Cayley - Hamilton theorem and quadratic forms:-

Unit - 2

CALEY - HAMILTON THEOREM Every square matrix satisfies its characteristic equation

1\. If $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Caley Hamilton theorem and hence find A^{-1}

Soln Given matrix.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

The characteristic equation matrix of A

$$A-\lambda I = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{bmatrix}$$

the characteristic equation of A is $|A-\lambda I|=0$

$$\begin{bmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)[(3-\lambda)(-2-\lambda)-0] - 1[5(-2-\lambda)+3] + 2[0+1(3-\lambda)] = 0$$

$$(2-\lambda)[-6-3\lambda+2\lambda+\lambda^2) - (-10-5\lambda+3) + 6-2\lambda = 0$$

$$(2-\lambda)(-6-3\lambda+2\lambda+\lambda^2) - (-10-5\lambda+13) + 6-2\lambda = 0$$

$$(2-\lambda)(\lambda^2-\lambda-6) + 5\lambda + 7 + 6 - 2\lambda = 0$$

$$2\lambda^2 - 2\lambda - 12 - \lambda^3 + \lambda^2 + 6\lambda + 3\lambda + 13 = 0$$

$$-\lambda^3 + 3\lambda^2 + 7\lambda + 1 = 0$$

$$\lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0$$

By Caley Hamilton theorem

$$A^3 - 3A^2 7A - I = 0$$

$$A^2 = A \cdot A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+5-2 & 2+3+0 & 4+3-4 \\ 10+15-3 & 5+9+0 & 10+9-6 \\ -2+0+2 & -1+0-0 & -2+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 7 & 5 & 13 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$$

122
12552
24
50208
25104
301248

$$= \begin{bmatrix} 14+25-3 & 7+15+0 & 14+15-6 \\ 44+70-13 & 22+42+0 & 44+42-26 \\ 0-5-2 & 0-3+0 & 0-3-4 \end{bmatrix}$$

$$= \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

$$A^3 - 3A^2 - 7A - I = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -17 \end{bmatrix} - \begin{bmatrix} 21 & 15 & 9 \\ 66 & 42 & 39 \\ 0 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 14 \\ 35 & 21 & 21 \\ -7 & 0 & -14 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ caley Hamilton theorem is satisfied

Now

$$A^3 - 3A^2 - 7A - I = 0$$

$$I = A^3 - 3A^2 - 7A$$

multiplying with A^{-1} o.b.s

$$A^{-1} = A^2 - 3A - 7I$$

$$A^{-1} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 3 & 6 \\ 15 & 9 & 9 \\ -3 & 0 & -6 \end{bmatrix} - \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 2 & -3 \\ 7 & -2 & 4 \\ 3 & -1 & 1 \end{bmatrix}$$

2. Find the inverse of the following matrices by using C-H-T. and also verify C-H-T

i) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$

iv) $\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ v) $\begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

Solu] i): Given matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(2-\lambda)-1] + 1[0-2] + 0 = 0$$

$$(1-\lambda)[2-2\lambda-\lambda+\lambda^2-1] - 2 = 0$$

$$(1-\lambda)[\lambda^2 - 3\lambda+1] - 2 = 0$$

$$\lambda^2 - 3\lambda + 1 - \lambda^3 + 3\lambda^2 - \lambda - 2 = 0$$

$$-\lambda^3 + 4\lambda^2 - 4\lambda - 1 = 0$$

$$\lambda^3 - 4\lambda^2 + 4\lambda + 1 = 0$$

By calcy hamilton theorem

$$A^3 - 4A^2 + 4A + I = 0$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

wrong

$$= \begin{bmatrix} 1+0+0 & 0+0+2 & 2+0+4 \\ -1-1+0 & 0+1+1 & -2+1+2 \\ 0-1+0 & 0+1+2 & 0+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 6 \\ -2 & 2 & 1 \\ -1 & 3 & 5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-0-2 & -1-2-1 & 0-2-2 \\ 2+0+6 & -2+2+3 & 0+2+6 \\ 6+0+10 & -6+1+5 & 0+1+10 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix}$$

$$A^3 - 4A^2 + 4A + I = \begin{bmatrix} -1 & -4 & -4 \\ 8 & 3 & 8 \\ 16 & 0 & 11 \end{bmatrix} - \begin{bmatrix} -4 & -16 & -16 \\ 32 & 12 & 32 \\ 64 & 0 & 44 \end{bmatrix} \begin{bmatrix} 4 & -8 & -24 \\ +8 & 8 & 12 \\ 24 & 4 & 20 \end{bmatrix}$$

$$+ \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ calcy hamilton theorem is satisfied

$$I = -A^3 + 4A^2 - 4A$$

Multiplying with A^{-1}

$$A^{-1} = -A^2 + 4A - 4I$$

$$A^{-1} = -\begin{bmatrix} 1 & -2 & -1 \\ 2 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 4 & -4 & 0 \\ 0 & 4 & 4 \\ 8 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4-4 & 2-4+0 & 1+0+0 \\ -2+0-0 & -2+4-4 & 3+4+0 \\ -6+8-0 & -1+4+0 & -5+8-4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 & 1 \\ -2 & -2 & 7 \\ 2 & 3 & -1 \end{bmatrix}$$

iv) Given matrix

$$A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

characteristic equation of A is matrix

$$A - \lambda I = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left[(-1-\lambda)(-1-\lambda) - 4\right] - 2(-6(-1-\lambda) - 12) - 2(-12 - 6(-1-\lambda))$$

$$(7-\lambda)[+1+\lambda+\lambda+\lambda^2-4]-2(6+6\lambda-12)-2(-12+6+6\lambda)=0$$

$$(7-\lambda)[\lambda^2+2\lambda-3]-2(6\lambda-6)-2(6\lambda-6)=0$$

$$7\lambda^2+14\lambda-21-\lambda^3-2\lambda^2+3\lambda-12\lambda+12-12\lambda+12=0$$

$$-\lambda^3+5\lambda^2-7\lambda+3=0$$

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

By caley Hamilton theorem

$$A^3-5A^2+7A-3I=0$$

$$A^2=\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}$$

$$=\begin{bmatrix}49-12-12&14-2-4&-14+4+2\\-42+6+12&-12+1+4&12-2-2\\42-12-6&12-2-2&-12+4+1\end{bmatrix}$$

$$=\begin{bmatrix}25&8&-8\\-24&-7&8\\24&8&-7\end{bmatrix}$$

$$A^3=A^2.A=\begin{bmatrix}25&8&-8\\-24&-7&8\\24&8&-7\end{bmatrix}\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}$$

$$=\begin{bmatrix}175-48-48&50-8-16&-50+16+8\\-168+42+48&-48+7+16&+48-14-8\\168-48-42&48-8-14&-48+16+7\end{bmatrix}$$

$$=\begin{bmatrix}79&26&-26\\-78&-25&26\\78&26&-25\end{bmatrix}$$

$$7A=7\begin{bmatrix}7&2&-2\\-6&-1&2\\6&2&-1\end{bmatrix}=\begin{bmatrix}49&14&-14\\-42&-7&14\\42&14&-7\end{bmatrix}$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - \begin{bmatrix} 125 & 40 & -40 \\ -120 & -135 & 40 \\ 120 & 40 & -35 \end{bmatrix} + \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix}$$

$$- \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 0$$

$$= \begin{bmatrix} 79-125+49-3 & 26-40+14-0 & -26+40-14-0 \\ -78+120-42-0 & -25+35-7-3 & 26-40+14+0 \\ 78-120+42+0 & 26-40+14-0 & -25+35-7+3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ caley Hamilton theorem is ~~not~~ verified

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^3 - 5A^2 + 7A = 3I \rightarrow (1)$$

Multiplying (1) with A^{-1}

$$A^2 - 5A + 7I = 3A^{-1}$$

$$3A^{-1} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} - \begin{bmatrix} 35 & 10 & -10 \\ -30 & -25 & 10 \\ 30 & 10 & -5 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 25-35+7 & 8-10+0 & -8+10+0 \\ -24+30+0 & -7+5+7 & 8-10+0 \\ 24-30+0 & 8-10+0 & -7+5+7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -3 & -2 & 2 \\ 6 & 5 & -2 \\ -6 & -2 & 5 \end{bmatrix}$$

iii) Given matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(5-\lambda)(5-\lambda)-1\right] - 1\left[-1(5-\lambda)+1\right] + 1\left[1-(5-\lambda)\right] = 0$$

$$(3-\lambda)\left[25-5\lambda-5\lambda+\lambda^2-1\right] - 1\left[-5+\lambda+1\right] + \left[1-5+\lambda\right] = 0$$

$$(3-\lambda)\left[24-10\lambda+\lambda^2\right] - 1[\lambda-4) + 1(\lambda-4) = 0$$

$$(\lambda^2-10\lambda+24)(3-\lambda) \cancel{-(\lambda-4)+(\lambda-4)} = 0$$

$$3\lambda^2 - 30\lambda + 72 - \lambda^3 + 10\lambda^2 - 24\lambda = 0$$

$$-\lambda^3 + 13\lambda^2 - 54\lambda + 72 = 0$$

$$\lambda^3 - 13\lambda^2 + 54\lambda - 72 = 0$$

By Caley Hamilton theorem

$$A^3 - 13A^2 + 54A - 72I = 0$$

$$A^2 = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 9-1+1 & 3+5-1 & 3-1+5 \\ -3-5-1 & -1+25+1 & -1-5-5 \\ 3+1+5 & 1-5-5 & 1+1+25 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 27-7+7 & 9+35-7 & 9-7+35 \\ -27-25-11 & -9+125+11 & -9-25-55 \\ 27+9+27 & 9-45-27 & 9+9+135 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix}$$

$$A^3 - 13A^2 + 54A - 72I$$

$$= \begin{bmatrix} 27 & 37 & 37 \\ -63 & 127 & -89 \\ 63 & -63 & 153 \end{bmatrix} - \begin{bmatrix} 117 & 91 & 91 \\ -117 & 325 & -143 \\ 117 & -117 & 351 \end{bmatrix} + \begin{bmatrix} 162 & 54 & 54 \\ -54 & 270 & -54 \\ 54 & -54 & 270 \end{bmatrix} - \begin{bmatrix} 72 & 0 & 0 \\ 0 & 72 & 0 \\ 0 & 0 & 72 \end{bmatrix}$$

$$= \begin{bmatrix} 27-117+162-72 & 37-91+54-0 & 37-91+54-0 \\ -63+117-54+0 & 127-325+270-72 & -89-325-54+0 \\ 63-117+54+0 & -63+117-54+0 & 153-351+270-72 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Coley Hamilton theorem is verified

$$A^3 - 13A^2 + 54A - 72I = 0$$

$$A^3 - 13A^2 + 54A = 72I$$

$$A^2 - 13A + 54I = 72A^{-1}$$

$$72A^{-1} = \begin{bmatrix} 9 & 7 & 7 \\ -9 & 25 & -11 \\ 9 & -9 & 27 \end{bmatrix} - 13\begin{bmatrix} 39 & 13 & 13 \\ -13 & 65 & -13 \\ 13 & -13 & 65 \end{bmatrix} + \begin{bmatrix} 54 & 0 & 0 \\ 0 & 54 & 0 \\ 0 & 0 & 54 \end{bmatrix}$$

$$72A^{-1} = \begin{bmatrix} 9-39+54 & 7-13+0 & 7-13+0 \\ -9+13+0 & 25-65+54 & -11+13+0 \\ 9-13+0 & -9+13+0 & 27-65+54 \end{bmatrix}$$

$$A^{-1} = \frac{1}{72}\begin{bmatrix} 24 & -6 & -6 \\ 4 & 14 & 2 \\ -4 & 4 & 16 \end{bmatrix}$$

ii) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(-1-\lambda)(-1-\lambda) - 4] - 2[2(-1-\lambda)-12] + 3(2-3(-1-\lambda)) = 0$$

$$(1-\lambda)[1+\lambda+\lambda+\lambda^2-4] - 2[-2-2\lambda-12] + 3(2+3+3\lambda) = 0$$

$$(1-\lambda)[\lambda^2+2\lambda-3] - 2(-2\lambda-14) + 3(3\lambda+5) = 0$$

$$\lambda^2+2\lambda-3-\lambda^3-2\lambda^2+3\lambda+4\lambda+28+9\lambda+15 = 0$$

$$-\lambda^3-\lambda^2+18\lambda+40 = 0$$

By caley Hamilton theorem

$$A^3 + A^2 - 18A - 40I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8-3 \\ 2-2+12 & 4+1+4 & 6-4-4 \\ 3+2-3 & 6-1-1 & 9+4+1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+24 & 28-3+8 & 42-12-8 \\ 12+18-6 & 24-9-2 & 36+36+2 \\ 2+8+42 & 4-4+14 & 6+16-14 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 22 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$A^3 + A^2 - 18A - 40I$$

$$= \begin{bmatrix} 44 & 33 & 22 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 44+14-18+40 & 33+3-36-0 & 22+8-54+0 \\ 24+12-36-0 & 13+9+18-40 & 74-2-72-0 \\ 52+2-54-0 & 14+4-18+0 & 8+14+18-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ since Caley Hamilton theorem is verified

$$A^3 + A^2 - 18A - 40I = 0$$

$$A^3 + A^2 - 18A = 40I$$

$A^2 + A - 18I = 40A^{-1}$

$$40A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 14+1-18 & 3+2-0 & 8+3+0 \\ 12+2-0 & 9-1-18 & -2+4+0 \\ 2+3+0 & 4+1+0 & 14-1-18 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

v) Given matrix

$$A = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 8-\lambda & -8 & 2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(-3-\lambda)(1-\lambda) - 8] + 8[4(1-\lambda)+6] + 2[-16-3(-3-\lambda)] = 0$$

$$(8-\lambda)[-3-\lambda+3\lambda+\lambda^2-8] + 8[4-4\lambda+6] + 2(-16+9+3\lambda) = 0$$

$$(8-\lambda)[\lambda^2+2\lambda-11] + 8(10-4\lambda) + 2(3\lambda-7) = 0$$

$$8\lambda^2 + 16\lambda - 88 - \lambda^3 - 2\lambda^2 + 11\lambda + 80 - 32\lambda + 6\lambda - 14 = 0$$

$$-\lambda^3 + 6\lambda^2 + \lambda - 22 = 0$$

$$\lambda^3 - 6\lambda^2 - \lambda + 22 = 0$$

By caley Hamilton theorem

$$A^3 - 6A^2 - A + 22I = 0$$

$$A^2 = \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 64-32+6 & -64+24-8 & 16+16+2 \\ 32-12-6 & -32+9+8 & 8+6-2 \\ 24-16+3 & -24+12-4 & 6+8+1 \end{bmatrix}$$

$$= \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 38 & -48 & 34 \\ 14 & -15 & 12 \\ 11 & -16 & 15 \end{bmatrix} \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 304-192+102 & -304+144-136 & 76+96+34 \\ 112-60+36 & 112+45-48 & 28+30+12 \\ 88-64+45 & -88+48-60 & 22+32+15 \end{bmatrix}$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & 109 & 70 \\ 69 & -100 & 69 \end{bmatrix}$$

$$A^3 - 6A^2 - A + 22I = 0$$

$$= \begin{bmatrix} 214 & -296 & 206 \\ 88 & 109 & 70 \\ 69 & -100 & -69 \end{bmatrix} - \begin{bmatrix} 228 & -288 & 204 \\ 84 & -90 & 72 \\ 66 & -96 & 90 \end{bmatrix} - \begin{bmatrix} 8 & -8 & 2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + \begin{bmatrix} 22 & 0 & 0 \\ 0 & 22 & 0 \\ 0 & 0 & 22 \end{bmatrix}$$

$$= \begin{bmatrix} 914-228-8+22 & 1-296+288+8+0 & 206-204-2+0 \\ 88-84-4+0 & 109+90+3+22 & 70-72+2+0 \\ 69-66-3+0 & -100+96+4+6 & 69-90-1+22 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Caley Hamilton theorem is verified.

$$A^3 - 6A^2 - A + 22I = 0$$

$$22I = -A^3 + 6A^2 + A$$

Multiplying with A^{-1}

$$22A^{-1} = -A^2 + 6A + I$$

$$22A^{-1} = \begin{bmatrix} -38 & 48 & -34 \\ -14 & 15 & -12 \\ -11 & 16 & -15 \end{bmatrix} + \begin{bmatrix} 48 & -48 & 12 \\ 24 & -18 & -12 \\ 18 & -24 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$22A^{-1} = \begin{bmatrix} -38+48+1 & 48-48+0 & -34+12+0 \\ -14+24+0 & 15-18+1 & -12-12+0 \\ -11+18+0 & 16-24+0 & -15+6+1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{22}\begin{bmatrix} 11 & 0 & -22 \\ 10 & -2 & -24 \\ 7 & -8 & -8 \end{bmatrix}$$

vi) Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I|=0 \rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix}=0$$

$$(1-\lambda)\left[(4-\lambda)(6-\lambda)-25\right]-2\left[2(6-\lambda)-15\right]+3\left[10-3(4-\lambda)\right]=0$$

$$(1-\lambda)\left[24-6\lambda-4\lambda+\lambda^2-25\right]-2\left[12-2\lambda-15\right]+3\left[10-12+3\lambda\right]=0$$

$$(1-\lambda)\left[\lambda^2-10\lambda-1\right]-2\left[-2\lambda-3\right]+3\left[3\lambda-2\right]=0$$

$$\lambda^2-10\lambda-1-\lambda^3+10\lambda^2+\lambda+4\lambda+\cancel{6}+9\lambda\cancel{-6}=0$$

$$-\lambda^3+11\lambda^2+4\lambda-1=0$$

$$\lambda^3-11\lambda^2-4\lambda+1=0$$

By calcy Hamilton theorem

$$A^3-11A^2-4A+I=0$$

$$A^2=A\cdot A=\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$$

$$=\begin{bmatrix}1+4+9 & 2+8+15 & 3+10+18\\ 2+8+15 & 4+16+25 & 6+20+30\\ 3+10+18 & 6+20+30 & 9+25+36\end{bmatrix}=\begin{bmatrix}14&25&31\\25&45&56\\31&56&70\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}14&25&31\\25&45&56\\31&56&70\end{bmatrix}\begin{bmatrix}1&2&3\\2&4&5\\3&5&6\end{bmatrix}$$

$$=\begin{bmatrix}14+50+93 & 28+100+155 & 42+125+186\\ 25+90+168 & 50+180+280 & \cancel{125}\ 75+225+336\\ 31+112+210 & 62+224+350 & 93+280+420\end{bmatrix}$$

$$=\begin{bmatrix}157&283&353\\283&510&636\\353&636&793\end{bmatrix}$$

A^3-11A^2-4A+I

$$= \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 253 & 636 & 793 \end{bmatrix} - \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 770 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 157-154-4+1 & 283-275-8+0 & 353-341-12+0 \\ 283-275-8+0 & 510-495-16+1 & 636-616-20+0 \\ 253-341-12+0 & 636-616-20+0 & 793-770-24+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ coley Homilton theorem is verified

$$A^3-11A^2-4A+I=0$$

$$I = -A^3+11A^2+4A$$

multiplying with A^{-1}

$$A^{-1} = -A^2+11A+4I$$

$$A^{-1} = \begin{bmatrix} -14 & -25 & -31 \\ -25 & -45 & -56 \\ -31 & -56 & -70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -14+11+4 & -25+22+0 & -31+33+0 \\ -25+22+0 & -45+44+4 & -56+55+0 \\ -31+33+0 & -56+55+0 & -70+66+4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

3. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ express $2A^5 - 3A^4 + A^2 - 4I$ as a linear polynomial in A

note
29/12/2018

Solu| Given matrix

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(2-\lambda) + 1 = 0$$

$$6 - 2\lambda - 3\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 5\lambda + 7 = 0$$

By caley Hamilton theorem

$$A^2 - 5A + 7I = 0$$

$$A^2 = 5A - 7I$$

$$A^3 = 5A^2 - 7A$$

$$A^4 = 5A^3 - 7A^2$$

$$A^5 = 5A^4 - 7A^3$$

$$2A^5 - 3A^4 + A^2 - 4I = 2[5A^4 - 7A^3] - 3A^4 + A^2 - 4I$$

$$= 7A^4 - 14A^3 + A^2 - 4I$$

$$= 7[5A^3 - 7A^2] - 14A^3 + A^2 - 4I$$

$$= 21A^3 - 48A^2 - 4I$$

$= 21[5A^2 - 7A] - 48A^2 - 4I$

$= 57A^2 - 147A - 4I$

$= 57(5A - 7I) - 147A - 4I$

$= 138A - 403I$

4. If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a polynomial

A) Given matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

The characteristic matrix of A is

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}$$

the characteristic equation of A is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(3-\lambda) + 2 = 0$

$3 - 3\lambda - \lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 4\lambda + 5 = 0$

By Cayley Hamilton theorem

$A^2 - 4A + 5I = 0$

$A^2 = 4A - 5I$

$A^3 = 4A^2 - 5A$

$A^4 = 4A^3 - 5A^2$

$A^5 = 4A^4 - 5A^3$

$A^6 = 4A^5 - 5A^4$

Given equation

$A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$A^6 - 14 = (4A^5 - 5A^4) - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= 3[4A^3 - 5A^2] \cancel{- 12A^3} + 14A^2$

$= -15A^2 + 14A^2$

$= -A^2 = -4A + 5I$

Quadratic forms:

* A homogeneous expression of the second degree in any no. of variables is called a quadratic form.

Ex: 1. $3x^2 + 5xy - 2y^2$ is a quadratic form in x & y

2. $x^2 + 2y^2 - 3z^2 + 2xy - 3yz + 5zx$ is a quadratic form in three variables.

* An expression of the form $Q = X^T AX = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ where A_{ij} are constants is called a quadratic form in n variables.

Matrix of a Quadratic form

Every quadratic form Q can be expressed as $Q = X^T AX$. The symmetric matrix A is called the matrix of the quadratic form Q and $|A|$ is called the discriminant of the quadratic form

Note:

* If $|A| = 0$ the quadratic form is singular.

* Ex: To write the matrix of quadratic form follow the diagram given below

write the co-efficients of square terms along the diagonal and divide the co-efficients of the product terms, xy, yz, zx by 2 and write them at the appropriate places.

Ex: $Q = 7x^2 + 8xy + 9yz + 2xz + 3y^2 - 5z^2$

$Q = 7xx + 4xy + 4xy + \frac{9}{2}yz + 2z + xz + 3yy - 5zz$

$= 7xx + 4xy + xz$

$+ 4yx + 3yy + \frac{9}{2}yz$

$+ zx + \frac{9}{2}zy - 5zz$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & x^2 & \frac{xy}{2} & \frac{xz}{2} \\ y & \frac{yx}{2} & y^2 & \frac{yz}{2} \\ z & \frac{zx}{2} & \frac{zy}{2} & z^2 \end{array}$$

$$A = \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & 9/2 \\ 1 & 9/2 & -5 \end{bmatrix}$$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $X^T = [x \ y \ z]$;

$$Q = X^T A X = [x\ y\ z] \begin{bmatrix} 7 & 4 & 1 \\ 4 & 3 & 9/2 \\ 1 & 9/2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

write the symmetric matrix of the following Q.F

1. $x^2 + 2y^2 - 7z^2 - 4xy - 6xz$
2. $2x^2 - 3y^2 + 5z^2 - 6xy - yz + 4zx$
3. $4xy + 6yz + 8zx$
4. $x^2 - y^2 + z^2 + 7xy + 9yz + 11zx$

4. $Q = x^2 - y^2 + z^2 + 7xy + 9yz + 11zx$

$Q = xx - yy + zz + \frac{7}{2}xy + \frac{9}{2}yz + \frac{11}{2}zx$

$= xx + \frac{7}{2}xy + \frac{11}{2}zx$

$+ \frac{7}{2}yx - yy + \frac{9}{2}yz$

$+ \frac{11}{2}yz + \frac{9}{2}zy + zz$

$$A = \begin{bmatrix} 1 & 7/2 & \frac{11}{2} \\ \frac{7}{2} & -1 & \frac{9}{2} \\ \frac{11}{2} & \frac{9}{2} & 1 \end{bmatrix}$$

3. $Q = 4xy + 6yz + 8zx$

$Q = 2xy + 3yz + 4zx$

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 0 & 3 \\ 4 & 3 & 0 \end{bmatrix}$$

2. $Q = 2x^2 - 3y^2 + 5z^2 - 6xy - yz + 4zx$

$2xx - 3yy + 5zz - 3xy - \frac{1}{2}yz + 2zx$

$Q = 2xx - 3xy + 2zx$

$-3yx - 3yy - \frac{1}{2}yz$

$+2zx - \frac{1}{2}zy + 5zz$

$$A = \begin{bmatrix} 2 & -3 & 2 \\ -3 & -3 & -1/2 \\ 2 & -1/2 & 5 \end{bmatrix}$$

1. $x^2 + 2y^2 - 7z^2 - 4xy - 6xz$

$Q = xx + 2yy - 7zz - 2xy - 3xz$

$Q = xx - 2xy - 3xz$

$-2xy + 2yy + 0 \cdot yz$

$-3xz + 0 \cdot yz - 7zz$

$$A = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 2 & 0 \\ -3 & 0 & -7 \end{bmatrix}$$

Date 31/12/2018

Write the Quadratic form of corresponding to the matrix

1) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 2) $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$ 3) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}$

4) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ 5) $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}, X^T = [x \ y \ z] \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

quadratic form $Q = X^T A X$

$$= [x \ y \ z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x+2y+3z \quad 2x+3z \quad 3x+3y+z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(x+2y+3z) + y(2x+3z) + z(3x+3y+z)$$

$$= x^2+2xy+3zx+2xy+3zx+3zy+z^2$$

$$= x^2+z^2+4xy+6zx+3zy$$

3) Given matrix

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix}, X^T = [x \ y \ z] ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form

$$Q = X^T A X$$

$$= [x \ y \ z] \begin{bmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 5 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x+2y+5z \quad 2x+3z \quad 5x+3y+4z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= x(x+2y+5z) + y(2x+3z) + z(5x+3y+4z)$$

$$= x^2+2xy+5zx+2xy+3zy+5zx+3zy+4z^2$$

$$= x^2+4z^2+4xy+10zx+6yz$$

Rank of a Quadratic form

Let X^TAX be a quadratic form the rank $R(A)$ is called the rank of the quadratic form. If 'r' is less than n, $|A| = 0$ (or) A is singular then the quadratic form is called "singular" otherwise "non-singular"

Canonical Form (or) Normal form of a Quadratic form

Let X^TAX be a quadratic form in n variables then there exsist a real non-singular linear transformation $X = PY$ which transforms X^TAX to another quadratic form of type $Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \cdots + \lambda_n y_n^2$ then Y^TDY is called the canonical form of quadratic form of X^TAX

Here $D = diag(\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_n)$

Index of a Real Quadratic Form

The number of positive terms in canonical form of quadratic form is known as the index of the quadratic form and is denoted by 's'

Signature of a Quadratic form

If 'r' is the rank of the quadratic form and 's' is the index of the quadratic form then $2s-r$ is called the signature of the quadratic form X^TAX.

Nature of Quadratic Forms

⇒ Positive Definite

The Quadratic form X^TAX in 'n' variables is said to be positive Definite if all the Eigen values of A are positive (or) If $r=n$ and $s=n$ i.e., $r=s=n$

⇒ Negative Definite

The quadratic form X^TAX in n variables is said to be negative definite if $r=n$ and $s=0$ (or) If all the eigen values of A are negative.

⇒ Positive - Semi - Definite

The Quadratic form X^TAX in n variables is said to be positive semi definite if $r<n$ & $s=r$ (or) If all the eigen values of $A \geq 0$ and atleast one eigen value is zero

⇒ Negative - Semi - Definite

The Quadratic form X^TAX in n variables is said to be negative semidefinite if $r<n$ & $s=0$ (or) If all the eigen values of $A \leq 0$. and atleast one eigen value is zero

⇒ In-Definite

In all other cases, if all the eigen values of A are positive and negative then the quadratic form is called in-definite

1. Identify the nature of the Quadratic forms

i) $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$

ii) $x^2+4xy+6xz-y^2+2yz+4z^2$

iii) $x^2+y^2+2z^2-2xy+2xz$

iv) $2x^2+19y^2+62z^2+8xy+8yz+6zx$

Solu) i) Given Quadratic form

$$Q=x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$$

$$Q=X^TAX,\quad A=\begin{bmatrix}1&-2&1\\-2&4&-2\\1&-2&1\end{bmatrix}$$

characteristic equation of A is

$$|A-\lambda I|=0$$

$$\begin{vmatrix}1-\lambda&-2&1\\-2&4-\lambda&-2\\1&-2&1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)[(4-\lambda)(1-\lambda)-4]+2[-2(1-\lambda)+2]+1(4-(4-\lambda))=0$$

$$(1-\lambda)[\cancel{4}-\lambda-4\lambda+\lambda^2\cancel{-4}]+2[\cancel{-2}+2\lambda\cancel{+2}]\cancel{+4}\cancel{-4}+\lambda=0$$

$$(1-\lambda)[\lambda^2-5\lambda]+4\lambda+\lambda=0$$

$$\lambda^2\cancel{-5\lambda}-\lambda^3+5\lambda^2\cancel{+5\lambda}=0$$

$$-\lambda^3+6\lambda^2=0$$

$$\lambda^3=6\lambda^2$$

$$\lambda=6\;;\;\lambda=0,0,6$$

Eigen values two are zeroes and the remaining is positive

Hence given Quadratic form is positive semidefinite

(iv) Given Quadratic form

$$Q = 2x^2+9y^2+6z^2+8xy+8yz+6zx$$

$$Q = X^T A X \; ; \quad A = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 9 & 4 \\ 3 & 4 & 6 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 4 & 3 \\ 4 & 9-\lambda & 4 \\ 3 & 4 & 6-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(9-\lambda)(6-\lambda)-16] - 4[4(6-\lambda)-12] + 3[16-3(9-\lambda)] = 0$$

$$(2-\lambda)[54-6\lambda-9\lambda+\lambda^2-16] - 4[24-4\lambda-12] + 3[16-27+3\lambda] = 0$$

$$(2-\lambda)[38-15\lambda+\lambda^2] - 4[12-4\lambda] + 3[3\lambda-11] = 0$$

$$2\lambda^2-30\lambda+76-\lambda^3+15\lambda^2-38\lambda-48+16\lambda+9\lambda-33 = 0$$

$$-\lambda^3+17\lambda^2-43\lambda-5 = 0$$

$$\lambda^3-17\lambda^2+43\lambda+5 = 0$$

14) Given quadratic form

$$Q = x^2 + 4xy + 6xz - y^2 + 2yz + 4z^2$$

$$Q = X^T A X, \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(-1-\lambda)(4-\lambda) - 1\right] - 2\left[2(4-\lambda) - 3\right] + 3\left[2 - 3(-1-\lambda)\right] = 0$$

$$(1-\lambda)\left[-4 - 4\lambda + \lambda + \lambda^2 - 1\right] - 2\left[8 - 2\lambda - 6\right] + 3\left[2 + 3 + 3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2 - 3\lambda - 5\right] - 2\left[-2\lambda + 2\right] + 3\left[3\lambda + 5\right] = 0$$

$$\lambda^2 - 3\lambda - 5 - \lambda^3 + 3\lambda^2 + 5\lambda + 4\lambda - 4 + 9\lambda + 15 = 0$$

$$-\lambda^3 + 4\lambda^2 + 15\lambda + 6 = 0$$

$$\lambda^3 - 4\lambda^2 - 15\lambda - 6 = 0$$

iii) Given Quadratic form

$$Q = x^2 + y^2 + 2z^2 - 2xy + 2xz$$

$$Q = X^T A X \quad A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)(2-\lambda) - 0] + 1[-1(2-\lambda) - 0] + 1[0 - (1-\lambda)] = 0$$

$$(1-\lambda)[(1-\lambda)(2-\lambda) - 0] + 1[-2+\lambda] + \lambda - 1 = 0$$

$$(1-\lambda)[2 - 2\lambda - \lambda + \lambda^2 - 0] + [-2+\lambda] + \lambda - 1 = 0$$

$$(1-\lambda)[\lambda^2 - 3\lambda + 2] - 2 + \lambda + \lambda - 1 = 0$$

$$\lambda^2 - 3\lambda + 2 - \lambda^3 + 3\lambda^2 - \cancel{2\lambda} - 3 + \cancel{2\lambda} = 0$$

$$-\lambda^3 + 4\lambda^2 - 3\lambda - 1 = 0$$

$$\lambda^3 - 4\lambda^2 + 3\lambda + 1 = 0$$

2. Given matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix} \quad X^T = [x, y, z] \; ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^TAX$

$$= [x \; y \; z] \begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [2x+y+5z \quad x+3y-2z \quad 5x-2y+4z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x(2x+y+5z) \quad y(x+3y-2z) \quad z(5x-2y+4z)]$$

$$= 2x^2+xy+5zx+xy+3y^2-2zy+5zx-2zy+4z^2$$

$$= 2x^2+3y^2+4z^2+2xy+10zx-4zy$$

4. Given matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \quad X^T = [x \; y \; z] \; ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^TAX$

$$= [x \; y \; z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x+2y+3z \quad 2x+y+3z \quad 3x+3y+z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x^2+2xy+3zx+2xy+y^2+3zy+3zx+3yz+z^2$$

$$= x^2+y^2+z^2+4xy+6zx+6zy.$$

5. Given matrix

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \quad X^T = [x \ y \ z] \ ; \ X \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Quadratic form $Q = X^T A X$

$$= [x \ y \ z] \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [5y - z \quad 5x + y + 6z \quad -x + 6y + 2z] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x5y - zx + 5xy + y^2 + 6zy - xz + 6yz + 2z^2]$$

$$y^2 + 2z^2 + 10xy + 12yz - 2xz$$

Date 9/1/19

1. Reduce the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to a diagonal part and interrupt the result in terms of Quadratic forms also find the rank signature, Index.

Solu) $A = I_3 A I_3$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 & 2 \\ 0 & 7 & -1 \\ 0 & -1 & 7 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 + R_1 \\ R_3 \to 3R_3 - R_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & -3 & 21 \end{bmatrix} \begin{matrix} C_2 \to 3C_2 + C_1 \\ C_3 \to 3C_3 - C_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -1 & 0 & 3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & -3 \\ 0 & 0 & 144 \end{bmatrix} R_3 \to 7R_3 + R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} C_3 \to 7C_3 + C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

$D = P^T A P$

$D = \text{dia}(6, 21, 1008)$

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 1008 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ -6 & 3 & 21 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix}$$

Quadratic form $= X^T A X$

$$= [x \; y \; z] \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$= 6x^2 + 3y^2 + 3z^2 - 4xy + 4xz - 2yz$

Non-singular transformation corresponding to the matrix P is $X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & -6 \\ 0 & 3 & 3 \\ 0 & 0 & 21 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= [y_1 + y_2 - 6y_3, \; 3y_2 + 3y_3 \; 21y_3]$

$x = y_1 + y_2 - 6y_3 ; \; y = 3y_2 + 3y_3 ; \; z = 21y_3$

canonical form $= Y^T D Y = 6y_1^2 + 21y_2^2 + 1008y_3^2$

Rank of A is $\rho(A) = 3$ (rank of diagonal matrix non zero rows)

Index $= s = 3$ (no. of positive terms)

Signature $= 2s - r = 2(3) - 3 = 3$

2\. Find the rank, signature, index of the quadratic form

i) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_1x_3$ by reducing it into canonical form also write the linear transformation which brings about the normal reduction solu) Given

Quadratic form

$= 2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_1x_3$

Given quadratic form into matrix

$$A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

$A = I_3 A I_3$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & -2 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} \begin{matrix} C_2 \to C_2 - 3C_1 \\ C_3 \to C_3 + C_1 \end{matrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 0 & 81 \end{bmatrix} R_3 \to -17R_3 - 2R_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 1377 \end{bmatrix} C_3 \to -17C_3 - 2C_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} A \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -2 \\ 0 & 0 & -17 \end{bmatrix}$$

$D = P^T A P$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & 1377 \end{bmatrix} \quad P^T = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & -2 & -17 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -2 \\ 0 & 0 & -17 \end{bmatrix}$$

Quadratic form $= X^TAX$

$$= [x\ y\ z] \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{matrix} 2x^2 + 6xy - 2zx \\ +6xy + y^2 - 4zy \\ -2xz - 4yz - 3z^2 \end{matrix}$$

$= [2x + 6y - 2z \quad 6x + y - 4z \quad -2x - 4y - 3z]$

$2x^2 + y^2 = 3z^2 + 12xy - 4zx - 8zy$

Non singular transformation corresponding to the matrix P is $X = PY$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & -11 \\ 0 & 1 & -9 \\ 0 & 0 & -17 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [y_1 - 3y_2 - 11y_3 \quad y_2 - 2y_3 \quad -17y_3]$$

$x = y_1 - 3y_2 - 11y_3$; $y = y_2 - 2y_3$; $z = -17y_3$

canonical form $= Y^T D Y = 2y_1^2 - 17y_2^2 + 1377y_3^2$

Rank of A is $\rho(A) = 3$

Index $s = 2$

signature $= 2s - r = 2(2) - 3 = 1$

Date 3/1/2019

3. Reduction to Normal form by orthogonal transformation.

Working Rule:

1. write the co-efficient matrix 'A' associated with the given quadratic form
2. Find the Eigen values of A
3. write the canonical form using $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$
4. Form the matrix P containing the normalized Eigen vectors of A as column vectors. then $X = PY$ gives the required orthogonal transformation which reduces quadratic form to canonical form.

1 Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation.

Sol) Given quadratic form

$$Q = 3x^2+2y^2+3z^2-2xy-2yz$$

The matrix form

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(3-\lambda)-1\right]+1\left[-(3-\lambda)-0\right]+0=0$$

$$(3-\lambda)\left[6-3\lambda-2\lambda+\lambda^2-1\right]+\left[-3+\lambda\right]=0$$

$$(3-\lambda)\left[\lambda^2-5\lambda+5\right]-3+\lambda=0$$

$$3\lambda^2-15\lambda+15-\lambda^3+5\lambda^2-5\lambda-3+\lambda=0$$

$$-\lambda^3+8\lambda^2-19\lambda+12=0$$

$$\lambda^3-8\lambda^2+19\lambda-12=0$$

$$\begin{array}{r|rrrr} 1 & 1 & -8 & 19 & -12 \\ & 0 & 1 & -7 & 12 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2-7\lambda+12)=0$$

$$(\lambda-1)\left[\lambda^2-4\lambda-3\lambda+12\right]=0$$

$$(\lambda-1)\left[(\lambda(\lambda-4)-3(\lambda-4)\right]=0$$

$$(\lambda-1)(\lambda-4)(\lambda-3)=0$$

$$\lambda = 1, 4, 3$$

The are the characteristic roots 1, 4, 3 case (i)

If $\lambda = 1$ $(A - \lambda I)x = 0$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to 2R_2 + R_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rank $= 2$, $n = 3$.

$n - r = 3 - 2 = 1$ L.I.S

$2x - y = 0$; $y - 2z = 0$; $z = k$.

$2x - 2k = 0$ $y = 2k$

$x = k$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

case (ii)

If $\lambda = 4$ $(A - \lambda I)x = 0$

$$\begin{bmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$; $n - r = 3 - 2 = 1$ L.I.S

$-x-y=0$; $-y-z=0$ $z=K$

$-x+K=0$ $\quad -y-K=0$

$x=+K$ $\quad -y=K$

$y=-K$

$$X_3=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -K \\ -K \\ K \end{bmatrix}=\begin{bmatrix} +1 \\ -1 \\ 1 \end{bmatrix}$$

Eigen

$$x_1=\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \; x_2=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \; x_3=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

case (iii)

If $\lambda=3$; then $(A-\lambda I)x=0$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \; R_3 \to R_3-R_1 \quad =\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A)=2$; $n=3$

$n-r=3-2$, L.I.S

$-y=0$; $-x+y-z=0$; $z=K$

$-x-0-K=0$

$-x=K$

$x=-K$

$$X_2=\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} -K \\ 0 \\ K \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}K$$

Here we observed that X_1, X_2, X_3 are mutually $\perp$ler

$X_1 X_2 = X_2 \cdot X_3 = X_3 X_1 = 0$

The normalized vectors are

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \quad e_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad e_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$P = [e_1 \; e_2 \; e_3] = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -1/\sqrt{3} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & 1/\sqrt{3} \end{bmatrix}$$

$$D = P^T A P$$

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Dat
4|1|1

$$D = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 1/\sqrt{6} \\ \frac{-1}{\sqrt{2}} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{6} - 2/\sqrt{6} + 0 & -1/\sqrt{6} + 4/\sqrt{6} - 1/\sqrt{6} & 0 - 2/\sqrt{6} + 3/\sqrt{6} \\ -3/\sqrt{2} + 0 + 0 & 1/\sqrt{2} + 0 - 1/\sqrt{2} & 0 + 0 + 3/\sqrt{2} \\ 3/\sqrt{3} + 1/\sqrt{3} + 0 & -1/\sqrt{3} - 2/\sqrt{3} - 1/\sqrt{3} & 0 + 1/\sqrt{3} + 3/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & \frac{-1}{\sqrt{2}} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

solu 2

$$= \begin{bmatrix} 1/\sqrt{6} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -3/\sqrt{2} & 0 & 3/\sqrt{2} \\ 4/\sqrt{3} & -4/\sqrt{3} & 4/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6} + \frac{4}{6} + \frac{1}{6} & -1/\sqrt{12} + 0 + 1/\sqrt{12} & 1/\sqrt{18} - 2/\sqrt{18} + 1/\sqrt{18} \\ \frac{-3}{\sqrt{12}} + 0 + 3/\sqrt{12} & 3/2 + 0 + 3/2 & -3/\sqrt{6} + 0 + 3/\sqrt{6} \\ 4/\sqrt{18} - 8/\sqrt{18} + 4/\sqrt{18} & -4/\sqrt{6} + 0 + 4/\sqrt{6} & \frac{4}{3} + \frac{4}{3} + \frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{diag}(1, 3, 4)$$

orthogonal transformation

$$X = PY$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$x = \frac{y_1}{\sqrt{6}} - \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{3}}$$

$$y = \frac{2y_1}{\sqrt{6}} - y_3/\sqrt{3}$$

$$z = \frac{y_1}{\sqrt{6}} + \frac{y_2}{\sqrt{2}} + \frac{y_3}{\sqrt{3}}$$

H.W
Date 4/1/19

2. Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal reduction

3. $x_1^2+3x_2^2+3x_3^2-2x_2x_3$

4. $2x^2+2y^2+2z^2-2xy-2yz-2zx$

$1/\sqrt{3}$
$-1/\sqrt{3}$
$1/\sqrt{3}$

solu) 2. Given Quadratic form

$$Q.F = 3x^2+5y^2+3z^2-2yz+2zx-2xy$$

The matrix form

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$[(3-\lambda)[3(5-\lambda)-1]+1[-3+1]+1[1-(5-\lambda)]=0$$

$$(3-\lambda)[15-3\lambda-1]+1[-2]+1[1-5+\lambda]=0$$

$$(3-\lambda)[14-3\lambda]-2+\lambda-4=0$$

$$42-14\lambda-9\lambda+3\lambda^2+\lambda-6=0$$

$$3\lambda^2-22\lambda+36=0$$

14
3
112

12
44
72
32

$$(3-\lambda)[(5-\lambda)(3-\lambda)-1]+1[-(3-\lambda)+1]+1[1-(5-\lambda)]=0$$

$$(3-\lambda)[15-3\lambda-5\lambda+\lambda^2-1]+[-3+\lambda+1]+1-5+\lambda=0$$

$$(3-\lambda)[\lambda^2-8\lambda+14]+\lambda-2+\lambda-4=0$$

$$3\lambda^2-24\lambda+42-\lambda^3+8\lambda^2-14\lambda+2\lambda-6=0$$

$$-\lambda^3+11\lambda^2-36\lambda+36=0$$

$$\lambda^3-11\lambda^2+36\lambda-36=0.$$

$$\begin{array}{c|cccc} 3 & 1 & -11 & 36 & -36 \\ & 0 & 3 & -24 & 36 \\ \hline & 1 & -8 & 12 & \boxed{0} \end{array}$$

$$\lambda^2-8\lambda+12=0$$

$$(\lambda-3)[\lambda^2-6\lambda-2\lambda+12]=0$$

$$(\lambda-3)[\lambda(\lambda-6)-2(\lambda-6)]=0$$

$$(\lambda-2)(\lambda-3)(\lambda-6)=0$$

$$\lambda=2,3,6$$

case (ii)

If $\lambda=2$ $\quad (A-\lambda I)X=0$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2+R_1 \\ R_3 \to R_3-R_1 \end{array} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2, \ n = 3$

$n - r = 3 - 2 = 1$, L.I.S

$x - y + z = 0$; $2y = 0$; let $z = k$

$x - 0 + k = 0$ $\qquad y = 0$

$x = -k$

$$\therefore X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

case (iii)

If $\lambda = 6$ $\quad (A - \lambda I)X = 0$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \ R_3 \to 2R_3 - 4R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & +1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \ R_2 \to \frac{R_2}{-2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2, \ n = 3$ $\qquad$ L.I.S

$\therefore n - r = 3 - 2 = 1$

$-3x - y + z = 0$; $y + 2z = 0$; $z = k$

$-3x + 2k + k = 0$ $\qquad y + 2k = 0$

$-3x = -3k$ $\qquad y = -2k$

$x = k$

$$\therefore X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

case(ii)

If $\lambda = 8$ $\quad (A-\lambda I)X = 0$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_2 \to R_2 + R_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 + R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rho(A) = 2$
$n = 3$
$n - r = 3-2$
$= 1$
$\therefore$ L.I.S

$x - y = 0$; $y - z = 0$; $z = k$

$x - k = 0$ $\quad y - k = 0$
$x = k.$ $\quad y = k$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} ; X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ; X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

we observed that X_1, X_2, X_3 are mutually perpendicular

The normalized vectors

$$e_1 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad e_2 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \quad e_3 = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$P = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$D = P^TAP$

$$D = \begin{bmatrix} -\frac{1}{\sqrt2} & 0 & 1/\sqrt2 \\ 1/\sqrt3 & 1/\sqrt3 & 1/\sqrt3 \\ 1/\sqrt6 & -\frac{2}{\sqrt6} & 1/\sqrt6 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt2} & 1/\sqrt3 & 1/\sqrt6 \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ 1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{\sqrt2}+0+\frac{1}{\sqrt2} & \cancel{\frac{1}{\sqrt2}}+0-\cancel{\frac{1}{\sqrt2}} & -\frac{1}{\sqrt2}+0+\frac{3}{\sqrt2} \\ \frac{3}{\sqrt3}-\cancel{\frac{1}{\sqrt3}}+\cancel{\frac{1}{\sqrt3}} & -\frac{1}{\sqrt3}+\frac{5}{\sqrt3}-\frac{1}{\sqrt3} & \cancel{\frac{1}{\sqrt3}}-\cancel{\frac{1}{\sqrt3}}+\frac{3}{\sqrt3} \\ \frac{3}{\sqrt6}+\frac{2}{\sqrt6}+\frac{1}{\sqrt6} & -\frac{1}{\sqrt6}-\frac{10}{\sqrt6}-\cancel{\frac{1}{\sqrt6}} & \frac{1}{\sqrt6}+\frac{2}{\sqrt6}+\frac{3}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ 0 & \frac{1}{\sqrt3} & \frac{-2}{\sqrt6} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2}{\sqrt2} & 0 & \frac{2}{\sqrt2} \\ \frac{3}{\sqrt3} & \frac{3}{\sqrt3} & \frac{3}{\sqrt3} \\ \frac{6}{\sqrt6} & \frac{-12}{\sqrt6} & \frac{6}{\sqrt6} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt2} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ \frac{1}{\sqrt2} & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{2}+0+\frac{2}{2} & \cancel{\frac{-2}{6}}+0+\cancel{\frac{2}{6}} & \cancel{\frac{-2}{12}}+0+\cancel{\frac{2}{12}} \\ \cancel{\frac{-3}{6}}+0+\cancel{\frac{3}{6}} & \frac{3}{9}+\frac{3}{9}+\frac{3}{9} & \cancel{\frac{3}{18}}-\cancel{\frac{6}{18}}+\cancel{\frac{3}{18}} \\ \cancel{\frac{-6}{12}}+0+\cancel{\frac{6}{12}} & \cancel{\frac{6}{18}}-\cancel{\frac{12}{18}}+\cancel{\frac{6}{18}} & \frac{6}{36}+\frac{24}{36}+\frac{6}{36} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \operatorname{diag}(2,3,6)$$

orthogonar transformation

$X = PY$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \\ 0 & 1/\sqrt3 & -2/\sqrt6 \\ 1/\sqrt2 & 1/\sqrt3 & 1/\sqrt6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore x = -\frac{y_1}{\sqrt2} + \frac{y_2}{\sqrt3} + \frac{y_3}{\sqrt6}$; $y = \frac{y_2}{\sqrt3} - \frac{2y_3}{\sqrt6}$; $z = \frac{y_1}{\sqrt2} + \frac{y_2}{\sqrt3} + \frac{y_3}{\sqrt6}$

4. Given quadratic form

$$Q.F = 2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$$

matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(2-\lambda)(2-\lambda)-1] + 1[-(2-\lambda)-1] - 1[1+(2-\lambda)] = 0$$

$$(2-\lambda)[4-2\lambda-2\lambda+\lambda^2-1] + [-2+\lambda-1] - [1+2-\lambda] = 0$$

$$(2-\lambda)[\lambda^2-4\lambda+3] + [\lambda-3] - [3-\lambda] = 0$$

$$(2-\lambda)[\lambda^2-4\lambda+3] + \lambda - 3 - 3 + \lambda = 0$$

$$2\lambda^2 - 8\lambda \cancel{+6} - \lambda^3 + 4\lambda^2 - 3\lambda + 2\lambda \cancel{-6} = 0$$

$$-\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda[\lambda^2 - 3\lambda - 3\lambda + 9] = 0$$

$$\lambda[\lambda(\lambda-3) - 3(\lambda-3)] = 0$$

$$\lambda(\lambda-3)(\lambda-3) = 0$$

$$\lambda = 0, 3, 3$$

case (i)

If $\lambda = 0$ $(A - \lambda I)X = 0$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} R_3 \to R_3 + R_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 2$; $n = 3$

$n - r = 3 - 2 = 1$ L.I.S

$2x - y - z = 0$; $3y - 3z = 0$; $z = k$

$2x - k - k = 0$ $\quad$ $3y - 3k = 0$

$x = k$ $\quad$ $3y = 3k$

$y = k$

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} k$$

case (ii)

If $\lambda = 3$; $(A - \lambda I)x = 0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$r(A) = 1$; $n = 3$

$n - r = 3 - 1 = 2$ L.I.S

$-x - y - z = 0$; $y = k_1$; $z = k_2$

$-x - k_1 - k_2 = 0$

$x + (k_1 + k_2) = 0$

$x = -(k_1 + k_2)$

$$x_2 = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix} \text{ (or) } = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} k_1 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} k_2$$

imp

DIAGONALISATION

if square matrix of A order n has n linearly indipendent eign vectors then a matrix B can be found such that ~~BA~~ inward ~~A B~~ AB is a diagonal matrix

$$B = [X_1, X_2, X_3]$$

here

X_1, X_2, X_3 are the eign vectors of the given matrix.

Q. diagonalise the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Sol^n: The C H matrix of A is

$$A - \lambda I$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix}$$

The ch eqⁿ of A

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(2-\lambda)(-1-\lambda)-1] - 1[-1(-1-\lambda)-0] - 2[-1-0] = 0$

$(1-\lambda)[-2-2\lambda+\lambda+\lambda^2-1] - 1(1+\lambda) + 2 = 0$

$(1-\lambda)[\lambda^2-\lambda-3] - 1 - \lambda + 2 = 0$

$(1-\lambda)(\lambda^2-\lambda-3) - \lambda + 1 = 0 \Rightarrow (1-\lambda)(\lambda^2-\lambda-3) - \lambda + 1$

$\Rightarrow (1-\lambda)(\lambda^2-\lambda-3) + (1-\lambda)$

$\lambda^2 - \lambda - 3 - \lambda^3 + \lambda^2 + 3\lambda - \lambda + 1 = 0$

$(1-\lambda)(\lambda^2-\lambda-3)$

$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$\lambda = 1 \Rightarrow 1 - 2 - 1 + 2 = 0$

$X = [x_1, x_2, x_3]$

$$\begin{array}{r|l} & \lambda^2 - \lambda - 2 \\ (\lambda - 1) & \lambda^3 - 2\lambda^2 - \lambda + 2 \\ & \underline{\lambda^3 - \lambda^2} \\ & -\lambda^2 - \lambda + 2 \\ & \underline{-\lambda^2 + \lambda} \\ & -2\lambda + 2 \\ & \underline{-2\lambda + 2} \\ & 0 \end{array}$$

$(\lambda - 1)(\lambda^2 - \lambda - 2) = 0$

$\lambda - 1 = 0$ | $\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = -1, 1, 2.$

The eigen roots of A are $-1, 1, 2$

Case I

if $\lambda = -1$ then $|A - \lambda I|x = 0$

$$\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x + y - 2z = 0$ — ①

$-x + 3y + z = 0$ — ②

$y = 0$ — ③

Put $y = 0$ in eqn ① & eqn ②,

$2x - 2z = 0$ — ④

$-x + z = 0$ — ⑤

multiply eqn ⑤ with ② + eqn ④

$2x - 2z = 0$

$-2x + 2z$

$-x + z = 0$

$x = z$

let $z = k$

$x = k,\ z = k$

$y = 0$

o.m.f

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

case - II

if $\lambda = 1$ then $|A - \lambda I| = 0$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$y - 2z = 0$ — (1)

$-x + y + z = 0$ — (2)

$y - 2z = 0$ — (3)

let

$z = k$

$$\boxed{\begin{matrix} \cancel{y - 2z = 0} \\ y - 2z = 0 \\ - \quad + \end{matrix}}$$

$y - 2z = 0$

$y - 2k = 0$

$+2k = +y$

$y = 2k$

$-x + 2k + k = 0$

$-x + 3k = 0$

$x = 3k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Case-(iii)

if $\lambda = 2$ then $|A - \lambda I| = 0$

$$\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + y - 2z = 0$

$-x + z = 0$

$y - 3z = 0$

Let, $z = k$

$-x + k = 0$

$+x = +k$

$x = k$

$-k + y - 2k = 0$

$y - 3k = 0$

$y = 3k$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

Hence eign vectors are

$$\left[\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right]$$

$$B = [x_1, x_2, x_3] = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \operatorname{adj} B$$

$$|B| = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2-3) - 3(0-3) + 1(0-2)$$

$$= -1 + 9 - 2$$

$$= 6 \neq 0.$$

Cofactor of $1 = (2-3) = -1$

" " $3 = -(0-3) = 3$

" " $1 = (0-2) = -2$.

" " $0 = -(3-1) = -2$

" " $2 = (1-1) = 0$

" " $3 = -(1-3) = 2$

" " $1 = (9-2) = 7$

" " $1 = -(3-0) = -3$

" " $1 = 2-0 = 2$.

co - factor matrix of $B = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 2 \\ 7 & -3 & 2 \end{bmatrix}$

$$\operatorname{adj} B = \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \operatorname{adj} B = \frac{1}{6} \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} AB = \frac{1}{6} \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & 3 \\ -2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -1+2+0 & -1+4+7 & 2-2-7 \\ 3-0-0 & 3+0-3 & -6+0+3 \\ -2-2+0 & -2+4+2 & 4+2-2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 2 & -7 \\ 3 & 0 & -3 \\ -4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1+0-7 & 3+4-7 & 1+6-7 \\ 3+0-3 & 9+0-3 & 3+0-3 \\ -4+0+4 & -12+8+4 & -4+12+4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

J.M.C

1/2/18

Diagonalisation by orthogonal transformation.

~~sappdse~~ suppose, a is a real symetry matrix then. a characteristis matrix of a will not be linearly indipendent. and also waise orthogonal. if we normalise each characteristics vector or eign vectors (X) we devide each component of X by the square root of the sum of the squares of all elements, write all normalized eign vectors to form normalized ~~moto~~ model matrix B then it can be easily shown that B is an orthogonal matrix and :

B^{-1} equal to B transpose.

therefore the symelerity transform

$$B^{-1}AB = D$$

Where D is the diogonal matrix.

this tranformation ~~B~~ transpose AB is equal to D, is known as orthogonal transformation.

Q. Calculation of powers of a matrix.

let, A be the given matrix of ~~AB~~ order 3.

We know that

Using

$D = B^{-1}AB$

$D^2 = (B^{-1}AB)(B^{-1}AB)$

$= (B^{-1}A)(BB^{-1})(AB)$

$= (B^{-1}A)(I)(AB)$

$D^2 = B^{-1}A^2B.$

III'ly

$D^3 = B^{-1}A^3B.$

$\vdots$

$D^n = B^{-1}A^nB$

$(BD^nB^{-1}) = B(B^{-1}A^nB)B^{-1}$

$= A^n$

$A^n = (BD^nB^{-1})$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$D^n = \begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix}$$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}} \\ \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} \end{bmatrix}$

$B = [X_1\ X_2\ X_3].$

Q. $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by O.T.

Sol: The CH matrix of A is

$A - \lambda I$

d.mg

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix}$$

The CH. eqn of A is

$$|A - \lambda I| = 0$$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)^2 - 4] - 1[1(1-\lambda) - 2] - 1[-2-(-1+\lambda)]$$

$$(2-\lambda)[1-2\lambda+\lambda^2-4] - 1[-\lambda-1] - 1[-2+1-\lambda] = 0$$

$$(2-\lambda)(\lambda^2-2\lambda-3] + \lambda+1+1+\lambda = 0$$

$$2\lambda^2 - 4\lambda - 6 - \lambda^3 + 2\lambda^2 + 3\lambda + 2\lambda + 2 = 0$$

$$-\lambda^3 + 4\lambda^2 + \lambda - 4 = 0$$

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

$$\lambda^3 - \lambda^2 - 3\lambda^2 + 3\lambda - 4\lambda + 4 = 0$$

$$\lambda^2(\lambda-1) - 3\lambda(\lambda-1) - 4(\lambda-1) = 0$$

$$(\lambda-1)(\lambda^2-3\lambda-4) = 0$$

$$\lambda^2 - 4\lambda + \lambda - 4 = 0$$

$$\lambda(\lambda-4) + 1(\lambda-4) = 0$$

$$(\lambda+1)(\lambda-4) = 0$$

$\lambda = -1, 1, 4.$

$$\boxed{\lambda = -1, 1, 4}$$

U.M.S

The CH roots of the eq^n is $-1, 1, 4$.

Case ①

if $\lambda = -1$, then

$$A - \lambda I = 0$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & -2 \\ -1 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & -5 & 5 \end{bmatrix} \begin{matrix} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + y - z = 0$$

$$5y - 5z = 0$$

$\rho(A) = 2, \quad n = 3, \quad n - r = 3 - 2 = 1$ L.I.S

Let,

$$z = K$$

$$5y - 5K = 0$$

$$5y = 5K$$

$$y = K$$

$$3x + K - K = 0$$

$$x = 0$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ K \\ K \end{bmatrix} = K\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Case (2)

if $\lambda = 1$ then

$$A - \lambda I = 0$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y - z = 0$$

$$-y - z = 0$$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

Let, $z = k_1$,

$$-y - k_1 = 0$$

$$-y = k_1$$

$$y = -k_1$$

$$x - k_1 - k_1 = 0$$

$$x - 2k_1 = 0$$

$$x = 2k_1$$

J.ms

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case. 3 if $\lambda = 4$ then

$$A - \lambda I = 0$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & -5 & -5 \end{bmatrix} \begin{matrix} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & -1 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$-2x + y - z = 0$$

$$-5y - 5z = 0$$

$\rho(A) = 2$, $n = 3$, $n - r = 3 - 2 = 1$ L.I.S

let $z = k_1$

$$-5y - 5k_1 = 0$$

$$-5y = 5k_1$$

$$y = -k_1$$

$$-2x - k_1 - k_1 = 0$$

$$-2x - 2k_1 = 0$$

$$-2x = 2k_1$$

$$x = -k_1$$

$$X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

eign vector

$$B = [x_1\ x_2\ x_3]$$

$$= \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$\frac{0}{\sqrt{0^2+1^2+1^2}}$ $\quad \frac{2}{\sqrt{4+1+1}}$ $\quad \frac{1}{\sqrt{1^2+1^2+1^2}}$

$B^T = B^{-1}$

We observed that eigen vectors are pair-wise orthogonal.

$$B = \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$\frac{0}{\sqrt2}$ $\quad \frac{1}{\sqrt2}$ $\quad \frac{2}{\sqrt6}$ $\quad \frac{-1}{\sqrt6}$

$$B^{-1} = B^T = \begin{bmatrix} 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 0 & \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{-1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 + \frac{1}{\sqrt2} - \frac{1}{\sqrt2} & 0 + \frac{1}{\sqrt2} - \frac{2}{\sqrt2} & 0 - \frac{2}{\sqrt2} + \frac{1}{\sqrt2} \\ \frac{4}{\sqrt6} - \frac{1}{\sqrt6} - \frac{1}{\sqrt6} & \frac{2}{\sqrt6} - \frac{1}{\sqrt6} - \frac{2}{\sqrt6} & \frac{-2}{\sqrt6} + \frac{2}{\sqrt6} + \frac{1}{\sqrt6} \\ \frac{2}{\sqrt3} + \frac{1}{\sqrt3} + \frac{1}{\sqrt3} & \frac{1}{\sqrt3} + \frac{1}{\sqrt3} + \frac{2}{\sqrt3} & \frac{-1}{\sqrt3} - \frac{2}{\sqrt3} - \frac{1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{-1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt6} & \frac{-1}{\sqrt3} \end{bmatrix}$$

d - mp

$$\begin{bmatrix} 0-\frac{1}{2}-\frac{1}{2} & 0+\frac{1}{\sqrt2}-\frac{1}{\sqrt2} & 0-\frac{1}{\sqrt6}+\frac{1}{\sqrt6} \\ 0-\frac{1}{\sqrt2}+\frac{1}{\sqrt2} & \frac{4}{6}+\frac{1}{6}+\frac{1}{6} & \frac{2}{\sqrt{18}}-\frac{1}{\sqrt{18}}-\frac{1}{\sqrt{18}} \\ 0+\frac{4}{\sqrt6}-\frac{4}{\sqrt6} & \frac{8}{\sqrt{18}}-\frac{4}{\sqrt{18}}-\frac{4}{\sqrt{18}} & \frac{4}{3}+\frac{4}{3}+\frac{4}{3} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = D(-1, 1, 4)$$

Recall that if A is a symmetric $n \times n$ matrix, then A has real eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated), and $\mathbb{R}^n$ has an orthonormal basis $v_1, \ldots, v_n$, where each vector v_i is an eigenvector of A with eigenvalue λ_i. Then

$$A = PDP^{-1}$$

where P is the matrix whose columns are $v_1, \ldots, v_n$, and D is the diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$. Since the vectors $v_1, \ldots, v_n$ are orthonormal, the matrix P is orthogonal, i.e. $P^T P = I$, so we can alternately write the above equation as

$$A = PDP^T. \tag{1}$$

A singular value decomposition (SVD) is a generalization of this where A is an $m \times n$ matrix which does not have to be symmetric or even square.

1 Singular values

Let A be an $m \times n$ matrix. Before explaining what a singular value decomposition is, we first need to define the singular values of A.

Consider the matrix $A^T A$. This is a symmetric $n \times n$ matrix, so its eigenvalues are real.

Lemma 1.1. *If λ is an eigenvalue of $A^T A$, then $\lambda \geq 0$.*

Proof. Let x be an eigenvector of $A^T A$ with eigenvalue λ. We compute that

$$\|Ax\|^2 = (Ax) \cdot (Ax) = (Ax)^T Ax = x^T A^T Ax = x^T(\lambda x) = \lambda x^T x = \lambda \|x\|^2.$$

Since $\|Ax\|^2 \geq 0$, it follows from the above equation that $\lambda \|x\|^2 \geq 0$. Since $\|x\|^2 > 0$ (as our convention is that eigenvectors are nonzero), we deduce that $\lambda \geq 0$. □

Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of $A^T A$, with repetitions. Order these so that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $\sigma_i = \sqrt{\lambda_i}$, so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Definition 1.2. The numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ defined above are called the **singular values** of A.

Proposition 1.3. *The number of nonzero singular values of A equals the rank of A.*

Proof. The rank of any square matrix equals the number of nonzero eigenvalues (with repetitions), so the number of nonzero singular values of A equals the rank of A^TA. By a previous homework problem, A^TA and A have the same kernel. It then follows from the "rank-nullity" theorem that A^TA and A have the same rank. □

Remark 1.4. In particular, if A is an $m \times n$ matrix with $m < n$, then A has at most m nonzero singular values, because $\operatorname{rank}(A) \leq m$.

The singular values of A have the following geometric significance.

Proposition 1.5. *Let A be an $m \times n$ matrix. Then the maximum value of $\|Ax\|$, where x ranges over unit vectors in $\mathbb{R}^n$, is the largest singular value σ_1, and this is achieved when x is an eigenvector of A^TA with eigenvalue σ_1^2.*

Proof. Let $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A^TA with eigenvalues σ_i^2. If $x \in \mathbb{R}^n$, then we can expand x in this basis as

$$x = c_1v_1 + \cdots + c_nv_n \tag{2}$$

for scalars $c_1, \ldots, c_n$. Since x is a unit vector, $\|x\|^2 = 1$, which (since the vectors $v_1, \ldots, v_n$ are orthonormal) means that

$$c_1^2 + \cdots + c_n^2 = 1.$$

On the other hand,

$$\|Ax\|^2 = (Ax) \cdot (Ax) = (Ax)^T(Ax) = x^TA^TAx = x \cdot (A^TAx).$$

By (2), since v_i is an eigenvalue of A^TA with eigenvalue σ_i^2, we have

$$A^TAx = c_1\sigma_1^2v_1 + \cdots + c_n\sigma_n^2v_n.$$

Taking the dot prodoct with (2), and using the fact that the vectors $v_1, \ldots, v_n$ are orthonormal, we get

$$\|Ax\|^2 = x \cdot (A^TAx) = \sigma_1^2c_1^2 + \cdots + \sigma_n^2c_n^2.$$

Since σ_1 is the largest singular value, we get

$$\|Ax\|^2 \leq \sigma_1^2(c_1^2 + \cdots + c_n^2).$$

Equality holds when $c_1 = 1$ and $c_2 = \cdots = c_n = 0$. Thus the maximum value of $\|Ax\|^2$ for a unit vector x is σ_1^2, which is achieved when $x = v_1$. □

One can similarly show that σ_2 is the maximum of $\|Ax\|$ where x ranges over unit vectors that are orthogonal to v_1 (exercise). Likewise, σ_3 is the maximum of $\|Ax\|$ where x ranges over unit vectors that are orthogonal to v_1 and v_2; and so forth.

2 Definition of singular value decomposition

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let r denote the number of nonzero singular values of A, or equivalently the rank of A.

Definition 2.1. A singular value decomposition of A is a factorization

$$A = U\Sigma V^T$$

where:

- U is an $m \times m$ orthogonal matrix.
- V is an $n \times n$ orthogonal matrix.
- Σ is an $m \times n$ matrix whose i^{th} diagonal entry equals the i^{th} singular value σ_i for $i = 1, \ldots, r$. All other entries of Σ are zero.

Example 2.2. If $m = n$ and A is symmetric, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. The singular values of A are given by $\sigma_i = |\lambda_i|$ (exercise). Let $v_1, \ldots, v_n$ be orthonormal eigenvectors of A with $Av_i = \lambda_i v_i$. We can then take V to be the matrix whose columns are $v_1, \ldots, v_n$. (This is the matrix P in equation (1).) The matrix Σ is the diagonal matrix with diagonal entries $|\lambda_1|, \ldots, |\lambda_n|$. (This is almost the same as the matrix D in equation (1), except for the absolute value signs.) Then U must be the matrix whose columns are $\pm v_1, \ldots, \pm v_n$, where the sign next to v_i is $+$ when $\lambda_i \geq 0$, and $-$ when $\lambda_i < 0$. (This is almost the same as P, except we have changed the signs of some of the columns.)

3 How to find a SVD

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, and let r denote the number of nonzero singular values. We now explain how to find a SVD of A.

Let $v_1, \ldots, v_n$ be an orthonormal basis of $\mathbb{R}^n$, where v_i is an eigenvector of $A^T A$ with eigenvalue σ_i^2.

Lemma 3.1. *(a)* $\|Av_i\| = \sigma_i$.

(b) If $i \neq j$ then Av_i and Av_j are orthogonal.

Proof. We compute

$$(Av_i)\cdot(Av_j) = (Av_i)^T(Av_j) = v_i^T A^T A v_j = v_i^T \sigma_j^2 v_j = \sigma_j^2 (v_i \cdot v_j).$$

If $i = j$, then since $\|v_i\| = 1$, this calculation tells us that $\|Av_i\|^2 = \sigma_j^2$, which proves (a). If $i \neq j$, then since $v_i \cdot v_j = 0$, this calculation shows that $(Av_i)\cdot(Av_j) = 0$. □

Theorem 3.2. *Let A be an $m \times n$ matrix. Then A has a (not unique) singular value decomposition $A = U\Sigma V^T$, where U and V are as follows:*

- *The columns of V are orthonormal eigenvectors $v_1, \ldots, v_n$ of $A^T A$, where $A^T A v_i = \sigma_i^2 v_i$.*
- *If $i \leq r$, so that $\sigma_i \neq 0$, then the i^{th} column of U is $\sigma_i^{-1} A v_i$. By Lemma 3.1, these columns are orthonormal, and the remaining columns of U are obtained by arbitrarily extending to an orthonormal basis for $\mathbb{R}^m$.*

Proof. We just have to check that if U and V are defined as above, then $A = U\Sigma V^T$. If $x \in \mathbb{R}^n$, then the components of $V^T x$ are the dot products of the rows of V^T with x, so

$$V^T x = \begin{pmatrix} v_1 \cdot x \\ v_2 \cdot x \\ \vdots \\ v_n \cdot x \end{pmatrix}.$$

Then

$$\Sigma V^T x = \begin{pmatrix} \sigma_1 v_1 \cdot x \\ \sigma_2 v_2 \cdot x \\ \vdots \\ \sigma_r v_r \cdot x \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

When we multiply on the left by U, we get the sum of the columns of U, weighted by the components of the above vector, so that

$$\begin{aligned} U\Sigma V^T x &= (\sigma_1 v_1 \cdot x)\sigma_1^{-1} A v_1 + \cdots + (\sigma_r v_r \cdot x)\sigma_r^{-1} A v_r \\ &= (v_1 \cdot x) A v_1 + \cdots + (v_r \cdot x) A v_r. \end{aligned}$$

Since $Av_i = 0$ for $i > r$ by Lemma 3.1(a), we can rewrite the above as

$$\begin{aligned} U\Sigma V^T x &= (v_1 \cdot x)Av_1 + \cdots + (v_n \cdot x)Av_n \\ &= Av_1 v_1^T x + \cdots + Av_n v_n^T x \\ &= A(v_1 v_1^T + \cdots v_n v_n^T)x \\ &= Ax. \end{aligned}$$

In the last line, we have used the fact that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then $v_1 v_1^T + \cdots + v_n v_n^T = I$ (exercise). □

Example 3.3. (from Lay's book) *Find a singular value decomposition of*

$$A = \begin{pmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{pmatrix}.$$

Step 1. We first need to find the eigenvalues of $A^T A$. We compute that

$$A^T A = \begin{pmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{pmatrix}.$$

We know that at least one of the eigenvalues is 0, because this matrix can have rank at most 2. In fact, we can compute that the eigenvalues are $\lambda_1 = 360$, $\lambda_2 = 90$, and $\lambda_3 = 0$. Thus the singular values of A are $\sigma_1 = \sqrt{360} = 6\sqrt{10}$, $\sigma_2 = \sqrt{90} = 3\sqrt{10}$, and $\sigma_3 = 0$. The matrix Σ in a singular value decomposition of A has to be a 2×3 matrix, so it must be

$$\Sigma = \begin{pmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{pmatrix}.$$

Step 2. To find a matrix V that we can use, we need to solve for an orthonormal basis of eigenvectors of $A^T A$. One possibility is

$$v_1 = \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2/3 \\ -1/3 \\ 2/3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}.$$

(There are seven other possibilities in which some of the above vectors are multiplied by -1.) Then V is the matrix with v_1, v_2, v_3 as columns, that is

$$V = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}.$$

Step 3. We now find the matrix U. The first column of U is

$$\sigma_1^{-1}Av_1 = \frac{1}{6\sqrt{10}}\begin{pmatrix}18\\6\end{pmatrix} = \begin{pmatrix}3/\sqrt{10}\\1/\sqrt{10}\end{pmatrix}.$$

The second column of U is

$$\sigma_2^{-1}Av_2 = \frac{1}{3\sqrt{10}}\begin{pmatrix}3\\9\end{pmatrix} = \begin{pmatrix}1/\sqrt{10}\\-3/\sqrt{10}\end{pmatrix}.$$

Since U is a 2×2 matrix, we do not need any more columns. (If A had only one nonzero singular value, then we would need to add another column to U to make it an orthogonal matrix.) Thus

$$U = \begin{pmatrix}3/\sqrt{10} & 1/\sqrt{10}\\1/\sqrt{10} & -3/\sqrt{10}\end{pmatrix}.$$

To conclude, we have found the singular value decomposition

$$\begin{pmatrix}4 & 11 & 14\\8 & 7 & -2\end{pmatrix} = \begin{pmatrix}3/\sqrt{10} & 1/\sqrt{10}\\1/\sqrt{10} & -3/\sqrt{10}\end{pmatrix}\begin{pmatrix}6\sqrt{10} & 0 & 0\\0 & 3\sqrt{10} & 0\end{pmatrix}\begin{pmatrix}1/3 & -2/3 & 2/3\\2/3 & -1/3 & -2/3\\2/3 & 2/3 & 1/3\end{pmatrix}^T.$$

4 Applications

Singular values and singular value decompositions are important in analyzing data.

One simple example of this is "rank estimation". Suppose that we have n data points $v_1, \ldots, v_n$, all of which live in $\mathbb{R}^m$, where n is much larger than m. Let A be the $m \times n$ matrix with columns $v_1, \ldots, v_n$. Suppose the data points satisfy some linear relations, so that $v_1, \ldots, v_n$ all lie in an r-dimensional subspace of $\mathbb{R}^m$. Then we would expect the matrix A to have rank r. However if the data points are obtained from measurements with errors, then the matrix A will probably have full rank m. But only r of the singular values of A will be large, and the other singular values will be close to zero. Thus one can compute an "approximate rank" of A by counting the number of singular values which are much larger than the others, and one expects the measured matrix A to be close to a matrix A' such that the rank of A' is the "approximate rank" of A.

For example, consider the matrix

$$A' = \begin{pmatrix}1 & 2 & -2 & 3\\-4 & 0 & 1 & 2\\3 & -2 & 1 & -5\end{pmatrix}$$

The matrix A' has rank 2, because all of its columns are points in the subspace $x_1+x_2+x_3=0$ (but the columns do not all lie in a 1-dimensional subspace). Now suppose we perturb A' to the matrix

$$A=\begin{pmatrix} 1.01 & 2.01 & -2 & 2.99 \\ -4.01 & 0.01 & 1.01 & 2.02 \\ 3.01 & -1.99 & 1 & -4.98 \end{pmatrix}$$

This matrix now has rank 3. But the eigenvalues of A^TA are

$$\sigma_1^2 \approx 58.604, \quad \sigma_2^2 \approx 19.3973, \quad \sigma_3^2 \approx 0.00029, \quad \sigma_4^2 = 0.$$

Since two of the singular values are much larger than the others, this suggests that A is close to a rank 2 matrix.

For more discussion of how SVD is used to analyze data, see e.g. Lay's book.

5 Exercises (some from Lay's book)

1. (a) Find a singular value decomposition of the matrix $A=\begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}$.

   (b) Find a unit vector x for which $\|Ax\|$ is maximized.

2. Find a singular value decomposition of $A=\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

3. (a) Show that if A is an $n \times n$ symmetric matrix, then the singular values of A are the absolute values of the eigenvalues of A.

   (b) Give an example to show that if A is a 2×2 matrix which is not symmetric, then the singular values of A might not equal the absolute values of the eigenvalues of A.

4. Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$. Let v_1 be an eigenvector of A^TA with eigenvalue σ_1^2. Show that σ_2 is the maximum value of $\|Ax\|$ where x ranges over unit vectors in $\mathbb{R}^n$ that are orthogonal to v_1.

5. Show that if $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then

$$v_1v_1^T + \cdots + v_nv_n^T = I.$$

6. Let A be an $m \times n$ matrix, and let P be an orthogonal $m \times m$ matrix. Show that PA has the same singular values as A.