# Recurrence relation (R.R)

generating functions :-

* The generating function of a sequence $a_0, a_1, a_2, a_3$ ----- $a_n$ of a real numbers is written as, the series. ~~the~~ given below.

$$G(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n$$

$$G(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Find the generating function for the sequence $1, 3, 3^2, 3^3$ ------ (or) Find the generating function for the sequence $\{a_n\}$ with $a_n = 3^n$.

Sol:- given series $1, 3, 3^2, 3^3$ ----

$$a_n = 3^n,$$

The generating function of given series is $G(z) = \sum_{n=0}^{\infty} 3^n z^n$

Find the generating function for the sequence $1, 2, 3, 4$

Sol:- given series, $1, 2, 3, 4$

$$a_n = n+1 .$$

The generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} (n+1) z^n.$$

find the generating function of the following sequences

(i) $0, 1, -2, 3, -4, \ldots$

(ii) $0, 2, 6, 12, 20, 30, 42, \ldots$

Sol:- (i) given series, $0, 1, -2, 3, -4, \ldots$

$$a_n = (-1)^{n+1} \cdot n$$

The generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} (-1)^{n+1} n\, z^n$$

$(-1)^{n+1} \cdot n = 0 \quad n=0$

$(-1)^{1+1} \cdot 1 \; (-1)^{n+1} \cdot n = 1 \quad n=1$

$(-1)^{2+1} \cdot 2 \; (-1)^{n+1} \; n = -2$

$(-1)^{3+1} \cdot 3 \Rightarrow n = 3$

$(-1)^{4+1} \cdot 4 \Rightarrow n = -4$

(ii) given series, $0, 2, 6, 12, 20, 30, 42$

$$a_n = \frac{2n(n+1)}{2}$$

the generating function for the given series is

$$G(z) = \sum_{n=0}^{\infty} \frac{2n(n+1)}{2} z^n$$

$n=0 \Rightarrow 0$

$n=1 \Rightarrow \frac{\cancel{2} \cdot 2}{\cancel{2}} = 2$

$n=2 \Rightarrow \frac{\cancel{2}(2)(3)}{\cancel{2}} = 6$

$n=3 \Rightarrow \frac{\cancel{2} \cdot 3 \cdot 4}{\cancel{2}} = 12$

$n=4 \Rightarrow \frac{\cancel{2} \cdot 4 \cdot 5}{\cancel{2}} = 20$

$n=5 \Rightarrow \frac{\cancel{2} \cdot 5 \cdot 6}{\cancel{2}} = 30$

$n=6 \Rightarrow \frac{\cancel{2} \cdot 6 \cdot 7}{\cancel{2}} = 42$

| | Sequence $(a_n)$ | generating function $G(z)$ |
|---|---|---|
| 1 | $a^n$ | $\frac{1}{1-az}$ |
| 2 | $ka^n$ | $\frac{k}{1-az}$ |
| 3 | $bna^n$ | $\frac{baz}{(1-az)^2}$ |
| 4 | $1$ | $\frac{1}{1-z}$ |
| 5 | $n+1$ | $\frac{1}{(1-z)^2}$ |
| 6 | $\frac{1}{n!}$ | $e^z$ |
| 7 | $\frac{(-1)^{n+1}}{n}$ | $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots$ |
| 8 | $^nc_k$ | $(1+x)^n$ |
| 9 | $^nc_k\, a^n$ | $(1+ax)^n$ |
| 10 | $^{n-k+1}c_k = {}^{n+k-1}c_{n-1}$ | $\frac{1}{(1-x)^n}$ |
| 11 | $(-)^k\, {}^{n+k-1}c_k = (-1)^k (n+k-1)c_{n-1}$ | $\frac{1}{(1-x)^n}$ |

problems :-

using generating function to solve the recurrence relation using generating function $a_n = 3a_{n-1} + 2, \; n \geq 1$ with $a_0 = 1$

**Sol**: given $a_n = 3a_{n-1} + 2 \quad n \geq 1$ with $a_0 = 1$

Taking bothsides $\sum_{n=0}^{\infty} z^n$

$$\sum_{n=0}^{\infty} a_n z^n = 3\sum_{n=0}^{\infty} a_{n-1} z^n + 2\sum_{n=0}^{\infty} 1 z^n$$

$$\sum_{n=1}^{\infty} a_n z^n = 3z \sum_{n=1}^{\infty} a_{n-1} z^{n-1} + 2\sum_{n=1}^{\infty} z^n$$

$0 \cdot z^n$

$a_0$

$$(G(z) - a_0) = 3z\,G(z) + 2\frac{1}{(1-z)}$$

$$G(z) - 1 - 3z\,G(z) = \frac{2z}{1-z}$$

$$G(z)[1-3z] = \frac{2z}{1-z} + 1$$

$$G(z) = \frac{2z + 1 - z}{(1-z)(1-3z)}$$

$$G(z) = \frac{z+1}{(1-z)(1-3z)}$$

$3\sum_{n=1}^{\infty} a_{n-1} \frac{z^n z}{z}$

$3z^2 \quad \frac{z^n z^{-1}}{z}$

$z^{n-1}$

$G(z) = a_0 + a_1 z + \cdots$

$G(z) - a_0 = a_1 z + a_2 z^2$

$G(z) - a_0 - a_1 z = a_2 z^2$

$$G(z) = \frac{z+1}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)} \longrightarrow (1)$$

$$\frac{z+1}{\cancel{(1-z)(1-3z)}} = \frac{A(1-3z) + B(1-z)}{\cancel{(1-z)(1-3z)}}$$

$$z+1 = A(1-3z) + B(1-z) \longrightarrow (2)$$

put $z = 1$ in eq (2) we get

$\cancel{2} = A(\cancel{-2}) + 0$

$\boxed{A = -1}$

put $z = \frac{1}{3}$ in eq (2) we get

$\frac{1}{3} + 1 = 0 + B(1 - \frac{1}{3})$

$\cancel{4}^{2}/\cancel{3} = B\frac{\cancel{2}}{\cancel{3}}$

$\boxed{B = 2}$

$$G(z) = \frac{z+1}{(1-z)(1-3z)} = \frac{-1}{1-z} + \frac{2}{1-3z}$$

$$G(z) = \frac{-1}{1-z} + \frac{2}{1-3z}$$

$$G(z) = -1\left(\frac{1}{1-z}\right) + 2\left(\frac{1}{1-3z}\right)$$

$$a_n = -1(1) + 2(3^n)$$

$$\boxed{a_n = -1 + 2(3^n)}$$

(2) Using the method of generating function to solve recurrence relation of

$a_n - 2a_{n-1} - 3a_{n-2} = 0,\ n \geq 2;$ with $a_0 = 3, a_1 = 1$

Sol:- given,

$a_n - 2a_{n-1} - 3a_{n-2} = 0\ n \geq 2,$

$$\sum_{n=2}^{\infty} a_n z^n - 2\sum_{n=2}^{\infty} a_{n-1} z^n - 3\sum_{n=2}^{\infty} a_{n-2} z^n = 0$$

$$\left(G(z)-a_0-a_1 z\right)-2z\sum_{n=2}^{\infty}a_{n-1}z^{n-1}-3z^2\sum_{n=2}^{\infty}a_{n-2}z^{n-2}=0$$

$$\left(G(z)-a_0-a_1 z\right)-2z\left(G(z)-a_0\right)-3z^2G(z)=0$$

$$\left(G(z)-3-z\right)-2z\left(G(z)-3\right)-3z^2G(z)=0$$

$$G(z)\left[-3z^2-2z+1\right]-3-z+6z=0$$

$$G(z)\left[-3z^2-2z+1\right]-3+5z=0$$

$$G(z)=\frac{3-5z}{(-3z^2-2z+1)}$$

$$G(z)=\frac{3-5z}{(1+\cancel{2z})(1-3z)}$$

$$G(z)=\frac{3-5z}{(1+z)(1-3z)}=\frac{A}{(1+z)}+\frac{B}{(1-3z)}\longrightarrow(1)$$

$$\frac{3-5z}{(1+\cancel{z)(1-3z)}}=\frac{A(1-3z)+B(1+z)}{(1+z)\cancel{(1-3z)}}$$

$$3-5z=A(1-3z)+B(1+z)\longrightarrow(2)$$

put $z=-1$ in eq (2), we get

$$3-5(-1)=A(1-3(-1))+B(1-1)$$

$$3+5=A(1+3)+B(0)$$

$$\overset{2}{\cancel{8}}=A(\cancel{4})$$

$$\boxed{A=2}$$

Put $z=1/3$ in eq (2), we get

$$3-5(1/3)=A(1-3(1/3))+B(1+1/3)$$

$3 - \frac{5}{3} = A(1-\frac{3}{3}) + B($

$3 - \frac{5}{3} = A(1-1) + B(\frac{4}{3})$

$\frac{9-5}{3} = B(\frac{4}{3})$

$\frac{4}{3} = B(\frac{4}{3})$

$\boxed{B=1}$

$$G(z) = \frac{2}{1+z} + \frac{1}{1-3z}$$

$$G(z) = 2\left(\frac{1}{1-(-z)}\right) + \frac{1}{1-3z}$$

$$a_n = 2(-1^n) + (3^n)$$

$$a_n = -2 + 3^n$$

Recurrence relation :-

An Equation that Express $a_n$ in terms of one or more of the previous terms of the sequence $a_0, a_1, a_2 \cdots a_n$ is called a recurrence relation for the sequence $\{a_n\}$.

1) Find the first five terms of the sequence define by each of the following recurrence relation and initial conditions

(i) $a_n = a_n^2 - 1$, $a_1 = 2$

(ii) $a_n = n a_{n-1} + n^2 a_{n-2}$ $\quad a_0 = 1, a_1 = 1$

(iii) $a_n = a_{n-1} + a_{n-3}$ $\quad a_0 = 1, a_1 = 2, a_2 = 0$

(i) given R.R is $a_n = a_{n-1}^2$

sol:

put $a_1 = 2$

$a_2 = a_{2-1}^2$

$a_2 = a_1^2$

$a_2 = 4$

$a_3 = a_2^2 = 16$

$a_4 = a_3^2 = (16)^2 = 256$.

$a_5 = a_4^2 = (256)^2 = (65536)$

$a_6 = a_5^2 = (65536)^2 = 4294967296$.

(ii) $a_n = n a_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

given R-R is $a_n = n a_{n-1} + n^2 a_{n-2}$

put $n = 2$

$a_2 = 2 a_{2-1} + 2^2 a_{2-2}$

$a_2 = 2a_1 + 4 a_0$

$a_2 = 2(1) + 4(1)$

$a_2 = 2 + 4$

$a_2 = 6$

$\Rightarrow a_3 = 3 a_{3-1} + 3^2 a_{3-2}$

$= 3 a_2 + 9 a_1$

$= 3(6) + 9(1)$

$= 18 + 9 \Rightarrow 27$

$a_4 = 4a_{4-1} + (4)^2 a_{4-2}$

$= 4a_3 + 16a_2$

$= 4(27) + 16(6)$

$= 108 + 96$

$= 204$

$a_5 = 5a_{5-1} + (5)^2 a_{5-2}$

$= 5a_4 + 25a_3$

$= 5(204) + 25(27)$

$= 1020 + 675$

$a_5 = 1695.$

$a_6 = 6(a_5) + 36a_4$

$= 6(1695) + 36(204)$

$= 17514$

(iii) given RR is, $a_n = a_{n-1} + a_{n-3}$, $a_0 = 1, a_1 = 2, a_2 = 0$

put $n = 3$

$\rightarrow a_3 = a_{3-1} + a_{3-3}$

$a_3 = a_2 + a_0$

$a_3 = 0 + 1$

$a_3 = 1$

$\rightarrow a_4 = a_{4-1} + a_{4-3}$

$a_4 = a_3 + a_1$

$a_4 = 1 + 2$

$a_4 = 3$

$\rightarrow a_5 = a_{5-1} + a_{5-3}$

$= a_4 + a_2 \Rightarrow 3 + 0 \Rightarrow \underline{\underline{3}}$

$\rightarrow a_6 = a_{6-1} + a_{6-3}$

$= a_5 + a_3$

$= 3 + 1$

$a_7 = 4$

$\rightarrow a_7 = a_{7-1} + a_{7-3}$

$= a_6 + a_4$

$= 4 + 3$

$a_7 = 7$

By using an iterative approach find the solutions to each of these recurrence relation with the given initial conditions

(i) $a_n = a_{n-1} + 2$, $a_0 = 3$

(ii) $a_n = a_{n-1} + n$, $a_0 = 1$

(iii) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$

(iv) $a_n = 3a_{n-1} + 1$, $a_0 = 1$

(i) given R.R is $a_n = a_{n-1} + 2$

put $n = 1$

$a_1 = a_0 + 2$

$a_1 = 3 + 2$

$a_1 = 5$

put $n = 2$

$a_2 = a_1 + 2$

$a_2 = 5 + 2$

$= 7$

put $n=3$

$a_3 = a_2 + 2$ $\qquad\qquad$ $3+0\times 2$

$a_3 = 9$ $\qquad\qquad$ $3+1\times 2$

$3+2\times 2$

$\vdots$ $\qquad\qquad$ $3+3\times 2$

$3+4\times 2$

$\vdots$

$3+n\cdot 2$

$a_n = 3+2n$

$\rightarrow$ it will satisify from, 0, 1, 2

(ii) given, $a_n = a_{n-1} + n$

given, R.R is $a_n = a_{n-1} + n$

put $n=1$

$a_1 = a_{1-1} + 1$

$= a_0 + 1$

$= 1+1$

$= 2$

put $n=2$

$a_2 = a_{2-1} + 2$

$= a_1 + 2$

$= 2+2$

$= 4$

put $n=3$

$a_3 = a_{3-1} + 3$

$a_3 = a_2 + 3$

$= 4+3$

$a_3 = 7$

put $n=4$

$$a_4 = a_{4-1}+4$$
$$= a_3+4$$
$$= 7+4$$
$$= 11$$

$1+\frac{n+1}{2}\cdot n$

$\Rightarrow$ put $n=1 \Rightarrow 1+\frac{1+1}{2}\cdot 1 \Rightarrow 2$

$\Rightarrow$ put $n=2 \Rightarrow 1+\frac{2+1}{2}\cdot 2 \Rightarrow 4$

$\Rightarrow$ put $n=3 \Rightarrow 1+\frac{3+1}{2}\cdot 3 \Rightarrow 7$

$\Rightarrow$ put $n=4 \Rightarrow 1+\frac{4+1}{2}\cdot 4 \Rightarrow 11$

$$a_n = 1+\left(\frac{n+1}{2}\right)\cdot n$$

(iii)

given R.R is $a_n = a_{n-1}+2n+3$, $a_0=4$

put $n=1$

$$a_1 = a_{1-1}+2(1)+3$$
$$= a_0+2+3$$
$$= a_0+5$$
$$= 4+5$$
$$= 9$$

put $n=2$

$$a_2 = a_{2-1}+2(2)+3$$
$$= a_1+4+3$$
$$= a_1+7$$
$$= 9+7$$
$$a_2 = 16$$

put $n=3$

$$a_3 = a_{3-1}+2(3)+3$$
$$= a_2+6+3$$
$$= a_2+9 = 16+9$$
$$= 25.$$

put $n=4$

$a_4 = a_{4-1} + 2(4) + 3$

$= a_3 + 8 + 3$

$= a_3 + 11$

$= 25 + 11$

$= 36$

$\Rightarrow a_0 = n^2 + 0\times4 + 4 \Rightarrow n=0$

$\Rightarrow a_1 = n^2 + 1\times4 + 4 \Rightarrow n=1$

$\Rightarrow a_2 = n^2 + 2\times4 + 4 \Rightarrow n=2$

$\Rightarrow a_3 = n^2 + 3\times4 + 4 \Rightarrow n=3$

$\vdots$

$a_n = n^2 + n4 + 4$

$a_n = n^2 + 4n + 4$

(iv) given R.R is $a_n = 3a_{n-1} + 1$ , $a_0 = 1$

put $n=1$

$a_1 = 3a_{1-1} + 1$

$= 3a_0 + 1$

$= 3(1) + 1$

$= 3 + 1$

$a_1 = 4$

put $n=2$

$a_2 = 3a_{2-1} + 1$

$= 3a_1 + 1$

$= 3(4) + 1$

$= 13$

put $n=3$

$a_3 = 3a_{3-1} + 1$

$a_3 = 3a_2 + 1$

$= 3(13) + 1$

$= 39 + 1$

$= 40$

put $n = 4$

$a_4 = 3a_3 + 1$

$= 3(40) + 1$

$= 121$

$\vdots$

$a_n = \frac{3^{n+1} - 1}{2}$

put $n=3 \Rightarrow a_n = \frac{3^{3+1} - 1}{2}$

$a_n = \frac{3^4 - 1}{2}$

$a_n = \frac{80}{2}$

$a_n = 40$

put $n = 4, \Rightarrow a_n = \frac{3^{4+1} - 1}{2}$

$a_n = \frac{3^5 - 1}{2}$

$a_n = 121.$

put $n=0 \Rightarrow a_n = \frac{3^{n+1} - 1}{2}$

$a_n = \frac{3^{0+1} - 1}{2}$

$a_n = \frac{3-1}{2} = 1$

put $n=1 \Rightarrow a_n = \frac{3^{1+1} - 1}{2}$

$= \frac{3^2 - 1}{2}$

$= \frac{9-1}{2}$

$= 4$

put $n=2 \Rightarrow a_n = \frac{3^{2+1} - 1}{2}$

$= \frac{3^3 - 1}{2}$

$= \frac{27-1}{2}$

$= 13$

Characteristic roots : consider the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $c_1, c_2, c_3 \cdots c_k$ are real numbers,

→ The characteristic equation of Recurrence relation, $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - a_c = 0$

→ The solutions of characteristic equation are three types

(1) if roots are real & different then the solution is,

$$a_n = c_1 r_1^n + c_2 r_2^n$$

(2) if roots are real & equal then solution is,

$$a_n = (c_1 + c_2 n) r^n$$

(3) if roots are complex roots, then solution is,

$$a_n = r^n [c_1 \cos n\theta + c_2 \sin n\theta]$$

Problems :-

1. solve the Recurrence relation, $a_n = 5a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$.

Sol :- given R.R is,

$$a_n = 5a_{n-1} + 6a_{n-2}$$

By simplifying,

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

characteristic eq of R.R is $r^2 - 5r + 6 = 0$

roots, $r = 2, 3$

$\therefore$ the given roots are real and different, then the solution is

$$a_n = c_1 r_1^n + c_2 r_2^n$$

$$a_n = c_1(2)^n + c_2(3)^n$$

Now, put $n=0$

$$a_0 = c_1 2^0 + c_2 3^0$$

$$1 = c_1 + c_2 \longrightarrow (1)$$

Now, put $n=1$

$$a_1 = c_1 2^1 + c_2 3^1$$

$$0 = 2c_1 + 3c_2 \longrightarrow (2)$$

Now (1) & (2) becomes,

$$c_1 + c_2 = 1 \times 2$$

$$2c_1 + 3c_2 = 0 \times 1$$

$$2c_1 + 2c_2 = 2$$

$$2c_1 + 3c_2 = 0$$

$$- \quad - \quad -$$

$$-c_2 = 2$$

$$\boxed{c_2 = -2}$$

from, $-2 + c_1 = 1$

$$c_1 = 1 + 2$$

$$\boxed{c_1 = 3}$$

$\therefore$ $a_n = 3 \cdot 2^n - 2 \cdot 3^n$.

2) solve the recurrence relation of

$a_n - 6a_{n-1} + 9a_{n-2} = 0 \quad n \geq 2 \quad a_0 = 5, a_1 = 12,$

$n > i = 2$

<u>Sol</u> :- Given $a_0 = 5$

$a_1 = 12$

R.R is $a_n - 6a_{n-1} + 9a_{n-2} = 0$

characteristic equation

$r^2 - 6r + 9 = 0$

$r^2 - 3r - 3r + 9 = 0$

$r(r-3) - 3(r-3) = 0$

$(r-3)(r-3) = 0$

roots $= 3, 3$

the given roots are real and equal

the solution will be

$a_n = (c_1 + c_2 n)3^n$

put $n=0$, $a_0 = (c_1 + c_2(0))3^0$

$5 = c_1 \longrightarrow (1)$

put $n=1$, $a_1 = (c_1 + c_2(1))3^1$

$12 = (c_1 + c_2)3 \longrightarrow (2)$

$\cancel{c_1 = 15}$

$5 = c_1$

$12 = (c_1 + c_2)3$

$15 + 3c_2 = 12$

$3c_2 = 12 - 15$

$3c_2 = -3$

$c_2 = -1$

$a_n = (5-1)3^n$

3) solve the recurrence relation :

$a_n = 8a_{n-1} - 16a_{n-2}$ for $n \geq 2$, $a_0 = 16$, $a_1 = 80$.

sol :- given R.R is

$a_n = 8a_{n-1} - 16a_{n-2}$

By simplifying,

$a_n - 8a_{n-1} + 16a_{n-2} = 0$

characteristic eq of R.R is $r^2 - 8r + 16 = 0$

$r^2 - 8r + 16 = 0$

$r^2 - 4r - 4r + 16 = 0$

$r(r-4) - 4(r-4) = 0$

$(r-4)(r-4) = 0$

$r = 4, 4$

The given roots are real and equal

the solution will be,

$a_n = (c_1 + c_2 n) 4^n$

put $n = 0$

$a_0 = (c_1 + c_2(0)) 4^0$

$16 = c_1$

put $n = 1$.

$a_1 = (c_1 + c_2(1)) 4^1$

$80 = (16 + c_2) 4$

$64 + 4c_2 = 80$

$4c_2 = 80 - 64$

$c_2 = 4$

$a_n = (16+4)4^n$

(4) solve the recurrence relation

$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3,4,5 \ldots$

with $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

Sol:- given recurrence relation is

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

By simplifying,

$$a_n - 2a_{n-1} + a_{n-2} + 2a_{n-3} = 0$$

characteristic eq of recurrence relation is

$$r^3 - 2r^2 - r + 2 = 0$$

| | | | |
|---|---|---|---|
| 1 | -2 | -1 | 2 |
| 0 | 1 | -1 | -2 |
| 1 | -1 | -2 | 0 |

roots $= 1, 2, -1$

The roots are real & different $r^2 - r - 2 = 0$

The solution will be

$r^2 - 2r + r - 2 = 0$

$r(r-2) + 1(r-2) = 0$

$$a_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n$$

$(r-2)(r+1) = 0$

$$a_n = c_1(1)^n + c_2(2)^n + c_3(-1)^n$$

$r = 2, -1$

$$a_n = c_1(1)^n + c_2(-1)^n + c_3(2)^n$$

put $n = 0$

$$a_0 = c_1(1)^0 + c_2(-1)^0 + c_3(2)^0$$

$$a_0 = c_1 1^0 + c_2(-1)^0 + c_3(2)^0$$

$$3 = c_1 + c_2 + c_3 \longrightarrow (1)$$

put $n = 1$

$$a_1 = c_1(1)^1 + c_2(-1)^1 + c_3(2)^1$$

$a_x =$

$6 = c_1 - c_2 + 2c_3 \longrightarrow (2)$

put $n=2$

$a_2 = c_1(1)^2 + c_2(-1)^2 + c_2(2)^2 \longrightarrow (2)$

$0 = c_1 + c_2 + 4c_3 \longrightarrow (3)$

solve (1) & (2)

$$\begin{aligned} c_1 + \cancel{c_2} + c_3 &= 3 \\ c_1 - \cancel{c_2} + 2c_3 &= 6 \\ \hline 2c_1 + 3c_3 &= 9 \longrightarrow (4) \end{aligned}$$

solve (2) & (3)

$$\begin{aligned} c_1 - \cancel{c_2} + 2c_3 &= 6 \\ c_1 + \cancel{c_2} + 4c_3 &= 0 \\ \hline 2c_1 + 6c_3 &= 6 \longrightarrow (5) \end{aligned}$$

solve (4) & (5)

$$\begin{aligned} 2c_1 + 3c_3 &= 9 \\ 2c_1 + 6c_3 &= 6 \\ \hline -3c_3 &= 3 \\ c_3 &= -1 \end{aligned}$$

sub $c_3$ value in (5)

$2c_1 + 6(-1) = 6$

$2c_1 = 12$

$\boxed{c_1 = 6}$

sub $c_1$ & $c_3$ in (2)

$6 - c_2 - 2 = 6$ $\qquad \boxed{c_3 = -1}$

$-c_2 = 2$

$\boxed{c_2 = -2}$

## Solutions of inhomogenous recurrence relation

→ A linear inhomogenous or non homogenous recurreence realtion with constant coefficents of degree k is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + G(n)$, where $c_1, c_2$ up to $c_k$ are real numbers and equal $G(n)$ is a function not identically zero depending only on $(n)$

Particular solution for $G(n)$ :-

| $G(n)$ | P.I |
|---|---|
| ① constant c | constant d |
| ② linear function $(c_0 + c_1 n)$ | $d_0 + d_1 k$ |
| ③ $m^{th}$ degree polynomial $c_0 + c_1 n + c_2 n^2 + \cdots + c_m n^m$ | $m^{th}$ degree polynomial $d_0 + d_1 k + d_2 k^2 + \cdots + d_m k^m$ |
| ④ $r^n$ $r \in R$ | $d r^n$ |

1) Solve the recurrence relation $a_n = 3a_{n-1} + 2^n$, $a_0 = 1$, $n \geq 1$

Sol :- Given $a_n = 3a_{n-1} + 2^n$

it is a non homogenous linear Equation,

$$a_n - 3a_{n-1} = 2^n$$

general solution

$$a_n - 3a_{n-1} = 0$$

The charcteristic Equation of given Eq,

$$r - 3 = 0$$

roots 1

$$r = 3$$

The roots are real solution will be

$$a_n = c_1(3)^n$$

Put $n=0$

$$a_0 = c_1(3)^0$$

$$1 = c_1$$

$$\boxed{c_1 = 1}$$

$$a_n = (3)^n$$

Now, we can PI,

$$PI = 2^n$$

$$d2^n - 3d\,2^{n-1} = 2^n$$

$$\cancel{2^n}\left(d - \frac{3d}{2}\right) = \cancel{2^n}$$

$$2d - 3d = 2$$

$$-d = 2$$

$$\boxed{d = -2}$$

this is of the form $dr^n$

$$dr^n = (-2)\,2^n \Rightarrow PI$$

Now, $a_n = G + PI$

$$a_n = (3)^n + (-2)\,2^n$$