\ Q_8\}^T$

Global load vector $\{F\} = \{F_{1x}\ F_{1Y}, F_{2x}\ F_{2Y}\ F_{3x}\ F_{3Y}\ F_{4x}\ F_{4Y}\}^T$

∴ The Global finite element equation or equilibrium equation in matrix form is $\{F\} = [K]\{\delta\}$

$$\begin{Bmatrix} F_{1x} \\ F_{1Y} \\ F_{2x} \\ F_{2Y} \\ F_{3x} \\ F_{3Y} \\ F_{4x} \\ F_{4Y} \end{Bmatrix} = 10^5 \begin{bmatrix} 1.475 & 0.374 & -1 & 0 & -0.475 & -0.374 & 0 & 0 \\ 0.374 & 0.303 & 0 & 0 & -0.374 & -0.303 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.25 & 0 & -1.25 & 0 & 0 \\ -0.475 & -0.374 & 0 & 0 & 1.475 & 0.374 & -1 & 0 \\ -0.374 & -0.303 & 0 & -1.25 & 0.374 & 1.553 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix}$$

The known boundary conditions are, $Q_1 = Q_2 = 0$, $Q_4 = 0$, $Q_7 = Q_8 = 0$

$F_{2x} = 10 \times 10^3 N$, $F_{3x} = 0$, $F_{3Y} = -15 kN = -15 \times 10^3 N$

The Global finite element equation after applying the boundary Conditions is

$$\begin{Bmatrix} F_{1x} \\ F_{1Y} \\ 10\times10^3 \\ F_{2Y} \\ 0 \\ -15\times10^3 \\ F_{4x} \\ F_{4Y} \end{Bmatrix} = 10^5 \begin{bmatrix} 1.475 & 0.374 & -1 & 0 & -0.475 & -0.374 & 0 & 0 \\ 0.374 & 0.303 & 0 & 0 & -0.374 & -0.303 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.25 & 0 & -1.25 & 0 & 0 \\ -0.475 & -0.374 & 0 & 0 & 1.475 & 0.374 & -1 & 0 \\ -0.374 & -0.303 & 0 & -1.25 & 0.374 & 1.553 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ Q_3 \\ 0 \\ Q_5 \\ Q_6 \\ 0 \\ 0 \end{Bmatrix}$$

The rows & column corresponding to DOF 1, 2, 4, 7, 8 which corresponds to fixed support are neglected.

The reduced finite element equations are

$$10\times10^{3} = [1\times Q_3]\times10^{5} \quad \text{——} (1)$$

$$Q_3 = 0.1\,mm$$

$$0 = 10^{5}\left[1.475\,Q_5 + 0.374\,Q_6\right] \quad \text{——} (2)$$

$$-15\times10^{3} = 10^{5}\left[0.374(Q_5) + 1.553(Q_6)\right] \quad \text{——}$$

$$\cancel{-0.15 = 1.553\,Q_6} \quad -0.15 = 0.374\,Q_5 + 1.553\,Q_6 \quad \text{——} (3)$$

$$\cancel{Q_6 = -0.096\,mm}$$

Solve eq (2) & (3), $Q_5 = 0.026\,mm$

$Q_6 = -0.102\,mm.$

$\therefore$ Nodal displacement vector $\{\delta\} = \{0 \quad 0 \quad 0.1 \quad 0 \quad 0.026 \quad -0.102 \quad 0 \quad 0\}^{T}$ mm

ii) <u>Support reactions,</u>

from the Global finite element equation.

$$F_{1x} = 10^{5}\left[0+0-Q_3+0-0.475\,Q_5-0.374\,Q_6+0+0\right]$$

$$F_{1x} = 10^{5}\left[-0.1-0.475(0.026)+0.374(0.102)\right]$$

$F_{1x} = -7420.2\,N$ (or) $7.420\,kN$ (Compression).

$$F_{1y} = 10^{5}\left[0+0+0+0-0.374\,Q_5-0.303\,Q_6\right]$$

$$= 10^{5}\left[-0.374(0.026)+0.303(0.102)\right]$$

$F_{1y} = 2118.2\,N$ (or) $2.118\,kN$ (Tensile)

$$F_{2y} = \left[-1.25\,Q_6\right]10^{5}$$

$$= \left[-1.25\times(-0.102)\right]10^{5}$$

$F_{2y} = 12750\,N$ (or) $12.75\,kN$ (Tensile)

$$F_{4x} = 10^{5}\left[-0.026\right]$$

$F_{4x} = -2600\,N$ (or) $2.6\,kN$ (Compressive)

$F_{4Y} = 10^5 [0] = 0$

Nodal force vector is $\{F\} = \{-7.42 \;\; 2.118 \;\; 10 \;\; 12.75 \;\; 0 \;\; -15 \;\; -2.6 \;\; 0\}^T$ kN

iii) Element stresses :-

The stress in each element can be determine using

$$\sigma = \frac{E_e}{L_e} [-l \;\; -m \;\; l \;\; m] \{q\}$$

The connectivity of element ① is 1-2 Consequently the nodal displacement vector for element ① is given by

$$\{q\}_1 = \{Q_1 \;\; Q_2 \;\; Q_3 \;\; Q_4\}^T$$

∴ Stress in element ①,

$$\sigma_1 = \frac{200 \times 10^3}{1000} [-1 \;\; 0 \;\; 1 \;\; 0] \begin{Bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{Bmatrix}$$

$$\sigma_1 = 200 (0.1) = 20 \; N/mm^2 \quad \text{(Tensile)}$$

Connectivity of element ② is 2-3 Consequently the nodal displacement vector for element ② is given by

$$\{q_2\} = \{Q_3 \;\; Q_4 \;\; Q_5 \;\; Q_6\}^T$$

∴ Stress in element ②

$$\sigma_2 = \frac{200 \times 10^3}{800} [0 \;\; -1 \;\; 0 \;\; 1] \begin{Bmatrix} 0.1 \\ 0 \\ 0.026 \\ -0.102 \end{Bmatrix}$$

$$\sigma_2 = 250 (-0.102)$$

$$\sigma_2 = -25.5 \; N/mm^2 \text{ (or) } 25.5 \; N/mm^2 \text{ (Compressive)}$$

Connectivity of element ③ is 1-3 Consequently the nodal displacement vector for element ③ is given by

$$\{q\}_3 = \{Q_1 \;\; Q_2 \;\; Q_5 \;\; Q_6\}^T$$

∴ Stress in element ③

$$\sigma_3 = \frac{200 \times 10^3}{1280.624} [-0.780 \;\; -0.624 \;\; 0.780 \;\; 0.624] \begin{Bmatrix} 0 \\ 0 \\ 0.026 \\ -0.102 \end{Bmatrix}$$

$\sigma_3 = 156.17\left[0.780(0.026) - 0.624(0.102)\right]$

$\sigma_3 = -6.77$ (or) $6.77\ N/mm^2$ (Compressive)

Connectivity of element (4) is 4-3 Consequently the nodal displacement vector for element (4) is

$$\{q\}_4 = \{Q_7\ Q_8\ Q_5\ Q_6\}$$

$\therefore$ stress in element (4),

$$\sigma_4 = \frac{200\times10^3}{1000}\begin{bmatrix}-1 & 0 & 1 & 0\end{bmatrix}\begin{Bmatrix}0\\0\\0.026\\-0.102\end{Bmatrix}$$

$$\sigma_4 = 200(0.026)$$

$$\sigma_4 = 5.2\ N/mm^2 \text{ (Tensile)}$$

Results :-

1. Nodal displacements

$Q_1 = Q_2 = Q_4 = Q_7 = Q_8 = 0$; $Q_3 = 0.1mm$

$Q_5 = 0.026mm$

$Q_6 = -0.102mm$

2. Support reactions.

$\{F\} = \{-7.42\ \ 2.118\ \ 10\ \ 12.75\ \ 0\ \ -15\ \ -2.6\ \ 0\}^T$ KN

3. Element stresses.

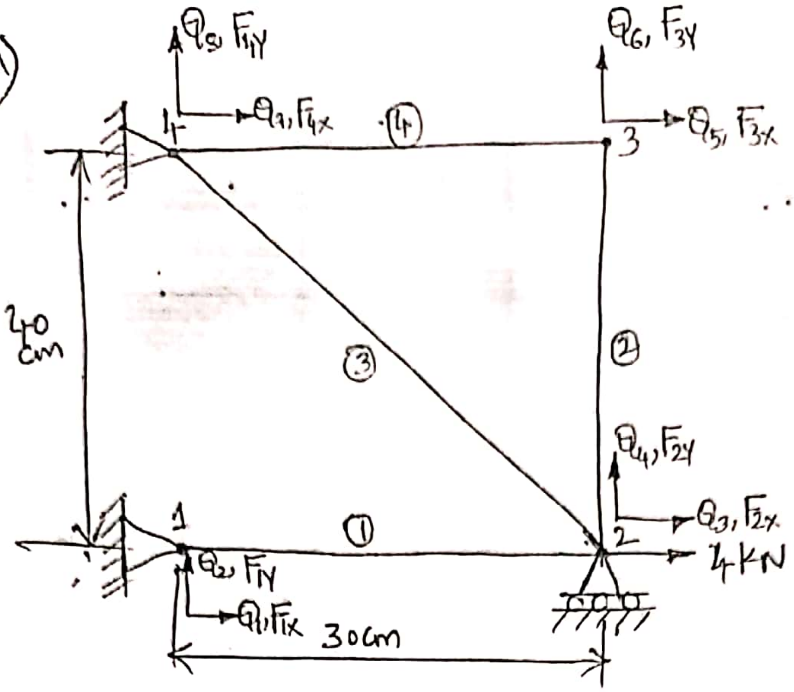
$\sigma_1 = 20N/mm^2$

$\sigma_2 = -25.5N/mm^2$

$\sigma_3 = -6.77N/mm^2$

$\sigma_4 = 5.2N/mm^2$

15M)



Assume, $E = 200\,GPa$ (for mild steel)

$A = 200\ mm^2$

A) Take node-1 as the origin, the coordinates of various nodes are follows.

Nodal Coordinates table:

Node No.	Nodal Coordinates (mm)	
	x	y
1	0	0
2	300	0
3	300	400
4	0	400

Element Connectivity table

Element	Element Node 1	Element Node 2
1	1	2
2	2	3
3	~~4~~ 2	~~2~~ 4
4	4	3

Length of element (1), $L_1 = 300mm$

Length of element ②, $L_2 = 400\,mm$

Length of element ③, $L_3 = \sqrt{(0-300)^2 + (400-0)^2} = 500\,mm$

Length of element ④, $L_4 = 300\,mm$.

Directions Cosine table, where $l = \cos\theta$, $m = \sin\theta$

Element No.	Le(mm)	θ	l	m
1	300	0	1	0
2	400	90	0	1
3	500	53.13	0.6	0.799≃0.8
4	300	0	1	0

$\tan\theta = \frac{400}{300}$

$\theta = 53.13°$

i) Nodal displacements

$A_1 = A_2 = A_3 = A_4 = 200\,mm^2$

$E_1 = E_2 = E_3 = E_4 = 200\times10^3\,N/mm^2$

Element stiffness matrix is given by

$$[K] = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Element stiffness matrix for element ① is

$$[K]_1 = \frac{200\times200\times10^3}{300}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

$$[K]_1 = 10^3\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 133.33 & 0 & -133.33 & 0 \\ 0 & 0 & 0 & 0 \\ -133.33 & 0 & 133.33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

Element stiffness matrix for element ② is.

$$[K]_2 = \frac{200 \times 200 \times 10^3}{400}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K]_2 = 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & -100 \\ 0 & 0 & 0 & 0 \\ 0 & -100 & 0 & 100 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

Element stiffness matrix for element (3) is

$$[K]_3 = \frac{200 \times 200 \times 10^3}{500}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

$$[K]_3 = 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 28.8 & 38.4 & -28.8 & -38.4 \\ 38.4 & 57.2 & -38.4 & -57.2 \\ -28.8 & -38.4 & 28.8 & 38.4 \\ -38.4 & -57.2 & 38.4 & 57.2 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array}$$

Element stiffness matrix for element (4) is

$$[K]_4 = \frac{200 \times 200 \times 10^3}{300} = [K]_4 = 10^3 \begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 133.33 & 0 & -133.33 & 0 \\ 0 & 0 & 0 & 0 \\ -133.33 & 0 & 133.33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}$$

The global stiffness matrix [K] is obtained by assembling the element stiffness matrices.

$$[K] = 10^3 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 133.33 & 0 & -133.33 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -133.33 & 0 & 162.13 & 38.4 & 0 & 0 & -28.8 & -38.4 \\ 0 & 0 & 38.4 & 151.2 & 0 & -100 & -38.4 & -51.2 \\ 0 & 0 & 0 & 0 & 133.33 & 0 & -133.33 & 0 \\ 0 & 0 & 0 & -100 & 0 & 100 & 0 & 0 \\ 0 & 0 & -28.8 & -38.4 & -133.33 & 0 & 162.13 & 38.4 \\ 0 & 0 & -38.4 & -51.2 & 0 & 0 & 38.4 & 51.2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

The Global finite element equation or equilibrium equation in matrix form is $\{F\} = [K]\{\delta\}$.

$$
\begin{Bmatrix} F_{1x} \\ F_{1Y} \\ F_{2x} \\ F_{2Y} \\ F_{3x} \\ F_{3Y} \\ F_{4x} \\ F_{4Y} \end{Bmatrix} = 10^3 \begin{bmatrix}
133.33 & 0 & -133.33 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-133.33 & 0 & 162.13 & 38.4 & 0 & 0 & -28.8 & -38.4 \\
0 & 0 & 38.4 & 151.2 & 0 & -100 & -38.4 & -57.2 \\
0 & 0 & 0 & 0 & 133.33 & 0 & -133.33 & 0 \\
0 & 0 & 0 & -100 & 0 & 100 & 0 & 0 \\
0 & 0 & -28.8 & -38.4 & -133.33 & 0 & 162.13 & 38.4 \\
0 & 0 & -38.4 & -57.2 & 0 & 0 & 38.4 & 57.2
\end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix}
$$

The known boundary conditions are, $Q_1 = Q_2 = 0$, $Q_4 = 0$, $Q_7 = Q_8 = 0$.

$F_{2x} = 4\times10^3 N$; $F_{3x} = F_{3Y} = 0$,

$$
\begin{Bmatrix} F_{1x} \\ F_{1Y} \\ 4\times10^3 \\ F_{2Y} \\ 0 \\ 0 \\ F_{4x} \\ F_{4Y} \end{Bmatrix} = 10^3 \begin{bmatrix}
133.33 & 0 & -133.33 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-133.33 & 0 & 162.13 & 38.4 & 0 & 0 & -28.8 & -38.4 \\
0 & 0 & 38.4 & 151.2 & 0 & -100 & -38.4 & -57.2 \\
0 & 0 & 0 & 0 & 133.33 & 0 & -133.33 & 0 \\
0 & 0 & 0 & -100 & 0 & 100 & 0 & 0 \\
0 & 0 & -28.8 & -38.4 & -133.33 & 0 & 162.13 & 38.4 \\
0 & 0 & -38.4 & -57.2 & 0 & 0 & 38.4 & 57.2
\end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{Bmatrix}
$$

The rows & columns corresponding to DOF 1, 2, 4, 7, 8 which correspond to fixed support are neglected. The reduced finite element equations are,

$$4\times10^3 = 10^3 [162.13\, Q_3]$$

$$Q_3 = 0.0246 \text{mm}$$

$$0 = (133.33\, Q_5)\, 10^3$$

$$Q_5 = 0$$

$$0 = 100\, Q_6$$

$$Q_6 = 0$$

ii) Support reactions

$F_{1x} = 10^3[-133.33(0.0246)]$.

$\therefore F_{1x} = -3279\,N$ (or) $3.279\,kN$ (Compressive)

$F_{2y} = 10^3[38.4(0.0246)]$

$F_{2y} = 944.64\,N$ (or) $0.944\,kN$ (Tensile)

$F_{4x} = 10^3[-28.8(0.0246)]$

$= -708.4\,N$ (or) $0.708\,kN$ (Compressive)

$F_{4y} = 10^3[-38.4(0.0246)] = -944.64\,N$ (or) $0.944\,kN$ (Compressive)

iii) Element stresses

The stress in each element can be determine using

$$\sigma = \frac{E_e}{l_e}[-l \; -m \; l \; m]\{q\}$$

The connectivity of element ① is 1-2

$\{q\}_1 = \{Q_1 \; Q_2 \; Q_3 \; Q_4\}^T$

$\therefore$ stress in element ①

$$\sigma_1 = \frac{200\times10^3}{300}[-1 \; 0 \; 1 \; 0]\begin{Bmatrix}0\\0\\0.0246\\0\end{Bmatrix}$$

$\sigma_1 = 16.4\,N/mm^2$

The connectivity of element ② is 2-3.

$\{q\}_2 = \{Q_3 \; Q_4 \; Q_5 \; Q_6\}^T$

$\therefore$ stress in element ②

$$\sigma_2 = \frac{200\times10^3}{400}[0 \; -1 \; 0 \; 1]\begin{Bmatrix}0.0246\\0\\0\\0\end{Bmatrix}$$

$\sigma_2 = 500(0) = 0$

The connectivity of element ③ is 2-4

$\{q\}_3 = \{Q_3 \; Q_4 \; Q_7 \; Q_8\}$

$$\sigma_3 = \frac{200\times10^3}{500}[-0.6 \; -0.8 \; 0.6 \; 0.8]\begin{Bmatrix}0.0246\\0\\0\\0\end{Bmatrix}$$

$\sigma_3 = 400[-0.6\times0.0246]$

$\sigma_3 = -5.904\,N/mm^2$

$\sigma_3 = 5.904\,N/mm^2$ (~~Compressive~~)

The connectivity of element ④ is 4-3.

$\{q\}_4 = \{Q_5 \; Q_6 \; Q_7 \; Q_8\}^T$

$\sigma_4 = 0$.