Unit-5

- ❖ Introduction of Test of Hypothesis
- One Tail test and two tail test
- Type I and Type II errors
- Hypothesis concerning one and two means & proportions
- ❖ Test significance (Small samples)
- Hypothesis concerning one and two means by Students t-Test
- F-test
- χ^2- Test

In many circumstances, we are to make decisions about population on the basis of only sample information.

- Example: A drug chemist is to decide whether a new drug is really effective in curing a disease.

Statistical Hypothesis: To arrive at decisions about the population on the basis of sample information, we make assumptions about the population parameters involved. Such an assumption is called a **Statistical Hypothesis** which may or may not be true.

- Example: The majority of men in the city are smokers

The procedure which enables us to decide on the basis of sample results whether a hypothesis is true or not, is called **Test of Hypothesis** or **Test of significance.**

Procedure for Testing Hypothesis

- **step 1:** Statement of hypothesis (or assumption)

There are two types of hypothesis : (i) Null Hypothesis

(ii) Alternate Hypothesis

Null Hypothesis: It is a statement about population parameter. It is denoted by H_0. A Null Hypothesis is the hypothesis which asserts that there is no significance difference between the statistic and the population parameter.

Example: $H_0 : \mu = \mu_0$

Alternate Hypothesis: Any hypothesis which contradicts the null hypothesis is called an Alternate Hypothesis, usually denoted by H_1.

Example: If $H_0 : \mu = \mu_0$

then $H_1 : \mu \neq \mu_0$

or $H_1 : \mu > \mu_0$

or $H_1 : \mu < \mu_0$

The alternate hypothesis $H_1 : \mu \neq \mu_0$ is known as two-tailed alternative and

$H_1 : \mu > \mu_0$ is known as right tailed and $H_1 : \mu < \mu_0$ is known as left tailed alternate hypothesis.

The setting of alternative hypothesis is very important to decide whether we have to use a single tailed (right or left) or two- tailed test.

- **Step 2:** Specification of the level of significance:

The level of significance denoted by α is the confidence with which we rejects or accepts the Null hypothesis H_0.

Example: 5% L.O.S. in a test procedure indicates that there are 5 cases out of 100 that we reject the null hypothesis when it is true. i.e. 95% confident that we have made right decision.

- **Step 3:** Identification of the Test statistic

There are several tests of significance by z -test(Normal) , t- test, F- test etc…. First we have to select the right test depending on the nature of the information given in the problem. Then we construct the test criterion and select the appropriate probability distribution.

- **Step 4:** Critical Region : The critical region is formed based on following factors.

i) Distribution of the statistic [i.e., whether the statistic follows the normal, t or F distribution.

ii) Form of alternative hypothesis.

Two sided (two-tailed)

left tailed   Right tailed

- **Step 5: Making decision :** By comparing the computed value of statistic and the critical value decision will be taken for accepting or rejecting H_0.

If the computed value (value from step 3)< critical value then we accept H_0, otherwise we reject H_0.

Errors in sampling: we decide to accept or to reject the null hypothesis on population after examining a sample from it. As such we have two types of errors.

(i) Type I error : Reject H_0 when it is true. If the null hypothesis H_0 is true but it rejected by test procedure, then the error made is called Type I error or α error. α = P(Type I error) = P(Rejecting a good population) Producer's risk

(ii) Type II error: Accept H_0 when it is wrong. If the null hypothesis H_0 is false but it accepted by test procedure, then the error made is called Type II error or β error. β = P(Type II error) = P(Accepting a bad population) Consumer's risk

<u>Two tailed test at level of significance α</u>

When Alternative Hypothesis is $\mu \neq \mu_0$.

The critical region is distributed on both

sides of mean equally.

Thus,the critical area under the right-tail = The critical area under the left-tail

= half of the total critical area

= $\frac{1}{2}$ probability of rejection = $\frac{\alpha}{2}$

Hence we find critical statistic with $z_{\alpha/2}$.

Left tailed test at level of significance α Right tailed test at level of significance α

When Alternative Hypothesis is $\mu > \mu_0$ or $\mu < \mu_0$, in a test statistical hypothesis be one tailed then the test is called one tailed test.

The critical region lies entirely in the right tail or left tail of the sampling distribution with area equal to level of significance α.

Hence we find critical statistic with z_α.

Test significance of large samples: If the sample size is $n \geq 30$ then we consider such samples as large samples. For large samples the sampling distribution of a statistic is approximately a normal distribution.

<u>Model 1</u>:Suppose we wish to test the hypothesis that the probability of success in such trails is p. Assuming it to be true, the mean μ and the standard deviation σ of the sampling distribution of number of successes are $np \ and \ \sqrt{npq}$ respectively.

If x be the observed number of successes in the sample and Z is standard normal variate then $Z = \frac{x-\mu}{\sigma}$.

Problem 1: A die is tossed 960 times and it falls with 5 upwards 184 times. Is the die unbiased at a level of significance of 0.01?

Solution: Given $n = 960,$ p = P(getting 5 in one trail) = $\frac{1}{6} \implies q = 1 - \frac{1}{6} = \frac{5}{6} \implies \mu = np = 960 \times \frac{1}{6} = 160 \ and \ \sigma = \sqrt{960 \times \frac{1}{6} \times \frac{5}{6}} = 11.55$, $x =$ number of successes = 184

Null Hypothesis H_0: The die is Unbiased

Alternative Hypothesis H_1: The die is Biased.

Level of significance: $\alpha = 0.01 \Rightarrow z_{\alpha/2} = z_{0.005} = 2.58$ (t-distribution table at ∞)

The test statistic is $Z = \frac{x-\mu}{\sigma} = \frac{184-160}{11.55} = 2.08$

Decision: As $|Z| = 2.08 < 2.58 \ (z_{0.005})$ the Null Hypothesis H_0 has to be accepted at 1% level of significance and conclude that the die is unbiased.

Problem 2: A coin is tossed 400 times and it returns head 216 times. Test the hypothesis that the coin is unbiased. Use a 0.05 level of significance.

Solution: Given $n = 400,$ p = P(getting head in one trail) = $\frac{1}{2} \implies q = 1 - \frac{1}{2} = \frac{1}{2} \implies \mu = np = 400 \times \frac{1}{2} = 200 \ and \ \sigma = \sqrt{400 \times \frac{1}{2} \times \frac{1}{2}} = 10$, $x =$ number of successes = 216

Null Hypothesis H_0: The coin is Unbiased

Alternative Hypothesis H_1: The coin is Biased.

Level of significance: $\alpha = 0.05 \Rightarrow z_{\alpha/2} = z_{0.025} = 1.96$

The test statistic is $Z = \frac{x-\mu}{\sigma} = \frac{216-200}{10} = 1.6$

Decision: As $|Z| = 1.6 < 1.96\ (z_{0.025})$ the Null Hypothesis H_0 has to be accepted at 5% level of significance and conclude that the coin is unbiased.

Modal 2: Test Significance of a Single Mean – Large Samples

Let a random sample size 'n' has the sample mean $\bar{x}$, and μ be the population mean. Also the population mean μ has a specified value μ_0.

Procedure for calculating test statistic:

- When the S.D. (σ) of population is known then test statistic $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$.
- When the S.D. (σ) of population is not known then we take s.d. of sample (s) will be used in test statistic formula. i.e., $z = \frac{\bar{x}-\mu}{s/\sqrt{n}}$.

The confidence interval for the mean of the population corresponding to the given sample is $\bar{x} \pm z_\alpha \frac{\sigma}{\sqrt{n}}$.

Problem 2.1: A sample of 400 items is taken from a population whose standard deviation is 10. The mean of the sample is 40. Test whether the sample has come from a population with mean 38. Also calculate 95% confidence interval for the population.

Solution: Given $n = 400,\ \sigma = 10,\ \ \bar{x} = 40,\ \mu = 38\ and\ \alpha = 1 - 0.95 = 0.05$

Null Hypothesis H_0: $\mu = 38$

Alternative Hypothesis H_1: $\mu \neq 38$

Level of significance $\alpha = 0.05 \Rightarrow\ z\alpha/_2 = 1.96$ (from Normal table)

The test statistic $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{40-38}{10/\sqrt{400}} = 4$

Decision: Since $z > z\alpha/_2$, we reject the null hypothesis H_0.

The confidence interval is $\left(\bar{x} \pm z\alpha/_2 \frac{\sigma}{\sqrt{n}}\right) = \left(40 - 1.96 * \frac{10}{\sqrt{400}},\ 40 + 1.96 * \frac{10}{\sqrt{400}}\right) = (39.02,\ 40.98)$

Problem 2.2: An ambulance service claims that it taken on the average less than 10 minutes to reach its destination in emergency calls. A sample of 36 calls has a mean of 11 minutes and the variance of 16 minutes. Test the claim at 0.05 level of significance.

Solution: Given $\mu = 10,\ n = 36, \bar{x} = 11, \sigma = \sqrt{16} = 4\ and\ \alpha = 0.05$

Null Hypothesis H_0: μ =10

Alternative Hypothesis H_1: $\mu < 10$

Level of significance $\alpha = 0.05 \Rightarrow\ z_\alpha = 1.645$ (from Normal table)

The test statistic $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{11-10}{4/\sqrt{36}} = 1.5$

Decision: Since $z < z_\alpha$, we accept the null hypothesis H_0.

Problem 2.3: A sample of 900 members has a mean of 3.4 cms and S.D. 2.61 cms. Is this sample has been taken from a large population of mean 3.25 cms and S.D. 2.61 cms. If the population is normal and its mean is unknown find the 95% fiducial (confidence) limits of true mean.

Solution: Given $\mathrm{n} = 900, \bar{x} = 3.4,\ s = 2.61,\ \mu = 3.25,\ \sigma = 2.61\ and\ \alpha = 0.05$

Null Hypothesis H_0: $\mu = 3.25$

Alternative Hypothesis H_1: $\mu \neq 3.25$

Level of significance $\alpha = 0.05 \Rightarrow\ z_{\alpha/2} = 1.96$ (from Normal table)

The test statistic $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{3.4-3.25}{2.61/\sqrt{900}} = 1.724$

Decision: Since $z < z_{\alpha/2}$, we accept the null hypothesis H_0.

The confidence interval is $\left(\bar{x} \pm z\alpha_{/2}\frac{\sigma}{\sqrt{n}}\right) = \left(3.4 - 1.96 * \frac{2.61}{\sqrt{900}},\ 3.4 + 1.96 * \frac{2.61}{\sqrt{900}}\right) = (3.2295,\ 3.57)$

Modal 3: Test Significance of a two Mean – Large Samples

Let $\overline{x_1}$ and $\overline{x_2}$ be the sample means of two independent large random samples of sizes n_1 and n_2 drawn from two populations having means μ_1 and μ_2 and standard deviations σ_1 and σ_2.

To test whether the two population means are not equal, Procedure for calculating test statistic:

- When the S.D. (σ) of population is known then test statistic $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$.
- When the S.D. (σ) of population is not known then we take s.d. of sample (s) will be used in test statistic formula. i.e., $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}}$ OR $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sigma\sqrt{\frac{1}{n_1^2}+\frac{1}{n_2^2}}}$ where $\sigma = \frac{n_1S_1^2+n_2S_2^2}{n_1+n_2}$

Problem 3.1: The means of two large samples of sizes 1000 and 2000 members are 67.5 and 68 inches respectively. Can the samples be regarded as drawn from the same population of S.D. 2.5 inches.

Solution: Let μ_1 and μ_2 be the means of the two populations.

Given $n_1 = 1000, n_2 = 2000,\ \overline{x_1} = 67.5, \overline{x_2} = 68,\ \ \sigma = 2.5\ and\ \alpha = 1 - 0.95 = 0.05$ (say)

Null Hypothesis H_0: $\mu_1 = \mu_2$

Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$

Level of significance $\alpha = 0.05 \Rightarrow\ z\alpha_{/2} = 1.96$ (from Normal table)

The test statistic $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}} = \frac{|67.5-68|-0}{\sqrt{\frac{2.5^2}{1000}+\frac{2.5^2}{2000}}} = \frac{0.5}{0.0968} = 5.16$

Decision: Since $z > z\alpha_{/2}$, we reject the null hypothesis H_0.

Problem 3.2: In a survey of buying habits, 400 women shoppers are chosen at random in super market A located in a certain section of the city. Their average weekly food expenditure is Rs 250 with a S.D. of Rs. 40. For 400 women shoppers chosen at random in super market B in another section of the city, the average weekly food expenditure is Rs. 220 with a S.D. of Rs. 55. Test at 1% level of significance whether the average weekly food expenditure of the two populations of shoppers are equal.

Solution: Let μ_1 and μ_2 be the means of the two populations.

Given $n_1 = 400, \overline{x_1} = 250, S_1 = 40$

$n_2 = 400,\ \overline{x_2} = 220, S_2 = 55\ and\ \alpha = 1\% = 0.01$

Null Hypothesis H_0: $\mu_1 = \mu_2$

Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$

Level of significance $\alpha = 0.01 \Rightarrow\ z\alpha_{/2} = z_{0.005} = 2.58$ (from Normal table)

The test statistic $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} = \frac{250-220-0}{\sqrt{\frac{40^2}{400}+\frac{55^2}{400}}} = \frac{30}{3.4} = 8.82$

Decision: Since $z > z\alpha_{/2}$, we reject the null hypothesis H_0.

Problem 3.3: The average marks scored by 32 boys is 72 with a S.D. of 8. While that for 36 girls is 70 with a S.D. of 6. Does this indicate that the boys perform better than girls at level of significance 0.05 ?

Solution: Let μ_1 and μ_2 be the means of the two populations.

Given $n_1 = 32, \overline{x_1} = 72, S_1 = 8$

$n_2 = 36,\ \overline{x_2} = 70, S_2 = 6\ and\ \alpha = 0.05$

Null Hypothesis H_0: $\mu_1 = \mu_2$

Alternative Hypothesis H_1: $\mu_1 > \mu_2$

Level of significance $\alpha = 0.05 \Rightarrow\ z_\alpha = z_{0.05} = 1.645$ (from Normal table)

The test statistic $z = \frac{|\overline{x_1}-\overline{x_2}|-|\mu_1-\mu_2|}{\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}} = \frac{72-70-0}{\sqrt{\frac{8^2}{32}+\frac{6^2}{36}}} = \frac{2}{\sqrt{3}} = 1.1547$

Decision: Since $z < z_\alpha$, we accept the null hypothesis H_0.

Modal 4: Test significance for single proportion

Suppose a large random sample of size n has a sample proportion p of members possessing a certain attribute (i.e. proportion of successes).

To test the hypothesis that the proportion p in the population has a specified value P. The test statistic $z = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$ where Q = 1-P

Problem 4.1: In a big city 325 men out of 600 men were found to be smokers. Does this information support the conclusion that the majority of men in this city are smokers ?

Solution: Given $n = 600,\ p = \frac{325}{600} = 0.5417$

Null Hypothesis H_0 : The number of smokers and non-smokers are equal in the city. i.e., $\mathrm{P} = \frac{1}{2} \Rightarrow$ $Q = 1 - \frac{1}{2} = \frac{1}{2}$

Alternative Hypothesis H_1 : $P > \frac{1}{2}$ (majority of men)

Level of significance : $\alpha = 0.05$ (say) $\Rightarrow$ $Z_\alpha = Z_{0.05} = 1.645$

Test statistic: $z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.5417-0.5}{\sqrt{\frac{0.5\times0.5}{600}}} = 2.04$

Decision: Since $z > z_\alpha$, we reject the Null hypothesis.

Problem 4.2: Experience had shown that 20% of a manufactured product is of the top quality. In one day's production of 400 articles only 50 are of top quality. Test the hypothesis at 0.05 level.

Solution: Given $\mathrm{P} = 20\% = 0.2,\ n = 400,\ p = \frac{50}{400} = 0.125$

Null Hypothesis H_0 : $\mathrm{P} = 0.2 \Rightarrow Q = 1 - 0.2 = 0.8$

Alternative Hypothesis H_1 : $P \neq 0.2$

Level of significance : $\alpha = 0.05 \Rightarrow$ $Z\alpha_{/2} = Z_{0.025} = 1.96$

Test statistic: $z = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.125-0.2}{\sqrt{\frac{0.2\times0.8}{400}}} = -3.75 \Rightarrow$ $|z| = 3.75$

Decision: Since $|z| > z\alpha_{/2}$, we reject the Null hypothesis.

Problem 4.3: Among 900 people in a state 90 are found to be chapati eaters. Construct 99% confidence interval for the true proportion.

Solution: Given $n = 900,\ p = \frac{90}{900} = 0.1 \Rightarrow q = 1 - p = 0,$

$\alpha = 1 - 0.99 = 0.01 \Rightarrow$ $z\alpha_{/2} = z_{0.005} = 2.576$

Now $\sqrt{\frac{pq}{n}} = \sqrt{\frac{0.1\times0.9}{900}} = 0.01$

Confidence interval is $\left(p - z\alpha_{/2}\sqrt{\frac{pq}{n}}, p + z\alpha_{/2}\sqrt{\frac{pq}{n}}\right)$

$$= (0.1 - 2.576 \times 0.01,\ 0.1 + 2.576 \times 0.01) = (0.07,\ 0.13)$$

Modal 5: Test significance for two proportions

Let p_1 and p_2 be the sample proportions in two large random samples of sizes n_1 and n_2 drawn from two populations having proportions P_1 and P_2. The test statistic $z = \frac{|p_1-p_2|-|P_1-P_2|}{\sqrt{\frac{P_1Q_1}{n_1}+\frac{P_2Q_2}{n_2}}}$ where Q = 1-P

OR $z = \frac{|p_1-p_2|}{\sqrt{\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}}}$ where q = 1 – p OR $z = \frac{|p_1-p_2|}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$ where $p = \frac{n_1p_1+n_2p_2}{n_1+n_2}$

Problem 5.1: In two large populations, there are 30% and 25% respectively of fair haired people. Is this difference likely to be hidden in samples of 1200 and 900 respectively from the two populations.

Solution: Given $n_1 = 1200,\ n_2 = 900,\ p_1 = 30\% = 0.3,\ p_2 = 25\% = 0.25$

Null Hypothesis $H_0 : P_1 = P_2$

Alternative Hypothesis $H_1 : P_1 \neq P_2$

Level of significance : $\alpha = 0.05$ (say) $\Rightarrow$ $Z\alpha_{/2} = Z_{0.025} = 1.96$

Test statistic: $z = \frac{|p_1 - p_2|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.3 - 0.25}{\sqrt{\frac{0.3 \times 0.7}{1200} + \frac{0.25 \times 0.75}{900}}} = \frac{0.05}{0.0195} = 2.55$

Decision: Since $z > z\alpha_{/2}$, we reject the Null hypothesis.

Problem 5.2:In an investigation on the machine performance the following results are obtained.

Test whether there is any significance difference in the performance of two machines at 0.05 level.

Solution: Given $n_1 = 375,\ n_2 = 450,\ p_1 = \frac{17}{375} = 0.045,\ p_2 = \frac{22}{450} = 0.049$

Null Hypothesis $H_0 : P_1 = P_2$ Alternative Hypothesis $H_1 : P_1 \neq P_2$

Level of significance : $\alpha = 0.05$ (say) $\Rightarrow$ $Z\alpha_{/2} = Z_{0.025} = 1.96$

Test statistic: $z = \frac{|p_1 - p_2|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{|0.045 - 0.049|}{\sqrt{\frac{0.045 \times 0.955}{375} + \frac{0.049 \times 0.951}{450}}} = \frac{0.004}{0.0147} = 0.27$

Decision: Since $z < z\alpha_{/2}$, we accept the Null hypothesis.

Problem 5.3:A machine puts out 16 imperfect articles in a sample of 500 articles. After the machine is overhauled it puts out 3 imperfect articles in a sample of 100 articles. Has the Machine improved ?

Solution: Given $n_1 = 500,\ n_2 = 100,\ p_1 = \frac{16}{500} = 0.032,\ p_2 = \frac{3}{100} = 0.03$

Null Hypothesis $H_0 : P_1 = P_2$

Alternative Hypothesis $H_1 : P_1 > P_2$

Level of significance : $\alpha = 0.05$ (say) $\Rightarrow$ $Z\alpha_{/2} = Z_{0.025} = 1.96$

Test statistic: $z = \frac{|p_1 - p_2|}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}} = \frac{0.032 - 0.03}{\sqrt{\frac{0.032 \times 0.968}{500} + \frac{0.03 \times 0.97}{100}}} = \frac{0.002}{0.0187} = 0.106$

Decision: Since $z < z\alpha_{/2}$, we accept the Null hypothesis.

Test significance for small samples: when size of sample is <30 then we say that sample is small sample.

We can use normal distribution to test for a specified population mean or difference of two population means as in large sample tests only when the sample is drawn from a normal population whose S.D is known.

If a population is normally distributed, the sampling distribution of the sample mean for any sample size is also normally distributed whether S.D. is known or not.

Degree of freedom (ν): The difference between the total number of observations in the sample and the number of independent constraints imposed on observations.

t – distribution OR Student's t-distribution

- It is used for testing of hypothesis when the sample size is small and population S.D. is not known.
- Modal 6: Test Significance of a Single Mean If $\{x_1,\ x_2,\ \ldots x_n\}$ be any random sample of size n drawn from a normal population with mean μ and variance σ^2, then the test statistic t is defined by $t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}}$ where $\bar{x}$ is sample mean and $S^2 = \frac{1}{n-1}\sum(x_i - \bar{x})^2$ and degree of freedom is $\nu = n-1$.
- we find critical value from t- distribution table at given level of significance at degree of freedom $\nu = n-1$.

Problem 6.1:The average breaking strength of the steel rods is specified to be 18.5 thousand pounds. To test this sample of 14 rods were tested. The mean and standard deviations obtained were 17.85 and 1.955 respectively. Is the result of experiment significant ?

Solution: Given sample size $n = 14,\ \bar{x} = 17.85,\ s = 1.955,\ \mu = 18.5$

And $degree\ of\ freedom,\ \nu = n - 1 = 13.$

Null Hypothesis, H_0: $\mu = 18.5$ Alternative Hypothesis, H_1: $\mu \neq 18.5$

Level of significance, $\alpha = 0.05$ (say) $z_{\frac{\alpha}{2}}(\nu) = z_{0.025}(13) = 2.16$

Test Statistic: $t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}} = \frac{17.85-18.5}{1.955/\sqrt{13}} = \frac{-0.65}{0.542} = -1.199$

Decision: Since $|t| = 1.199 < 2.16$, we accept the null Hypothesis at 5% L.O.S.

Problem 6.2: A random sample of 10 boys had the following I.Q's: 70, 120, 110, 101, 88, 83, 95, 98, 107 and 100. i) Do these data support the assumption mean I.Q. of 100 ? ii) Find a reasonable range in which most of the mean I.Q values of samples of 10 boys lie.

Solution: Given $n = 10$,

The sample is $\{70,120,110,101,88,83,95,98,107,100\}$

$\Rightarrow\ \bar{x}$ =mean of sample= $\frac{\sum x}{n} = \frac{972}{10} = 97.2$

$$\Rightarrow\ S^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{1}{9}\begin{pmatrix} 27.2^2 + 22.8^2 + 12.8^2 + 3.8^2 + 9.2^2 + 14.2^2 \\ +2.2^2 + 0.8^2 + 9.8^2 + 2.8^2 \end{pmatrix} = \frac{1833.6}{9} = 203.73$$

$\Rightarrow S = \sqrt{203.73} = 14.27$

Degree of freedom, $\nu = n - 1 = 9$

i) Null Hypothesis H_0: $\mu = 100$

Alternative Hypothesis H_1: $\mu \neq 100$

Level of significance, $\alpha = 0.05$ (say) $z_{\frac{\alpha}{2}}(\nu) = z_{0.025}(13) = 2.16$

Test Statistic: $t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}} = \frac{97.2-100}{14.27/\sqrt{9}} = \frac{-2.8}{4.756} = -0.588$

Decision: Since $|t| = 0.588 < 2.16$, we accept the null Hypothesis at 5% Level of Significance.

Problem 6.3: Eight students were given a test in Statistics and after one month coaching they were given another test of the similar nature. The following table gives the increase in their marks in the second test over the first. Do the marks indicate that the students have gained from the coaching?

Student No.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Increase in marks	4	-2	6	-8	12	5	-7	2

Solution: Given $n = 8$,

The sample is $\{4, -2{,}110{,}101{,}88{,}83{,}95{,}98{,}107{,}100\}$

$\Rightarrow$ $\bar{x}$ =mean of sample$= \frac{\sum x}{n} = \frac{972}{10} = 97.2$

$$\Rightarrow S^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{1}{9}\begin{pmatrix} 27.2^2 + 22.8^2 + 12.8^2 + 3.8^2 + 9.2^2 + 14.2^2 \\ +2.2^2 + 0.8^2 + 9.8^2 + 2.8^2 \end{pmatrix} = \frac{1833.6}{9} = 203.73$$

$\Rightarrow S = \sqrt{203.73} = 14.27$

Degree of freedom, $\nu = n - 1 = 9$

i) Null Hypothesis H_0: $\mu = 100$

Alternative Hypothesis H_1: $\mu \neq 100$

Level of significance, $\alpha = 0.05$ (say) $z_{\frac{\alpha}{2}}(\nu) = z_{0.025}(9) = 2.262$

Test Statistic: $t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}} = \frac{97.2-100}{14.27/\sqrt{9}} = \frac{-2.8}{4.756} = -0.588$

Decision: Since $|t| = 0.588 < 2.262$, we accept the null Hypothesis at 5% Level of Significance.

Problem 6.4: Prices of shares of a company on the different days in a month were found to be 66, 65, 69, 70, 69, 71, 70, 63, 63 ,64 and 68. Discuss whether the mean price of the shares in the month is 65.

Solution: Given $\mu = 65$, $n = 11$, sample $= \{66{,}65{,}69{,}70{,}69{,}71{,}70{,}63{,}63{,}64{,}68\}$

$\Rightarrow$ $\bar{x}$ =mean of sample$= \frac{\sum x}{n} = \frac{738}{11} = 67.09$

$$\Rightarrow S^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{1}{10}\begin{pmatrix} 1.09^2 + 2.09^2 + 1.91^2 + 2.91^2 + 1.91^2 + 3.91^2 \\ +4.09^2 + 4.09^2 + 3.09^2 + 0.91^2 \end{pmatrix} = \frac{80.441}{10} = 8.0441$$

$\Rightarrow S = \sqrt{8.0441} = 2.836$

Degree of freedom, $\nu = n - 1$ =10

i) Null Hypothesis H_0: $\mu = 65$

Alternative Hypothesis H_1: $\mu \neq 65$

Level of significance, $\alpha = 0.05$ (say) $z_{\frac{\alpha}{2}}(\nu) = z_{0.025}(10) = 2.228$

Test Statistic: $t = \frac{\bar{x}-\mu}{S/\sqrt{n-1}} = \frac{67.09-65}{2.836/\sqrt{10}} = \frac{2.09}{0.897} = 2.33$

Decision: Since $|t| = 2.33 > 2.228$, we reject the null Hypothesis at 5% Level of Significance.

- Modal 7: Test Significance of difference of Means

Let $\bar{x}$, $\bar{y}$ be the means of two independent samples of sizes $n_1 \& n_2$ drawn from two normal populations having means $\mu_1 \& \mu_2$. To test whether the two populations means are equal (i.e., to test whether the difference $\mu_1 - \mu_2$ is significant).

The test statistic is $t = \frac{\bar{x}-\bar{y}}{S.\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$, where $S = \sqrt{\frac{n_1 s_1^2+n_2 s_2^2}{n_1+n_2-2}}$, $\bar{x} = \frac{1}{n_1}\sum x_i$, $\bar{y} = \frac{1}{n_2}\sum y_i$

OR $S = \sqrt{\frac{[\sum(x_i-\bar{x})^2+\sum(y_i-\bar{y})^2]}{n_1+n_2-2}}$ follows t-distribution with $(n_1 + n_2 - 2)$ degree of freedom.

Problem 7.2: A group of 5 patients treated with medicine A weigh 42, 39, 48, 60 and 41 kgs. Second group of 7 patients from the same hospital treated with medicine B weigh 38, 42, 56, 64, 68, 69 and 62 kgs. Do you agree with the claim that meadicine B increases the weight significantly.

Solution: Given sample sizes $n_1 = 5$, $n_2 = 7$

$$\bar{x} = \frac{\sum x}{5} = \frac{230}{5} = 46$$

$$\bar{y} = \frac{\sum y}{7} = \frac{399}{7} = 57$$

x	$x-\bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
42	-4	16	38	-19	361
39	-7	49	42	-15	225
48	2	4	56	-1	1
60	14	196	64	7	49
41	-5	25	68	11	121
			69	12	144
			62	5	25
230		290	399		926

$S^2 = \frac{\sum(x-\bar{x})^2+\sum(y-\bar{y})^2}{n_1+n_2-2} \quad = \frac{290+926}{5+7-2} \quad = \frac{1216}{10} = 121.6 \Rightarrow S = 11.03, \ \ \nu = 5+7-2 = 10$

Null Hypothesis, H_0: $\mu_1 = \mu_2$ Alternative Hypothesis, H_1: $\mu_1 > \mu_2$

Level of significance, $\alpha = 0.05$ (say) , $z_\alpha(\nu) = z_{0.05}(10) = 1.812$

Test Statistic: $t = \frac{\bar{x}-\bar{y}}{S.\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{46-57}{11.03\sqrt{\frac{1}{5}+\frac{1}{7}}} = \frac{-11}{6.4585} = -1.703$

Decision: since $|t| = 1.703 < 1.812$, we accept the null hypothesis.

Problem 7.3: Two horses A and B were tested according to the time (in seconds) to run a particular track with the following results. Test whether the two horses have the same running capacity.

Horse A	**28**	**30**	**32**	**33**	**33**	**29**	**34**
Horse B	29	30	30	24	27	29	

Solution: Calculation of sample means and S.D.s

$$\bar{x} = \frac{\sum x}{7} = \frac{219}{7} = 31.286$$

$$\bar{y} = \frac{\sum y}{6} = \frac{169}{6} = 28.16$$

x	**x-$\bar{x}$**	$(x-\bar{x})^2$	**y**	$y-\bar{y}$	$(y-\bar{y})^2$
28	-3.286	10.8	29	0.84	0.7056
30	-1.286	1.6538	30	1.84	3.3856
32	0.714	0.51	30	1.84	3.3856
33	1.714	2.94	24	-4.16	17.3056
33	1.714	2.94	27	-1.16	1.3456
29	-2.286	5.226	29	0.84	0.7056
34	2.714	7.366			
219		31.4358	169		26.8336

$S^2 = \frac{\sum(x-\bar{x})^2+\sum(y-\bar{y})^2}{n_1+n_2-2} \quad = \frac{31.4358+26.8336}{7+6-2} \quad = \frac{58.2694}{11} = 5.23 \Rightarrow S = 2.3, \ \ \nu = 7+6-2 = 11$

Null Hypothesis $H_0: \mu_1 = \mu_2$

Alternative Hypothesis $H_1: \mu_1 \neq \mu_2$

Level of significance, $\alpha = 0.05$ (say) $z_{\frac{\alpha}{2}}(\nu) = z_{0.025}(11) = 2.201$

Test Statistic: $t = \frac{\bar{x}-\bar{y}}{S.\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{31.286-28.16}{2.3\sqrt{\frac{1}{7}+\frac{1}{6}}} = 2.443$

Decision: Since $|t| = 2.443 > 2.201$, we reject the null Hypothesis at 5% Level of Significance.

- Modal 9: Test Significance for variances

If s_1^2 and s_2^2 are the variances of two samples of sizes n_1 and n_2 respectively, then the population variances are given by $n_1 s_1^2 = (n_1 - 1)S_1^2$ and $n_2 s_2^2 = (n_2 - 1)S_2^2$.

The quantities $\nu_1 = n_1 - 1$ and $\nu_2 = n_2 - 1$ are called degrees of freedom of these estimates.

The test statistic is $F = \frac{S_1^2}{S_2^2}\ \ OR\ \frac{S_2^2}{S_1^2}$ according as $S_1^2 > S_2^2$ or $S_2^2 > S_1^2$ follows F-distribution with degree of freedom $(n_1 - 1, n_2 - 1)$.

We take the greater of the two variances in the numerator and the other in denominator. When F is close to one then the two variances are nearly same.

Conclusion: If the test statistic value F > critical value from F table then we reject the null Hypothesis.

Problem 9.1: It is known that the mean diameters of rivets produced by two firms A and B are practically the same, but the standard deviations may differ. For 22 rivets produced by firm A, the S.D. is 2.9 mm, while for 16 rivets manufactured by firm B, the S.D. is 3.8 mm. Compute the statistic you would use to test whether the products of firm A have the same variability as those of firm B and test its significance.

Solution: Given $n_1 = 22, n_2 = 16,\ s_1 = 2.9$ and $s_2 = 3.8$

$\Rightarrow\ S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{185.02}{21} = 8.81$ and $S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{231.04}{15} = 15.40$

Null Hypothesis $H_0:\ \sigma_1^2 = \sigma_2^2$

Alternative Hypothesis $H_1:\ \sigma_1^2 \neq \sigma_2^2$

Level of significance $\alpha = 0.05$ (say) then $F_\alpha(\nu_2, \nu_1) = F_{0.05}(15,21) = 2.33$

Test statistic : $F = \frac{S_2^2}{S_1^2}$ since $S_2^2 > S_1^2$

$$= \frac{15.40}{8.81} = 1.74$$

Decision: Since Test statistic value $F = 1.74 < F_\alpha = 2.33$ we accept the null Hypothesis.

Problem 9.2: The time taken by workers in performing a job by method I and method II is given below. Do the data show that the variances of time distribution from population from which these sample are drawn do not differ significantly ?

Method I	**20**	**16**	**26**	**27**	**23**	**22**	**-**
Method II	**27**	**33**	**42**	**35**	**32**	**34**	**38**

Solution: Given $n_1 = 6, n_2 = 7$

$$\bar{x} = \frac{\sum x}{n_1} = \frac{134}{6} = 22.3 \qquad \bar{y} = \frac{\sum y}{n_2} = \frac{241}{7} = 34.4$$

x	$\boldsymbol{x - \bar{x}}$	$\boldsymbol{(x - \bar{x})^2}$	**y**	$\boldsymbol{y - \bar{y}}$	$\boldsymbol{(y - \bar{y})^2}$
20	-2.3	5.29	27	-7.4	54.76
16	-6.3	39.69	33	-1.4	1.96
26	3.7	13.69	42	7.6	57.76
27	4.7	22.09	35	0.6	0.36
23	0.7	0.49	32	-2.4	5.76
22	-0.3	0.09	34	-0.4	0.16
			38	3.6	12.96
134		81.34	241		133.72

$$S_1^2 = \frac{\sum(x - \bar{x})^2}{n_1 - 1} = \frac{81.34}{5} = 16.26 \qquad S_2^2 = \frac{\sum(y - \bar{y})^2}{n_2 - 1} = \frac{133.72}{6} = 22.29$$

Null Hypothesis H_0: $\sigma_1^2 = \sigma_2^2$

Alternative Hypothesis H_1: $\sigma_1^2 \neq \sigma_2^2$

Level of significance $\alpha = 0.05$ (say) then $F_\alpha(\nu_2, \nu_1) = F_{0.05}(6,5) = 4.39$

Test statistic : $F = \frac{S_2^2}{S_1^2}$ since $S_2^2 > S_1^2$

$$= \frac{22.29}{16.26} = 1.37$$

Decision: Since Test statistic value $F = 1.37 < F_\alpha = 4.39$ we accept the null Hypothesis.

Modal 10: Test of Significance for Goodness of fit

This test is used to decide whether the discrepancy between theory and experiment is significant or not.

Let $O_1,\ O_2, \dots O_n$ be a set of observed frequencies and $E_1,\ E_2, \dots E_n$ are the corresponding set of expected frequencies.

The test statistic is $\chi^2 = \sum\left[\frac{(O_i-E_i)^2}{E_i}\right]$

If the calculated value of $\chi^2 >$ tabulated value of χ^2 [at α level in χ^2 distribution with $(n-1)$ degree of freedom] then H_0 is rejected [otherwise we accept].

Problem 10.1 : Four coins were tossed 160 times and the 0, 1, 2, 3 and 4 results were obtained 17, 52, 54, 31 and 6 times. Under the assumption that coins are balanced, find the expected frequencies of 0, 1, 2, 3 or 4 heads, and test the goodness of fit at level of significance 0.05.

Solution: Given N = 160, n = 4

p = P(getting head in one trial) = 0.5 , q = 1 – p = 0.5

No. of Heads	**0**	**1**	**2**	**3**	**4**
Observed frq.(O)	**17**	**52**	**54**	**31**	**6**
Expected frq. (E)	$160 \times 4_{C_0} = 10$	$160 \times 4_{C_1} = 40$	$160 \times 4_{C_2} = 60$	$160 \times 4_{C_3} = 40$	$160 \times 4_{C_4} = 10$
$O - E$	**7**	**12**	**- 6**	**- 9**	**- 4**
$\frac{(O-E)^2}{E}$	**4.9**	**3.6**	**0.6**	**2.025**	**1.6**

$$\chi^2 = \sum\left[\frac{(O-E)^2}{E}\right] = 12.725$$

Null Hypothesis H_0: Coins are biased

Alternative Hypothesis H_1: Coins are unbiased

Level of significance: $\alpha = 0.05,\ \chi^2_\alpha(\nu) = \chi^2_{0.05}(4) = 9.488$

Test statistic: $\chi^2 = \sum\left[\frac{(O_i-E_i)^2}{E_i}\right] = 12.725$

Decision: Since $\chi^2 > \chi^2_{0.05}(4)$, Null hypothesis is rejected.

Problem 10.2: A sample analysis of examination results of 500 students was made. It was found that 220 students had failed, 170 had secured a third class, 90 were placed in second class and 20 got a first class. Do these figures commensurate with the general examination result which is in the ratio of 4 : 3 : 2 : 1 for the various categories respectively.

Solution: Given observed results of 500 students, who secured fail, third class, second class and first class result are 220, 170, 90, 20.

Given expected result are in the ration 4 : 3 : 2 : 1.

Hence $\frac{4}{10} * 500, \frac{3}{10} * 500, \frac{2}{10} * 500$ and $\frac{1}{10} * 500$ i.e., 200, 150, 100 and 50 are expected result secured by 500 students.

Result	fail	third	second	First
Observed (O)	**220**	**170**	**90**	**20**
Expected (E)	**200**	**150**	**100**	**50**
$(\boldsymbol{O}-\boldsymbol{E})$	**20**	**20**	**- 10**	**- 30**
$\frac{(\boldsymbol{O}-\boldsymbol{E})^2}{\boldsymbol{E}}$	**2**	**2.67**	**1**	**18**

Null Hypothesis H_0: The observed result commensurate with the general exam result.

Alternative Hypothesis H_1: The observed result commensurate with the general exam result.

Level of significance: $\alpha = 0.05\ (say),\ \chi^2_\alpha(v) = \chi^2_{0.05}(3) = 7.815$

Test statistic: $\chi^2 = \sum\left[\frac{(O_i-E_i)^2}{E_i}\right] = 23.67$

Decision: Since $\chi^2 > \chi^2_{0.05}(4)$, Null hypothesis is rejected.