UNIT – V

DIFFERENTIAL EQUATIONS OF HIGHER ORDER

Introduction :

Differential Equations are extremely helpful to solve complex mathematical problems in almost every domain of Engineering, Science and Mathematics. Engineers will be integrating and differentiating hundreds of equations throughout their career, because these equations have a hidden answer to a really complex problem. Mathematicians and Researchers like Laplace, Fourier, Hilbert etc., have developed such equations to make our life easier. Various Transforms from Time Domain to Frequency Domain or vice versa in Engineering is only possible because of Differential Equations. In real life situations, people use such equations for solving complex fluid dynamics problems, and finding the right balance of weights and measures to build stuff like a Cantilever Truss. Other applications include free vibration analysis, Simple mass-spring system, Damped mass-spring system, forced vibration analysis, Resonant vibration analysis, simple harmonic motion, simple pendulum, pressure Change with altitude, velocity profile in fluid flow, vibration of springs, Discharge of a capacitor, Newton's second law of motion and many more.

Definition: An equation of the form $\frac{d^n y}{dx^n}+P_1(x)\frac{d^{n-1}y}{dx^{n-1}}+P_2(x)\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n(x)y=Q(x)$ where $P_1(x),P_2(x),P_3(x)\ldots P_n(x)$ and $Q(x)$ (functions of x) are continuous is called a linear differential equation of order n.

Linear Differential Equations With Constant Coefficients

Def: An equation of the form $\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\ldots+P_n y=Q(x)$ where $P_1,P_2,P_3\ldots P_n$, are real constants and $Q(x)$ is a continuous function of x is called an linear differential equation of order ' n' with constant coefficients.

Note:

1. Operator $D = \frac{d}{dx}$; $D^2=\frac{d^2}{dx^2}$; ... ... $D^n=\frac{d^n}{dx^n}$

$$D\,y = \frac{dy}{dx};\ D^2 y=\frac{d^2y}{dx^2};\ldots\ldots\ldots\ D^n y=\frac{d^n y}{dx^n}$$

2. Operator $\frac{1}{D}Q=\int Q\,dx$ i e $D^{-1}Q$ is called the integral of Q.

To find the general solution of $f(D).y = 0$:

Here $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots + P_n$ is a polynomial in D.

Now consider the auxiliary equation: $f(m) = 0$

i.e $f(m) = m^n + P_1 m^{n-1} + P_2 m^{n-2} + \ldots + P_n = 0$

where $P_1, P_2, P_3 \ldots P_n$ are real constants.

Let the roots of $f(m) = 0$ be $m_1, m_2, m_3 \ldots m_n$.

Depending on the nature of the roots we write the complementary function as follows:

Consider the following table

S.No	Roots of A.E $f(m) = 0$	Complementary function(C.F)
1.	$m_1, m_2, \ldots m_n$ are real and distinct.	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$
2.	$m_1, m_2, \ldots m_n$ and two roots are equal i.e., m_1, m_2 are equal and real (i.e repeated twice) &the rest are real and different.	$y_c = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots c_n e^{m_n x}$
3.	$m_1, m_2, \ldots m_n$ are real and three roots are equal i.e., m_1, m_2, m_3 are equal and real (i.e repeated thrice) &the rest are real and different.	$y_c = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + c_4 e^{m_4 x} + \ldots c_n e^{m_n x}$

4.	Two roots of A.E are complex say $\alpha + i\beta, \alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3e^{m_3x} + \ldots c_ne^{m_nx}$
5.	If $\alpha \pm i\beta$ are repeated twice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x\right] + c_5e^{m_5x} + \ldots c_ne^{m_nx}$
6.	If $\alpha \pm i\beta$ are repeated thrice & rest are real and distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2x + c_3x^2)\cos\beta x + (c_4 + c_5x + c_6x^2)\sin\beta x\right]$ $+c_7e^{m_7x} + \ldots c_ne^{m_nx}$
7.	If roots of A.E. irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.	$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \ldots\ldots + c_ne^{m_nx}$

Solved Problems

1. Solve $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = 0$

Sol : Given equation is of the form $f(D).y = 0$

Where $f(D) = (D^3 - 3D + 2)y = 0$

Now consider the auxiliary equation $f(m) = 0$

$f(m) = (m^3 - 3m + 2)y = 0 \Rightarrow (m-1)(m-1)(m+2) = 0$

$\Rightarrow m = 1, 1, -2$

Since m_1 and m_2 are equal and m_3 is -2

$y_c = (c_1 + c_2x)e^x + c_3e^{-2x}$

2. Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$

Sol : Given $f(D) = (D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$ …(1)

Auxiliary equation is $f(m) = 0$

$\Rightarrow m^4 - 2m^3 - 3m^2 + 4m + 4 = 0$ …(2)

By inspection $m + 1$ is its factor.

$(m+1)(m^3 - 3m^2 + 4) = 0$ …(3)

By inspection m+1 is factor of $(m^3 - 3m^2 + 4)$.

$\therefore (3)$ is $(m+1)(m+1)(m^2 - 4m + 4) = 0$

$\Rightarrow (m+1)^2(m-2)^2 = 0$

$\Rightarrow m = -1, -1, 2, 2$

Hence general solution of (1) is

$y = (c_1 + c_2x)e^{-x} + (c_3 + c_4x)e^{2x}$

3. Solve $(D^4 + 8D^2 + 16)\, y = 0$

Sol : Given $f(D) = (D^4 + 8D^2 + 16)\, y = 0$

Auxiliary equation $f(m) = (m^4 + 8\, m^2 + 16) = 0$

$(m^2 + 4)^2 = 0$

$(m + 2i)^2\, (m + 2i)^2 = 0$

$m = 2i, 2i, -2i, -2i$

Here roots are complex and repeated

Hence general solution is

$y_c = [(c_1+c_2x)cos2x + (c3 + c4x)sin2x)]$

4. **Solve $y'' + 6y' + 9y = 0$; $y(0) = -4$, $y1(0) = 14$**

Sol : Given equation is $y'' + 6y' + 9y = 0$

Auxiliary equation $f(D)y = 0 \Rightarrow D^2 + 6D + 9)\, y = 0$

A.equation $f(m) = 0 \Rightarrow (m^2 + 6m + 9) = 0$

$\Rightarrow m = -3, -3$

$yc = (c_1+c_2x)e^{-3x}$ -------------------> (1)

Differentiate of (1) w.r.to $x \Rightarrow y' = (c_1+c_2x)(-3e^{-3x}) + c2(e^{-3x})$

Given $y'(0) = 14 \Rightarrow c1 = -4$ & $c2 = 2$

Hence we get $y = (-4 + 2x)\,(e^{-3x})$

5. Solve $4y''' + 4y'' + y' = 0$

Sol : Given equation is $4y''' + 4y'' + y' = 0$

That is $(4D^3+4D^2+D)y = 0$

Auxiliary equation $f(m) = 0$

$$4m^3 + 4m^2 + m = 0$$

$$m(4m^2 + 4m + 1) = 0 \Rightarrow m(2m+1)^2 = 0$$

$$m = 0, -1/2, -1/2$$

$$y = c_1 + (c_2 + c_3 x)\, e^{-x/2}$$

6. Solve $(D^2 - 3D + 4)\, y = 0$

Sol : Given equation $(D^2 - 3D + 4)\, y = 0$

A.E. is $f(m) = 0$

$$m^2 - 3m + 4 = 0$$

$$m = \frac{3 \pm \sqrt{9-16}}{2} = \frac{3 \pm i\sqrt{7}}{2}$$

$$\alpha \pm i\beta = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

$$y = e^{\frac{3}{2}}\left(c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right)$$

To Find General solution of $f(D)\, y = Q(x)$

It is given by $y = y_c + y_p$

i.e. $y = C.F + P.I$

Where the P.I consists of no arbitrary constants and P.I of $f(D)\, y = Q(x)$

Is evaluated as

$$P.I = \frac{1}{f(D)} Q(x)$$

Depending on the type of function of $Q(x)$, P.I is evaluated .

1. Find $\frac{1}{D}(x^2)$

Sol : $\frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3}$

2. **Find Particular value of** $\frac{1}{D+1}(x)$

Sol : $\frac{1}{D+1}(x) = e^{-x}\int x e^{x} dx$ (By definition $\frac{1}{D+\alpha} Q = e^{\alpha x} \int Q\, e^{-\alpha x} dx$

$= e^{-x}(xe^{x} - e^{x})$

$= x - 1$

General methods of finding Particular integral :

P.I of $f(D)y = Q(x)$, when $\frac{1}{f(D)}$ is expressed as partial fractions.

1. Solve $(D^2 + a^2)y = \sec ax$

Sol : Given equation is …(1)

Let $f(D) = D^2 + a^2$

The AE is $f(m) = 0$ i.e $m^2 + a^2 = 0$ …(2)

The roots are $m = -ai, -ai$

$y_c = c_1\cos ax + c_2\sin ax$

$$y_p = \frac{1}{D^2 + a^2}\sec ax = \frac{1}{2ai}\left[\frac{1}{D - ai} - \frac{1}{D + ai}\right]\sec ax \quad \ldots(3)$$

$$\frac{1}{D - ai}\sec ax = e^{iax}\int \sec ax\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx$$

$$= e^{iax}\int (1 - i\tan ax)dx = e^{iax}\left[x + \frac{i}{a}\log\cos ax\right] \ldots(4)$$

Similarly we get $\frac{1}{D + ai}\sec ax = e^{-iax}\left[x - \frac{i}{a}\log\cos ax\right]$ …(5)

From (3),(4) and (5), we get

$$y_p = \frac{1}{2ai}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \frac{x(e^{iax} - e^{-iax})}{2ai} + \frac{1}{a^2}(\log\cos ax)\frac{(e^{iax} + e^{-iax})}{2}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax)$$

∴ The general solution of (1) is

$$y = y_c + y_p = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\log(\cos ax)$$

Rules For Finding P.I In Some Special Cases

Type 1: P.I of $f(D)y = Q(x)$ where $Q(x) = e^{ax}$, where 'a' is constant.

$$\text{Case1.P.I} = \frac{1}{f(D)}.Q(x) = \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

When $f(a) \neq 0$

i.e In $f(D)$, put $D = a$ and Particular integral will be calculated.

Case 2: If $f(a) = 0$ then the above method fails. Then if $f(D) = (D-a)^k \phi(D)$ (i.e 'a' is repeated root k times).

$$\text{Then } P.I = \frac{1}{\phi(a)}e^{ax}.\frac{1}{k!}x^k \text{ provided } \phi(a) \neq 0$$

Type 2: P.I of $f(D)y = Q(x)$ where $Q(x) = \sin ax$ or $Q(x) = \cos ax$ where 'a' is constant then $P.I = \frac{1}{f(D)}Q(x)$.

Working Rule :

Case 1: In $f(D)$ put $D^2 = -a^2 \ni f(-a^2) \neq 0$ then $P.I = \frac{\sin ax}{f(-a)^2}$

Case 2: If $f(-a^2) = 0$ then $D^2 + a^2$ is a factor of $\phi(D^2)$ and hence it is a factor of $f(D)$. Then let $f(D) = (D^2 + a^2)\phi(D^2)$.

$$\text{Then } \frac{\sin ax}{f(D)} = \frac{\sin ax}{(D^2 + a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\sin ax}{D^2 + a^2} = \frac{1}{\phi(-a^2)}\frac{-x\cos ax}{2a}$$

$$\frac{\cos ax}{f(D)} = \frac{\cos ax}{(D^2 + a^2)\phi(D^2)} = \frac{1}{\phi(-a^2)}\frac{\cos ax}{D^2 + a^2} = \frac{1}{\phi(-a^2)}\frac{x\sin ax}{2a}$$

Type 3: P.I for $f(D)y = Q(x)$ where $Q(x) = x^k$ where k is a positive integer, $f(D)$ can be expressed as $f(D) = [1 \pm \phi(D)]$

Express $\frac{1}{f(D)} = \frac{1}{[1 \pm \phi(D)]} = [1 \pm \phi(D)]^{-1}$

Hence $P.I = \frac{1}{[1 \pm \phi(D)]} Q(x)$

$= [1 \pm \phi(D)]^{-1} x^k$

Type 4: P.I of $f(D)y = Q(x)$ when $Q(x) = e^{ax} V$ where 'a' is a constant and V is function of x. where $V = sinax$ or $cosax$ or x^k

Then $P.I = \frac{1}{f(D)} Q(x)$

$= \frac{1}{f(D)} e^{ax} V$

$= e^{ax} \left[\frac{1}{f(D+a)} V \right] \& \frac{1}{f(D+a)} V$ is evaluated depending on V.

Type 5: P.I of $f(D)y = Q(x)$ when $Q(x) = xV$ where V is a function of x.

Then $P.I = \frac{1}{f(D)} Q(x)$

$= \frac{1}{f(D)} V$

$= \left[x - \frac{1}{f(D)} f'(D) \right] \frac{1}{f(D)} V$

Type 6: P.I. of $f(D)y = Q(x)$ where $Q(x) = x^m v$ where v is a function of x.

When $P.I. = \frac{1}{f(D)} \times Q(x) = \frac{1}{f(D)} x^m v$, where v = cosax or sinax

i. $P.I. = \frac{1}{f(D)} x^m \sin ax = I.P. \text{of } \frac{1}{f(D)} x^m e^{iax}$

ii. $P.I. = \frac{1}{f(D)} x^m \cos ax = R.P. \text{of } \frac{1}{f(D)} x^m e^{iax}$

Formulae

1. $\frac{1}{1-D} = (1 - D)^{-1} = 1 + D + D^2 + D^3 + ---------------$
2. $\frac{1}{1+D} = (1 + D)^{-1} = 1 - D + D^2 - D^3 + -----------$
3. $\frac{1}{(1-D)^2} = (1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + ----------$

4. $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + ---------$

5. $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + -----$

6. $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + ---------$

Solved Problems

1. Solve $(4D^2 - 4D + 1)y = 100$

Sol : A.E is $4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{-1}{2}$

$$C.F = (c_1 + c_2 x)e^{\frac{x}{2}}$$

$$\text{Now P.I} = \frac{100}{4D^2 - 4D + 1} = \frac{100e^{0x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100 \{ \text{since } 100e^{0x} = 100 \}$$

Hence the general solution is $y = C.F + P.F = (c_1 + c_2 x)e^{\frac{x}{2}} + 100$

2. Solve the differential equation $(D^2 + 4)y = \sinh 2x + 7$.

Sol : Auxillary equation is $m^2 + 4 = 0$

$$\Rightarrow m^2 = -4 \Rightarrow m = \pm 2i$$

$\therefore$ C.F is $y_c = c_1\cos 2x + c_2 \sin 2x$ …(1)

To find P.I :

$$y_p = \frac{1}{D^2+4}(\sinh 2x + 7)$$

$$= \frac{1}{D^2+4}\left(\frac{e^{2x}+e^{-2x}}{2} + 7e^0\right)$$

$$= \frac{1}{2}.\frac{e^{2x}}{D^2+4} + \frac{1}{2}\frac{e^{-2x}}{D^2+4} + 7\frac{e^0}{(D^2+4)}$$

$$= \frac{e^{2x}}{2(4+4)} + \frac{e^{-2x}}{2(4+4)} + \frac{7}{(0+4)}$$

$$= \frac{e^{2x}+e^{-2x}}{16} + \frac{7}{4} = \frac{1}{8}\sinh 2x + \frac{7}{4} \qquad \text{…(2)}$$

$$y = y_c + y_p$$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{8}\sinh 2x + \frac{7}{4}$$

3. Solve $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Sol : The given equation is

$$(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x \qquad \ldots(1)$$

This is of the form $f(D)y = e^{-2x} + 2\sinh x$

A.E is $f(m) = 0 \Rightarrow (m+2)(m-1)^2 = 0 \therefore m = -2, 1, 1$

The roots are real and one root is repeated twice.

$\therefore$ C.F is $y_c = c_1e^{-2x} + (c_2 + c_3x)e^x$.

$$P.I = \frac{e^{-2x} + 2\sinh x}{(D+2)(D-1)^2} = \frac{e^{-2x} + e^x - e^{-x}}{(D+2)(D-1)^2} = y_{p_1} + y_{p_2} + y_{p_3}$$

Now $y_{p_1} = \dfrac{e^{-2x}}{(D+2)(D-1)^2}$

Hence $f(-2) = 0$. Let $f(D) = (D-1)^2$. Then $\phi(2) \neq 0$ and $m = 1$

$$\therefore y_{p_1} = \frac{e^{-2x}x}{9} = \frac{xe^{-2x}}{9}$$

and $y_{p_2} = \dfrac{e^x}{(D+2)(D-1)^2}$. Here f(1)=0

$$= \frac{e^x x^2}{(3)2!} = \frac{x^2e^x}{6}$$

and $y_{p_3} = \dfrac{e^{-x}}{(D+2)(D-1)^2}$

Putting $D = -1$, we get $y_{p_3} = \dfrac{e^{-x}}{(1)(-2)^2} = \dfrac{e^{-x}}{4}$

$\therefore$ The general solution is $y = y_c + y_{p_1} + y_{p_2} + y_{p_3}$

i.e $y = c_1e^{-2x} + (c_c + c_3x)e^x + \dfrac{xe^{-2x}}{9} + \dfrac{x^2e^x}{6} - \dfrac{e^{-x}}{4}$

4. Solve the differential equation $(D^2 + D + 1)y = \sin 2x$.

Sol : A.E is $m^2 + m + 1 = 0$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\therefore y_c = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) \qquad \ldots(1)$$

To find P.I :

$$y_p = \frac{\sin 2x}{D^2 + D + 1} = \frac{\sin 2x}{-4 + D + 1}$$
$$= \frac{\sin 2x}{D - 3} = \frac{(D+3)\sin 2x}{D^2 - 9} = \frac{(D+3)\sin 2x}{-4 - 9}$$
$$= \frac{D\sin 2x + 3\sin 2x}{-13} = \frac{2\cos 2x + 3\sin 2x}{-13}$$

$$\therefore y = y_c + y_p = e^{\frac{-x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2} + c_2\sin\frac{x\sqrt{3}}{2}\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

5. Solve $(D^2 - 4)y = 2\cos^2 x$

Sol : Given equation is $(D^2 - 4)y = 2\cos^2 x$ …(1)

Let $f(D) = D^2 - 4$ A.E is $f(m) = 0$ i.e $m^2 - 4 = 0$

The roots are $m = 2, -2$. The roots are real and different.

$\therefore$ C.F $= y_c = c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I} = y_p = \frac{1}{D^2 - 4}(2\cos^2 x) = \frac{1}{D^2 - 4}(1 + \cos 2x)$$

$$= \frac{e^{0x}}{D^2 - 4} + \frac{\cos 2x}{D^2 - 4} = \text{P.I}_1 + \text{P.I}_2$$

$$\text{P.I}_1 = y_{p_1} = \frac{e^{0x}}{D^2 - 4} \text{ [Put D=0] } = \frac{e^{0x}}{-4} = -\frac{1}{4}$$

$$\text{P.I}_2 = y_{p_2} = \frac{\cos 2x}{D^2 - 4} = \frac{\cos 2x}{-8} \text{ [Put } D^2 = -2^2 = -4\text{]}$$

$\therefore$ The general solution of (1) is $y = y_c + y_{p_1} + y_{p_2}$

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$$

6. Solve $(D^2 + 1)y = \sin x \sin 2x$

Sol : Given D.E is $(D^2 + 1)y = \sin x \sin 2x$

A.E is $m^2 + 1 = 0 \Rightarrow m = \pm i$

The roots are complex conjugate numbers.

C.F is $y_c = c_1\cos x + c_2\sin x$

w.k.t 2sinAsinB=cos(A-B)–cos(A+B)

$$P.I = \frac{\sin x\sin 2x}{(D^2+1)} = \frac{1}{2}\frac{\cos x - \cos 3x}{(D^2+1)} = P.I_1 + P.I_2$$

Now $P.I_1 = \frac{1}{2}\frac{\cos x}{D^2+1}$

Put $D^2 = -1$ we get $D^2+1 = 0$

$\therefore P.I_1 = \frac{1}{2}\frac{x\sin x}{2} = \frac{x\sin x}{4}$ $\left[\because \text{Case of failure}: \frac{\cos ax}{D^2+a} = \frac{x}{2a}\sin ax\right]$

and $P.I_2 = -\frac{1}{2}\frac{\cos 3x}{D^2+1}$

Put $D^2 = -9$, we get

$$P.I_2 = -\frac{1}{2}\frac{\cos 3x}{-9+1} = \frac{\cos 3x}{16}$$

General solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1\cos x + c_2\sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16}$$

7. Solve the differential equation $(D^3 - 3D^2 - 10D + 24)y = x + 3$.

Sol : The given D.E is $(D^3 - 3D^2 - 10D + 24)y = x + 3$

A.E is $m^3 - 3m^2 - 10m + 24 = 0$

$\Rightarrow$m=2 is a root.

The other two roots are given by $m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow$m=2 (or) m = -1

One root is real and repeated, other root is real.

C.F is $y_c = e^{2x}(c_1 + c_2x) + c_3e^{-x}$

$$y_p = \frac{x+3}{(D^3-3D^2-10D+24)} = \frac{1}{24}\frac{x^3+3}{1+\left(\frac{D^3-3D^2-10D}{24}\right)}$$

$$= \frac{1}{24}\left[\frac{1+D^3-3D^2-10D}{24}\right]^{-1}(x+3)$$

$$= \frac{1}{24}\left[1-\left(\frac{D^3-3D^2-10D}{24}\right)\right](x+3)$$

$$= \frac{1}{24}\left[x+3+\frac{10}{24}\right] = \frac{24x+82}{576}$$

General solution is $y = y_c + y_p$

$$\Rightarrow y = e^{2x}(c_1+c_2x)+c_3e^{-x}+\frac{24x+82}{576}$$

8. Solve the differential equation $(D^2-4D+4)y = e^{2x}+x^2+\sin 3x$.

Sol : The A.E is $(m^2-4m+4)=0 \Rightarrow (m-2)^2=0 \Rightarrow m=2,2$

$\therefore y_c = (c_1+c_2x)e^{2x}$ …(1)

To find y_p : $y_p = \frac{1}{D^2-4D+4}(e^{2x}+x^2+\sin 3x)$

$$= \frac{e^{2x}}{(D-2)^2}+\frac{x^2}{(D-2)^2}+\frac{\sin 3x}{D^2-4D+4}$$

$$= \frac{x^2}{2!}e^{2x}+\frac{x^2}{4\left(1-\frac{D}{2}\right)^2}+\frac{\sin 3x}{-9-4D+4}$$

$$= \frac{x^2}{2}e^{2x}+\frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}x^2-\frac{(4D-5)\sin 3x}{(5+4D)}$$

$$= \frac{x^2}{2}e^{2x}+\frac{1}{4}\left(1+\frac{2D}{2}+\frac{3D^2}{4}\right)x^2-\frac{(4D-5)\sin 3x}{16D^2-25}$$

$$= \frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}-\frac{(12\cos 3x-5\sin 3x)}{-144-25}$$

$$= \frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}+\frac{(12\cos 3x-5\sin 3x)}{169} \quad \text{…(2)}$$

$$y = y_c+y_p = (c_1+c_2x)e^{2x}+=\frac{x^2}{2}e^{2x}+\frac{x^2}{4}+\frac{x}{2}+\frac{3}{8}+\frac{(12\cos 3x-5\sin 3x)}{169}$$

9. Solve the differential equation $(D^2+4)y = x\sin x$.

Sol : Auxiliary equation is $m^2+4-0 \Rightarrow m^2=(2i)^2$

$\therefore m = \pm 2i.$ The roots are complex and conjugate.

Hence Complementary Function, $y_c = c_1\cos 2x + c_2\sin 2x$

Particular integral, $y_p = \frac{1}{D^2+4} x\sin x$

$$= \text{I.P of } \frac{1}{D^2+4} xe^{ix}$$

$$= \text{I.P of} e^{ix} \frac{1}{(D+i)^2+4} x = \text{I.P of } e^{ix} \frac{1}{D^2+2Di+3} x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1} x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left(1-\frac{D^2+2Di}{3}+\dots\right) x$$

$$= \text{I.P of } \frac{e^{ix}}{3}\left(1-\frac{2}{3}Di\right) x \left[D^2(x)=0, \text{etc}\right]$$

$$= \text{I.P of } \frac{1}{3}(\cos x + i\sin x)\left(x - i\frac{2}{3}\right)$$

$$= \frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right)$$

Hence the general solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left(x\sin x - \frac{2}{3}\cos x\right)$$

where c_1 and c_2 are constants.

Other Method (using type 5): $y_p = \frac{1}{D^2+4} x\sin x$

$$= \left\{x - \frac{2D}{D^2+4}\right\}\frac{\sin x}{D^2+4}$$

$$= \frac{x\sin x}{3} - \frac{2(D\sin x)}{3(D^2+4)}$$

$$= \frac{x\sin x}{3} - \frac{2\cos x}{9}$$

Hence the general solution is

$$y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\left(x\sin x - \frac{2}{3}\cos x\right)$$

10. Solve the Differential equation $(D^2+5D+6)y = e^x$

Sol : Given equation is $(D^2+5D+6)y = e^x$

Here $Q(x) = e^{x}$

Auxiliary equation is $f(m) = m^2+5m+6=0$

$m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

$m=-2$ or $m=-3$

The roots are real and distinct

$C.F = y_c = c_1e^{-2x}+c_2e^{-3x}$

Particular Integral= $y_p = \frac{1}{f(D)}Q(x)$

$$=\frac{1}{D^2+5D+6}e^x = \frac{1}{(D+2)(D+3)}e^x$$

Put $D = 1$ in $f(D)$

$P.I = \frac{1}{(3)(4)}e^x$

Particular Integral = $y_p = \frac{1}{12}e^x$

General solution is $y = y_c + y_p$

$$y = c_1e^{-2x}+c_2e^{-3x}+\frac{e^x}{12}$$

11. Solve $y''-4y'+3y=4e^{3x}, y(0)=-1, y'(0)=3$

Sol : Given equation is $y''-4y'+3y=4e^{3x}$

i.e $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=4e^{3x}$ it can be expressed as

$D^2y-4Dy+3y=4e^{3x}$

$(D^2-4D+3)y=4e^{3x}$

Here $Q(x) = 4e^{3x}$; $f(D) = D^2-4D+3$

Auxiliary equation is $f(m) = m^2-4m+3 = 0$

$m^2-3m-m+3=0$

$$m(m-3)-1(m-3)=0 \Rightarrow m=3 \text{ or } 1$$

The roots are real and distinct.

$$C.F = y_c = c_1e^{3x}+c_2e^{x}$$

P.I= $y_p = \frac{1}{f(D)}Q(x)$

$$= \frac{1}{D^2 - 4D + 3} 4e^{3x}$$

$$= \frac{1}{(D-1)(D-3)} 4e^{3x}$$

Put $D = 3$

$$y_p = \frac{4e^{3x}}{(3-1)(D-3)} = \frac{4}{2} \frac{e^{3x}}{(D-3)} = 2\frac{x'}{1!} e^{3x} = 2xe^{3x}$$

General solution is $y = y_c + y_p$

$$y = c_1 e^{3x} + c_2 e^{x} + 2xe^{3x} \qquad \ldots(3)$$

Equation (3) differentiating with respect to 'x'

$$y' = 3c_1 e^{3x} + c_2 e^{x} + 2e^{3x} + 6xe^{3x} \qquad \ldots(4)$$

By data, $y(0) = -1$, $y'(0) = 3$

From (3), $\quad -1 = c1 + c2 \qquad \ldots(5)$

From (4), $\quad 3 = 3c1 + c2 + 2$

$$3c1 + c2 = 1 \qquad \ldots(6)$$

Solving (5) and (6) we get $c1 = 1$ and $c2 = -2$

$y = -2e^{x} + (1 + 2x)\, e^{3x}$

12. **Solve $y'' + 4y' + 4y = 4\cos x + 3\sin x, y(0) = 0, y'(0) = 0$**

Sol : Given differential equation in operator for

$$(D^2 + 4D + 4)y = 4\cos x + 3\sin x$$

A.E is $m^2 + 4m + 4 = 0$

$(m + 2)^2 = 0 \qquad$ then $m = -2, -2$

$\therefore$ C.F is $y_c = (c_1 + c_2 x)e^{-2x}$

P.I is= $y_p = \frac{4\cos x + 3\sin x}{(D^2 + 4D + 4)}$ put $D^2 = -1$

$$y_p = \frac{4\cos x + 3\sin x}{(4D + 3)} = \frac{(4D - 3)(4\cos x + 3\sin x)}{(4D - 3)(4D + 3)}$$

$$= \frac{(4D - 3)(4\cos x + 3\sin x)}{16D^2 - 9}$$

$$y_p = \frac{(4D - 3)(4\cos x + 3\sin x)}{-16 - 9}$$

$$= \frac{-16\sin x + 12\cos x - 12\cos x - 9\sin x}{-25} = \frac{-25\sin x}{-25} = \sin x$$

$\therefore$ General equation is $y = y_c + y_p$

$y = (c_1 + c_2x)e^{-2x} + \sin x$ ...(1)

By given data $y(0) = 0, c_1 = 0$ and

Differentiating (1) w.r.t 'x', $y' = (c_1 + c_2x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x$ ...(2)

given $y'(0) = 0$

Substitute in (2) $\Rightarrow -2c1 + c2 + 1 = 0$ $\therefore c2 = -1$

$\therefore$ Required solution is $y = -xe^{-2x} + \sin x$

13. **Solve $(D^2+9)y = \cos 3x$**

Sol : Given equation is $(D^2+9)y = \cos 3x$

A.E is $m^2+9 = 0$

$\therefore m = \pm 3i$

$y_c = C.F = c_1\cos 3x + c_2\sin 3x$

$$y_p = P.I = \frac{\cos 3x}{D^2+9} = \frac{\cos 3x}{D^2+3^2}$$

$$= \frac{x}{2(3)}\sin 3x = \frac{x}{6}\sin 3x$$

General equation is $y = y_c + y_p$

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x$$

14. Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

Sol : Given equation can be written as

$(D^3 + 2D^2 - D - 2)y = 1 - 4x^3$

A.E is $m^3 + 2m^2 - m - 2 = 0$

$(m^2 - 1)(m + 2) = 0$

$m^2 = 1$ or $m = -2$

$m = 1, -1, -2$

C.F= $c_1e^x + c_2e^{-x} + c_3e^{-2x}$

$$P.I = \frac{1}{(D^3 + 2D^2 - D - 2)}(1 - 4x^3) = \frac{-1}{2\left[1 - \frac{(D^3 + 2D^2 - D)}{2}\right]}(1 - 4x^3)$$

$$= \frac{-1}{2}\left[1 - \frac{(D^3 + 2D^2 - D)}{2}\right]^{-1}(1 - 4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{(D^3+2D^2-D)}{2}+\frac{(D^3+2D^2-D)^2}{4}+\frac{(D^3+2D^2-D)^3}{8}+\ldots\right](1-4x^3)$$

$$= \frac{-1}{2}\left[1+\frac{1}{2}\left(D^3+2D^2-D\right)+\frac{1}{4}\left(D^2-4D^3\right)+\frac{1}{8}\left(-D^3\right)\right]\left(1-4x^3\right)$$

$$= \frac{-1}{2}\left[1-\frac{5}{8}D^3+\frac{5}{4}D^2-\frac{1}{2}D\right](1-4x^3)$$

$$= \frac{-1}{2}\left[(1-4x^3)-\frac{5}{8}(-24)+\frac{5}{4}(-24x)-\frac{1}{2}(-12x^2)\right]$$

$$= \frac{-1}{2}\left[-4x^3+6x^2-30x+16\right]$$

$$=[2x^3-3x^2+15x-8]$$

The general solution is

$$y = C.F + P.I$$

$$y = c_1e^{x}+c_2e^{-x}+c_3e^{-2x}+[2x^3-3x^2+15x-8]$$

15. Solve $(D^3-7D^2+14D-8)y=e^x\cos 2x$

Sol : Given equation is

$$(D^3-7D^2+14D-8)y=e^x\cos 2x$$

A.E is $(m^3-7m^2+14m-8)=0$

$$(m-1)(m-2)(m-4)=0$$

Then $m = 1,2,4$

C.F= $c_1e^{x}+c_2e^{2x}+c_3e^{4x}$

P.I= $\dfrac{e^x\cos 2x}{(D^3-7D^2+14D-8)}$

$$= e^x \frac{1}{(D+1)^3-7(D+1)^2+14(D+1)-8}\cos 2x \quad \left[\because P.I=\frac{1}{f(D)}e^{ax}v=e^{ax}\frac{1}{f\left(D+a\right)}v\right]$$

$$= e^x \frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$= e^x \frac{1}{(D^3-4D^2+3D)}\cos 2x$$

$$= e^x \frac{1}{(-4D+3D+16)}\cos 2x$$ (Replacing $D2$ with -2^2)

$$= e^{x}\frac{1}{(16-D)}\cos 2x$$

$$= e^{x}\frac{16+D}{(16-D)(16+D)}\cos 2x$$

$$= e^{x}\frac{16+D}{256-D^{2}}\cos 2x$$

$$= e^{x}\frac{16+D}{256-(-4)^{2}}\cos 2x$$

$$= \frac{e^{x}}{260}(16\cos 2x - 2\sin 2x)$$

$$= \frac{2e^{x}}{260}(8\cos 2x - \sin 2x)$$

$$= \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$$

General solution is $y = y_c + y_p$

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{4x} + \frac{e^{x}}{130}(8\cos 2x - \sin 2x)$$

16. Solve $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

Sol : Given $(D^2 - 4D + 4)y = x^2\sin x + e^{2x} + 3$

A.E is $(m^2 - 4m + 4) = 0$

$(m-2)^2 = 0$ then m=2,2

C.F= $(c_1 + c_2x)e^{2x}$

$$\text{P.I}= \frac{x^2\sin x + e^{2x} + 3}{(D-2)^2} = \frac{1}{(D-2)^2}(x^2\sin x) + \frac{1}{(D-2)^2}e^{2x} + \frac{1}{(D-2)^2}(3)$$

$$\text{Now } \frac{1}{(D-2)^2}(x^2\sin x) = \frac{1}{(D-2)^2}(x^2) \quad (\text{I.P of } e^{ix})$$

$$= \text{I.P of } \frac{1}{(D-2)^2}(x^2)e^{ix}$$

$$= \text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

$$\text{I.P of } (e^{ix})\frac{1}{(D+i-2)^2}(x^2)$$

On simplification, we get

$$\frac{1}{(D+i-2)^2}(x^2 \sin x) = \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right]$$

and $\frac{1}{(D-2)^2}e^{2x} = \frac{x^2}{2}e^{2x}$,

$$\frac{1}{(D-2)^2}(3) = \frac{3}{4}$$

$$\text{P.I}=\frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

$y=y_c+y_p$

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$$

17. Solve the differential equation $(D^3+1)y = \cos(2x-1)$.

Sol : Given D.E is $(D^3+1)y = \cos(2x-1)$

The A.E is $m^3+1=0$

$\Longrightarrow (m+1)(m^2-m+1)=0 \quad [a^3+b^3 = (a+b)(a^2-ab+b^2)]$

$$\Longrightarrow m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$C.F = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I = \frac{1}{D^3+1}\cos(2x-1)$$

Putting $D^2 = a^2 = -4$ then we have

P.I= $\frac{1}{1-4D}\cos(2x-1) = \frac{1+4D}{1-16D^2}[\cos(2x-1)]$

Again putting $D^2 = a^2 = -4$ then we have

P.I= $\frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$

∴ General solution is

$$y = C.F + P.I$$

$$y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{65}[\cos(2x-1) - 8\sin(2x-1)]$$

Linear equations of second order with variable coefficients

An equation of the form $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$, where $P(x),Q(x),R(x)$ are real valued functions of 'x'is called linear equation of second order with variable coefficients.

Variation of Parameters :

This method is applied when P,Q in above equation are either functions of 'x' or real constants but R is a function of 'x'.

Working Rule :

1. Find C.F. Let C.F= $y_c=c_1u(x)+c_2u(x)$
2. Take $P.I = y_p = Au+Bv$ where $A = -\int\frac{vRdx}{uv'-vu'}$ and $B=\int\frac{uRdx}{uv'-vu'}$
3. Write the G.S. of the given equation $y=y_c+y_p$

1.Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2}+y=\text{cosecx}$

Sol : Given equation in the operator form is $(D^2+1)y=\text{cosecx}$ …(1)

A.E is $(m^2+1)=0$

$\therefore m=\pm i$

The roots are complex conjugate numbers.

C.F is $y_c=c_1\cos x+c_2\sin x$

Let $y_p = A\cos x + B\sin x$ be P.I. of (1)

$$u\frac{dv}{dx}-v\frac{du}{dx}=\cos^2x+\sin^2x=1$$

A and B are given by

$$A=-\int\frac{vRdx}{uv'-vu'}=-\int\frac{\sin x\,\text{cosec}x}{1}dx=-\int dx=-x$$

$$B=\int\frac{uRdx}{uv^1-vu^1}=\int\cos x.\text{cosec}x dx=\int\cot x dx=\log(\sin x)$$

$\therefore y_p=-x\cos x+\sin x.\log(\sin x)$

$\therefore$ General solution is $y = y_c+y_p$.

$$y=c_1\cos x+c_2\sin x-x\cos x+\sin x.\log(\sin x)$$

2.Solve $(D^2 - 2D + 2)y = e^x \tan x$ by method of variation of parameters.

Sol : A.E is $m^2 - 2m + 2 = 0$

$$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

We have $y_c = e^x(c_1\cos x + c_2\sin x) = c_1 e^x\cos x + c_2 e^x \sin x$

$$= c_1(u) + c_2(u)$$

where $u = e^x\cos x, v = e^x \sin x$

$$\frac{du}{dx} = e^x(-\sin x) + e^x\cos x, \frac{dv}{dx} = e^x\cos x + e^x\sin x$$

$$u\frac{dv}{dx} - v\frac{du}{dx} = e^x\cos x(e^x\cos x + e^x\sin x) - e^x\sin x(e^x\cos x - e^x\sin x)$$

$$= e^{2x}(\cos^2 x + \cos x\sin x - \sin x\cos x + \sin^2 x) = e^{2x}$$

Using variation of parameters,

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} = -\int \frac{e^x \tan x}{e^{2x}}(e^x \sin x)dx$$

$$= -\int \tan x\sin x dx = \int\left(\frac{\sin^2 x}{\cos x}dx\right) = \int \frac{(1-\cos^2 x)}{\cos x}dx$$

$$= \int (\sec x - \cos x)dx = \log(\sec x + \tan x) - \sin x$$

$$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}}dx$$

$$= \int \frac{e^x\cos x . e^x \tan x}{e^{2x}}dx = \int \sin x dx = -\cos x$$

General solution is given by $y = y_c + Au + Bv$

i.e $y = c_1e^x\cos x + c_2 e^x \sin x + [\log(\sec x + \tan x) - \sin x]e^x\cos x - e^x\cos x\sin x$

or $y = c_1e^x\cos x + c_2 e^x\sin x + [\log(\sec x + \tan x) - 2\sin x]e^x\cos x$

3.Solve the differential equation $(D^2 + 4)\, y = sec2x$ by the method of variation of parameters.

Sol : Given equation is $(D^2 + 4)\, y = sec2x$ …..(1)

$\therefore$ A.E is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$

The roots are complex conjugate numbers.

$\therefore yc = C.F = c_1 \cos 2x + c_2 \sin 2x$

Let $yp = P.I = A\cos 2x + B\sin 2x$

Here $u = \cos 2x, v = \sin 2x$ and $R = \sec 2x.$

$\therefore \frac{du}{dx}$ = -2 $\sin 2x$ and $\frac{dv}{dx}$ = 2 $\cos 2x$

$\therefore \quad u \frac{dv}{dx} - v\frac{du}{dx} = (\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x) = 2$

A and B are given by :

$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = -\int \frac{\sin 2x \sec 2x}{2} dx = -\frac{1}{2}\int \tan 2x\, dx = \frac{1}{2}\frac{\log|\cos 2x|}{2}$

$\Rightarrow A = \frac{\log|\cos 2x|}{4}$

$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = \int \frac{\cos 2x \sec 2x}{2} dx = \frac{1}{2}\int dx = \frac{x}{2}$

$\therefore y_p = P.I = \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

$\therefore$ The general solution is given by :

$y = y_c + y_p =$ C.F. + P.I

i.e., $y = c_1 \cos 2x + c_2 \sin 2x + \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$

4.Solve $(D^2 + a^2)y = \tan ax$ by the method variation of parameters.

Sol: Given $(D^2 + a^2)y = \tan ax$ i.e $\frac{d^2y}{dx^2} + a^2y = \tan ax$ --------(1)

Now compare equation (1) with $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + Q(x)y = R(x)$ then

$P = 0, Q(x) = a^2$ and $R(x) = \tan ax$

The solution of (1) is $y = C.F + P.I$

<u>Finding C.F :</u>

The A.E of (1) is $m^2 + a^2 = 0 \quad \Longrightarrow m = \pm ai$

$\therefore C.F = c_1 \cos ax + c_2 \sin ax = c_1 u + c_2 v$

Here $u = \cos ax$ and $v = \sin ax$

<u>Finding P.I :</u> $P.I = Au + Bv$

Where $A = \int \frac{vR}{uv' - vu'} dx = -\int \frac{\sin ax \tan ax}{a} = \frac{-1}{a}\int \frac{\sin^2 ax}{\cos ax} dx = \frac{-1}{a}\int \frac{1 - \cos^2 ax}{\cos ax} dx$

$$= \frac{-1}{a}\left[\int \sec ax\, dx - \int \cos ax\, dx\right]$$

$$A = \frac{-1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax$$

$$B = \int \frac{uR}{uv' - vu'}dx = \int \frac{cosaxtanax}{a} = \frac{1}{a}\int sinax\, dx = \frac{-1}{a^2}cosax$$

Therefore $P.I = Au + Bv = \left(\frac{-1}{a^2}log|secax + tanax| + \frac{1}{a^2}sinax\right)cosax + \left(\frac{-1}{a^2}cosax\right)sinax$

Therefore the general solution is $y = C.F + P.I$

i.e $y = c_1cosax + c_2sinax + \left(\frac{-1}{a^2}log|secax + tanax| + \frac{1}{a^2}sinax\right)cosax + \left(\frac{-1}{a^2}cosax\right)sinax$

Equations reducible to linear ODE with constant coefficients :

Cauchy-Euler Equations (Homogenous Linear Differential Equation)

A linear differential equation of the form

$a_0x^n\left({d^ny}/{dx^n}\right) + a_1x^{n-1}\left({d^{n-1}y}/{dx^{n-1}}\right) + \ldots\ldots + a_{n-1}x\left({dy}/{dx}\right) + a_n\,y = X$ -- (1)

i.e,$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}x\,D + a_n)y = X$, $where\ D = {d}/{dx}$ …… (2)

where $a_0, a_1, a_2, \ldots\ldots a_n$ are constants and X is either constant or a function of x only is called a homogenous linear differential equation .These are also known as Cauchy – Euler equations.

Method of solution of homogenous linear differential equation

$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}x\,D + a_n)y = X\ldots..$ (1)

In order to solve (1) introduce a new independent variable z such that

$x = e^x$ or $\log x = z$ so that ${1}/{x} = {dz}/{dx}$ …… (2)

Now , $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$, using (2) ……(3)

or $x\frac{dy}{dx} = \frac{dy}{dz}$ or $x\,D = x\frac{d}{dx} = \frac{d}{dz} = D_1$, say …….(4)

Again$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$

$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$, by (2)

or $x^2D^2 = x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = ({D_1}^2 - D_1)y = D_1(D_1 - 1)y$ … …. (5)

and so on , proceeding likewise , we can show that

$x^nD^n = x^n\frac{d^ny}{dx^n} = D_1(D_1 - 1)(D_1 - 2)\ldots\ldots(D_1 - n + 1)y$ …….(6)

Substituting the above values of $x, xD, x^2D^2, x^3D^3 \ldots\ldots x^nD^n$ in (1) and thus changing the independent variable from $x\ to\ z$, we have

$$[a_0D_1(D_1-1)\ldots.(D_1-n+1)+\cdots+a_{n-2}D_1(D_1-1)+\ a_{n-1}D_1+a_n]y=Z$$

$or\ f(D_1)y = Z$ ……..(7)

Where Z is now a function of Z only .

Working rule for solving linear homogenous differential equation

$(a_0x^nD^n+\ a_1x^{n-1}D^{n-1}+\cdots+\ a_{n-1}x\,D+\ a_n)y=X$ ………(1)

Step I: Put $x=\ e^z\ \ or\ \ z=\log x\,,\ where\ x>0$

Step II: Assume that $D_1=\frac{d}{dz}\ and\ D\equiv\ \frac{d}{dx}$. Then we have

$xD=\ D_1,\qquad x^2D^2=\ D_1(D_1-1),\ \ x^3D^3=\ D_1(D_1-1)(D_1-2)$ and so on.

Then (1) reduces to $f(D_1)y=Z,$ where Z is now function of z only ………(2)

Step III : (2) gives the general solution $y=\ \emptyset(z)$ ………(3)

Since $z=\ \log x$, the desired solution is $y=\ \emptyset(\log x\,)x>0$ ………(4)

Solved Problems:

1. Solve the following differential equations:

   i) $x^2y_2+y=3\,x^2$

   ii) $xy_3+\ y_2=\ {}^1/_x$

   iii) $(x^2D^2-3xD+4)y=2x^2$

   iv) $x^2D^2-2y=\ x^2+\ ({}^1/_x)$

Sol. i) Given $x^2y_2+y=3\,x^2$ or $((x^2D^2+1)y=3x^2$ where $D\equiv{}^d/_{dx}$

Let $x=\ e^z$ and $D_1\equiv{}^d/_{dz}$ so that $x^2D^2=\ D_1(D_1-1)$

$\therefore$ (1) $\Rightarrow$ $[D_1(D_1-1)+1]y=3e^{2z}$ or $({D_1}^2-\ D_1+1)y=3e^{2z}$

Its auxiliary equation is ${D_1}^2-\ D_1+1=0$ so that $D_1=\ (1\pm i\sqrt{3})/2$

C.F = $e^{z/2}\left[c_1\ \cos({}^{z\sqrt{3}}/_2)+\ c_2\sin({}^{z\sqrt{3}}/_2)\right]\ =\ (e^z)^{\frac{1}{2}}\left[c_1\ \cos({}^{z\sqrt{3}}/_2)+\ c_2\sin({}^{z\sqrt{3}}/_2)\right]$

$=x^{\frac{1}{2}}\left[c_1\cos\left\{\left(\frac{\sqrt{3}}{2}\right)\log x\right\}+\ c_2\sin\left\{\left(\frac{\sqrt{3}}{2}\right)\log x\right\}\right]$ as $x=\ e^z$

c_1 and $c_2$being arbitrary constants

P.I $=\frac{1}{{D_1}^2-D_1+1}\,3e^{2z}=3\,\frac{1}{2^2-2+1}e^{2z}=\ (e^z)^2=\ x^2$

Hence the general solution is $y=C.F+P.I$

i.e, $y\ =x^{\frac{1}{2}}\left[c_1\cos\left\{\left(\frac{\sqrt{3}}{2}\right)\log x\right\}+\ c_2\sin\left\{\left(\frac{\sqrt{3}}{2}\right)\log x\right\}\right]+x^2$

Sol.ii) Given$x^3\left(\frac{d^3y}{dx^3}\right)+\ x^2\left(\frac{d^2y}{dx^2}\right)=x\quad or\ \ (x^3D^3+x^2D^2)y=x\,,\quad D\equiv\frac{d}{dx}$ ……(1)

Let $x=\ e^z$ (or $z=\ \log x$) and $D_1\equiv\frac{d}{dz}$

So that $x^2D^2 = D_1(D_1-1)$, $x^3D^3 = D_1(D_1-1)(D_1-2)$. Then (1) transforms to

$[D_1(D_1-1)(D_2-2) + D_1(D_1-1)]y = e^z$ *or* $(D_1{}^3 - 2D_1{}^2 + D_1)y = e^z$

∴ Auxiliary equation is $D_1{}^3 - 2D_1{}^2 + D_1 = 0$ so that $D_1 = 0, 1, 1$

C.F = $c_1e^{0z} + (c_2 + c_3z)e^z = c_1 + (c_2 + c_3\log x)x$, *as* $e^z = x$ *and* $z = \log x$

P.I $= \frac{1}{D_1{}^3 - 2D_1{}^2 + D_1}e^z = \frac{1}{(D_1-1)^2}\frac{1}{D_1}e^z = \frac{1}{(D_1-1)^2}e^z$, as $\frac{1}{D_1}e^z = \int e^z dz = e^z$

$= \frac{z^2}{2!}e^z$

$= (x/2)(\log x)^2$, since $x = e^z$ *and* $z = \log x$

The required solution is $y = c_1 + (c_2 + c_3\log x)x + (x/2)(\log x)^2$,

c_1, c_2 and c_3 being arbitrary constants

Sol.iii) Given $(x^2D^2 - 3xD + 4)y = 2x^2$ ……(1)

Let $x = e^z$ $(or\ z = \log x)\ and\ D_1 \equiv d/dz$. Then (1) becomes

$\{D_1(D_1-1) - 3D_1 + 4\}y = 2e^{2z}$ or $(D_1-2)^2y = 2e^{2z}$

Its auxiliary equation is $(D_1-2)^2 = 0$ so that $D_1 = 2,2$

∴ C.F $= (c_1 + c_2z)e^{2z} = (c_1 + c_2z)(e^z)^2 = (c_1 + c_2\log x)x^2$, since $x = e^z$ and $z = \log x$

P.I $= \frac{1}{(D_1-2)^2}2e^{2z} = 2\frac{z^2}{2!}e^{2z} = z^2(e^z)^2 = (\log x)^2x^2$

Hence the required solution is y = C.F + P.I,

i.e, $y = (c_1 + c_2\log x)x^2 + (\log x)^2x^2$

Sol.iv) Given $(x^2D^2 - 2)y = x^2 + x^{-1}$ where $D \equiv d/dx$ ……….(1)

Let $x = e^z$ $(or\ z = \log x)\ and\ D_1 \equiv d/dz$. Then (1) becomes

$\{D_1(D_1-1) - 2\}y = e^{2z} + e^{-z}$ or $(D_1{}^2 - D_1 - 2)y = e^{2z} + e^{-z}$

Its auxiliary equation is $D_1{}^2 - D_1 - 2 = 0$ so that $D_1 = 2, -1$

∴ $C.F = c_1e^{2z} + c_2e^{-z} = c_1(e^z)^2 + c_2(e^z)^{-1} = c_1x^2 + c_2x^{-1}$,as $x = e^z$

P.I $= \frac{1}{D_1{}^2 - D_1 - 2}(e^{2z} + e^{-z}) = \frac{1}{(D_1-2)(D_1+1)}e^{2z} + \frac{1}{(D_1-2)(D_1+1)}e^{-z}$

$= \frac{1}{D_1-2}\frac{1}{2+1}e^{2z} + \frac{1}{D_1+1}\frac{1}{-1-2}e^{-z} = \frac{1}{3}\frac{z}{1!}e^{2z} - \frac{1}{3}\frac{z}{1!}e^{-z} = \frac{1}{3}\log x\,(x^2 + x^{-1})$, *as* $x = e^z$

∴ Solution is $y = c_1x^2 + c_2x^{-1} + \frac{1}{3}\log x\,(x^2 + x^{-1})$, where

c_1, c_2 are aribitrary constants

2. Solve $x^3\left(\frac{d^3y}{dx^3}\right) + 3x^2\left(\frac{d^2y}{dx^2}\right) + x\frac{dy}{dx} + y = \log x + x$

Sol. Given $(x^3D^3 + 3x^2D^2 + xD + 1)y = \log x + x$ where $D \equiv d/dx$ ……..(1)

Let $x = e^z$ $(or\ z = \log x)$ $and\ D_1 \equiv {}^{d}/_{dz}$. Then (1) becomes

$[D_1(D_1-1)(D_2-2)+3D_1(D_1-1)+D_1+1]y = z+e^z$ or $({D_1}^3+1)y = z+e^z$

Its auxiliary equation is $({D_1}^3+1) = 0$ so that $D_1 = -1, \left(\frac{1\pm i\sqrt{3}}{2}\right)$

C.F = $c_1e^{-z} + e^{\frac{z}{2}\left(c_2\cos\left(\frac{\sqrt{3}}{2}z\right)+c_3\sin\left(\frac{\sqrt{3}}{2}z\right)\right)}$

$= c_1x^{-1} + x^{\frac{1}{2}\left[c_2\cos\left(\frac{\sqrt{3}}{2}\right)\log x + c_3\sin\left(\frac{\sqrt{3}}{2}\right)\log x\right]}$

P.I = $\frac{1}{({D_1}^3+1)}(z+e^z) = \frac{1}{{D_1}^3+1}e^z + \frac{1}{{D_1}^3+1}z = \frac{1}{1^3+1}e^z + \left(1+{D_1}^3\right)^{-1}z$

$= \frac{1}{2}e^z + \left(1-{D_1}^3+\cdots..\right)z = \frac{e^z}{2}+z = \frac{x}{2}+\log x$

∴ The required solution is y= C.F+ P.I

$y = c_1x^{-1} + x^{\frac{1}{2}\left[c_2\cos\left(\frac{\sqrt{3}}{2}\right)\log x + c_3\sin\left(\frac{\sqrt{3}}{2}\right)\log x\right]} + \frac{x}{2} + \log x$

3. Solve $(x^2D^2 - xD + 2)y = x\log x$

Sol. Given $(x^2D^2 - xD + 2)y = x\log x$ where $D \equiv {}^{d}/_{dx}$ ………(1)

Let $x = e^z$ $(or\ z = \log x)$ $and\ D_1 \equiv {}^{d}/_{dz}$. Then (1) becomes

$[D_1(D_1-1) - D_1 + 2]y = ze^z$ or $\left({D_1}^2 - 2D_1 + 2\right)y = ze^z$

auxiliary equation is ${D_1}^2 - 2D_1 + 2 = 0$ giving $D_1 = 1 \pm i$

∴ C.F $= e^z(c_1\cos z + c_2\sin z) = x(c_1\cos(\log x) + c_2\sin(\log x))$, where c_1, c_2 are arbitrary constants

P.I = $\frac{1}{{D_1}^2-2D_1+2}ze^z = e^z\frac{1}{(D_1+1)^2-2(D_1+1)+2}z = e^z\frac{1}{{D_1}^2+1}.z$

$= e^z\left(1+{D_1}^2\right)^{-1}.z = e^z\left(1-{D_1}^2+\cdots\right)z = e^z.z = x\log x$, using (1)

∴ The required solution is y= C.F+ P.I

$y = x(c_1\cos(\log x) + c_2\sin(\log x)) + x\log x$

4. Solve $x^3\left(\frac{d^3y}{dx^3}\right) + 2x^2\left(\frac{d^2y}{dx^2}\right) + 2y = 10\left(x + {}^{1}/_{x}\right)$

Sol. Given $(x^3D^3 + 2x^2D^2 + 2)y = 10(x + x^{-1})$where $D \equiv {}^{d}/_{dx}$ ……..(1)

Let $x = e^z$ $(or\ z = \log x)$ $and\ D_1 \equiv {}^{d}/_{dz}$. Then (1) becomes

$[D_1(D_1-1)(D_2-2) + 2D_1(D_1-1) + 2]y = 10(e^z + e^{-z})$

$({D_1}^3 - {D_1}^2 + 2)y = 10e^z + 10e^{-z}$ …………(2)

A. E of (2) is ${D_1}^3 - {D_1}^2 + 2 = 0$ giving $D_1 = -1, 1 \pm i$

∴ C.F = $c_1e^{-z} + e^z[c_2\cos(z) + c_3\sin(z)] = c_1x^{-1} + x(c_2\cos\log x + c_3\sin\log x)$

P.I coressponding to 10 $e^z = 10 \frac{1}{(D_1+1)(D_1{}^2-2D_1+2)} e^z = 10 \frac{1}{2(1-2+2)} e^z = 5x$

P.I coressponding to 10 $e^{-z} = 10 \frac{1}{(D_1+1)(D_1{}^2-2D_1+2)} e^{-z} = 10 \frac{1}{(D_1+1)} \cdot \frac{1}{1+2+2} e^{-z}$

$= 2 \frac{1}{(D_1+1)} e^{-z} = 2e^{-z} \frac{1}{D_1-1+1} .1 = 2e^{-z} \frac{1}{D_1} .1 = 2e^{-z} z$

$= 2\, x^{-1} \log x$

∴ The required solution is y = C.F+ P.I

$y = c_1 x^{-1} + x(c_2 \cos \log x + c_3 \sin \log x) + 5x + x^{-1} \log x$

Equations reducible to homogeneous linear form- Legendre's l inear equation

A linear differential equation of the form

$$[a_0(a+bx)^n D^n + a_1(a+bx)^{n-1} D^{n-1} + \cdots. + a_{n-1}(a+bx)D + a_n]y = X, \ldots\ldots(1)$$

Where a,b,$a_0, a_1, a_2, \ldots\ldots a_n$ are constants and X is either a constant or a function of x only, is called Legendre's l inear equation. Note that the index of $(a+bx)$ and the order of derivative is same in each term of such equations.

Method of solution: To solve (1), introduce a new variable z such that

$$a + bx = e^z \quad or \quad log(a+bx) = z \ \ldots\ldots(2)$$

$$\text{Let } D_1 = \frac{d}{dz} \text{ and } D = \frac{d}{dx} \ldots\ldots\ldots(3)$$

$$\text{From (2), we have } \frac{dz}{dx} = \frac{b}{a+bx} \ldots\ldots(4)$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}, \text{ using (4)} \quad \ldots\ldots\ldots(5)$$

$$\Rightarrow (a+bx)\frac{dy}{dx} = b\frac{dy}{dz} \Rightarrow (a+bx)D = bD_1 \ldots\ldots\ldots(6)$$

$$\text{Again, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right), \text{ using (5)}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\frac{b}{a+bx}, \text{ using(4)}$$

$$\Rightarrow (a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) \Rightarrow (a+bx)^2 D^2 y = b^2(D_1^2 - D_1)y$$

$$\therefore (a+bx)^2 D^2 = b^2 D_1(D_1 - 1) \ \ldots\ldots\ldots(7)$$

Similarly, $(a+bx)^3 D^3 = b^3 D_1(D_1-1)(D_1-2) \ldots\ldots\ldots(8)$ and so on.

Proceeding likewise, we finally have

$$(a+bx)^n D^n = b^n D_1(D_1-1)(D_1-2)\ldots\ldots(D_1-n+1)$$

Substituting the above values of $(a+bx)^nD^n, \ldots \ldots .(a+bx)^2D^2, (a+bx)D$ etc in (1), we have

$[a_0b^nD_1(D_1-1)(D_1-2)\ldots\ldots.(D_1-n+1)+\cdots.+a_{n-1}bD+a_n]y=Z,\ldots\ldots\ldots(9)$

Which is a linear differential equation with constant coefficients in variables y and z; Z is now function of z only and is obtained by using transformation (2) by replacing x by $\frac{e^z-a}{b}$.

Let a solution of (1) $be\ y=F(z)$. Then, the required solution is given by

$y=F\big(log(a+bx)\big)$, as $log(a+bx)=z$

Working rule for solving Legendre's l inear equation, i.e.,

$[a_0(a+bx)^nD^n+a_1(a+bx)^{n-1}D^{n-1}+\cdots.+a_{n-1}(a+bx)D+a_n]y=X,\ldots\ldots.(1)$

Where a,b,$a_0,a_1,a_2,\ldots\ldots..a_n$ are constants and X is either a constant or a function of x only and D= $\frac{d}{dx}$

Step I : Introduce a new variable z such that

$$a+bx=e^z \quad or \quad log(a+bx)=z\ \ldots\ldots.(2)$$

Step II: Assume that $D_1=\frac{d}{dz}$ .Then, we have

$(a+bx)D=bD_1,\quad (a+bx)^2D^2=b^2D_1(D_1-1),(a+bx)^3D^3=b^3D_1(D_1-1)(D_1-2)$ and so on.

As a particular case , when $b=1$, we have

$(a+x)D=D_1$, $(a+x)^2D^2=D_1(D_1-1)$, $(a+x)^3D^3=D_1(D_1-1)(D_1-2)$ and so on.

Then (1) reduces to $f(D_1)y=Z$, where Z is now function of z only. $\ldots\ldots\ldots\ldots.(3)$

Step III : We now use the methods of Chapter 5 to slove (3) and get a solution of form

$$y=F(Z)\ldots\ldots\ldots(4)$$

Using (2), the required solution is given by $y=F\big(log(a+bx)\big)\ldots\ldots\ldots.(5)$

Solved Problems:

1. Solve $(1+x^2)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4coslog(1+x)$

Sol. Given $(1+x^2)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4coslog(1+x)$

$$[(1+x^2)^2D^2+(1+x)D+1]y=4coslog(1+x),\quad D=\frac{d}{dx}\quad \ldots\ldots(1)$$

Let $1+x=e^z$ or $\log(1+x)=z$. Also, let $D_1=\frac{d}{dz}\ldots\ldots\ldots.(2)$

Then, we have $(1+x)D=D_1,(1+x^2)^2D^2=D_1(D_1-1)$ and hence (1) gives

$[D_1(D_1-1)+D_1+1]=4cosz$ or $({D_1}^2+1)y=4cosz\ \ldots\ldots\ldots(3)$

Its auxiliary equation is ${D_1}^2+1=0$ so that $D_1=0\pm i$

$\therefore$ C.F. $=e^{0z}(c_1cosz+c_2sinz)=c_1coslog(1+x)+c_2sinlog(1+x)$, using (2)

Where c_1 and c_2 are arbitrary constants.

P.I. $=\frac{1}{D_1^2+1}4cosz=$ R.P. of $\frac{1}{D_1^2+1}4e^{iz}$, where R.P. stands for real part

$=$ R.P. of $\frac{1}{D_1^2+1}e^{iz}.4=$ R.P. of $e^{iz}\frac{1}{(D_1+i)^2+1}.4$

$=$ R.P. of $e^{iz}\frac{1}{D_1^2+2Di}.4=$ R.P. of $e^{iz}\frac{1}{2D_1i\left(1+\frac{D_1}{2i}\right)}.4$

$=$R.P. of $\frac{e^{iz}}{2i}\frac{1}{D_1}\left(1+\frac{D_1}{2}\right)^{-1}4=$ R.P. of $\frac{e^{iz}}{2i}\frac{1}{D_1}\left(1-\frac{D_1}{2i}+\cdots\ldots\right)4$n

$=$ R.P. of $e^{iz}\left(\frac{1}{2i}\right)(4z)=R.P.of\ (-2iz)(cosz+isinz)$, as $\frac{1}{i}=-i$

$=2z\sin z=2\log(1+x)\sin\log(1+x)$ as $\log(1+x)=z$.

$\therefore$Solution is $y=c_1\cos\log(1+x)+c_2\sin\log(1+x)+2\log(1+x)\sin\log(1+x)$

2. Solve $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4)$

Sol. Let $D=\frac{d}{dx}$

Given equation reduces to $\{(x+1)^2D^2+(x+1)D\}y=4x^2+14x+12\ldots\ldots(1)$

Let $x+1=e^z$ or $z=log(x+1)$ Also let $D_1=\frac{d}{dz}\ldots\ldots\ldots(2)$

Then $(x+1)D=D_1$ and $(x+1)^2D^2=D_1(D_1-1)$

Hence, (1) gives

$$\{D_1(D_1-1)+D_1\}y=4(e^z-1)^2+14(e^z-1)+12$$

$${D_1}^2y=4e^{2z}+6e^z+2\ \ldots\ldots(3)$$

Auxiliary equation of (3) *is* ${D_1}^2=0$. giving $D_1=0,0$.

$\therefore$ C.F. $=(c_1+c_2z)e^{0z}=c_1+c_2z=c_1+c_2\,log(x+1)$, using $z=log(x+1)$

P.I. $=\frac{1}{D_1^2}(4e^{2z}+6e^z+2)=\frac{1}{D_1}(2e^{2z}+6e^z+2z)=e^{2z}+6e^z+z^2=(e^z)^2+6e^z+z^2$

$$=(x+1)^2+6(x+1)+[log(x+1)]^2$$

Thus, P.I. $=x^2+8x+7+[log(x+1)]^2$

$\therefore$ Solution is $y=c_1+c_2\,log(x+1)+x^2+8x+7+[log(x+1)]^2$,

where c_1 and c_2 are arbitrary constants

3. Solve $(x+1)^2\frac{d^2y}{dx^2}-4(x+1)\frac{dy}{dx}+6y=6(x+1)$

Sol. Let $D=\frac{d}{dx}$

Given equation reduces to $\{(x+1)^2D^2 - 4(x+1)D + 6\}y = 6(x+1) \ldots .(1)$

Let $x+1 = e^z$ or $z = log(x+1)$ Also let $D_1 = \frac{d}{dz} \ldots\ldots\ldots.(2)$

Then $(x+1)D = D_1$ and $(x+1)^2D^2 = D_1(D_1-1)$. So (1) gives

$$\{D_1(D_1-1) - 4D_1 + 6\}y = 6e^z$$

$$(D_1^2 - 5D_1 + 6)y = 6e^z \ldots\ldots.(3)$$

Auxiliary equation of (3) is $D_1^2 - 5D_1 + 6 = 0$ giving $D_1 = 2,3$.

$\therefore$ C.F. $= c_1e^{2z} + c_2e^{3z} = c_1(e^z)^2 + c_2(e^z)^3 = c_1(x+1)^2 + c_2(x+1)^3$

P.I. $= \frac{1}{D_1^2-5D_1+6}6e^z = 6\frac{1}{1^2-(5x1)+6}e^z = 3e^z = 3(x+1),\ as\ x+1 = e^z$

$\therefore$ Solution is $y = c_1(x+1)^2 + c_2(x+1)^3 + 3(x+1)$

Where c_1 and c_2 are arbitrary constants

4. Solve $[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1,\ D = \frac{d}{dx}$.

Sol. Given $[(3x+2)^2D^2 + 3(3x+2)D - 36]y = 3x^2 + 4x + 1 \ \ldots\ldots(1)$

Let $3x+2 = e^z$ or $log(3x+2) = z$. Also let $D_1 = \frac{d}{dz} \ldots\ldots\ldots.(2)$

$(3x+2)D = 3D_1, (3x+2)^2D^2 = 3^2D_1(D_1-1)$. Then (1)gives

$$[3^2D_1(D_1-1) + 3.3D_1 - 36]y = 3\left\{\left(\frac{e^z-2}{3}\right)^2\right\} + 4\left(\frac{e^z-2}{3}\right) + 1$$

$$since\ 3x+2 = e^z \Rightarrow 3x = e^z - 2 \Rightarrow x = \frac{e^z-2}{3}$$

$$9[D_1(D_1-1) + D_1 - 4] = \tfrac{1}{3}(e^{2z} - 4e^z + 4) + \tfrac{4}{3}(e^z-2) + 1$$

$$9(D_1^2 - 4) = \tfrac{1}{3}e^{2z} - \tfrac{1}{3} \Rightarrow D_1^2 - 4 = \tfrac{1}{27}e^{2z} - \tfrac{1}{27}$$

Here auxiliary equation is $D_1^2 - 4 = 0$ so that $D_1 = 2,-2$.

$\therefore$ C. F. $= c_1e^{2z} + c_2e^{-2z} = c_1(e^z)^2 + c_2(e^z)^{-2} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$

P.I. corresponding to $\frac{1}{27}e^{2z} = \frac{1}{27}\frac{1}{D_1^2-4}e^{2z} = \frac{1}{27}\frac{1}{D_1-2}\frac{1}{D_1+2}e^{2z} = \frac{1}{27}\frac{1}{D_1-2}\frac{1}{2+2}e^{2z}$

$= \frac{1}{108}\frac{1}{D_1-2}e^{2z} = \frac{1}{108}\frac{z}{1!}e^{2z}$, as $\frac{1}{(D_1-a)^n}e^{az} = \frac{z^n}{n!}e^{az}$

$= \frac{1}{108}z(e^z)^2 = \frac{1}{108}(3x+2)^2log(3x+2)$, using (2)

P.I. corresponding to $-\frac{1}{27} = -\frac{1}{27}\frac{1}{D_1^2-4}.1 = -\frac{1}{27}\frac{1}{D_1^2-4}e^{0z} = -\frac{1}{27}\frac{1}{0^2-4}e^{0z} = \frac{1}{108}$

$\therefore$ Solution is $y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2log(3x+2) + 1]$

Where c_1 and c_2 are arbitrary constants