Unit - 4

16/11/19 Saturday

Partial Derivatives

1. Homogeneous function, Euler's Theorem, Total derivatives, chain rule, Jackobean, Functionally dependents;
Taylor's and ~~Machlarries~~ Maclaurin's Expansions with two variables.

Applications: Maxima and Minima with constants and without constants, Lagranges

(I)

③ If $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ (or) prove that $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \sin 2U$.

Sol:-

Given $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\tan U = \frac{x^3+y^3}{x+y}$$

$$\tan U = \frac{x^{\cancel{3}\,2}\left(1+\frac{y^3}{x^3}\right)}{\cancel{x}\left(1+\frac{y}{x}\right)}$$

$$\tan U = x^2\left[\frac{1+\left(\frac{y}{x}\right)^3}{\left(1+\frac{y}{x}\right)}\right]$$

$$\tan U = x^2 \cdot f\left(\frac{y}{x}\right)$$

→ Tan U is homogeneous of degree "2".

By Euler's theorem,

$$x \cdot \frac{\partial \tan U}{\partial x} + y \cdot \frac{\partial \tan U}{\partial y} = 2 \cdot \tan U$$

$$x \cdot \sec^2 U \cdot \frac{\partial U}{\partial x} + y \sec^2 U \frac{\partial U}{\partial y} = 2 \cdot \tan U$$

$$\sec^2 U\left(x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y}\right) = 2 \cdot \tan U$$

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \frac{2 \cdot \tan U}{\sec^2 U}$$

$$x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 2 \cdot \frac{\sin U}{\cos U} \times \cos^2 U$$

$$\boxed{x \cdot \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = \sin 2U}$$

④ If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u$

Sol:

Given $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right)$

$$u = \sin^{-1}\left(\frac{x\left(1+2\frac{y}{x}+3\frac{z}{x}\right)}{x^{4}\sqrt{1+\frac{y^8}{x^8}+\frac{z^8}{x^8}}}\right)$$

$$\sin u = x^{-3}\left[\frac{1+2\cdot\frac{y}{x}+3\left(\frac{z}{x}\right)}{\sqrt{1+\left(\frac{y}{x}\right)^8+\left(\frac{z}{x}\right)^8}}\right]$$

$$\sin u = x^{-3}\cdot f\left(\frac{y}{x}, \frac{z}{x}\right)$$

$\therefore \sin u$ is homogeneous of degree "-3".

By Euler's theorem,

$$x\cdot\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} + z\frac{\partial \sin u}{\partial z} = -3\sin u$$

$$x\cdot\cos u\cdot\frac{\partial u}{\partial x} + y\cdot\cos u\frac{\partial u}{\partial y} + z\cdot\cos u\frac{\partial u}{\partial z} = -3\sin u$$

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z}\right) = -3\sin u$$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u}$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z} = -3\tan u.$$

⑤ $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ show that $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = 3$.

Sol

Given $u = \log\left(\frac{x^4+y^4}{x+y}\right)$

$$e^u = \frac{x^{4}\left(1+\frac{y^4}{x^4}\right)}{x\left(1+\frac{y}{x}\right)}$$

$$e^u = x^3\left(\frac{1+(y/x)^4}{1+y/x}\right)$$

$$e^u = x^3\cdot f\left(\frac{y}{x}\right)$$

$\therefore e^u$ is homogeneous of degree "3".

By Euler's theorem,

$$x\cdot\frac{\partial e^u}{\partial x} + y\cdot\frac{\partial e^u}{\partial y} = 3\cdot e^u$$

$$x \cdot e^{u} \frac{\partial u}{\partial x} + y e^{u} \frac{\partial u}{\partial y} = 3 \cdot e^{u}$$

$$e^{u}\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 3 \cdot e^{u}$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 3.}$$

(7) $u = x f\left(\frac{y}{x}\right)$ prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$.

Given $u = x f\left(\frac{y}{x}\right)$

$\therefore$ u is the homogeneous of degree "1".

By Euler's theorem,

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = (1) u.$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u}$$

(1) $u = (x^{1/2} + y^{1/2})(x^{n} + y^{n})$ verify the Euler's theorem.

Given $u = (x^{1/2} + y^{1/2})(x^{n} + y^{n})$

$$u = x^{1/2}\left(1 + \frac{y^{1/2}}{x^{1/2}}\right) x^{n}\left(1 + \frac{y^{n}}{x^{n}}\right)$$

$$= x^{n+1/2}\left[\left(1 + \left(\frac{y}{x}\right)^{1/2}\right)\left(1 + \left(\frac{y}{x}\right)^{n}\right)\right]$$

$$u = x^{n+1/2} f\left(\frac{y}{x}\right)$$

$\therefore$ u is the homogeneous of degree "$n+\frac{1}{2}$".

By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u$$

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right) u}$$

We have to prove that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right) \cdot u$.

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left[(x^{1/2} + y^{1/2})(x^{n} + y^{n})\right]$$

$$= (x^{1/2} + y^{1/2})(n x^{n-1} + 0) + (x^{n} + y^{n})\left(\frac{1}{2} x^{-1/2} + 0\right)$$

$$= (x^{1/2} + y^{1/2})\, n\, x^{n-1} + (x^{n} + y^{n}) \frac{1}{2} x^{-1/2}$$

$$x \cdot \frac{\partial u}{\partial x} = n \cdot x^{n}(x^{1/2} + y^{1/2}) + \frac{1}{2} x^{1/2}(x^{n} + y^{n})$$

Similarly, $\frac{\partial u}{\partial y} = n \cdot y^{n-1}(x^{1/2}+y^{1/2}) + \frac{1}{2} y^{-1/2}(x^n+y^n)$

$$y \cdot \frac{\partial u}{\partial y} = n \cdot y^n (x^{1/2}+y^{1/2}) + \frac{1}{2} y^{1/2}(x^n+y^n)$$

L.H.S

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$= n \cdot x^n (x^{1/2}+y^{1/2}) + \frac{1}{2}x^{1/2}(x^n+y^n) + n y^n (x^{1/2}+y^{1/2}) + \frac{1}{2}y^{1/2}(x^n+y^n)$

$= n(x^{1/2}+y^{1/2})(x^n+y^n) + \frac{1}{2}(x^n+y^n)(x^{1/2}+y^{1/2})$

$= (x^n+y^n)(x^{1/2}+y^{1/2})(n+\frac{1}{2})$

$= (n+\frac{1}{2})u$

$= R.H.S$

$\therefore$ Euler's theorem verified.

② $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$. Verify the Euler's theorem.

Given $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$= \operatorname{cosec}^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^0\left[\operatorname{cosec}^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$u = x^0 f\left(\frac{y}{x}\right)$$

$\therefore u$ is homogeneous of degree "0".

By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$$
$$= (0)\,u. = 0.$$

We have to prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

L.H.S

$$\frac{\partial}{\partial x}(u) = \frac{\partial}{\partial x}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}(y)\left(\frac{-1}{x^2}\right)$$

$$= \frac{1}{y}\frac{1}{\sqrt{\frac{y^2-x^2}{y^2}}} + \frac{-y}{x^2}\frac{1}{\frac{x^2+y^2}{x^2}}$$

$$= \frac{1}{\cancel{y}\frac{\sqrt{y^2-x^2}}{\cancel{y}}} + -\frac{y}{x^2+y^2}$$

$$\frac{\partial U}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\Rightarrow x\cdot\frac{\partial U}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2}$$

$$\frac{\partial}{\partial y}(U) = \frac{\partial}{\partial y}\left[\sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\, x\left(\frac{-1}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}$$

$$= \frac{-x}{\cancel{y^2}\frac{\sqrt{y^2-x^2}}{\cancel{y}}} + \frac{1}{\cancel{x}\cdot\left(\frac{x^2+y^2}{x^{\cancel{2}}}\right)}$$

$$\frac{\partial U}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\Rightarrow y\frac{\partial U}{\partial y} = \frac{-x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$= \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

L.H.S

$$x\cdot\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}$$

$$= \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$= 0$

$= \underline{\text{R.H.S}}$

$\therefore$ Euler's theorem verified.

⑥ $U = \log\left(\frac{x^2+y^2}{xy}\right)$ verify the Euler's theorem.

Sol: Given $U = \log\left(\frac{x^2+y^2}{xy}\right)$

$$e^U = \frac{x^{\cancel{2}}\left(1+\frac{y^2}{x^2}\right)}{\cancel{x}y}$$

$$e^U = \frac{x\left(1+\left(\frac{y}{x}\right)^2\right)}{x\cdot(y/x)}$$

$$e^U = x^0 f\left(\frac{y}{x}\right)$$

$\therefore e^U$ is homogeneous of degree '0'.

By Euler's theorem, $x\cdot\frac{\partial U}{\partial x} + y\cdot\frac{\partial U}{\partial y} = n\cdot U$
$= (0)\,U = \underline{0}$

We have to prove that, $x \frac{\partial U}{\partial x} + y \cdot \frac{\partial U}{\partial y} = 0$.

$$\frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x}\left[\log\left(\frac{x^2+y^2}{xy}\right)\right]$$

$$= \frac{1}{\frac{x^2+y^2}{xy}}\left[\frac{xy(2x+0)-(x^2+y^2)\cdot y}{(xy)^2}\right]$$

$$= \frac{xy}{x^2+y^2}\left[\frac{xy(2x)-(x^2+y^2)y}{(xy)^4}\right]$$

$$= \frac{1}{x^2+y^2}\left[\frac{2x^2y - x^2y - y^3}{xy}\right]$$

$$= \frac{1}{x^2+y^2}\left(\frac{x^2y-y^3}{xy}\right)$$

$$= \frac{1}{x^2+y^2}\,\frac{\cancel{y}\,[x^2-y^2]}{x\cancel{y}}$$

$$\frac{\partial U}{\partial x} = \frac{x^2-y^2}{x(x^2+y^2)}$$

$$\Rightarrow x\cdot\frac{\partial U}{\partial x} = \frac{\cancel{x}\cdot(x^2-y^2)}{\cancel{x}(x^2+y^2)} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial}{\partial y}(U) = \frac{\partial}{\partial x}\left[\log\left(\frac{x^2+y^2}{xy}\right)\right]$$

$$= \frac{1}{\frac{x^2+y^2}{xy}}\left[\frac{xy(0+2y)-(x^2+y^2)x}{(xy)^4}\right]$$

$$= \frac{1}{x^2+y^2}\left[\frac{2xy^2-x^3-xy^2}{xy}\right]$$

$$= \frac{1}{x^2+y^2}\left(\frac{xy^2-x^3}{xy}\right)$$

$$= \frac{1}{x^2+y^2}\,\frac{\cancel{x}(y^2-x^2)}{\cancel{x}y}$$

$$\frac{\partial U}{\partial y} = \frac{y^2-x^2}{y(x^2+y^2)}$$

$$\Rightarrow y\cdot\frac{\partial U}{\partial y} = (\cancel{y})\cdot\frac{y^2-x^2}{\cancel{y}(x^2+y^2)} = \frac{y^2-x^2}{x^2+y^2}$$

<u>L.H.S</u>

$$x\frac{\partial U}{\partial x} + y\cdot\frac{\partial U}{\partial y}$$

$$= \frac{x^2-y^2}{x^2+y^2} + \frac{y^2-x^2}{x^2+y^2}$$

$$= \frac{\cancel{x^2}-\cancel{y^2}+\cancel{y^2}-\cancel{x^2}}{x^2+y^2}$$

$$= \frac{0}{x^2+y^2} = 0. = R.H.S$$

$\therefore$ Euler's theorem verified.

(8) $U = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$. verify the Euler's theorem.

Sol: Given $U = \dfrac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

$$U = \frac{x^{1/4}\left[1+\frac{y^{1/4}}{x^{1/4}}\right]}{x^{1/5}\left[1+\frac{y^{1/5}}{x^{1/5}}\right]}$$

$$U = x^{1/4}.x^{-1/5}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{1/4-1/5}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{\frac{5-4}{20}}\left[\frac{1+(y/x)^{1/4}}{1+(y/x)^{1/5}}\right]$$

$$U = x^{1/20}\, f\left(\frac{y}{x}\right)$$

$\therefore U$ is homogeneous of degree "$1/20$".

By Euler's theorem, $x.\dfrac{\partial U}{\partial x} + y.\dfrac{\partial U}{\partial y} = n.U$

$$x.\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \frac{1}{20}U.$$

We have to prove that,

$$x\frac{\partial U}{\partial x} + y.\frac{\partial U}{\partial y} = \frac{1}{20}U.$$

$$\frac{\partial}{\partial x}(U) = \frac{\partial}{\partial x}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4-1}+0\right) - (x^{1/4}+y^{1/4})\left(\frac{1}{5}x^{1/5-1}+0\right)}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial U}{\partial x} = \frac{\frac{1}{4}x^{-3/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{-4/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow x.\frac{\partial U}{\partial x} = \frac{\frac{1}{4}x^{-3/4+1}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{-4/5+1}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{1/4\, x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial}{\partial y}(U) = \frac{\partial}{\partial y}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{(x^{1/5}+y^{1/5})(0+\frac{1}{4}y^{1/4-1}) - (x^{1/4}+y^{1/4})(0+\frac{1}{5}y^{1/5-1})}{(x^{1/5}+y^{1/5})^2}$$

$$\frac{\partial U}{\partial y} = \frac{\frac{1}{4}\cdot y^{-3/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{-4/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$\Rightarrow y\frac{\partial U}{\partial y} = \frac{\frac{1}{4}y^{-3/4+1}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{-4/5+1}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

<u>L.H.S</u>

$$x\cdot\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2} + \frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4}) + \frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5}) - \frac{1}{5}y^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{\frac{1}{4}(x^{1/5}+y^{1/5})[x^{1/4}+y^{1/4}] - \frac{1}{5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$= \frac{(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})(\frac{1}{4}-\frac{1}{5})}{(x^{1/5}+y^{1/5})^2} = \frac{(x^{1/4}+y^{1/4})\cancel{(x^{1/5}+y^{1/5})}(\frac{1}{20})}{(x^{1/5}+y^{1/5})^{\cancel{2}}}$$

$$= \frac{1}{20}\left(\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}\right)$$

$$= \frac{1}{20}U.$$

$$= R.H.S$$

$\therefore$ Euler's theorem verified.

22/11/2019
Friday (II)

⑨ If $U = \frac{x^2 y}{x+y}$. show that $x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial y \partial x} = 2\cdot\frac{\partial U}{\partial x}$.

Sol:-

Given $U = \frac{x^2y}{x+y}$

$$U = \frac{x^2 y}{x(1+\frac{y}{x})} = x^2\left[\frac{\frac{y}{x}}{1+\frac{y}{x}}\right]$$

$$U = x^2\left[\frac{y/x}{1+y/x}\right]$$

$\therefore$ U is homogeneous of degree "2".

By Euler's theorem $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = 2U$.

diff. w. r. to 'x' partially

$$(1)\frac{\partial U}{\partial x} + x\frac{\partial^2 U}{\partial x^2} + (0)\frac{\partial U}{\partial y} + y\cdot\frac{\partial^2 U}{\partial x\partial y} = 2\frac{\partial U}{\partial x}$$

$$\frac{\partial U}{\partial x} + x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = 2\cdot\frac{\partial U}{\partial x}$$

$$x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = 2\frac{\partial U}{\partial x} - \frac{\partial U}{\partial x}$$

$$\boxed{x\frac{\partial^2 U}{\partial x^2} + y\frac{\partial^2 U}{\partial x\partial y} = 2\frac{\partial U}{\partial x}}$$

⑯ If $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$. prove that $x^2\frac{\partial^2 U}{\partial x^2} + 2xy\frac{\partial^2 U}{\partial x\partial y} + y^2\frac{\partial^2 U}{\partial y^2} =$
$\sin 4U - \sin 2U = 2\cos 3U \sin U$.

Sol:-

Given $U = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$

$$\tan U = \frac{x^3\left[1+\left(\frac{y}{x}\right)^3\right]}{x\left(1+\frac{y}{x}\right)}$$

$$\tan U = x^2\left[\frac{1+(y/x)^3}{1+(y/x)}\right]$$

$\therefore$ Tan U is homogeneous of degree "2".

By Euler's theorem, $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = nU$

$$x\frac{\partial}{\partial x}(\tan U) + y\cdot\frac{\partial}{\partial y}(\tan U) = 2\tan U. \rightarrow ①$$

$$x\cdot\sec^2 U\cdot\frac{\partial U}{\partial x} + y\sec^2 U\cdot\frac{\partial U}{\partial y} = 2\tan U$$

$$\sec^2 u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]=2\tan u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2.\frac{\sin u}{\cos u}\times\cos^2 u.$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u \longrightarrow (2)$$

diff. w. r. to 'x' partially.

$$(1)\frac{\partial u}{\partial x}+x.\frac{\partial^2 u}{\partial x^2}+y.\frac{\partial^2 u}{\partial x.\partial y}=\cos 2u\,(2)\frac{\partial u}{\partial x}.$$

$$\frac{\partial u}{\partial x}+x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\frac{\partial u}{\partial x}$$

$$\cancel{x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u.\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}}$$

$$\cancel{x\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial x\partial y}=(2\cos 2u-1)\frac{\partial u}{\partial x}}$$

$$x\frac{\partial u}{\partial x}+x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\cdot x\frac{\partial u}{\partial x}\rightarrow (3)$$

from (2),

lly,
$$y\cdot\frac{\partial u}{\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+xy\frac{\partial^2 u}{\partial y\partial x}=2\cos u\cdot y\cdot\frac{\partial u}{\partial y}\rightarrow (4)$$

Adding (3) & (4)

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)-\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos 2u\ \sin 2u-\sin 2u.$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=\cancel{\sin 2u\,(2\cos 2u}\ \sin 4u-\sin 2u$$

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+2xy\frac{\partial^2 u}{\partial x\partial y}=2\cos\left(\frac{4u+2u}{2}\right).\sin\left(\frac{4u-2u}{2}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2\cos 3u\sin u.$$

(17) If $u=\tan^{-1}\left(\frac{y^2}{x}\right)$ show that $x^2.\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u.\sin^2 u.$

Sol: Given $u=\tan^{-1}\left(\frac{y^2}{x}\right).$

$$\tan u=\frac{y^2}{x}.$$

$\tan u = \frac{xy^2}{x^2}$

$\tan u = x\left(\frac{y}{x}\right)^2 \Rightarrow \tan u = x \cdot f\left(\frac{y}{x}\right)$

$\therefore \tan u$ is homogeneous of degree '1'.

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

$$x \cdot \frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = \tan u. \rightarrow (1)$$

$$x \cdot \sec^2 u \cdot \frac{\partial u}{\partial x} + y\sec^2 u \cdot \frac{\partial u}{\partial y} = \tan u.$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cancel{\cos u}} \times \cos^{\cancel{2}} u$$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin u \cdot \cos u.$$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin 2u. \rightarrow (2)$$

diff. w. r. to "x" partially

$$(1)\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{\cancel{2}}\cos 2u(\cancel{2})\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \cos 2u \frac{\partial u}{\partial x}$$

$$x \cdot \frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = x \cdot \cos 2u \cdot \frac{\partial u}{\partial x} \rightarrow (3)$$

from (2),

Illy,
$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = y\cos 2u \frac{\partial u}{\partial x} \rightarrow (4)$$

(3) + (4) $\Rightarrow$

$$x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}(\cos 2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sin 2u \cdot (\cos 2u - 1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sin 2u(\cancel{1} - 2\sin^2 u - \cancel{1})$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{\cancel{2}}\sin 2u \sin^2 u(\cancel{2})$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\sin 2u \cdot \sin^2 u.$$

* (18) If $u = (x^2+y^2)^{1/3}$. Show that $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}$.

* (19) If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$. Then evaluate $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

*(20) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)^{1/2}$. Evaluate $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y}$.

(10) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$

Sol:-

Given $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \frac{x(1+y/x)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\sin u = x \cdot x^{-1/2}\left[\frac{1+y/x}{1+\sqrt{y/x}}\right]$$

$$\sin u = x^{1/2} f\left(\frac{y}{x}\right)$$

$\therefore \sin u$ is homogeneous of degree "$\frac{1}{2}$".

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n \cdot u$

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = \frac{1}{2}\sin u \rightarrow (1)$$

$$x \cdot \cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \rightarrow (2)$$

diff. w.r.to "x" partially.

$$(1)\frac{\partial u}{\partial x} + x\cdot\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot \frac{\partial u}{\partial x}$$

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}x\cdot\sec^2 u\cdot\frac{\partial u}{\partial x} \rightarrow (3)$$

from (2)

Hly,

$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{2}\sec^2 u \cdot y\frac{\partial u}{\partial y} \rightarrow (4)$$

(3) + (4)

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{1}{2}\sec^2 u - 1\right]\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{1}{2}\sec^2 u - 1\right)\frac{1}{2}\tan u.$$

$$= \frac{1}{4}\frac{1}{\cos^2 u}\cdot\frac{\sin u}{\cos u} - \frac{1}{2}\tan u$$

$$= \frac{1}{4}\frac{\sin u}{\cos^3 u} - \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{\sin u - 2\sin u\cos^2 u}{4\cos^3 u}$$

$$= \frac{\sin u(1-2\cos^2 u)}{4\cos^3 u}$$

$$= \frac{-\sin u(2\cos^2 u - 1)}{4\cos^3 u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u\cos 2u}{4\cos^3 u}$$

⑫ If $f(x,y) = \sqrt{x^2-y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$, prove that $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y)$

Given $f(x,y) = \sqrt{x^2-y^2}\,\sin^{-1}\left(\frac{y}{x}\right)$

$$f(x,y) = x^1\sqrt{1-\left(\frac{y}{x}\right)^2}\,\sin^{-1}\left(\frac{y}{x}\right)$$

$$f(x,y) = x^1 f\left(\frac{y}{x}\right)$$

$\therefore f$ is homogeneous of degree "1".

By using Euler's theorem, $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u$

$$x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = (1)\,f(x,y)$$

$$x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = f(x,y).$$

⑬ If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$. Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

Given $u = \cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$

$$\cos u = \frac{x(1+y/x)}{\sqrt{x}\left(1+\frac{\sqrt{y}}{\sqrt{x}}\right)}$$

$$\cos u = x\cdot x^{-1/2}\left(\frac{1+y/x}{1-\sqrt{y/x}}\right)$$

$$\cos u = x^{1/2}\cdot f\left(\frac{y}{x}\right)$$

$\therefore \cos u$ is homogeneous of degree "$\frac{1}{2}$".

By using Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,u$

$$x\cdot\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u \quad \rightarrow (1)$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u.$$

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

$$\boxed{x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0.}$$

(14) If $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$ show that $\frac{\partial u}{\partial x} = -\frac{y}{x}\cdot\frac{\partial u}{\partial y}$

Sol:- Given $u = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \frac{\cancel{\sqrt{x}}(1-\sqrt{y}/\sqrt{x})}{\cancel{\sqrt{x}}(1+\sqrt{y}/\sqrt{x})}$$

$$\sin u = x^0\left[\frac{1-\sqrt{y/x}}{1+\sqrt{y/x}}\right]$$

$$\sin u = x^0\cdot f(y/x)$$

$\therefore \sin u$ is homogeneous of degree "0".

By Euler's theorem, $x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\cdot u$

$$x\cdot\frac{\partial}{\partial x}(\sin u) + y\cdot\frac{\partial}{\partial y}(\sin u) = 0.$$

$$x\cdot\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 0.$$

$$x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$x\frac{\partial u}{\partial x} = -y\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{-y}{x}\frac{\partial u}{\partial y}}$$

(15) Show that $x\cdot\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\log u$ where $\log u = \frac{x^3+y^3}{3x+4y}$

Sol:- Given $\log u = \frac{x^3+y^3}{3x+4y}$

$$\log u = \frac{x^{\cancel{3}\,2}(1+y^3/x^3)}{\cancel{x}(3+4(y/x))}$$

$$\log u = x^2\left[\frac{1+(y/x)^3}{3+4(y/x)}\right]$$

$$\log u = x^2\cdot f(y/x)$$

$\therefore \log u$ is homogeneous of degree "2".

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot u$

$$x\frac{\partial}{\partial x}(\log u) + y\frac{\partial}{\partial y}(\log u) = 2\cdot\log u$$

$$x \cdot \frac{1}{u} \cdot \frac{\partial u}{\partial x} + y \frac{1}{u} \frac{\partial u}{\partial y} = 2 \log u$$

$$\boxed{x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \log u.}$$

(18) If $u = (x^2+y^2)^{1/3}$ show that $x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}$.

sol: Given $u = (x^2+y^2)^{1/3}$

$$u = \left[x^2 (1 + y^2/x^2)\right]^{1/3}$$

$$u = x^{2/3} \left(1 + (y/x)^2\right)^{1/3}$$

$$u = x^{2/3} . f(y/x)$$

$\therefore u$ is homogeneous of degree "$2/3$".

By Euler's theorem, $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{2}{3} u \rightarrow (1)$$

diff. w. r. to 'x' partially

$$(1) \frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} \frac{\partial u}{\partial x}$$

$$x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} \cdot x \frac{\partial u}{\partial x} \rightarrow (2)$$

Ily $\quad y \cdot \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = \frac{2}{3} y \cdot \frac{\partial u}{\partial x} \rightarrow (3)$

(2)+(3)

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{2}{3}\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$x^2 \cdot \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2}{3} - 1\right)\left(x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{2-3}{3}\right) \frac{2}{3} u$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{-2u}{9}.$$

(19)

Given $u = x^2 \cdot \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}(x/y)$

$$u = x^2 \tan^{-1}(y/x) - y^2 \cot^{-1}(y/x)$$

$$u = x^2 \left[\tan^{-1}(y/x) - \left(\frac{y}{x}\right)^2 \cot^{-1}(y/x)\right]$$

$$u = x^2 . f(y/x).$$

$\therefore u$ is homogeneous of degree "2".

By Euler's theorem, $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = n \cdot u$

$$x \cdot \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u \rightarrow (1)$$

diff. w. r. to 'x' partially

$$(1)\frac{\partial u}{\partial x} + x \cdot \frac{\partial^2 u}{\partial x^2} + y \cdot \frac{\partial^2 u}{\partial x \partial y} = 2 \cdot \frac{\partial u}{\partial x}.$$

$$x \cdot \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} = 2x \frac{\partial u}{\partial x} \rightarrow (2)$$

lly, $$y \cdot \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + xy \frac{\partial^2 u}{\partial x \partial y} = 2y \frac{\partial u}{\partial x} \rightarrow (3)$$

(2) + (3)

$$\Rightarrow x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y}$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = 2u.$$

(20) Given $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]^{1/2}$

$$\operatorname{cosec} u = \left[\frac{x^{1/2}(1+y^{1/2}/x^{1/2})}{x^{1/3}(1+y^{1/3}/x^{1/3})}\right]^{1/2}$$

$$\operatorname{cosec} u = \frac{x^{1/4}}{x^{1/6}} \left[\frac{1+(y/x)^{1/2}}{1+(y/x)^{1/3}}\right]^{1/2}$$

$2 \mid 6, 4$
$\quad 3, 2$

$$\operatorname{cosec} u = x^{1/4} \cdot x^{-1/6} f(y/x)$$

$$\operatorname{cosec} u = x^{1/12} f(y/x)$$

$\frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$

$\therefore \operatorname{cosec} u$ is homogeneous of degree '$1/12$'.

By Euler's theorem, $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \cdot u$

$$x \cdot \frac{\partial}{\partial x}(\operatorname{cosec} u) + y \frac{\partial}{\partial y}(\operatorname{cosec} u) = \frac{1}{12} \cdot \operatorname{cosec} u.$$

$$-x \cdot \operatorname{cosec} u \cdot \cot u \cdot \frac{\partial u}{\partial x} + y(-\operatorname{cosec} u \cdot \cot u)\frac{\partial u}{\partial y} = \frac{1}{12} \operatorname{cosec} u.$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{1}{12} \frac{\cancel{\operatorname{cosec} u}}{-\cancel{\operatorname{cosec} u} \cdot \cot u}$$

$$x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = \frac{-1}{12} \tan u \rightarrow (1)$$

diff. w. r. to "x" partially,

$$(1)\frac{\partial u}{\partial x} + x.\frac{\partial^2 u}{\partial x^2} + y.\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\frac{\partial u}{\partial x}$$

$$x.\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\ x.\frac{\partial u}{\partial x} \rightarrow ②$$

lly,
$$y.\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{-1}{12}\sec^2 u.\ y.\frac{\partial u}{\partial y} \rightarrow ③$$

② + ③

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \left[\frac{-1}{12}\sec^2 u - 1\right]\left(x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \left(\frac{-1}{12}\sec^2 u - 1\right)\left(\frac{-1}{12}\tan u\right)$$

$$= \frac{1}{144}.\frac{\sin u}{\cos^3 u} + \frac{1}{12}.\frac{\sin u}{\cos u}$$

$$= \frac{\sin u + 12\sin u\cos^2 u}{144.\cos^3 u}$$

$$= \frac{\sin u(1+12\cos^2 u)}{144.\cos^3 u}$$

$$= \frac{\sin u\left[1+12(1-\sin^2 u)\right]}{144\cos^3 u}$$

$$= \frac{\sin u\left[1+12-12\sin^2 u\right]}{144\cos^3 u}$$

$$= \frac{11\sin u - 12\sin^3 u}{144.\cos^3 u}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = \frac{11}{144}.\frac{\sin u}{\cos^3 u} - \frac{1}{12}\tan^3 u.$$

22/11/19 Friday

Total Derivative and Chain Rule:

$$u \begin{matrix} \nearrow x \searrow \\ \searrow y \nearrow \end{matrix} t \qquad\qquad u \begin{matrix} \nearrow x \searrow \\ \searrow y \nearrow \end{matrix} r,\theta.$$

① If $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$. show that $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

sol:- Given $u = \sin^{-1}(x-y)$, $x = 3t$, $y = 4t^3$ $\qquad u\begin{matrix} x \\ y\end{matrix} t$

∴ By using Total Derivative

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\sin^{-1}(x-y) = \frac{1}{\sqrt{1-(x-y)^2}}(1-0) = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left[\sin^{-1}(x-y)\right] = \frac{1}{\sqrt{1-(x-y)^2}}(0-1) = \frac{-1}{\sqrt{1-(x-y)^2}}$$

$$\frac{dx}{dt} = \frac{d}{dt}(3t) = 3 \quad , \quad \frac{dy}{dt} = \frac{d}{dt}(4t^3) = 12t^2.$$

$$\frac{du}{dt} = \frac{1}{\sqrt{1-(x-y)^2}}(3) + \frac{-1}{\sqrt{1-(x-y)^2}}(12t^2)$$

$$= \frac{3-12t^2}{\sqrt{1-(x-y)^2}}$$

$$= \frac{3-12t^2}{\sqrt{1-x^2-y^2+2xy}}$$

$$= \frac{3-12t^2}{\sqrt{1-9t^2-16t^6+24t^4}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16t^6+24t^4-9t^2+1}}$$

$$= \frac{3(1-4t^2)}{\sqrt{-16x^3+24x^2-9x+1}}$$

$$\begin{array}{r|rrrr} 1 & -16 & 24 & -9 & 1 \\ & 0 & -16 & 8 & -1 \\ \hline & -16 & 8 & -1 & \boxed{0} \end{array}$$

$(x-1)(-16x^2+8x-1)=0$

$(x-1)[-(16x^2-8x+1)]=0$

$(1-x)\,(4x-1)^2=0$

$$= \frac{3(1-4t^2)}{\sqrt{(1-x)\,(1-4x)^2}}$$

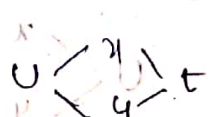
$$= \frac{3(1-4t^2)}{\sqrt{1-x}\,(1-4x)}$$

$$= \frac{3\cancel{(1-4t^2)}}{\sqrt{1-t^2}\,\cancel{(1-4t^2)}}$$

$$= \frac{3}{\sqrt{1-t^2}}$$

⑩ If $u = \tan^{-1}(y/x)$, $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ then find $\frac{du}{dt}$.

Sol: Given $u = \tan^{-1}(y/x)$



By using Total Derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$\frac{\partial}{\partial x}\left[\tan^{-1}(y/x)\right] = \frac{1}{1+\frac{y^2}{x^2}}\, y\left(\frac{-1}{x^2}\right) = \frac{-y}{\cancel{x^2}}\,\frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} = \frac{-y}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left[\tan^{-1}(y/x)\right] = \frac{1}{1+y^2/x^2}\,\frac{1}{x} = \frac{1}{\cancel{x}}\,\frac{1}{\frac{x^2+y^2}{x^{\cancel{2}}}} = \frac{x}{x^2+y^2}$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t - e^{-t}) = e^t - e^{-t}(-1) = e^t + e^{-t}.$$

$$\frac{dy}{dt} = \frac{d}{dt}(e^t + e^{-t}) = e^t + e^{-t}(-1) = e^t - e^{-t}.$$

$$\frac{du}{dt} = \frac{-y}{x^2+y^2}(e^t + e^{-t}) + \frac{x}{x^2+y^2}(e^t - e^{-t})$$

$$= \frac{-y(y) + x(x)}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$= \frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$= \frac{\cancel{e^{2t}} + \cancel{e^{-2t}} - 2 - \cancel{e^{2t}} - \cancel{e^{-2t}} - 2}{e^{2t} + e^{-2t} - \cancel{2} + e^{2t} + e^{-2t} + \cancel{2}}$$

$$= \frac{-\cancel{4}\ 2}{\cancel{2}(e^{2t} + e^{-2t})}$$

$$= \frac{-1}{\frac{e^{2t}+e^{-2t}}{2}} = \frac{-1}{\cosh 2t} = -\operatorname{sech} 2t.$$

(14) If $u = f(x^2 + 2yz, y^2 + 2zx)$ prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0.$

Sol: Given $u = f(x^2 + 2yz, y^2 + 2zx)$

$u = f(r, s)$ where $r = x^2 + 2yz$, $s = y^2 + 2zx$

$u < \begin{matrix} r \\ s \end{matrix} > x, y, z$

By using chain Rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} \cancel{+ \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z}$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r};\ \frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}} \qquad \boxed{u = f(r, s)}$$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2yz) = 2x + 0 = 2x.$

$\Rightarrow \frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2yz) = (0 + 2z) = 2z.$

$\Rightarrow \frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(x^2 + 2yz) = (0 + 2y) = 2y$

$\Rightarrow \frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(y^2 + 2zx) = (0 + 2z) = 2z.$

$\Rightarrow \frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(y^2 + 2zx) = (2y + 0) = 2y$

$\Rightarrow \frac{\partial s}{\partial z} = \frac{\partial}{\partial z}(y^2 + 2zx) = (0 + 2x) = 2x.$

$$\frac{\partial U}{\partial x} = \frac{\partial f}{\partial r}(2x) + \frac{\partial f}{\partial s}(2z)$$

$$\frac{\partial U}{\partial y} = \frac{\partial f}{\partial r}(2z) + \frac{\partial f}{\partial s}(2y)$$

$$\frac{\partial U}{\partial z} = \frac{\partial f}{\partial r}(2y) + \frac{\partial f}{\partial s}(2x)$$

Now, $(y^2 - zx)\frac{\partial U}{\partial x} + (x^2 - yz)\frac{\partial U}{\partial y} + (z^2 - xy)\frac{\partial U}{\partial z}$

$$= (y^2 - zx)\left(\frac{\partial f}{\partial r}2x + \frac{\partial f}{\partial s}2z\right) + (x^2 - yz)\left(\frac{\partial f}{\partial r}2z + \frac{\partial f}{\partial s}2y\right) + (z^2 - xy)\left(\frac{\partial f}{\partial r}2y + \frac{\partial f}{\partial s}2x\right)$$

$$= \cancel{2xy^2\frac{\partial f}{\partial r}} - \cancel{2x^2z\frac{\partial f}{\partial r}} + \cancel{2zy^2\frac{\partial f}{\partial s}} - \cancel{2z^2x\frac{\partial f}{\partial s}} + \cancel{2x^2z\frac{\partial f}{\partial r}} - \cancel{2z^2y\frac{\partial f}{\partial r}}$$
$$+ \cancel{2yx^2\frac{\partial f}{\partial s}} - \cancel{2y^2z\frac{\partial f}{\partial s}} + \cancel{2yz^2\frac{\partial f}{\partial r}} - \cancel{2xy^2\frac{\partial f}{\partial r}} + \cancel{2xz^2\frac{\partial f}{\partial s}} - \cancel{2x^2y\frac{\partial f}{\partial s}}$$

$= 0.$

② If z is a function of x and y where $x = e^{u} + e^{-v}$ and $y = e^{-u} - e^{v}$. show that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x.\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.

Sol:- Given $z = f(x,y)$, $x = e^{u} + e^{-v}$, $y = e^{-u} - e^{v}$.

By using chain Rule, $\quad z \left< \begin{matrix} x \\ y \end{matrix} \right. \rightarrow u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}$$

$\cancel{\frac{\partial z}{\partial x}} \Rightarrow$

$\frac{\partial x}{\partial u} = e^{u} + \cancel{e^{-v}}(0)$ $\quad$ $\frac{\partial y}{\partial u} = e^{-u}(-1) - 0 = -e^{-u}$

$= e^{u} + \cancel{0}$

$\frac{\partial x}{\partial v} = 0 + e^{-v}(-1)$ $\quad$ $\frac{\partial y}{\partial v} = 0 - e^{v} = -e^{v}$

$= -e^{-v}$

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}(e^{u}) + \frac{\partial f}{\partial y}(-e^{-u}) = \frac{\partial f}{\partial x}e^{u} - \frac{\partial f}{\partial y}e^{-u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}(-e^{-v}) + \frac{\partial f}{\partial y}(-e^{v}) = -\left[\frac{\partial f}{\partial x}e^{-v} + \frac{\partial f}{\partial y}e^{v}\right]$$

$$\therefore \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} e^{u} - \frac{\partial f}{\partial y} e^{-u} + \frac{\partial f}{\partial x} e^{-v} + \frac{\partial f}{\partial y} e^{v}$$

$$= (e^{u} + e^{-v}) \frac{\partial f}{\partial x} + (e^{v} - e^{-u}) \frac{\partial f}{\partial y}$$

$$= (e^{u} + e^{-v}) \frac{\partial f}{\partial x} - (e^{-u} - e^{v}) \frac{\partial f}{\partial y}$$

$$= x \cdot \frac{\partial z}{\partial x} - y \cdot \frac{\partial z}{\partial y}.$$

(3) If $u = f(y-z,\ z-x,\ x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Given $u = f(y-z,\ z-x,\ x-y)$

$u = f(a, b, c)$

Where $a = y-z$, $b = z-x$, $c = x-y$

By using chain Rule, $\quad u \begin{array}{c} a \\ b \\ c \end{array} x\,y\,z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial x} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial x} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial y} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial a} \cdot \frac{\partial a}{\partial z} + \frac{\partial u}{\partial b} \cdot \frac{\partial b}{\partial z} + \frac{\partial u}{\partial c} \cdot \frac{\partial c}{\partial z}$$

$$\frac{\partial u}{\partial a} = \frac{\partial f}{\partial a}, \quad \frac{\partial u}{\partial b} = \frac{\partial f}{\partial b}, \quad \frac{\partial u}{\partial c} = \frac{\partial f}{\partial c}$$

$\frac{\partial a}{\partial x} = \frac{\partial}{\partial x}(y-z) = 0$

$\frac{\partial a}{\partial y} = \frac{\partial}{\partial y}(y-z) = 1$

$\frac{\partial a}{\partial z} = \frac{\partial}{\partial z}(y-z) = -1$

$\frac{\partial b}{\partial x} = \frac{\partial}{\partial x}(z-x) = -1$

$\frac{\partial b}{\partial y} = \frac{\partial}{\partial y}(z-x) = 0$

$\frac{\partial b}{\partial z} = \frac{\partial}{\partial z}(z-x) = 1$

$\frac{\partial c}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1$

$\frac{\partial c}{\partial y} = \frac{\partial}{\partial y}(x-y) = \cancel{0}-1$

$\frac{\partial c}{\partial z} = \frac{\partial}{\partial z}(x-y) = 0$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial a}(0) + \frac{\partial f}{\partial b}(-1) + \frac{\partial f}{\partial c}(1) = -\frac{\partial f}{\partial b} + \frac{\partial f}{\partial c}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial a}(1) + \frac{\partial f}{\partial b}(0) + \frac{\partial f}{\partial c}(-1) = \frac{\partial f}{\partial a} - \frac{\partial f}{\partial c}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial a}(-1) + \frac{\partial f}{\partial b}(1) + \frac{\partial f}{\partial c}(0) = -\frac{\partial f}{\partial a} + \frac{\partial f}{\partial b}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= -\cancel{\frac{\partial f}{\partial b}} + \cancel{\frac{\partial f}{\partial c}} + \cancel{\frac{\partial f}{\partial a}} - \cancel{\frac{\partial f}{\partial c}} - \cancel{\frac{\partial f}{\partial a}} + \cancel{\frac{\partial f}{\partial b}}$$

$= 0.$

④ If $w = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$. Show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Sol: Given $w = f(x,y)$

and $x = r\cos\theta$, $y = r\sin\theta$.

By using chain Rule, $w < \begin{matrix} x \\ y \end{matrix} > r,\theta$.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x}, \quad \frac{\partial w}{\partial y} = \frac{\partial f}{\partial y}$$

$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$. | $\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(r\sin\theta) = \sin\theta$.

$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(r\cos\theta) = r(-\sin\theta) = -r\sin\theta$ | $\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(r\sin\theta) = r\cos\theta$.

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}(\cos\theta) + \frac{\partial f}{\partial y}\sin\theta \rightarrow ①$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y} r\cos\theta \rightarrow ②$$

$$① \Rightarrow \left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\sin^2\theta + \cancel{2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cdot\cos\theta}.$$

$$② \Rightarrow \left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 r^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2 r^2\cos^2\theta - 2r^2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cos\theta.$$

$$\left(\frac{\partial w}{\partial \theta}\right)^2 = r^2\left[\left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta - 2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\cdot\sin\theta\cos\theta\right]$$

$$\frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial f}{\partial y}\right)^2\cos^2\theta - \cancel{2\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}\sin\theta\cos\theta}$$

①+③ $\Rightarrow$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2[\cos^2\theta + \sin^2\theta] + \left(\frac{\partial f}{\partial y}\right)^2(\sin^2\theta + \cos^2\theta) \rightarrow ③$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

⑤ If f is the function u, v and $u = x^2 - y^2$, $v = 2xy$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right)$

Sol: Given $f = \theta(u,v)$

$u = x^2 - y^2$, $v = 2xy$

By using chain Rule, $f < \begin{matrix} u \\ v \end{matrix} > xy$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial U}\cdot\frac{\partial U}{\partial x} + \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial U}\cdot\frac{\partial U}{\partial y} + \frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial y}$$

$$\frac{\partial f}{\partial U} = \frac{\partial \phi}{\partial U}, \quad \frac{\partial f}{\partial V} = \frac{\partial \phi}{\partial V}$$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(x^2-y^2) = 2x \qquad \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}(x^2-y^2) = -2y \qquad \frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial f}{\partial x} = \frac{\partial \phi}{\partial U}(2x) + \frac{\partial \phi}{\partial V}(2y)$$

$$\frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial U}(-2y) + \frac{\partial \phi}{\partial V}(2x)$$

$$\cancel{\frac{\partial f}{\partial x} = 2\frac{\partial f}{\partial U}x + 2\cdot\frac{\partial f}{\partial V}y}$$

~~diff. w. r. to "x" partially~~

$$\cancel{\frac{\partial^2 \theta}{\partial x^2} = 2\frac{\partial f}{\partial U}(1) + 2x\frac{\partial^2 f}{\partial U\partial x} + 2y\frac{\partial^2 f}{\partial V\partial x} + 2\frac{\partial f}{\partial V}(0)}$$

$$\cancel{= 2\cdot\frac{\partial f}{\partial U} + 2x\frac{\partial^2 f}{\partial x\cdot\partial U} + 2y\frac{\partial^2 f}{\partial V\partial x}}$$

$$\cancel{\frac{\partial^2 \theta}{\partial y^2} = (-2y)\frac{\partial^2 f}{\partial U\partial y} + 2\frac{\partial f}{\partial U}(1) + 2x\cdot\frac{\partial^2 f}{\partial V^2}}$$

$$\frac{\partial f}{\partial x} = 2x\cdot\frac{\partial \theta}{\partial U} + 2y\frac{\partial \theta}{\partial V} \rightarrow (1)$$

$$\frac{\partial f}{\partial x} = 2\left[x\cdot\frac{\partial \theta}{\partial U} + y\cdot\frac{\partial \theta}{\partial V}\right]$$

$$\frac{\partial f}{\partial x} = 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right]\theta$$

$$\frac{\partial}{\partial x} = 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right] \rightarrow (2)$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$= 2\left[x\cdot\frac{\partial}{\partial U} + y\cdot\frac{\partial}{\partial V}\right] 2\left(x\cdot\frac{\partial \theta}{\partial U} + y\cdot\frac{\partial \theta}{\partial V}\right) \quad [\because \text{from (1) \& (2)}]$$

$$= 4\left(x^2\cdot\frac{\partial^2 \theta}{\partial U^2} + xy\frac{\partial^2 \theta}{\partial U\cdot\partial V} + xy\frac{\partial^2 \theta}{\partial U\cdot\partial V} + y^2\cdot\frac{\partial^2 \theta}{\partial V^2}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = 4\left(x^2\cdot\frac{\partial^2 \theta}{\partial U^2} + 2xy\frac{\partial^2 \theta}{\partial U\cdot\partial V} + y^2\frac{\partial^2 \theta}{\partial V^2}\right) \rightarrow (3)$$

⑥ If $U = x^2+y^2+z^2$ and $x = e^{2t}$, $y = e^{2t}\cdot\cos 3t$, $z = e^{2t}\sin 3t$ find $\frac{du}{dt}$.

Solⁿ

Given $U = x^2+y^2+z^2$

and $x = e^{2t}$, $y = e^{2t}\cdot\cos 3t$, $z = e^{2t}\sin 3t$

By using Total Derivatives $\qquad U < \begin{matrix} x \\ y \\ z \end{matrix} > t$

$$\frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt} + \frac{du}{dz}\cdot\frac{dz}{dt}.$$

$\frac{dU}{dx} = \frac{d}{dx}(x^2+y^2+z^2) = 2x$

$\frac{dx}{dt} = \frac{d}{dt}(e^{2t}) = 2\cdot e^{2t}$

$\frac{dU}{dy} = \frac{d}{dy}(x^2+y^2+z^2) = 2y$

$\frac{dy}{dt} = \frac{d}{dt}(e^{2t}\cos 3t)$

$= e^{2t}(-\sin 3t)3 + \cos 3t\, e^{2t}(2)$

$= -3e^{2t}\sin 3t + 2e^{2t}\cos 3t$

$\frac{dU}{dz} = \frac{d}{dz}(x^2+y^2+z^2) = 2z$.

$\frac{dz}{dt} = \frac{d}{dt}(e^{2t}\sin 3t)$

$= e^{2t}\cdot\cos 3t(3) + \sin 3t(e^{2t})2$

$= 3e^{2t}\cos 3t + 2e^{2t}\sin 3t$

$$\frac{du}{dt} = 2x(2\cdot e^{2t}) + 2y(-3e^{2t}\sin 3t + 2\cdot e^{2t}\cos 3t) + 2z(3e^{2t}\cos 3t + 2e^{2t}\sin 3t)$$

$= 4x\cdot e^{2t} - 6y e^{2t}\sin 3t + 4ye^{2t}\cos 3t + 6ze^{2t}\cos 3t + 4ze^{2t}\sin 3t.$

$= 4x\cdot e^{2t} - e^{2t}\sin 3t\,(6y - 4z) + e^{2t}\cos 3t\,(4y + 6z)$

$= 4x\cdot e^{2t} - e^{2t}\sin 3t(6e^{2t}\cos 3t - 4e^{2t}\sin 3t) + e^{2t}\cos 3t(4e^{2t}\cos 3t + 6e^{2t}\sin 3t)$

$= 4e^{2t}e^{2t} - \cancel{6e^{4t}\sin 3t\cos 3t} + 4e^{4t}\sin^2 3t + 4e^{4t}\cos^2 3t + \cancel{6e^{4t}\sin 3t\cos 3t}$

$= 4\cdot e^{4t}\,(1 + \underline{\sin^2 3t + \cos^2 3t})$

$= 4\cdot e^{4t}(1+1) \quad = 4\cdot e^{4t}(2) \quad = \underline{8\cdot e^{4t}}$

⑦ If $U = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$ then find $\frac{du}{dt}$.

Given $u = \sin\left(\frac{x}{y}\right)$

$x = e^t$, $y = t^2$

By using Total Derivative, $\qquad U < \begin{matrix} x \\ y \end{matrix} > t$

$$\frac{du}{dt} = \frac{dU}{dx}\cdot\frac{dx}{dt} + \frac{du}{dy}\cdot\frac{dy}{dt}$$

$\frac{dU}{dx} = \cos\frac{x}{y}\left(\frac{1}{y}\right)$

$\frac{dx}{dt} = e^t$

$\frac{dU}{dy} = \cos\left(\frac{x}{y}\right) x\left(\frac{-1}{y^2}\right)$

$\frac{dy}{dt} = 2t$.

$$\frac{dU}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right)e^t - \frac{x}{y^2}\cos\left(\frac{x}{y}\right)2t.$$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) - \frac{e^t}{t^4}\cos\left(\frac{e^t}{t^2}\right)2t$$

$$= \frac{e^t}{t^2}\left\{\cos\left(\frac{e^t}{t^2}\right)\left[1-\frac{2t}{t^2}\right]\right.$$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right)\left(\frac{t^2-2t}{t^2}\right)$$

$$\frac{du}{dt} = \frac{e^t(t^2-2t)}{t^4}\cos\left(\frac{e^t}{t^2}\right) \Rightarrow \frac{du}{dt} = \frac{e^t(t-2)}{t^3}\cdot\cos\left(\frac{e^t}{t^2}\right)$$

⑧ If $u = x^3+y^3$ where $x = a\cos t$, $y = b\sin t$ find $\frac{du}{dt}$.

Given $u = x^3+y^3$, $x = a\cos t$, $y = b\sin t$.

By using Total Derivative, $u < \begin{matrix} x \\ y \end{matrix} > t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}.$$

$$\frac{\partial u}{\partial x} = 3x^2 \qquad \frac{\partial u}{\partial y} = 3y^2$$

$$\frac{dx}{dt} = -a\sin t \qquad \frac{dy}{dt} = b\cos t.$$

$$\frac{du}{dt} = 3x^2(-a\sin t) + 3y^2\cdot b\cos t$$

$$= -3(x^2 a\sin t - y^2 b\cos t)$$

$$= -3(a^2\cos^2 t\cdot a\sin t - b^2\sin^2 t\cdot b\cdot\cos t)$$

$$= -3(a^3\cdot\sin t\cdot\cos^2 t - b^3\sin^2 t\cdot\cos t)$$

$$= -3\sin t\cdot\cos t\,(a^3\cos t - b^3\sin t)$$

$$= \frac{-3}{2}\cdot\sin 2t\,(a^3\cos t - b^3\sin t)$$

⑨ If $z = u^2+v^2$, $u = r\cos\theta$, $v = r\sin\theta$. find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial \theta}$.

Sol:- Given $z = u^2+v^2 = f(u,v)$

$u = r\cos\theta$. $v = r\sin\theta$.

By using chain Rule, $z < \begin{matrix} u \\ v \end{matrix} > r, \theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial r}$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial \theta} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial \theta}.$$

$\frac{\partial z}{\partial u} = 2u$	$\frac{\partial z}{\partial v} = 2v$
$\frac{\partial u}{\partial r} = \cos\theta$	$\frac{\partial u}{\partial \theta} = r(-\sin\theta)$
$\frac{\partial v}{\partial r} = \sin\theta$	$\frac{\partial v}{\partial \theta} = r\cos\theta$

$$\frac{\partial z}{\partial r} = 2u\cos\theta + 2v\,(-r\sin\theta)$$

$$= 2(r\cos\theta)\cos\theta - 2(r\sin\theta)(-r\sin\theta)$$

$$= 2r\cos^2\theta + 2r^2\sin^2\theta.$$

$$= 2r(\cos^2\theta + r.\sin^2\theta)$$

$$\frac{\partial z}{\partial \theta} = 2u\sin\theta + 2v\,(r\cos\theta)$$

$$= 2(r\cos\theta)(\sin\theta) + 2(r\sin\theta)\,r\cos\theta$$

$$= r.2\sin\theta.\cos\theta + r^2.2\sin\theta.\cos\theta$$

$$= r.\sin 2\theta + r^2.\sin 2\theta$$

$$= r\sin 2\theta\,(1+r)$$

(10) If $v = \tan^{-1}(y/x)$; $x = e^t - e^{-t}$, $y = e^t + e^{-t}$ find $\frac{du}{dt}$.

Given $v = \tan^{-1}(y/x)$

$x = e^t - e^{-t}$.

(11) If $z = \log(u^2+v)$, $u = e^{x^2+y^2}$, $v = x^2+y$ find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

Given $z = \log(u^2+v)$

$u = e^{x^2+y^2}$, $v = x^2+y$

By using chain Rule, $z < \begin{matrix} u \\ v \end{matrix} > x, y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}.$$

$$\frac{\partial z}{\partial u} = \frac{1}{u^2+v}(2u) \qquad \frac{\partial z}{\partial v} = \frac{1}{u^2+v}(1)$$

$$\frac{\partial u}{\partial x} = e^{x^2+y^2}(2x) \qquad \frac{\partial u}{\partial y} = e^{x^2+y^2}(2y)$$

$$\frac{\partial v}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 1$$

$$\frac{\partial z}{\partial x} = \frac{2u}{u^2+v}\cdot e^{x^2+y^2}(2x) + \frac{1}{u^2+v}(2x)$$

$$= \frac{2x}{u^2+v}\left(2u.e^{x^2+y^2}+1\right)$$

$$= \frac{2x}{(e^{x^2+y^2})^2 + x^2+y}\left(2.e^{x^2+y^2}.e^{(x^2+y^2)}+1\right)$$

$$= \frac{2x}{e^{2(x^2+y^2)} + x^2+y}\left[2.e^{2(x^2+y^2)}+1\right]$$

$$\frac{\partial z}{\partial y} = \frac{2u}{u^2+v} e^{x^2+y^2}(2y) + \frac{1}{u^2+v}(1)$$

$$= \frac{4yu \cdot e^{x^2+y^2}}{u^2+v} + \frac{1}{u^2+v}$$

$$= \frac{4y\, e^{x^2+y^2} \cdot e^{x^2+y^2} + 1}{(e^{x^2+y^2})^2 + x^2+y}$$

$$= \frac{4y \cdot e^{2(x^2+y^2)} + 1}{e^{2(x^2+y^2)} + x^2+y}$$

(12) If $u = f(r,s,t)$ and $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$ prove that,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Given $u = f(r,s,t)$

$r = \frac{x}{y}$, $s = \frac{y}{z}$; $t = \frac{z}{x}$

By using chain Rule, $u \left\langle \begin{matrix} r \\ s \\ t \end{matrix} \right\rangle x, y, z.$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$	$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}$
$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y}$	$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = x\left(\frac{-1}{y^2}\right)$	$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\frac{x}{y}\right) = 0$
$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{z}\right) = 0$	$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{z}\right) = \frac{1}{z}$	$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}\left(\frac{y}{z}\right) = y\left(\frac{-1}{z^2}\right)$
$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\frac{z}{x}\right) = z\left(\frac{-1}{x^2}\right)$	$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{x}\right) = 0$	$\frac{\partial t}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{x}\right) = \frac{1}{x}$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}\left(\frac{-z}{x^2}\right) = \frac{1}{y}\frac{\partial f}{\partial r} - \frac{z}{x^2}\cdot\frac{\partial f}{\partial t}$$

$$\Rightarrow x\cdot\frac{\partial u}{\partial x} = \frac{x}{y}\cdot\frac{\partial f}{\partial r} - \frac{z}{x}\cdot\frac{\partial f}{\partial t} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial f}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial f}{\partial t}(0)$$

$$\Rightarrow y\cdot\frac{\partial u}{\partial y} = -\frac{x}{y}\cdot\frac{\partial f}{\partial r} + \frac{y}{z}\cdot\frac{\partial f}{\partial s} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\cdot(0) + \frac{\partial f}{\partial s}\left(-\frac{y}{z^2}\right) + \frac{\partial f}{\partial t}\left(\frac{1}{x}\right)$$

$$\Rightarrow z \cdot \frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial f}{\partial s} + \frac{z}{x}\cdot\frac{\partial f}{\partial t} \rightarrow (3)$$

Adding (1)+(2)+(3)

$$\Rightarrow x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\cdot\frac{\partial u}{\partial z}$$

$$= \frac{x}{y}\cancel{\frac{\partial f}{\partial r}} - \frac{z}{x}\cancel{\frac{\partial f}{\partial t}} - \frac{x}{y}\cancel{\frac{\partial f}{\partial r}} + \frac{y}{z}\cancel{\frac{\partial f}{\partial s}} - \frac{y}{z}\cancel{\frac{\partial f}{\partial s}} + \frac{z}{x}\cdot\cancel{\frac{\partial f}{\partial t}}$$

$$= 0.$$

(15) If $u = f(r,s)$, $r = x+y$, $s = x-y$ show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\cdot\frac{\partial u}{\partial r}$.

Given $u = f(r,s)$

$r = x+y$, $s = x-y$

By using chain Rule, $u \langle {r \atop s} \rangle x,y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$

$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r} \qquad \frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1 \qquad \frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(x+y) = 1$$

$$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(x-y) = 1 \qquad \frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(x-y) = -1$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(1) = \frac{\partial f}{\partial r} + \frac{\partial f}{\partial s}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(1) + \frac{\partial f}{\partial s}(-1) = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial s}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial f}{\partial r} + \cancel{\frac{\partial f}{\partial s}} + \frac{\partial f}{\partial r} - \cancel{\frac{\partial f}{\partial s}}$$

$$= 2\cdot\frac{\partial f}{\partial r}$$

$$= 2\cdot\frac{\partial u}{\partial r}$$

(13) If $u = f[2x-3y, (3y-4z), 4z-2x]$ Prove that

$$\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

Given $u = f(2x-3y, 3y-4z, 4z-2x)$

$u = f(r, s, t)$

where $r = 2x-3y$, $s = 3y-4z$, $t = 4z-2x$.

By using chain Rule, $u \leftarrow \begin{matrix} r \\ s \\ t \end{matrix} \rightarrow x, y$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}.$$

$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial r}$	$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial s}$	$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial t}$
$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(2x-3y) = 2$	$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}(2x-3y) = -3$	$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}(2x-3y) = 0$
$\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(3y-4z) = 0$	$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(3y-4z) = 3$	$\frac{\partial s}{\partial z} = \frac{\partial}{\partial z}(3y-4z) = -4$
$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(4z-2x) = -2$	$\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}(4z-2x) = 0$	$\frac{\partial t}{\partial z} = \frac{\partial}{\partial z}(4z-2x) = 4$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}(2) + \frac{\partial f}{\partial s}(0) + \frac{\partial f}{\partial t}(-2)$$

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \rightarrow (1).$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}(-3) + \frac{\partial f}{\partial s}(3) + \frac{\partial f}{\partial t}(0)$$

$$\rightarrow \frac{1}{3}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}(0) + \frac{\partial f}{\partial s}(-4) + \frac{\partial f}{\partial t}(4)$$

$$\rightarrow \frac{1}{4}\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \rightarrow (3)$$

Adding (1) + (2) + (3)

$$\Rightarrow \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$$

$$= \cancel{\frac{\partial f}{\partial r}} - \cancel{\frac{\partial f}{\partial t}} - \cancel{\frac{\partial f}{\partial r}} + \cancel{\frac{\partial f}{\partial s}} - \cancel{\frac{\partial f}{\partial s}} + \cancel{\frac{\partial f}{\partial t}}$$

$= 0.$

$$\therefore \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0.$$

⑤ → Continuous :-

$$\frac{\partial f}{\partial y} = -2y\frac{\partial \theta}{\partial u} + 2x\frac{\partial \theta}{\partial v} \rightarrow ④$$

$$\frac{\partial f}{\partial y} = 2\left(x\cdot\frac{\partial \theta}{\partial v} - y\frac{\partial \theta}{\partial u}\right)$$

$$\frac{\partial f}{\partial y} = 2\cdot\left(x\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right)\theta$$

$$\frac{\partial}{\partial y} = 2\left(x\cdot\frac{\partial}{\partial v} - y\frac{\partial}{\partial u}\right) \rightarrow ⑤$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$= 2\left(x\frac{\partial}{\partial v} - y\cdot\frac{\partial}{\partial u}\right) 2\left(x\cdot\frac{\partial \theta}{\partial v} - y\frac{\partial \theta}{\partial u}\right)$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial v^2} - xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} - xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\cdot\frac{\partial^2 \theta}{\partial u^2}\right] \rightarrow ⑥$$

Adding ③ + ⑥

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial u^2} + 2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\cdot\frac{\partial^2 \theta}{\partial v^2}\right] + 4\left[x^2\cdot\frac{\partial^2 \theta}{\partial v^2} - 2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v} + y^2\frac{\partial^2 \theta}{\partial u^2}\right]$$

$$= 4\left[x^2\frac{\partial^2 \theta}{\partial u^2} + \cancel{2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v}} + y^2\frac{\partial^2 \theta}{\partial v^2} + x^2\frac{\partial^2 \theta}{\partial v^2} - \cancel{2xy\frac{\partial^2 \theta}{\partial u\cdot\partial v}} + y^2\cdot\frac{\partial^2 \theta}{\partial u^2}\right]$$

$$= 4\left[\frac{\partial^2 \theta}{\partial u^2}(x^2+y^2) + \frac{\partial^2 \theta}{\partial v^2}(x^2+y^2)\right]$$

$$= 4(x^2+y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2+y^2)\left(\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2}\right)$$

23/4/19
Saturday

Implicit Function:

① If $z = \sqrt{x^2+y^2}$ and $x^3+y^3+3axy = 5a^2$. Find the value of $\frac{dz}{dx}$ when $x=y=a$.

Given $z = \sqrt{x^2+y^2}$, $x^3+y^3+3axy = 5a^2$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \qquad z < \begin{matrix} x \\ y \end{matrix} > x$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$z = \sqrt{x^2+y^2}$

$$\frac{dz}{dx} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x) = \frac{x}{\sqrt{x^2+y^2}}$$

$$\frac{dz}{dy} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y) = \frac{y}{\sqrt{x^2+y^2}}$$

Given $x^3+y^3+3axy-5a^2=0$

differentiate with r.to 'x'.

$$3x^2+3y^2\frac{dy}{dx}+3a\left[(1)y+x.\frac{dy}{dx}\right]=0$$

$$x^2+y^2\frac{dy}{dx}+ay+ax.\frac{dy}{dx}=0$$

$$(y^2+ax)\frac{dy}{dx} = -(x^2+ay)$$

$$\frac{dy}{dx} = \frac{-(x^2+ay)}{y^2+ax}$$

$$\therefore \frac{dz}{dx} = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\left(\frac{-(x^2+ay)}{y^2+ax}\right)$$

$$= \frac{x}{\sqrt{x^2+y^2}} - \frac{y(x^2+ay)}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$= \frac{x(y^2+ax)-y(x^2+ay)}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$= \frac{xy^2+ax^2-x^2y-ay^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$= \frac{(x-a)y^2+(a-y)x^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

$$\frac{dz}{dx} = \frac{(x-a)y^2-(y-a)x^2}{\sqrt{x^2+y^2}(y^2+ax)}$$

at $x=y=a$

$$\frac{dz}{dx} = \frac{(a-a)a^2-(a-a)a^2}{\sqrt{a^2+a^2}(a^2+a^2)}$$

$$= \frac{0-0}{\sqrt{2a^2}(2a^2)}$$

$$\boxed{\frac{dz}{dx}=0.}$$

② If $U = x\log(xy)$ where $x^3+y^3+3xy=1$ find $\frac{du}{dx}$.

Given $U = x.\log(xy)$

$U < \begin{matrix} x \\ y \end{matrix} \to x.$

$$\frac{dU}{dx} = \frac{\partial U}{\partial x}.\frac{dx}{dx}+\frac{\partial U}{\partial y}.\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}$$

$U = x \cdot \log(xy)$

$$\frac{du}{dx} = x\cdot\left(\frac{1}{xy}\right)(y) + \log(xy)(1)$$

$$= 1 + \log(xy)$$

$$\frac{du}{dy} = x \cdot \frac{1}{xy}(x) = \frac{x}{y}.$$

Given $x^3 + y^3 + 3xy = 1$

diff. w. r. to 'x'.

$$3x^2 + 3y^2\frac{dy}{dx} + 3\left[x\cdot\frac{dy}{dx} + y(1)\right] = 0$$

$$x^2 + y^2\cdot\frac{dy}{dx} + x\cdot\frac{dy}{dx} + y = 0$$

$$(y^2+x)\frac{dy}{dx} = -(x^2+y)$$

$$\frac{dy}{dx} = \frac{-(x^2+y)}{y^2+x}.$$

$$\frac{du}{dx} = 1 + \log(xy) + \frac{x}{y}\cdot\left(-\frac{(x^2+y)}{y^2+x}\right)$$

$$= 1 + \log(xy) - \frac{x(x^2+y)}{y(y^2+x)}$$

$$= \log(xy) + \frac{y^3 + \cancel{xy} - x^3 - \cancel{xy}}{y(y^2+x)}$$

$$= \log(xy) + \cancel{y^3 - x}$$

③ If $z = x^2y$ and $x^2 + xy + y^2 = 1$, find $\frac{dz}{dx}$.

Given. $z = x^2y$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$z = x^2y$

$$\frac{\partial z}{\partial x} = y(2x) = 2xy$$

$$\frac{\partial z}{\partial y} = x^2(1) = x^2$$

Given $x^2 + xy + y^2 = 1$

diff. w. r. to 'x'.

$$2x + \cancel{2y}(x \cdot \frac{dy}{dx} + y(1)) + 2y = 0$$

$$2x + x \cdot \frac{dy}{dx} + 3y = 0$$

$$x \cdot \frac{dy}{dx} = -(2x+3y)$$

$$\frac{dy}{dx} = \frac{-(2x+3y)}{x}$$

$$\frac{dz}{dx} = 2xy + x^2 \frac{-(2x+3y)}{x}$$

$$= 2xy - x(2x+3y)$$

$$= 2xy - 2x^2 - 3xy = -2x^2 - xy.$$

$$= -(2x^2 + xy)$$

⑤ If $x^y = y^x$ then find $\frac{dy}{dx}$.

Given $x^y = y^x$

$$x^y - y^x = 0$$

$$f(x,y) = x^y - y^x \longrightarrow ①$$

$$\therefore \frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

differentiate Equ ① w.r.to 'x' partially.

$$\Rightarrow \frac{\partial f}{\partial x} = y \cdot x^{y-1} \cancel{\log x} - y^x \cdot \log y.$$

diff. w.r.to "y" Partially.

$$\Rightarrow \frac{\partial f}{\partial y} = x^y \cdot \log x - x \cdot y^{x-1}$$

$$\therefore \frac{dy}{dx} = -\frac{y \cdot x^{y-1} - y^x \cdot \log y}{x^y \cdot \log x - x \cdot y^{x-1}}$$

⑨ Find $\frac{dy}{dx}$ when $(\cos x)^y = (\sin y)^x$

Given $(\cos x)^y = (\sin y)^x$

$$(\cos x)^y - (\sin y)^x = 0.$$

$$f(x,y) = (\cos x)^y - (\sin y)^x \longrightarrow ①$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

diff. eq^n ① w. r. to 'x' partially,

$$\Rightarrow \frac{df}{dx} = y(\cos x)^{y-1}(-\sin x) - \sin y^{x}\cdot \log \sin y\ (\cancel{\cos y})\cos$$

$$= -y\sin x\,(\cos x)^{y-1} + \cos y(\sin y)^{x}\cdot \log \sin y$$

diff. eq^n ① w. respect to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^{y}\cdot \log \cos x\ (\cancel{-\sin x}) - x.(\sin y)^{x-1}\cos y$$

$$= \cancel{+}\sin x(\cos x)^{y}\cdot \log(\cos x) - \cancel{\cos}\, x\cdot \cos y\cdot(\sin y)^{x-1}$$

$$\therefore \frac{dy}{dx} = \frac{\cancel{-}[\cancel{+}y\sin x.(\cos x)^{y-1} + \cos y\cdot \sin y^{x}\cdot \log \sin y]}{\cancel{+}[\sin x\,(\cos x)^{y}\log(\cos x) + x.\cos y\,(\sin y)^{x-1}]}$$

$$= \cancel{+}\frac{(y\sin x(\cos x)^{y-1} + \cos y\,(\sin y)^{x}\cdot \log \sin y)}{\sin x.(\cos x)^{y}\cdot \log \cos x + x\cdot \cos y\cdot(\sin y)^{x-1}}$$

$$\therefore \frac{dy}{dx} \neq$$

diff. eq^n ① w. r. to 'x' partially

$$\Rightarrow \frac{df}{dx} = y\cdot(\cos x)^{y-1}(-\sin x) - (\sin y)^{x}\cdot \log \sin y$$

diff. eq^n ① w. r. to 'y' partially.

$$\Rightarrow \frac{df}{dy} = (\cos x)^{y}\cdot \log(\cos x) - x.(\sin y)^{x-1}\cdot \cos y$$

$$\therefore \frac{dy}{dx} = \frac{-[-y\cdot\sin x.(\cos x)^{y-1} - (\sin y)^{x}\cdot \log(\sin y)]}{(\cos x)^{y}\cdot \log(\cos x) - x\cdot\frac{(\sin y)^{x}}{\sin y}\cdot \cos y}$$

$$= \frac{y\cdot \tan x\cdot(\cos x)^{y} + (\cos x)^{y}\cancel{y}\cdot \log \sin y}{(\cos x)^{y}\log(\cos x) - x(\sin y)^{x}\cdot \cot y}$$

$$= \frac{\cancel{(\cos x)^{y}}[y\tan x + \cancel{y}\cdot\log\cdot(\sin y)]}{\cancel{(\cos x)^{y}}[\log \cos x - x.\cot y]}$$

$$= \frac{y\tan x + \log(\sin y)}{\log(\cos x) - x.\cot y}$$

④ If $x^3+3x^2y+6xy^2+y^3=1$. Find $\frac{dy}{dx}$

Given that $x^3+3x^2y+6xy^2+y^3=1$

$x^3+3x^2y+6xy^2+y^3-1=0$

$f(x,y) = x^3+3x^2y+6xy^2+y^3-1 \rightarrow ①$

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

diff. equⁿ① w. r. to 'x' partially

$$\Rightarrow \frac{\partial f}{\partial x} = 3x^2+3y(2x)+6y^2(1)+0-0$$

$$= 3x^2+6xy+6y^2$$

diff. equⁿ① w. r. to 'y' partially.

$$\Rightarrow \frac{\partial f}{\partial y} = 0+3x^2(1)+6x(2y)+3y^2-0$$

$$= 3x^2+12xy+3y^2$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2+6xy+6y^2)}{3x^2+12xy+3y^2}$$

$$= \frac{-\cancel{3}(x^2+2xy+2y^2)}{\cancel{3}(x^2+4xy+y^2)}$$

$$= \frac{-(x^2+2xy+2y^2)}{x^2+4xy+y^2}$$

⑥ If $x^3+y^3-3axy=0$. Find $\frac{dy}{dx}$.

Given that $x^3+y^3-3axy=0$

$f(x,y) = x^3+y^3-3axy \rightarrow ①$

diff. equⁿ① w. r. to 'x' partially.

$$\frac{\partial f}{\partial x} = 3x^2+0-3ay(1) = 3x^2-3ay$$

diff. w. r. to 'y' partially.

$$\frac{\partial f}{\partial y} = 0+3y^2-3ax(1) = 3y^2-3ax.$$

$$\therefore \frac{dy}{dx} = \frac{-(3x^2-3ay)}{(3y^2-3ax)} = \frac{-(x^2-ay)}{y^2-ax}.$$

⑦ ~~prove that~~ find $\frac{dy}{dx}$ If $y^3-3ax^2+x^3=0$.

sol:

Given that $y^3-3ax^2+x^3=0$

$$f(x,y)=y^3-3ax^2+x^3 \rightarrow ①$$

diff. equⁿ① w. r. to 'x' partially.

$$\frac{\partial f}{\partial x}=0-3a(2x)+3x^2 = 3x^2-6ax$$

diff. equⁿ① w. r. to 'y' partially.

$$\frac{\partial f}{\partial y}=3y^2-0+0 = 3y^2$$

$$\therefore \frac{dy}{dx}=\frac{-(3x^2-6ax)}{3y^2}=\frac{-\cancel{3}(x^2-2ax)}{\cancel{3}y^2}=\frac{2ax-x^2}{y^2}$$

⑧ find $\frac{dy}{dx}$ when $x^y+y^x=c$.

Given that $x^y+y^x=c$

$$x^y+y^x-c=0$$

$$f(x,y)=x^y+y^x-c \rightarrow ①$$

diff. equⁿ① w. r. to 'x' partially.

$$\frac{\partial f}{\partial x}=y\cdot x^{y-1}+y^x\log y-0 = yx^{y-1}+y^x\cdot\log y$$

diff. equⁿ① w. r. to 'y' partially

$$\frac{\partial f}{\partial y}=x^y\cdot\log x+x\cdot y^{x-1}-0 = x^y\log x+x\cdot y^{x-1}$$

$$\therefore \frac{dy}{dx}=\frac{-(yx^{y-1}+y^x\cdot\log y)}{x^y\log x+x\cdot y^{x-1}}$$

20/11/19 Tuesday Taylor's (Expansion) theorem:

* Expand the following functions.

① $f(x,y) = e^x \sin y$

By Maclaurin's expansion,

$$f(x,y) = f(0,0) + [x \cdot f_x(0,0) + y \cdot f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy f_{xy}(0,0)] + \dots$$

Now, $f(x,y) = e^x \sin y$

$\Rightarrow f(0,0) = e^0 \cdot \sin(0) = (1)(0) = 0.$

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \sin y \cdot e^x \Rightarrow f_x(0,0) = \sin(0) e^0 = 0.$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = e^x \cdot \cos y \Rightarrow f_y(0,0) = e^0 \cdot \cos(0) = 1$

$\Rightarrow f_{xx} = \frac{\partial^2 f}{\partial x^2} = \sin y \cdot e^x \Rightarrow f_{xx}(0,0) = \sin(0)\, e^{(0)} = 0$

$\Rightarrow f_{yy} = \frac{\partial^2 f}{\partial y^2} = e^x \cdot (-\sin y) \Rightarrow f_{yy}(0,0) = -e^0 \sin(0) = 0.$

$\Rightarrow f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cdot \cos y \Rightarrow f_{xy}(0,0) = e^0 \cdot \cos 0 = 1$

$\therefore e^x \sin y = 0 + [x(0) + y(1)] + \frac{1}{2!}[x^2(0) + y^2(0) + 2xy(1)] + \dots$

$= 0 + 0 + y + 0 + 0 + \frac{1}{\cancel{2}}(\cancel{2}xy) + \dots$

$= y + xy + \dots$

② $f(x,y) = \tan^{-1}(y/x)$ in powers of $(x-1)$ and $(y-1)$ up to third degree terms. Hence compute $f(1.1, 0.9)$ approximately.

By Taylor's expansion at the (a,b) is

$$f(x,y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + (y-a)^2 f_{yy}(a,b) + 2(x-a)(y-b) f_{xy}(a,b)] + \dots$$

$$\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + (y-b)^3 f_{yyy}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)] + \dots$$

at $(1,1)$.

$$f(x,y) = f(1,1) + [(x-1)f_x(1,1) + (y-1)f_y(1,1)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1) f_{xy}(1,1)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,1) + (y-1)^3 f_{yyy}(1,1) + 3(x-1)^2(y-1) \cdot f_{xxy}(1,1) + 3(x-1)(y-1)^2 f_{xyy}(1,1)] + \dots \longrightarrow ①$$

We have $f(x,y) = \tan^{-1}(y/x)$

$\Rightarrow f(1,1) = \tan^{-1}(1/1) = \tan^{-1}(1) = \pi/4$.

$f_x = \frac{df}{dx} = \frac{1}{1+(y/x)^2} y\left(\frac{-1}{x^2}\right) = \frac{-y}{y^2+x^2} \Rightarrow f_x(1,1) = \frac{-1}{1+1} = \frac{-1}{2}$

$f_y = \frac{df}{dy} = \frac{1}{1+y^2/x^2} \frac{1}{x}(1) = \frac{x}{y^2+x^2} \Rightarrow f_y(1,1) = \frac{1}{1+1} = \frac{1}{2}$.

$f_{xx} = \frac{d^2f}{dx^2} = (-y)\frac{-1}{(x^2+y^2)^2}(2x) \Rightarrow f_{xx}(1,1) = -1\,\frac{-1}{(1+1)^2}(2(1))$

$= \frac{\cancel{2}}{\cancel{4}_2} = \frac{1}{2}$

$f_{yy} = \frac{d^2f}{dy^2} = x\cdot\frac{-1}{(x^2+y^2)^2}(2y) = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow f_{yy}(1,1) = \frac{\cancel{2}}{(2)^2} = \frac{-1}{2}$.

$f_{xy} = \frac{d^2f}{dxdy} = \cancel{\frac{1}{1+(y/x)^2}(y)\left(\frac{-1}{x^2}\right)}\ \frac{(y^2+x^2)(-1) - (-y)(2y+0)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$

$= \cancel{\frac{-(x^2+y^2+2xy)}{(x^2+y^2)^2}} \neq \cancel{\frac{-(x+y)^2}{(x^2+y^2)^2}}$

$\cancel{\Rightarrow f_{xy}(1,1) = \frac{-(1+1)^2}{(1+1)^2} \neq \frac{-4}{-8}}$

$\Rightarrow f_{xy}(1,1) = \frac{-1-1+2}{(1+1)^2} = 0$.

$f_{xxx} = \frac{(x^2+y^2)^2(2y)(1) - 2xy\,2(x^2+y^2)(2x)}{[(x^2+y^2)^2]^2}$

$= \frac{2y(x^2+y^2)^2 - 8x^2y\cdot(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{xxx}(1,1) = \frac{2(1)(1+1)^2 - 8(1)(1)(1+1)}{(1+1)^4} = \frac{8-16}{16} = \frac{-8}{16} = \frac{-1}{2}$.

$f_{yyy} = \frac{(x^2+y^2)^2(-2x)(1) + 2xy\,2(x^2+y^2)(0+2y)}{[(x^2+y^2)^2]^2}$

$= \frac{-2x(x^2+y^2)^2 + 8xy^2(x^2+y^2)}{(x^2+y^2)^4}$

$\Rightarrow f_{yyy}(1,1) = \frac{-2(4) + 8(1+1)}{16} = \frac{-8+16}{16} = \frac{8}{16} = 1/2$.

$f_{xxy} = 2x\left[\frac{(x^2+y^2)^2(+1) - y\cdot 2(x^2+y^2)\,2y}{[(x^2+y^2)^2]^2}\right]$

$= 2x\left[\frac{(x^2+y^2) - 4y^2}{(x^2+y^2)^3}\right]$

$\Rightarrow f_{xxy}(1,1) = \cancel{x}\left[\frac{2-4}{8_y}\right] = \frac{-2}{4} = \frac{-1}{2}.$

$|||^{ly} = f_{xyy}(1,1) = \frac{1}{2}.$

from (1),

$\tan^{-1}(y/x) = \pi/4 + [(x-1)(\frac{-1}{2}) + (y-1)(\frac{1}{2})] + \frac{1}{2!}[(x-1)^2(\frac{1}{2}) + (y-1)^2(\frac{-1}{2}) + 2(x-1)(y-1)(0)]$

$f(x,y) =$

$+ \frac{1}{3!}[(x-1)^3 \cdot (\frac{-1}{2}) + (y-1)^3(\frac{1}{2}) + 3(x-1)^2(y-1)(-\frac{1}{2}) + 3(x-1)(y-1)^2(\frac{1}{2})]$

$= \frac{\pi}{4} + \frac{1}{2}[-(x-1) + (y-1)] + \frac{1}{2!}\frac{1}{2}[(x-1)^2 - (y-1)^2] + \frac{1}{3!}\frac{1}{2}[(x-1)^3$
$+ (y-1)^3 + 3(x-1)^2(y-1) + 3(x-1)(y-1)^2].$

$= \frac{\pi}{4} + \frac{1}{2}(-(x-1) + (y-1)) + \frac{1}{4}[(x-1)^2 - (y-1)^2] + \frac{1}{12}[(x-1)^3 + (y-1)^3$
$+ 3(x-1)^2(y-1) + 3(x-1)(y-1)^2] + \ldots$

$f(1.1, 0.9) = \frac{\pi}{4} + \frac{1}{2}[-(1.1-1) + (0.9-1)] + \frac{1}{4}[(1.1-1)^2 - (0.9-1)^2] + \frac{1}{12}[(1.1-1)^3$
$+ (0.9-1)^3 + 3(1.1-1)^2(0.9-1) + 3(1.1-1)(0.9-1)^2]$

$= \frac{3.14}{4} + \frac{1}{2}[-0.2] + \frac{1}{4}[0.00] + \frac{1}{12}\{-3.009\}$

$= 0.785 - 0.1 + \frac{1}{4}(0) + \frac{1}{12}[0.001 - 0.001 + 3($

$= 0.68533.$

④ $f(x,y) = e^x \log(1+x)$

Sol:- $f(x,y) = e^x \log(1+x)$

$f(x,y) = f(0,0) + [x.f_x(0,0) + y.f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) +$
$y^2 f_{yy}(0,0)]$

We have,

$f(x,y) = e^y \log(1+x) \Rightarrow f(0,0) = e^0.[\log(1)] = 0.$

$f_x = \frac{\partial f}{\partial x} = e^y \frac{1}{1+x} \Rightarrow f_x(0,0) = e^0 \frac{1}{1+0} = 1$

$f_y = \frac{\partial f}{\partial y} = \log(1+x)e^y \Rightarrow f_y(0,0) = 0.$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = e^y \frac{-1}{(1+x)^2} \Rightarrow f_{xx}(0,0) = e^0 \frac{-1}{(1+0)^2} = -1.$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{1}{1+x} e^y \Rightarrow f_{xy}(0,0) = \frac{1}{1+0} e^{(0)} = 1.$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \log(1+x) e^y \Rightarrow f_{yy}(0,0) = 0.$

$$e^y \log(1+x) = 0 + [x(1) + y(0)] + \frac{1}{2!}[x^2(-1) + 2xy(1) + y^2(0)] + .$$

$$= x + \frac{1}{2}(-x^2 + 2xy) + \cdots$$

$$= x - \frac{x^2}{2} + xy + \cdots$$

③ $f(x,y) = e^x \log(1+y)$.

Given $f(x,y) = e^x \log(1+y)$

By Maclaurin's expansion

$$f(x,y) = f(0,0) + [x \cdot f_x(0,0) + y\, f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0) + 2xy\, f_{xy}(0,0)] + . \rightarrow ①$$

$f(x,y) = e^x \log(1+y) \Rightarrow f(0,0) = e^0 \log(1+0) = 0.$

$f_x = \frac{\partial f}{\partial x} = \log(1+y)\, e^x \Rightarrow f_x(0,0) = \log(1+0)\, e^0 = 0.$

$f_y = \frac{\partial f}{\partial y} = e^x \cdot \frac{1}{1+y} \Rightarrow f_y(0,0) = e^0 \frac{1}{1+0} = 1.$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \log(1+y)\, e^x \Rightarrow f_{xx}(0,0) = \log(1+0)\, e^0 = 0$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = e^x \cdot \frac{-1}{(1+y)^2} \Rightarrow f_{yy}(0,0) = e^0 \cdot \frac{-1}{(1+0)^2} = -1$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = e^x \cdot \frac{1}{1+y} \Rightarrow f_{xy}(0,0) = e^0 \frac{1}{1+0} = 1$

from ①,

$$e^x \cdot \log(1+y) = 0 + [x \cdot (0) + y(1)] + \frac{1}{2!}[x^2(0) + y^2(-1) + 2xy(1)] + \cdots$$

$$= y + \frac{1}{2}[-y^2 + 2xy] + \cdots$$

$$= y - \frac{y^2}{2} + xy + \cdots$$

⑨ Expand $x^2y + 3y - 2$ in power of $(x-1)$ and $(y+2)$ using Taylor's theorem.

By Taylor's expansion,

$$f(x,y) = f(a,b) + [(x-a) f_x(a,b) + (y-b) f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \cdots$$

$$= f(1,-2) + [(x-1) f_x(1,-2) + (y+2) f_y(1,-2)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,-2) + 2(x-1)(y+2) f_{xy}(1,-2) + (y+2)^2 f_{yy}(1,-2)] + \cdots \rightarrow ①$$

We have $f(x,y) = x^2y + 3y - 2 \Rightarrow f(1,-2) = -2-6-2 = \underline{-10}$

$f_x = \frac{\partial f}{\partial x} = y(2x) + 0 - 0 \Rightarrow f_x(1,-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^2(1) + 3(1) - 0 \Rightarrow f_y(1,-2) = 1+3 = 4$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2y(1) \Rightarrow f_{xx}(1,-2) = -4$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x(1) \Rightarrow f_{xy}(1,-2) = 2$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0 + 0 \Rightarrow f_{yy}(1,-2) = 0.$

from ①,

$$x^2y + 3y - 2 = -10 + [(x-1)(-4) + (y+2)(+4)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \cdots$$

$$= -10 - 4[(x-1) - (y+2)] + \frac{-4}{2}[(x-1)^2 - (x-1)(y+2)] + \cdots$$

$$= -10 - 4[(x-1) - (y+2)] - 2[(x-1)^2 - (x-1)(y+2)] + \cdots$$

⑧ Show that $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

and hence deduce that $\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$

By Maclaurin's expansion,

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

We have $f(x) = \log(1+e^x) \Rightarrow f(0) = \log(1+e^0) = \log 2.$

$f'(x) = \frac{1}{1+e^x}.e^x \Rightarrow f'(0) = \frac{e^0}{1+e^0} = \frac{1}{2}$

$$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2} \Rightarrow f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x.2(1+e^x)e^x}{[(1+e^x)^2]^2} = \frac{\cancel{(1+e^x)}[(1+e^x)e^x - 2e^{2x}]}{(1+e^x)^{\cancel{4}\,3}}$$

$$= \frac{e^x + e^{2x} - 2e^{2x}}{(1+e^x)^3} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$\Rightarrow f'''(0) = \frac{e^0 - e^0}{(1+e^0)^3} = 0.$$

$$f^{iv}(x) = \frac{(1+e^x)^3\{e^x - e^{2x}(2)\} - (e^x - e^{2x})\,3(1+e^x)^2 e^x}{[(1+e^x)^3]^2}$$

$$f^{IV}(x) = \frac{(1+e^x)^2\left[1+e^x(e^x-2e^{2x}) - 3e^x(e^x-e^{2x})\right]}{(1+e^x)^4}$$

$$\Rightarrow f^{IV}(0) = \frac{(1+e^0)(e^0-2\cdot e^0) - 3\cdot e^0(e^0-e^0)}{(1+e^0)^4}$$

$$= \frac{2\cdot(1-2)-3(1)(1-1)}{(1+1)^4} = \frac{-2-0}{16} = \frac{-2}{16} = \frac{-1}{8}.$$

$$\log(1+e^x) = \log 2 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\frac{1}{4} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(\frac{-1}{8}\right) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$$

diff. w. r. to 'x'

$$\frac{1}{1+e^x}\cdot e^x = 0 + \frac{1}{2} + \frac{1}{8}(2x) - \frac{4x^3}{192} + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$$

⑤ $f(x,y) = e^{xy}$ in powers of $(x-1)$ and $(y-1)$.

By Taylor's expansion,

$$f(x,y) = f(1,1) + \left[(x-1)f_x(1,1) + (y-1)f_y(1,1)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,1) + (y-1)^2 f_{yy}(1,1) + 2(x-1)(y-1)f_{xy}(1,1)\right] + \cdots$$

We have $f(x,y) = e^{xy} \Rightarrow f(1,1) = e^{(1)(1)} = e.$

$f_x = \frac{\partial f}{\partial x} = e^{xy}(y) \Rightarrow f_x(1,1) = e^{(1)(1)}(1) = e.$

$f_y = \frac{\partial f}{\partial y} = e^{xy}(x) \Rightarrow f_y(1,1) = e^{(1)(1)}(1) = e.$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = y\cdot e^{xy}(y) \Rightarrow f_{xx}(1,1) = (1)e^{(1)(1)}(1) = e.$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = x\cdot e^{xy}(x) \Rightarrow f_{yy}(1,1) = (1)e^{(1)(1)}(1) = e.$

$f_{xy} = \frac{\partial^2 f}{\partial x\partial y} = e^{xy}(1) + y\cdot e^{xy}(x) \Rightarrow f_{xy}(1,1) = e + e = 2e$

$$e^{xy} = e + \left[(x-1)e + (y-1)e\right] + \frac{1}{2!}\left[(x-1)^2 e + (y-1)^2 e + 2(x-1)(y-1)2e\right] + \cdots$$

$$= e + e\left[(x-1)+(y-1)\right] + \frac{e}{2!}\left[(x-1)^2 + (y-1)^2 + 4(x-1)(y-1)\right] + \cdots$$

⑥ $f(x,y) = e^x \cos y$ about $(1, \pi/4)$.

By Taylor's Expansion,

$$f(x,y) = f(1,\pi/4) + \left[(x-1) f_x(1,\pi/4) + (y-\pi/4) f_y(1,\pi/4)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/4) + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4) + (y-\pi/4)^2 f_{yy}(1,\pi/4)\right] + \cdots$$

We have $f(x,y) = e^x \cos y \Rightarrow f(1,\pi/4) = e^1 \cos \pi/4 = \frac{e}{\sqrt{2}}$.

$f_x = \frac{\partial f}{\partial x} = \cos y \cdot e^x \Rightarrow f_x(1,\pi/4) = \cos \pi/4 \, e^{(1)} = \frac{e}{\sqrt{2}}$,

$f_y = \frac{\partial f}{\partial y} = e^x \cdot (-\sin y) \Rightarrow f_y(1,\pi/4) = -e^{(1)} \sin \pi/4 = \frac{-e}{\sqrt{2}}$.

$f_{xx} = \cos y \cdot e^x \Rightarrow f_{xx}(1,\pi/4) = \cos \pi/4 \cdot e^{(1)} = \frac{e}{\sqrt{2}}$

$f_{xy} = e^x(-\sin y) \Rightarrow f_{xy}(1,\pi/4) = -e^{(1)} \sin \pi/4 = \frac{-e}{\sqrt{2}}$

$f_{yy} = -e^x \cdot \cos y \Rightarrow f_{yy}(1,\pi/4) = -e^{(1)} \cos \pi/4 = \frac{-e}{\sqrt{2}}$

$$e^x \cdot \cos y = \frac{e}{\sqrt{2}} + \left[(x-1)\frac{e}{\sqrt{2}} + (y-\pi/4)\left(\frac{-e}{\sqrt{2}}\right)\right] + \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y-\pi/4)\left(\frac{-e}{\sqrt{2}}\right) + (y-\pi/4)^2\left(\frac{-e}{\sqrt{2}}\right)\right] + \cdots$$

$$= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}}\left[(x-1) - (y-\pi/4)\right] + \frac{1}{2!}\frac{e}{\sqrt{2}}\left[(x-1)^2 \cancel{+} - 2(x-1)(y-\pi/4) - (y-\pi/4)^2\right] + \cdots$$

$$= \frac{e}{\sqrt{2}} + \frac{e}{\sqrt{2}}\left[(x-1) - (y-\pi/4)\right] + \frac{e}{2\sqrt{2}}\left[(x-1)^2 - 2(x-1)(y-\pi/4) - (y-\pi/4)^2\right] + \cdots$$

⑦ $f(x,y) = \sin xy$ in powers of $(x-1)$ and $(y-\pi/2)$ up to second degree terms.

By Taylor's Expansion,

$$f(x,y) = f(1,\pi/2) + \left[(x-1) f_x(1,\pi/2) + (y-\pi/2) f_y(1,\pi/2)\right] + \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/2) + 2(x-1)(y-\pi/2) f_{xy}(1,\pi/2) + (y-\pi/2)^2 f_{yy}(1,\pi/2)\right] + \cdots$$

We have,

$f(x,y) = \sin xy \Rightarrow f(1,\pi/2) = \sin \pi/2 = 1$

$f_x = \frac{\partial f}{\partial x} = \cos xy \, (y) \Rightarrow f_x(1,\pi/2) = (\pi/2) \cos \pi/2 = 0$.

$f_y = \frac{\partial f}{\partial y} = \cos xy (x) \Rightarrow f_y(1,\pi/2) = (1) \cos \pi/2 = 0$.

$f_{xx} = y \cdot (-\sin xy)(y) \Rightarrow f_{xx}(1,\pi/2) = \pi/2 \cdot \pi/2 (-\sin \pi/2) = \frac{\pi^2}{4}(-1) = \frac{-\pi^2}{4}$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = y.(-\sin xy)(x) + \cos xy\,(1)$$

$$\Rightarrow f_{xy}(1, \pi/_2) = \pi/_2 - \sin\pi/_2(1) + \cos\pi/_2$$
$$= -\pi/_2(1) + 0 = -\pi/_2$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = x.(-\sin xy)(x) = (1) - \sin\pi/_2(1) = -1$$

$$\sin xy = 1 + [(x-1)0 + (y-\pi/_2)0] + \frac{1}{2!}\left[(x-1)^2(-\pi^2/_4) + 2(x-1)(y-\pi/_2)(-\pi/_2) + (y-\pi/_2)^2(-1)\right] + \ldots$$

$$= 1 + [0+0] + \frac{1}{2}\left[(x-1)^2(-\pi^2/_4) \mp \cancel{2}(x-1)(y-\pi/_2)\pi/_2 \mp (y-\pi/_2)^2\right] + \ldots$$

$$\sin xy = 1 - \frac{1}{2}\left[(x-1)^2\pi^2/_4 + (x-1)(y-\pi/_2)\pi/_2 + (y-\pi/_2)^2\right] + \ldots$$

28/11/19 Thursday

Jacobian:

② If $x = r\sin\theta\cos\phi$, $y = r\sin\theta.\sin\phi$, $z = r\cos\theta$. show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta.$$

Sol:- $x = r\sin\theta\cos\phi$ $\quad y = r\sin\theta.\sin\phi$ $\quad z = r\cos\theta$

$$\begin{matrix} x \\ y \\ z \end{matrix} \Big> r\,\theta\,\phi$$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$x = r\sin\theta.\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta.\cos\phi(1)$

$\frac{\partial x}{\partial \theta} = r\cos\phi\,\cos\theta)$

$\frac{\partial x}{\partial \phi} = r\sin\theta.(-\sin\phi)$

$y = r\sin\theta.\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta.\sin\phi$

$\frac{\partial y}{\partial \theta} = r\sin\phi.\cos\theta$

$\frac{\partial y}{\partial \phi} = r\sin\theta.\cos\phi$

$z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial z}{\partial \theta} = r(-\sin\theta)$

$\frac{\partial z}{\partial \phi} = 0.$

$$\frac{\partial(x\,y\,z)}{\partial(r\,\theta\,\phi)} = \begin{array}{c} \quad(+) \qquad\qquad (-) \qquad\qquad (+) \\ \begin{vmatrix} \sin\theta.\cos\phi & r\cos\theta\cos\phi & -r\sin\theta.\sin\phi \\ \sin\theta.\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \end{array}$$

$= \sin\theta\cdot\cos\phi\,[0 + r^2\sin^2\theta\cdot\cos\phi] - r\cos\theta\cdot\cos\phi\,[0 - r\sin\theta\cos\theta\cos\phi]$

$\quad - r\sin\theta\cdot\sin\phi\,[-r\sin^2\theta\cdot\sin\phi - r\cdot\cos^2\theta\cdot\sin\phi]$

$= r^2\sin^3\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\cos^2\theta\cdot\cos^2\phi - r\sin\theta\cdot\sin\phi\,\{(-r\sin\phi)[\sin^2\theta + \cos^2\theta]\}$

$= r^2\sin^3\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\cos^2\theta\cdot\cos^2\phi + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\cdot\cos^2\phi\,[\sin^2\theta + \cos^2\theta] + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\cdot\cos^2\phi\,(1) + r^2\sin\theta\cdot\sin^2\phi$

$= r^2\sin\theta\,[\cos^2\phi + \sin^2\phi]$

$= r^2\sin\theta.$

③ If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ show that $\dfrac{\partial(uvw)}{\partial(xyz)} = 0$.

Sol: $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ $\qquad uvw \leftarrow x,y,z$ (or) $u,v,w \rightarrow xyz$

$$\frac{\partial(uvw)}{\partial(xyz)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$u = \dfrac{x}{y-z}$

$\dfrac{\partial u}{\partial x} = \dfrac{1}{y-z}$

$\dfrac{\partial u}{\partial y} = x\cdot\dfrac{-1}{(y-z)^2}$

$\dfrac{\partial u}{\partial z} = x\cdot\dfrac{-1}{(y-z)^2}(0-1)$

$= \dfrac{x}{(y-z)^2}$

$v = \dfrac{y}{z-x}$

$\dfrac{\partial v}{\partial x} = y\cdot\dfrac{-1}{(z-x)^2}(-1)$

$= \dfrac{y}{(z-x)^2}$

$\dfrac{\partial v}{\partial y} = \dfrac{1}{z-x}$

$\dfrac{\partial v}{\partial z} = y\dfrac{-1}{(z-x)^2}$

$w = \dfrac{z}{x-y}$

$\dfrac{\partial w}{\partial x} = z\,\dfrac{-1}{(x-y)^2}$

$\dfrac{\partial w}{\partial y} = z\cdot\dfrac{-1}{(x-y)^2}(-1) = \dfrac{z}{(x-y)^2}$

$\dfrac{\partial w}{\partial z} = \dfrac{1}{x-y}$

$$\frac{\partial(uvw)}{\partial(xyz)} = \begin{vmatrix} \overset{(+)}{\frac{1}{y-z}} & \overset{(-)}{\frac{-x}{(y-z)^2}} & \overset{(+)}{\frac{x}{(y-z)^2}} \\ \frac{y}{(z-x)^2} & \frac{1}{z-x} & \frac{-y}{(z-x)^2} \\ \frac{-z}{(x-y)^2} & \frac{z}{(x-y)^2} & \frac{1}{x-y} \end{vmatrix}$$

$= \dfrac{1}{y-z}\left[\dfrac{1}{(x-y)(z-x)} + \dfrac{yz}{(x-y)^2(z-x)^2}\right] + \dfrac{x}{(y-z)^2}\left[\dfrac{y}{(x-y)(z-x)^2}\right.$

$\left. - \dfrac{zy}{(x-y)^2(z-x)^2}\right] + \dfrac{x}{(y-z)^2}\left[\dfrac{yz}{(z-x)^2(x-y)^2} + \dfrac{z}{(x-y)^2(z-x)}\right]$

$$= \frac{1}{y-z}\,\frac{1}{x-y}\,\frac{1}{z-x}\left[1+\frac{yz}{(x-y)(z-x)}\right] + \frac{xy}{(x-y)(y-z)^2(z-x)^2}\left[1-\frac{z}{x-y}\right]$$

$$+\frac{xz}{(y-z)^2(z-x)(x-y)^2}\left[\frac{y}{z-x}+1\right]$$

$$= \frac{1}{(x-y)(y-z)(z-x)}\left[\frac{(x-y)(z-x)+yz}{(x-y)(z-x)}\right] + \frac{xy}{(x-y)(y-z)^2(z-x)^2}\left[\frac{x-y-z}{x-y}\right]$$

$$+\frac{xz}{(y-z)^2(z-x)(x-y)^2}\left[\frac{y+z-x}{z-x}\right]$$

$$= \frac{1}{(x-y)^2(y-z)(z-x)^2}\left[xz-x^2-yz+xy+yz\right] + \frac{xy}{(x-y)^2(y-z)^2(z-x)^2}(x-y-z)$$

$$+\frac{xz}{(y-z)^2(z-x)^2(x-y)^2}\left[y+z-x\right]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\left[(y-z)(xz-x^2+xy)+xy(x-y-z)+xz(y+z-x)\right]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\left[\cancel{xyz}-\cancel{x^2y}+\cancel{xy^2}-\cancel{xz^2}+\cancel{zx^2}-\cancel{xyz}+\cancel{x^2y}-\cancel{xy^2}-\cancel{xyz}+\cancel{xyz}+\cancel{xz^2}-\cancel{x^2z}\right]$$

$$= \frac{1}{(x-y)^2(y-z)^2(z-x)^2}\,(0)$$

$$= 0.$$

① If $r = \sqrt{x^2+y^2}$. $\theta = \tan^{-1}(y/x)$. evaluate $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Sol:- $r=\sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}(y/x)$

$\tan\theta = y/x$

$r,\theta < \begin{matrix} x \\ y\end{matrix}$

$\begin{matrix} r \\ \theta\end{matrix} > xy$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y}\end{vmatrix}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x)$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\,y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2}(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

$$= \frac{\cancel{x^2+y^2}}{\sqrt{x^2+y^2}\,\cancel{(x^2+y^2)}}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{\sqrt{x^2+y^2}}$$

⑫ If $U = \frac{yz}{x}$, $V = \frac{zx}{y}$, $W = \frac{xy}{z}$ show that $\frac{\partial(xyz)}{\partial(UVW)} = \frac{1}{4}$.

$UVW \to x, y, z$

$$\frac{\partial(U,V,W)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$\frac{\partial U}{\partial x} = yz\left(\frac{-1}{x^2}\right)$, $\frac{\partial U}{\partial y} = \frac{z}{x}$, $\frac{\partial U}{\partial z} = \frac{y}{x}$

$\frac{\partial V}{\partial x} = \frac{z}{y}$, $\frac{\partial V}{\partial y} = zx\left(\frac{-1}{y^2}\right)$, $\frac{\partial V}{\partial z} = \frac{x}{y}$

$\frac{\partial W}{\partial x} = \frac{y}{z}$, $\frac{\partial W}{\partial y} = \frac{x}{z}$, $\frac{\partial W}{\partial z} = xy\left(\frac{-1}{z^2}\right)$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ y/z & x/z & \frac{-xy}{z^2} \end{vmatrix}$$

$$\cancel{= \frac{-yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right]}$$

$$= \begin{vmatrix} \frac{-yz}{x^2} & \frac{zx}{x^2} & \frac{xy}{x^2} \\ \frac{zy}{y^2} & \frac{-zx}{y^2} & \frac{xy}{y^2} \\ \frac{yz}{z^2} & \frac{xz}{z^2} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{(yz)(zx)(xy)}{x^2\,y^2\,z^2}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{x^2y^2z^2}}{\cancel{x^2y^2z^2}}\left[-1(1-1) - 1(-1-1) + 1(1+1)\right]$$

$$= 0 + 2 + 2 = 4 \Rightarrow \boxed{\frac{\partial(xyz)}{\partial(UVW)} = \frac{1}{4}}$$

We know that,

$$\frac{\partial(u\,v\,w)}{\partial(x\,y\,z)} \cdot \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1$$

$$4 \cdot \frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1$$

$$\boxed{\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = 1/4}$$

(14) $u = x+y+z$; $uv = y+z$; $uvw = z$ show that $\dfrac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = u^2v$.

$u = x+y+z$	$uv = y+z$	$uvw = z$
$u = x+uv$	$uv = y+uvw$	$z = uvw$.
$x = u-uv$	$y = uv-uvw$	

$x\ y\ z < u,\ v,\ w$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u-uv$	$y = uv-uvw$	$z = uvw$
$\frac{\partial x}{\partial u} = 1-v$	$\frac{\partial y}{\partial u} = v-vw$	$\frac{\partial z}{\partial u} = vw$
$\frac{\partial x}{\partial v} = 0-u$	$\frac{\partial y}{\partial v} = u-uw$	$\frac{\partial z}{\partial v} = uw$
$\frac{\partial x}{\partial w} = 0$	$\frac{\partial y}{\partial w} = 0-uv$	$\frac{\partial z}{\partial w} = uv$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$=(1-v)\left[(u-uw)uv + u^2vw\right] + u\left[(v-vw)uv + uv^2w\right] + 0$$

$$=(1-v)\left[u^2v - \cancel{u^2vw} + \cancel{u^2vw}\right] + u\left[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}\right]$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= u^2v.$$

(16) $y_1 = 1-x_1$; $y_2 = x_1(1-x_2)$; $y_3 = x_1 x_2 (1-x_3)$ find $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)}$.

Sol:- $y_1 = 1-x_1$ $\quad y_2 = x_1 - x_1 x_2$ $\quad y_3 = x_1 x_2 - x_1 x_2 x_3$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$y_1 y_2 y_3 \left< \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \right.$

$y_1 = 1-x_1$

$\frac{\partial y_1}{\partial x_1} = -1$

$\frac{\partial y_1}{\partial x_2} = 0$

$\frac{\partial y_1}{\partial x_3} = 0$

$y_2 = x_1 - x_1 x_2$

$\frac{\partial y_2}{\partial x_1} = 1-x_2$

$\frac{\partial y_2}{\partial x_2} = 0 - x_1$

$\frac{\partial y_2}{\partial x_3} = 0$

$y_3 = x_1 x_2 - x_1 x_2 x_3$

$\frac{\partial y_3}{\partial x_1} = x_2 - x_2 x_3$

$\frac{\partial y_3}{\partial x_2} = x_1 - x_1 x_3$

$\frac{\partial y_3}{\partial x_3} = 0 - x_1 x_2$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} -1 & 0 & 0 \\ 1-x_2 & -x_1 & 0 \\ x_2 - x_2 x_3 & x_1 - x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= -1\left(x_1^2 x_2 - 0\right) - 0 + 0$$

$$= -x_1^2 x_2$$

(17) $u = x+y+z$; $u^2 v = y+z$; $u^3 w = z$ prove that $\frac{\partial(uvw)}{\partial(x y z)} = u^{-5}$.

Sol:- $u = x+y+z$ $\quad u^2 v = y+z$ $\quad u^3 w = z$

$u = x + u^2 v$ $\quad u^2 v = y + u^3 w$ $\quad z = u^3 w$

$x = u - u^2 v$ $\quad y = u^2 v - u^3 w$

$x y z \left< \begin{matrix} u \\ v \\ w \end{matrix} \right.$

$$\frac{\partial(x y z)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u - u^2 v$

$\frac{\partial x}{\partial u} = 1 - v.2u$

$y = u^2 v - u^3 w$

$\frac{\partial y}{\partial u} = 2uv - 3u^2 w$

$z = u^3 w$

$\frac{\partial z}{\partial u} = w.3u^2$

$\frac{\partial x}{\partial v} = 0-u^2$ | $\frac{\partial y}{\partial v} = u^2-0$ | $\frac{\partial z}{\partial v} = 0$

$\frac{\partial x}{\partial w} = 0$ | $\frac{\partial y}{\partial w} = 0-u^3$ | $\frac{\partial z}{\partial w} = u^3$.

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} 1-2uv & -u^2 & 0 \\ 2uv-3u^2w & u^2 & -u^3 \\ 3u^2w & 0 & u^3 \end{vmatrix} \quad R_1 \to R_1+R_2+R_3.$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3u^2w & u^2 & -u^3 \\ 3u^2w & 0 & u^3 \end{vmatrix}$$

$$= u^3 \begin{vmatrix} 1 & 0 & 0 \\ 2uv-3u^2w & u^2 & -1 \\ 3u^2w & 0 & 1 \end{vmatrix}$$

$$= u^3 [1(u^2+0) - 0 + 0]$$

$$= u^3 (v^2)$$

$$\frac{\partial(xyz)}{\partial(uvw)} = u^5.$$

We know that, $\frac{\partial(u.v.w)}{\partial(xyz)} \cdot \frac{\partial(xyz)}{\partial(uvw)} = 1$

$$\frac{\partial(uvw)}{\partial(xyz)} \cdot u^5 = 1$$

$$\frac{\partial(uvw)}{\partial(xyz)} = \frac{1}{u^5}$$

$$\boxed{\frac{\partial(uvw)}{\partial(xyz)} = u^{-5}}$$

(18) If $u^3+v^3 = x+y$; $u^2+v^2 = x^3+y^3$ prove that $\frac{\partial(uv)}{\partial(xy)}$

Sol: Let us take $f_1 = u^3+v^3-x-y$

$$f_2 = u^2+v^2-x^3-y^3$$

$f_1 = u^3+v^3-x-y$ | $f_2 = u^2+v^2-x^3-y^3$

$\frac{\partial f_1}{\partial u} = 3u^2$ | $\frac{\partial f_2}{\partial u} = 2u$

$\frac{\partial f_1}{\partial v} = 3v^2$ | $\frac{\partial f_2}{\partial v} = 2v$

$\frac{\partial f_1}{\partial x} = -1$ | $\frac{\partial f_2}{\partial x} = -3x^2$

$\frac{\partial f_1}{\partial y} = -1$ | $\frac{\partial f_2}{\partial y} = -3y^2$

We know that $\frac{\partial(UV)}{\partial(xy)} = (-1)^2 \dfrac{\frac{\partial(f_1 f_2)}{\partial(xy)}}{\frac{\partial(f_1 f_2)}{\partial(UV)}}$

$\frac{\partial(f_1 f_2)}{\partial(xy)} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$ $\qquad$ $\frac{\partial(f_1 f_2)}{\partial(UV)} = \begin{vmatrix} 3U^2 & 3V^2 \\ 2U & 2V \end{vmatrix}$

$= 6U^2V - 6UV^2$

$= \cancel{f} \; 3y^2 - 3x^2$

$\frac{\partial(UV)}{\partial(xy)} = \frac{3y^2 - 3x^2}{6U^2V - 6UV^2} = \frac{1}{2}\frac{(y^2 - x^2)}{(U^2V - UV^2)}$

④ If $U = x(1-y)$, $V = xy$ prove that $\frac{\partial(UV)}{\partial(xy)} \times \frac{\partial(xy)}{\partial(UV)} = 1$.

$U = x(1-y) \quad V = xy$ $\qquad$ $UV < \begin{matrix} x \\ y \end{matrix}$

$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$ $\qquad$ $\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix}$

$U = x(1-y)$ | $V = xy$

$\frac{\partial U}{\partial x} = 1-y$ | $\frac{\partial V}{\partial x} = y$

$\frac{\partial U}{\partial y} = -x$ | $\frac{\partial V}{\partial y} = x$

$= (1-y)x + xy$

$= x - \cancel{xy} + \cancel{xy}$

$= x$

$U = x - xy \qquad V = xy$

$U = x - V \qquad y = \frac{V}{x}$

$x = U + V \qquad y = \frac{V}{U+V}$

$xy < \begin{matrix} U \\ V \end{matrix}$

$J^1 = \frac{\partial(xy)}{\partial(UV)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix}$

$x = U + V$ | $y = \frac{V}{U+V}$

$\frac{\partial x}{\partial U} = 1$ | $\frac{\partial y}{\partial U} = V\frac{-1}{(U+V)^2} = \frac{-V}{(U+V)^2}$

$\frac{\partial x}{\partial V} = 1$ | $\frac{\partial y}{\partial V} = \frac{(U+V)(1) - V(0+1)}{(U+V)^2} = \frac{U}{(U+V)^2}$

$$\frac{\partial(xy)}{\partial(uv)} = \begin{vmatrix} 1 & 1 \\ \frac{-v}{(u+v)^2} & \frac{u}{(u+v)^2} \end{vmatrix}$$

$$= \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x-xy+xy} = \frac{1}{x}$$

$$J \cdot J' = x \cdot \frac{1}{x} = 1$$

$$\therefore \frac{\partial(uv)}{\partial(xy)} \cdot \frac{\partial(xy)}{\partial(uv)} = 1$$

(7) If $x = r\cos\theta$, $y = r\sin\theta$. Show that $\frac{\partial(xy)}{\partial(r\theta)} \cdot \frac{\partial(r\theta)}{\partial(xy)} = 1$

Sol:- $x = r\cos\theta$ $y = r\sin\theta$ $xy < {r \atop \theta}$

$$J = \frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial \theta} = r\cdot\cos\theta$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= r(1) = \underline{r}$

$x = r\cos\theta$ $y = r\sin\theta$.

S.O.B. S.O.B

$x^2 = r^2\cos^2\theta$ $y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$

$y/x = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\theta = \tan^{-1}(y/x)$

$r\theta < {x \atop y}$

$$J' = \frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} 2x$

$= \frac{x}{\sqrt{x^2+y^2}}$

$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y\left(\frac{-1}{x^2}\right)$

$= \frac{-y}{x^2+y^2}$

$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2} \frac{1}{x}$

$= \frac{x}{x^2+y^2}$

$$J' = \frac{\partial(r\,\theta)}{\partial(x\,y)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{+y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{\sqrt{x^2+y^2}\,(x^2+y^2)} + \frac{y^2}{\sqrt{x^2+y^2}\,(x^2+y^2)}$$

$$= \frac{\cancel{x^2+y^2}}{\sqrt{x^2+y^2}\,\cancel{(x^2+y^2)}}$$

$$= \frac{1}{r}$$

$$\therefore J \cdot J' = \cancel{r} \cdot \frac{1}{\cancel{r}} = 1$$

$$\therefore \frac{\partial(x\,y)}{\partial(r\,\theta)} \cdot \frac{\partial(r\,\theta)}{\partial(x\,y)} = 1$$

⑤ If $x = UV$; $y = \frac{U}{V}$ prove that $\frac{\partial(x\,y)}{\partial(UV)} \times \frac{\partial(UV)}{\partial(xy)} = 1$.

sol: $x = UV \quad y = \frac{U}{V}$ $\qquad x\ y < \begin{matrix} U \\ V \end{matrix}$

$$J = \frac{\partial(x\,y)}{\partial(UV)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} \end{vmatrix} \qquad \frac{\partial(xy)}{\partial(UV)} = \begin{vmatrix} V & U \\ \frac{1}{V} & \frac{-U}{V^2} \end{vmatrix}$$

$x = UV$

$\frac{\partial x}{\partial U} = V$

$\frac{\partial x}{\partial V} = U$

$y = \frac{U}{V}$

$\frac{\partial y}{\partial U} = \frac{1}{V}$

$\frac{\partial y}{\partial V} = U\,\frac{-1}{V^2}$

$$= \frac{-\cancel{V}U}{V^{\cancel{2}}} - \frac{U}{V}$$

$$J = \frac{-2U}{V}$$

$x = UV \qquad\qquad y = \frac{U}{V}$

$U = \frac{x}{V} \qquad\qquad V = \frac{U}{y} \Rightarrow V = \frac{x/V}{y}$

$U = \frac{x}{U/y} \qquad\qquad V = \frac{x}{yV}$

$U = \frac{xy}{U} \qquad\qquad V^2 = \frac{x}{y} \Rightarrow V = \frac{\sqrt{x}}{\sqrt{y}}$

$U^2 = xy \Rightarrow U = \sqrt{xy}$

$$J' = \frac{\partial(UV)}{\partial(x\,y)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} \qquad UV < \begin{matrix} x \\ y \end{matrix}$$

$U = \sqrt{xy}$

$\frac{\partial U}{\partial x} = \frac{1}{2\sqrt{xy}}\,y$

$\frac{\partial U}{\partial y} = \frac{1}{2\sqrt{xy}}\,x$

$V = \sqrt{\frac{x}{y}}$

$\frac{\partial V}{\partial x} = \frac{1}{\sqrt{y}}\,\frac{1}{2\sqrt{x}}$

$\frac{\partial V}{\partial y} = \frac{-\sqrt{x}}{2y\sqrt{y}}$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{y}{2\sqrt{xy}} & \frac{x}{2\sqrt{xy}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$

$$= \frac{-\cancel{y}\cancel{\sqrt{x}}}{4\cancel{y}\cancel{\sqrt{x}}y} - \frac{x}{4(\sqrt{xy})^2}$$

$$= \frac{-1}{4y} - \frac{\cancel{x}}{4\cancel{x}y}$$

$$= \frac{-1}{4y} - \frac{1}{4y} = \frac{-\cancel{2}}{\cancel{4}_2 y} = \underline{\underline{\frac{-1}{2y}}}$$

$$= \frac{-1}{2\,u/v} = \frac{-v}{2u}.$$

$$J \cdot J' = \frac{-\cancel{2u}}{\cancel{v}} \times \frac{-\cancel{v}}{\cancel{2u}}$$

$$= 1$$

$$\therefore \frac{\partial(xy)}{\partial(uv)} \times \frac{\partial(uv)}{\partial(xy)} = 1$$

⑥ If $x = r\cos\theta$, $y = r\sin\theta$ show that $\frac{\partial(xy)}{\partial(r\theta)} = r$.

Sol:

$x = r\cos\theta \qquad y = r\sin\theta \qquad xy < {r \atop \theta}$

$$J = \frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} \end{vmatrix}$$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial\theta} = r(-\sin\theta)$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial y}{\partial\theta} = r\cos\theta$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$\boxed{\frac{\partial(xy)}{\partial(r\theta)} = r.}$$

$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 = r^2\cos^2\theta \qquad y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2+y^2 = r^2$

$r = \sqrt{x^2+y^2}$

$y/x = \frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta}$

$\tan\theta = y/x \Rightarrow \theta = \tan^{-1}(y/x)$

$r, \theta < {x \atop y}$

$$J' = \frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix}$$

Extra:

$r = \sqrt{x^2+y^2}$

$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}x)$

$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}(\cancel{2}y)$

$\theta = \tan^{-1}(y/x)$

$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\, y\left(\frac{-1}{x^2}\right) = \frac{-y}{x^2+y^2}$

$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2}$

$$\frac{\partial(r\theta)}{\partial(xy)} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{x^2+y^2\sqrt{x^2+y^2}} + \frac{y^2}{x^2+y^2\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{x^2+y^2\sqrt{x^2+y^2}} = \frac{1}{r}.$$

⑧ If $x = r\cos\theta$; $y = r\sin\theta$, $z = z$ evaluate $\frac{\partial(xyz)}{\partial(r\theta z)}$

$x = r\cos\theta$, $y = r\sin\theta$ $z = z$

$xyz < \begin{matrix} r \\ \theta \\ z \end{matrix}$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix}$$

$x = r\cos\theta$	$y = r\sin\theta$	$z = z$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$	$\frac{\partial z}{\partial r} = 0$
$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$	$\frac{\partial y}{\partial \theta} = r\cos\theta$	$\frac{\partial z}{\partial \theta} = 0$
$\frac{\partial x}{\partial z} = 0$	$\frac{\partial y}{\partial z} = 0$	$\frac{\partial z}{\partial z} = 1$

$$\frac{\partial(xyz)}{\partial(r\theta z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\theta[r\cos\theta - 0] + r\sin\theta[\sin\theta - 0]$$

$$= r\cos^2\theta + r\sin^2\theta.$$

$$= r[\cos^2\theta + \sin^2\theta]$$

$$= r.$$

⑨ If $u = 2xy$ $v = x^2 - y^2$ $x = r\cos\theta$, $y = r\sin\theta$ evaluate $\frac{\partial(uv)}{\partial(r\theta)}$.

Sol: $u = 2xy$ $v = x^2 - y^2$ $x = r\cos\theta$, $y = r\sin\theta$

$uv < \begin{matrix} x \\ y \end{matrix} > r\theta.$

$$\frac{\partial(uv)}{\partial(r\theta)} = \frac{\partial(uv)}{\partial(xy)} \cdot \frac{\partial(xy)}{\partial(r\theta)}$$

$$\frac{\partial(uv)}{\partial(xy)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix}$$

$$= -4y^2 - 4x^2$$

$$= \underline{\underline{-4(x^2+y^2)}}.$$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$	$\frac{\partial v}{\partial x} = 2x$
$\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial y} = -2y$

$$\frac{\partial(xy)}{\partial(r\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= \underline{r}$$

$$= \sqrt{x^2+y^2}$$

$x = r\cos\theta$	$y = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = r(-\sin\theta)$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\therefore \frac{\partial(uv)}{\partial(r\theta)} = -4(x^2+y^2)\sqrt{x^2+y^2} = \frac{-4(x^2+y^2)^{3/2}}{(or)}$$

$$= -4(r^2)^{3/2} = \underline{\underline{-4r^3}}$$

(10) If $x = \sqrt{vw}$ $y = \sqrt{wu}$ $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$. Then evaluate $\frac{\partial(xyz)}{\partial(r\theta\phi)}$

$$\frac{\partial(xyz)}{\partial(r\theta\phi)} = \frac{\partial(xyz)}{\partial(uvw)} \cdot \frac{\partial(uvw)}{\partial(r\theta\phi)}$$

$x, y, z \to uvw \to r\theta\phi$

$$\frac{\partial(xyz)}{\partial(uvw)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = \sqrt{vw}$	$y = \sqrt{wu}$	$z = \sqrt{uv}$
$\frac{\partial x}{\partial u} = 0$	$\frac{\partial y}{\partial u} = \frac{1}{2\sqrt{wu}}$	$\frac{\partial z}{\partial u} = \frac{1}{2\sqrt{uv}}$
$\frac{\partial x}{\partial v} = \frac{1}{2\sqrt{vw}}$	$\frac{\partial y}{\partial v} = 0$	$\frac{\partial z}{\partial v} = \frac{1}{2\sqrt{uv}}$
$\frac{\partial x}{\partial w} = \frac{1}{2\sqrt{vw}}$	$\frac{\partial y}{\partial w} = \frac{1}{2\sqrt{wu}}$	$\frac{\partial z}{\partial w} = 0$

$$\frac{\partial(x\,y\,z)}{\partial(u\,v\,w)} = \begin{vmatrix} 0 & \frac{1}{2\sqrt{vw}} & \frac{1}{2\sqrt{vw}} \\ \frac{1}{2\sqrt{wu}} & 0 & \frac{1}{2\sqrt{wu}} \\ \frac{1}{2\sqrt{uv}} & \frac{1}{2\sqrt{uv}} & 0 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{vw}} \cdot \frac{1}{2\sqrt{wu}} \cdot \frac{1}{2\sqrt{uv}} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{8uvw}\left[-1(\cdot 0-1)+1(1-0)\right]$$

$$= \frac{1}{8uvw}(\cdot 1+1)$$

$$= \frac{1}{\underset{4}{\cancel{8}}uvw}(\cancel{2}) = \underline{\underline{\frac{1}{4uvw}}}.$$

$$\frac{\partial(u\,v\,w)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} & \frac{\partial u}{\partial \phi} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} & \frac{\partial v}{\partial \phi} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial \phi} \end{vmatrix}$$

$u = r\sin\theta\cos\phi$

$\frac{\partial u}{\partial r} = \sin\theta\cdot\cos\phi$

$\frac{\partial u}{\partial \theta} = r\cos\phi\cos\theta$

$\frac{\partial u}{\partial \phi} = r\sin\theta(-\sin\phi)$

$v = r\sin\theta\sin\phi$

$\frac{\partial v}{\partial r} = \sin\theta\sin\phi$

$\frac{\partial v}{\partial \theta} = r\sin\phi\cos\theta$

$\frac{\partial v}{\partial \phi} = r\sin\theta\cos\phi$

$w = r\cos\theta.$

$\frac{\partial w}{\partial r} = \cos\theta$

$\frac{\partial w}{\partial \theta} = r(-\sin\theta)$

$\frac{\partial w}{\partial \phi} = 0$

$$\frac{\partial(u\,v\,w)}{\partial(r\,\theta\,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & -r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\cdot r \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \sin\theta\sin\phi & -\cos\theta\sin\phi & \sin\theta\cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\Big[\sin\theta\cos\phi\,(0+\sin^2\theta\cdot\cos\phi) - \cos\theta\cos\phi\,(0-\sin\theta\cos\theta\cos\phi)$$
$$- \sin\theta\sin\phi\,(-\sin^2\theta\cdot\sin\phi + \cos^2\theta\cdot\sin\phi)\Big]$$

$$= r^2\left[\sin^3\theta\cdot\cos^2\phi + \sin\theta\cdot\cos^2\theta\cdot\cos^2\phi + \sin^3\theta\cdot\sin^2\phi - \sin\theta\cos^2\theta\,\sin^2\phi\right]$$

$$= r^2\left[\sin^3\theta(\cos^2\phi+\sin^2\phi) + \sin\theta\cos^2\theta\,(\cos^2\phi - \sin^2\phi)\right]$$

$$= r^2\left[\sin^3\theta + \sin\theta\cdot\cos^2\theta.\right.$$

(11) $y_1 = \frac{x_2 x_3}{x_1}$; $y_2 = \frac{x_3 x_1}{x_2}$; $y_3 = \frac{x_1 x_2}{x_3}$ show that $\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = 4$.

Sol:-

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial y_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$y_1, y_2, y_3 < x_1, x_2, x_3$

$$y_1 = \frac{x_2 x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_1} = x_2 x_3\left(\frac{-1}{x_1^2}\right)$$

$$\frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1}$$

$$\frac{\partial y_1}{\partial x_3} = \frac{x_2}{x_1}$$

$$y_2 = \frac{x_3 x_1}{x_2}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2}$$

$$\frac{\partial y_2}{\partial x_2} = x_3 x_1\left(\frac{-1}{(x_2)^2}\right)$$

$$\frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

$$y_3 = \frac{x_1 x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3}$$

$$\frac{\partial y_3}{\partial x_2} = \frac{x_1}{x_3}$$

$$\frac{\partial y_3}{\partial x_3} = x_1 x_2\left(\frac{-1}{x_3^2}\right)$$

$$\frac{\partial(y_1 y_2 y_3)}{\partial(x_1 x_2 x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_1 x_3}{x_1^2} & \frac{x_1 x_2}{x_1^2} \\ \frac{x_2 x_3}{x_2^2} & \frac{-x_1 x_3}{x_2^2} & \frac{x_1 x_2}{x_2^2} \\ \frac{x_2 x_3}{x_3^2} & \frac{x_1 x_3}{x_3^2} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2}\cdot\frac{1}{x_2^2}\cdot\frac{1}{x_3^2}(x_2x_3)(x_1x_3)(x_1x_2)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{\cancel{x_1^2x_2^2x_3^2}}{\cancel{x_1^2x_2^2x_3^2}}\left[-1(1-1)-1(-1-1)+1(1+1)\right]$$

$$= -1(0)-1(-2)+1(2)$$

$$= 0+2+2$$

$$\frac{\partial(y_1y_2y_3)}{\partial(x_1,x_2x_3)} = \underline{\underline{4}}.$$

(13) $U = \frac{y^2}{2x}$, $V = \frac{x^2+y^2}{2x}$ find $\frac{\partial(UV)}{\partial(xy)}$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix} \qquad UV < \begin{matrix} x \\ y \end{matrix}$$

$U = \frac{y^2}{2x}$

$\frac{\partial U}{\partial x} = \frac{y^2}{2}\left(\frac{-1}{x^2}\right)$

$\frac{\partial U}{\partial y} = \frac{1}{\cancel{2}x}(\cancel{2}y) = y/x$

$V = \frac{x^2+y^2}{2x}$

$\frac{\partial V}{\partial x} = \frac{2x(2x+0)-(x^2+y^2)2}{(2x)^2} = \frac{4x^2-2x^2-2y^2}{4x^2} = \frac{\cancel{2}(x^2-y^2)}{\cancel{4}_2x^2}$

$= \frac{x^2-y^2}{2x^2}$

$\frac{\partial V}{\partial y} = \frac{1}{2x}(0+2y) = \frac{1}{\cancel{2}x}(\cancel{2}y) = y/x$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{-y^2}{2x^2} & y/x \\ \frac{x^2-y^2}{2x^2} & y/x \end{vmatrix}$$

$$= \frac{1}{2x^2}\cdot y/x \begin{vmatrix} -y^2 & 1 \\ x^2-y^2 & 1 \end{vmatrix}$$

$$= \frac{y}{2x^3}\left[-\cancel{y^2}-x^2+\cancel{y^2}\right]$$

$$= \frac{-\cancel{x^2}y}{2x^{\cancel{3}}} = \underline{\underline{\frac{-y}{2x}}}$$

(15) $U = xyz$, $V = xy+yz+zx$, $W = x+y+z$ show that

$$\frac{\partial(UVW)}{\partial(xyz)} = (x-y)(y-z)(z-x).$$

$$UVW \leftarrow \begin{matrix} x \\ y \\ z \end{matrix}$$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} & \frac{\partial W}{\partial z} \end{vmatrix}$$

$U = xyz$ | $V = xy + yz + zx$ | $W = x + y + z$

$\frac{\partial U}{\partial x} = yz$ | $\frac{\partial V}{\partial x} = y + 0 + z = y+z$ | $\frac{\partial W}{\partial x} = 1+0+0 = 1$

$\frac{\partial U}{\partial y} = xz$ | $\frac{\partial V}{\partial y} = x + z + 0 = x+z$ | $\frac{\partial W}{\partial y} = 0+1+0 = 1$

$\frac{\partial U}{\partial z} = xy$ | $\frac{\partial V}{\partial z} = 0 + y + x = x+y$ | $\frac{\partial W}{\partial z} = 0+0+1 = 1$

$$\frac{\partial(UVW)}{\partial(xyz)} = \begin{vmatrix} yz & zx & xy \\ y+z & z+x & x+y \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(z+\cancel{x}-\cancel{x}-y) - zx(\cancel{y}+z-x-\cancel{y}) + xy(y+\cancel{z}-\cancel{z}-x)$$

$$= yz(z-y) - zx(z-x) + xy(y-x)$$

$$= yz^2 - y^2z - z^2x + zx^2 + xy^2 - x^2y$$

$$= x^2($$

(19) If $x^2+y^2+U^2-V^2=0$ and $UV+xy=0$, prove that $\frac{\partial(UV)}{\partial(xy)} = \frac{x^2-y^2}{U^2+V^2}$

Let us take $f_1 = x^2+y^2+U^2-V^2$, $f_2 = UV+xy$.

$f_1 f_2 \to U, V, x, y$

$\frac{\partial f_1}{\partial x} = 2x$ | $\frac{\partial f_2}{\partial x} = y$

$\frac{\partial f_1}{\partial y} = 2y$ | $\frac{\partial f_2}{\partial y} = x$

$\frac{\partial f_1}{\partial U} = 2U$ | $\frac{\partial f_2}{\partial U} = V$

$\frac{\partial f_1}{\partial V} = -2V$ | $\frac{\partial f_2}{\partial V} = U$

We know that,

$$\frac{\partial(UV)}{\partial(xy)} = (-1)^2 \frac{\frac{\partial(f_1 f_2)}{\partial(xy)}}{\frac{\partial(f_1 f_2)}{\partial(UV)}}$$

$$\frac{\partial(f_1 f_2)}{\partial(xy)} = \begin{vmatrix} 2x & 2y \\ y & x \end{vmatrix}$$

$$= 2x^2 - 2y^2$$

$$= 2(x^2-y^2)$$

$$\frac{\partial(f_1 f_2)}{\partial(UV)} = \begin{vmatrix} 2U & -2V \\ V & U \end{vmatrix}$$

$$= 2U^2 + 2V^2$$

$$= 2(U^2+V^2)$$

$$\therefore \frac{\partial(UV)}{\partial(xy)} = (-1)^2 \frac{\cancel{\partial}(x^2-y^2)}{\cancel{\partial}(U^2+V^2)} = \frac{x^2-y^2}{U^2+V^2}.$$

3/12/2019
Tuesday

Functional Dependence

② If $U = \frac{x+y}{1-xy}$ and $V = \tan^{-1}x + \tan^{-1}y$.

$$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$\begin{matrix} U \\ V \end{matrix} > xy.$

$$U = \frac{x+y}{1-xy}$$

$$\frac{\partial U}{\partial x} = \frac{(1-xy)(1)-(x+y)(0-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$= \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(1-xy)(1)-(x+y)(0-x)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$= \frac{1+x^2}{(1-xy)^2}.$$

$$V = \tan^{-1}x + \tan^{-1}y$$

$$\frac{\partial V}{\partial x} = \frac{1}{1+x^2} + 0.$$

$$= \frac{1}{1+x^2}$$

$$\frac{\partial V}{\partial y} = \cancel{\tan^{-1}x}\cdot 0 + \frac{1}{1+y^2}$$

$$= \frac{1}{1+y^2}.$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2}[1-1]$$

$$= \frac{1}{(1-xy)^2}(0).$$

$$\boxed{\frac{\partial(UV)}{\partial(xy)} = 0.}$$

$\therefore$ U and V are functionally dependent.

That is, there is a relation b/w U and V.

$V = \tan^{-1}x + \tan^{-1}y$

$\tan V = \tan[\tan^{-1}x + \tan^{-1}y]$

$= \dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\cdot\tan(\tan^{-1}y)}$

$= \dfrac{x+y}{1-xy}$

$\tan V = U$

② If $U = x+y+z$, $U^2V = y+z$, $U^3W = z$.

$U = x+y+z \qquad U^2V = y+z \qquad U^3W = z$

$U = x + U^2V \qquad U^2V = y + U^3W \qquad z = U^3W.$

$x = U - U^2V \qquad y = U^2V - U^3W$

$$J = \frac{\partial(x\,y\,z)}{\partial(U\,V\,W)} = \begin{vmatrix} \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} & \frac{\partial x}{\partial W} \\ \frac{\partial y}{\partial U} & \frac{\partial y}{\partial V} & \frac{\partial y}{\partial W} \\ \frac{\partial z}{\partial U} & \frac{\partial z}{\partial V} & \frac{\partial z}{\partial W} \end{vmatrix} \qquad \begin{matrix} x \\ y \\ z \end{matrix} \Big\rangle\, U, V, W$$

$x = U - U^2V$

$\frac{\partial x}{\partial U} = 1 - V(2U) = 1 - 2UV$

$\frac{\partial x}{\partial V} = 0 - U^2(1) = -U^2$

$\frac{\partial x}{\partial W} = 0.$

$y = U^2V - U^3W$

$\frac{\partial y}{\partial U} = V(2U) - W(3U^2) = 2UV - 3U^2W$

$\frac{\partial y}{\partial V} = U^2(1) - 0 = U^2$

$\frac{\partial y}{\partial W} = 0 - U^3 = -U^3$

$z = U^3W$

$\frac{\partial z}{\partial U} = W(3U^2) = 3U^2W$

$\frac{\partial z}{\partial V} = 0$

$\frac{\partial z}{\partial W} = U^3(1) = U^3$

$$J = \frac{\partial(x\,y\,z)}{\partial(U\,V\,W)} = \begin{vmatrix} 1-2UV & -U^2 & 0 \\ 2UV - 3U^2W & U^2 & -U^3 \\ 3U^2W & 0 & U^3 \end{vmatrix}$$

$$= U^2 \cdot U^3 \begin{vmatrix} 1-2UV & -1 & 0 \\ 2UV - 3U^2W & 1 & -1 \\ 3U^2W & 0 & 1 \end{vmatrix}$$

$$= u^5\left[1-2uv(1+0)+1(2uv-3u^2w+3u^2w)+0\right]$$

$$= u^5\left[1-\cancel{2uv}+\cancel{2uv}-\cancel{3u^2w}+\cancel{3u^2w}\right]$$

$$\frac{\partial(xyz)}{\partial(uvw)} = u^5.$$

$\therefore$ x,y,z are not functionally dependent.

Hence there is no relation between x,y and z.

(4) If $U = \dfrac{x-y}{x+y}$, $V = \dfrac{xy}{(x+y)^2}$

Sol:- $U = \dfrac{x-y}{x+y}$ $\quad V = \dfrac{xy}{(x+y)^2}$

$$J = \frac{\partial(U\,V)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

$\begin{matrix} U \\ V \end{matrix} > xy$

$$U = \frac{x-y}{x+y}$$

$$\frac{\partial U}{\partial x} = \frac{(x+y)(1-0)-(x-y)(1+0)}{(x+y)^2}$$

$$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial U}{\partial y} = \frac{(x+y)(0-1)-(x-y)(0+1)}{(x+y)^2}$$

$$= \frac{-x-\cancel{y}-x+\cancel{y}}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$V = \frac{xy}{(x+y)^2}$$

$$\frac{\partial V}{\partial x} = \frac{(x+y)^2 y - xy\,2(x+y)}{[(x+y)^2]^2}$$

$$= \frac{x^2y+y^3+\cancel{2xy^2}-2x^2y-\cancel{2xy^2}}{(x+y)^4}$$

$$= \frac{y^3-x^2y}{(x+y)^4}$$

$$\frac{\partial V}{\partial y} = \frac{(x+y)^2 x - xy\,2(x+y)}{((x+y)^2)^2}$$

$$= \frac{(x^2+y^2+2xy)x - 2x^2y - 2xy^2}{(x+y)^4}$$

$$= \frac{x^3+xy^2+\cancel{2x^2y}-\cancel{2x^2y}-2xy^2}{(x+y)^4}$$

$$= \frac{x^3-xy^2}{(x+y)^4}$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{y(y^2-x^2)}{(x+y)^4} & \frac{x(x^2-y^2)}{(x+y)^4} \end{vmatrix}$$

$$= \frac{2x\,y}{(x+y)^2}\begin{vmatrix} 1 & -1 \\ \frac{y^2-x^2}{(x+y)^2} & \frac{x^2-y^2}{(x+y)^2} \end{vmatrix}$$

$$= \frac{2xy}{(x+y)^2}\left[\frac{x^2-y^2}{(x+y)^2}+\frac{y^2-x^2}{(x+y)^2}\right]$$

$$= \frac{2xy}{(x+y)^2}\left[\frac{\cancel{x^2}-\cancel{y^2}+\cancel{y^2}-\cancel{x^2}}{(x+y)^2}\right]$$

$$= \frac{2xy}{(x+y)^2}(0)$$

$$\boxed{\therefore \frac{\partial(UV)}{\partial(xy)} = 0}$$

⑤ $U = xy + yz + zx, \quad V = x^2+y^2+z^2, \quad \omega = x+y+z.$

$$J = \frac{\partial(UV\omega)}{\partial(xyz)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \\ \frac{\partial \omega}{\partial x} & \frac{\partial \omega}{\partial y} & \frac{\partial \omega}{\partial z} \end{vmatrix}$$

U, V, ω < x, y, z

$U = xy+yz+zx$	$V = x^2+y^2+z^2$	$\omega = x+y+z$
$\frac{\partial U}{\partial x} = y+0+z$	$\frac{\partial V}{\partial x} = 2x$	$\frac{\partial \omega}{\partial x} = 1$
$\frac{\partial U}{\partial y} = x+z.$	$\frac{\partial V}{\partial y} = 2y$	$\frac{\partial \omega}{\partial y} = 1$
$\frac{\partial U}{\partial z} = y+x.$	$\frac{\partial V}{\partial z} = 2z$	$\frac{\partial \omega}{\partial z} = 1$

$$\frac{\partial(UV\omega)}{\partial(xyz)} = \begin{vmatrix} y+z & x+z & y+x \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$= y+z\,(2y-2z) - (x+z)(2x-2z) + (y+x)(2x-2y)$

$= \cancel{2y^2} - \cancel{2yz} + \cancel{2yz} - \cancel{2z^2} - \cancel{2x^2} + \cancel{2zx} - \cancel{2zx} + \cancel{2z^2} + \cancel{2xy} - \cancel{2y^2} + \cancel{2x^2} \cancel{-2xy}$

$= 0.$

$$\boxed{\therefore \frac{\partial(UV\omega)}{\partial(xyz)} = 0}$$

① If $U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $V = \sin^{-1}x + \sin^{-1}y$. Show that U,V are functionally dependent.

Sol:-

$$J = \frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} \end{vmatrix}$$

U,V < x, y

$$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\frac{\partial U}{\partial x} = \sqrt{1-y^2} + y.\frac{1}{\cancel{2}\sqrt{1-x^2}}(-\cancel{2}x)$$

$$= \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial U}{\partial y} = x.\frac{1}{\cancel{2}\sqrt{1-y^2}}(-\cancel{2}y) + \sqrt{1-x^2}$$

$$= \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2}$$

$$V = \sin^{-1}x + \sin^{-1}y$$

$$\frac{\partial V}{\partial x} = \frac{1}{\sqrt{1-x^2}} + 0$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\partial V}{\partial y} = 0 + \frac{1}{\sqrt{1-y^2}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial(UV)}{\partial(xy)} = \begin{vmatrix} \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \frac{-xy}{\sqrt{1-y^2}} + \sqrt{1-x^2} \\ \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \end{vmatrix}$$

$$= \frac{1}{\sqrt{1-x^2}}\,\frac{1}{\sqrt{1-y^2}} \begin{vmatrix} \sqrt{1-x^2}.\sqrt{1-y^2} - xy & -xy + \sqrt{1-x^2}\sqrt{1-y^2} \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\left[\sqrt{1-y^2-\cancel{x^2}+x^2y^2} - xy + \cancel{xy} - \sqrt{1-y^2-\cancel{x^2}+x^2y^2}\right]$$

$$= \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\ (0)$$

$$= 0.$$

$$\boxed{\therefore \frac{\partial(UV)}{\partial(xy)} = 0}$$

$\therefore$ U,V are functionally dependent.

i.e., there is a relation b/w U and V.

$U = x\sqrt{1-y^2} + y\sqrt{1-x^2}$ $x = \sin y \Rightarrow y = \sin^{-1}x$

$y = \sin x \Rightarrow x = \sin^{-1}y$

$= \sin y\sqrt{1-\sin^2 x} + \sin x\sqrt{1-\sin^2 y}$

$= \sin y \cdot \cos x + \sin x \cdot \cos y.$

$= \sin(x+y)$

$= \sin(\sin^{-1}y + \sin^{-1}x)$

$\boxed{U = \sin V}$

Maxima And Minima: (without constraints)

(2) $x^3y^2(1-x-y)$

Sol:- Let $f(x,y) = x^3y^2(1-x-y)$

$f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$\frac{\partial f}{\partial x} = y^2(3x^2) - y^2 4x^3 - y^3(3x^2)$

$= 3x^2y^2 - 4x^3y^2 - 3x^2y^3$

$\frac{\partial f}{\partial y} = x^3(2y) - x^4(2y) - x^3 3y^2$

$= 2x^3y - 2x^4y - 3x^3y^2$

we have $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$

$x^2y^2(3-4x-3y) = 0$

$x=0, y=0, 4x+3y-3=0$

$2x^3y - 2x^4y - 3x^3y^2 = 0$

$x^3y(2-2x-3y) = 0$

$x=0, y=0, (2x+3y-2) = 0.$

if $x=0$, $2x+3y-2=0$

$3y-2=0$

$\boxed{y = 2/3}$

$(0, 2/3)$

if $y=0$, $2x+3y-2=0$

$2x-2=0$

$\boxed{x=1}$

$(1,0)$

If $4x+3y-3=0$, $x=0$

$3y-3=0$

$\boxed{y=1}$

$(0,1)$

If $4x+3y-3=0$, $y=0$

$4x-3=0$

$\boxed{x=3/4}$

$(3/4, 0)$

If $4x+3y-3=0$, $2x+3y-2=0$

$$\begin{array}{r} 4x+\cancel{3y}-3=0 \\ 2x+\cancel{3y}-2=0 \\ \hline 2x-1=0 \end{array}$$

$\boxed{x=1/2}$

$4(1/2)+3y-3=0$

$2+3y-3=0$

$\boxed{y=1/3}$

$(1/2, 1/3)$

$\therefore$ The stationary points are $(0, 2/3)$, $(1,0)$, $(0,1)$, $(3/4, 0)$, $(1/2, 1/3)$.

$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2y - 8x^3y - 9x^2y^2$

$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3y$

At the point $(0, 2/3)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$.

At the point $(1,0)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$.

At the point $(0,1)$

$r=0$, $s=0$, $t=0$, $rt-s^2=0$

At the point $(3/4, 0)$

$r=0$, $s=0$, $t = 2(3/4)^3 - 2(3/4)^4$, $rt-s^2=0$

$= \frac{27}{128}$.

At the point $(1/2, 1/3)$

$r = 6(1/2)(1/3)^2 - 12(1/2)^2(1/3)^2 - 6(1/2)(1/3)^3$

$= \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{1}{9} = -\frac{1}{9}$

$s = 6(1/2)^2(1/3) - 8(1/2)^3(1/3) - 9(1/2)^2(1/3)^2$

$= \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = -\frac{1}{12}$

$t = 2(1/2)^3 - 2(1/2)^4 - 6(1/2)^3(1/3)$

$= \frac{1}{4} - \frac{1}{8} - \frac{1}{4} = -\frac{1}{8}$.

$\Rightarrow rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(\frac{-1}{12}\right)^2$

$= \frac{1}{72} - \frac{1}{144} = \frac{2-1}{144} = \frac{1}{144} > 0$

$rt - s^2 > 0, \quad r = -\frac{1}{9} < 0.$

$\therefore$ The function has maximum at the point $(1/2, 1/3)$.

Maximum value is $f = x^3y^2(1-x-y)$

$= (1/2)^3(1/3)^2(1 - 1/2 - 1/3)$

$= \frac{1}{72}\left(\frac{6-3-2}{6}\right)$

$= \frac{1}{72}\left(\frac{1}{6}\right) = \frac{1}{432}$.

(4) $\sin x + \sin y + \sin(x+y)$

Let $f(x,y) = \sin x + \sin y + \sin(x+y)$

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$

We have $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$\cos x + \cos(x+y) = 0$

$2\cos\left(\frac{x+x+y}{2}\right)\cdot\cos\left(\frac{x-x-y}{2}\right) = 0$

$\cos\left(\frac{2x+y}{2}\right)\cdot\cos(-y/2) = 0$

$\cos\frac{2x+y}{2} = 0, \quad \cos y/2 = 0$

$\frac{2x+y}{2} = \cos^{-1}(0) \quad y/2 = \cos^{-1}(0)$

$\frac{2x+y}{2} = \frac{\pi}{2}, \frac{3\pi}{2}\ldots \quad y/2 = \pi/2, 3\pi/2\ldots$

$2x+y = \pi, 3\pi\ldots \quad y = \pi, 3\pi\ldots$

$\cos y + \cos(x+y) = 0$

$2\cos\left(\frac{y+x+y}{2}\right)\cos\left(\frac{y-x-y}{2}\right) = 0$

$\cos\left(\frac{x+2y}{2}\right)\cdot\cos(-x/2) = 0$

$\cos\left(\frac{x+2y}{2}\right) = 0 \quad \cos x/2 = 0$

$\frac{x+2y}{2} = \cos^{-1}(0) \quad x/2 = \cos^{-1}(0)$

$\frac{x+2y}{2} = \pi/2, 3\pi/2\ldots \quad x/2 = \pi/2, 3\pi/2\ldots$

$x+2y = \pi, 3\pi\ldots \quad x = \pi, 3\pi\ldots$

$2x+y = \pi, \quad 2x+y = 3\pi, \quad y = \pi, \quad y = 3\pi$

$x+2y = \pi, \quad x+2y = 3\pi, \quad x = \pi, \quad x = 3\pi.$

if $2x+y=\pi,\ x+2y=\pi$

$\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$2x+y=\pi$
$2x+4y=2\pi$
$-3y=-\pi$
$\boxed{y=\pi/3}$

$2x+\pi/3=\pi$
$2x=\pi-\pi/3=2\pi/3$
$\boxed{x=\pi/3}$

if $2x+y=\pi,\ x+2y=3\pi$

$\left(-\frac{\pi}{3}, \frac{5\pi}{3}\right)$

if $2x+y=3\pi,\ x=3\pi$
$6\pi+y=3\pi \Rightarrow \boxed{y=-3\pi}$
$(3\pi,-3\pi)$

if $2x+y=\pi,\ x=\pi$

$2\pi+y=\pi$ $\quad$ $2x-\pi=\pi$
$\boxed{y=-\pi}$ $\quad$ $2x=2\pi$
$\boxed{x=\pi}$

$(\pi,-\pi)$

if $2x+y=\pi,\ x=3\pi$
$6\pi+y=\pi \Rightarrow \boxed{y=-5\pi}$
$2x-5\pi=\pi \Rightarrow 2x=6\pi$
$\boxed{x=3\pi}$
$(3\pi,-5\pi)$

if $2x+y=3\pi,\ x+2y=\pi$

$\left(\frac{5\pi}{3}, -\frac{\pi}{3}\right)$

if $y=\pi,\ x+2y=3\pi$
$x+2\pi=3\pi \Rightarrow \boxed{x=\pi}$
(π,π)

if $2x+y=3\pi,\ x+2y=3\pi$

(π,π)

if $y=3\pi,\ x+2y=\pi$
$x+6\pi=\pi \Rightarrow \boxed{x=-5\pi}$
$(-5\pi,3\pi)$

if $2x+y=3\pi,\ x=\pi$

(π,π)

if $y=3\pi,\ x+2y=3\pi$
$x+6\pi=3\pi \Rightarrow \boxed{x=-3\pi}$
$(-3\pi,3\pi)$

if $y=\pi,\ x+2y=\pi$
$x+2\pi=\pi \Rightarrow \boxed{x=-\pi}$
$(-\pi,\pi)$

$\therefore$ The stationary points are $\left(\frac{\pi}{3},\frac{\pi}{3}\right), \left(-\frac{\pi}{3},\frac{5\pi}{3}\right), (\pi,-\pi), (3\pi,-5\pi), \left(\frac{5\pi}{3},-\frac{\pi}{3}\right)$
$(\pi,\pi), (\pi,\pi), (-\pi,\pi), (\pi,\pi), (-5\pi,3\pi), (-3\pi,3\pi), (3\pi,-3\pi)$

$$r=\frac{\partial^2 f}{\partial x^2}=-\sin x-\sin(x+y)=-\sin x-\sin(x+y)$$

$$s=\frac{\partial^2 f}{\partial x\,\partial y}=-\sin(x+y)$$

$$t=\frac{\partial^2 f}{\partial y^2}=-\sin y-\sin(x+y)$$

At $\left(\frac{\pi}{3},\frac{\pi}{3}\right)$

$$r=-\sin\frac{\pi}{3}-\sin\left(\frac{\pi}{3}+\frac{\pi}{3}\right)$$
$$=-\frac{\sqrt{3}}{2}-\sin\frac{2\pi}{3}=-\frac{\sqrt{3}}{2}-\sin\frac{\pi}{3}=\frac{-\sqrt{3}}{2}-\frac{\sqrt{3}}{2}=\underline{-\sqrt{3}}$$

$s = -\sin(\pi/3+\pi/3) = -\sin 2\pi/3 = -\sin \pi/3 = -\frac{\sqrt{3}}{2}.$

$t = -\sin\pi/3 - \sin(\pi/3+\pi/3)$

$= -\frac{\sqrt{3}}{2} - \sin 2\pi/3 = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}.$

$rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4}$

$= \frac{12-3}{4} = \frac{9}{4} > 0.$

$\therefore rs - t^2 > 0, \quad \because r < 0. \quad \because \boxed{r = -\sqrt{3}.}$

$\therefore$ The function has maximum at point $(\pi/3, \pi/3)$.

$\therefore$ Maximum value $f = \sin x + \sin y + \sin(x+y)$

$= \sin\pi/3 + \sin\pi/3 + \sin(\pi/3+\pi/3)$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin 2\pi/3.$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \sin\pi/3.$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}.$

At $(-\pi/3, 5\pi/3)$

$r = -\sin(-\pi/3) - \sin(-\pi/3 + 5\pi/3)$

$= \sin\pi/3 - \sin(4\pi/3)$

$= \frac{\sqrt{3}}{2} - \sin(\pi+\pi/3)$

$= \frac{\sqrt{3}}{2} + \sin\pi/3$

$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$

$s = -\sin(-\pi/3+5\pi/3)$

$= -\sin(4\pi/3)$

$= -\sin(\pi+\pi/3)$

$= \sin\pi/3$

$= \frac{\sqrt{3}}{2}$

$t = -\sin(5\pi/3) - \sin(-\pi/3+5\pi/3)$

$= -\sin(2\pi - \pi/3) - \sin(4\pi/3)$

$= \sin\pi/3 - \sin(\pi+\pi/3)$

$= \sin\pi/3 + \sin\pi/3 = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$

$rt - s^2$

$= (\sqrt{3})(\sqrt{3}) - \left(\frac{\sqrt{3}}{2}\right)^2$

$= 3 - \frac{3}{4} = \frac{12-3}{4} = \frac{9}{4} > 0.$

$\therefore rt - s^2 > 0$, $r = \sqrt{3} > 0.$

$\therefore$ The function has minimum at the point $(-\pi/3, 5\pi/3)$

$\therefore$ Minimum value $f = \sin x + \sin y + \sin(x+y)$

$= \sin(-\pi/3) + \sin(5\pi/3) + \sin(-\pi/3 + 5\pi/3)$

$= -\sin \pi/3 + \sin(2\pi - \pi/3) + \sin(4\pi/3)$

$= -\frac{\sqrt{3}}{2} - \sin \pi/3 + \sin(\pi + \pi/3)$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \sin \pi/3$

$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$

$= \frac{-3\sqrt{3}}{2}.$

At the points $(\pi, -\pi)$, $(3\pi, -5\pi)$, (π, π), $(\pi\ \pi)$, $(-\pi, \pi)$, (π, π), $(-5\pi, 3\pi)$
$(-3\pi, 3\pi)$, $(3\pi, -3\pi)$.

$rt - s^2 = 0.$

$\therefore$ We need further investigation.

(7) $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Let $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$

$\frac{\partial f}{\partial x} = y + a^3\left(\frac{-1}{x^2}\right) + 0 = y - \frac{a^3}{x^2}$

$\frac{\partial f}{\partial y} = x + 0 + a^3\left(\frac{-1}{y^2}\right) = x - \frac{a^3}{y^2}.$

we have $\frac{\partial f}{\partial x} = 0$ $\qquad$ $\frac{\partial f}{\partial y} = 0$

$y - \frac{a^3}{x^2} = 0$ $\qquad$ $x - \frac{a^3}{y^2} = 0.$

$y = \frac{a^3}{x^2}$ $\qquad$ $x = \frac{a^3}{y^2}$

Sub y value in $x = \frac{a^3}{y^2}$

$$x - \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = 0$$

$$x - \frac{a^{\cancel{6}}}{(a^3)^{\cancel{2}}} x^4 = 0$$

$$x - \frac{x^4}{a^3} = 0$$

$$a^3 x - x^4 = 0$$

$$x(a^3 - x^3) = 0$$

$$x = 0, \quad (a^3 - x^3) = 0 \qquad \left[a^3 - b^3 = (a-b)(a^2 + ab + b^2)\right]$$

$$x = 0, \quad (a - x) = 0$$

$$\boxed{x = a}$$

Sub $x = a$ in $y = \frac{a^3}{x^2}$

$$y = \frac{a^{\cancel{3}}}{\cancel{a^2}}$$

$$\boxed{y = a}$$

$\therefore x = a, \ y = a$.

$\therefore$ The stationary point is (a, a).

At (a, a) $\quad r = \frac{d^2 f}{dx^2} = \frac{a^3}{x^{\cancel{4}}{}_3}(2\cancel{x}) = \frac{2a^3}{x^3}$

$$S = \frac{d^2 f}{dx\,dy} = 1$$

$$t = \frac{d^2 f}{dy^2} = \frac{a^3}{y^{\cancel{4}}{}_3}(2\cancel{y}) = \frac{2a^3}{y^3}$$

At the point (a, a)

$$r = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2, \quad S = 1, \quad t = \frac{2\cancel{a^3}}{\cancel{a^3}} = 2.$$

$$rt - s^2$$
$$= (2)(2) - (1)^2$$
$$= 4 - 1 = 3 > 0.$$

$rt - s^2 > 0, \quad r = 2 > 0.$

$\therefore$ The function has minimum value at the point (a, a).

$\therefore$ Minimum value is $f = (a)(a) + \frac{a^3}{a} + \frac{a^3}{a}$

$$= a^2 + a^2 + a^2$$
$$= \underline{3a^2}$$