M-III Unit - V

Higher order partial Differential Equations.

partial Differential eqn :- A D.E which involves partial derivatives is called a partial differential eqn

Ex :- (i) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$ (ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ are P.D.E

order of P.D.E The order of the P.D.E is the highest ordered partial derivative in the equation.

Degree of P.D.E The Degree of the P.D.E is the highest power of the highest ordered derivative.

Ex :- $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$ order = 2, Degree = 1

If degree of P.D.E is one then it is called Linear P.D.E.

Note If z is a function of two independent variables x and y, then we shall use the following notation for the partial derivatives of z.

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

In this chapter we shall consider only Linear partial differential equation of higher order with Constant Coefficients. It can be divided into two parts

1. **Homogeneous Linear P.D.E** (each term order is same, degree is one)

Ex :- $\frac{\partial^3 u}{\partial x^3} + 2 \frac{\partial^3 u}{\partial x^2 \partial y} + 4 \frac{\partial^3 u}{\partial y^3} = \log(x+2y)$

2. **Non Homogeneous Linear P.D.E** (Having different order terms, degree is one)

Ex :- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 3 \frac{\partial u}{\partial x} - 4 \frac{\partial u}{\partial y} = \sin(x+y)$

Homogeneous Linear Partial differential eqn with Constant Coefficients :- An eqn of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = Q(x, y) \quad \text{--- (1)}$$

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where $a_0, a_1, a_2, \ldots, a_n$ are constants, is called a homogeneous linear partial differential equation with constant coefficients of n^{th} order.

Here all the partial derivatives are of the n^{th} order.

Here we take $\boxed{\frac{\partial}{\partial x} = D, \quad \frac{\partial}{\partial y} = D'}$

$\therefore$ The symbolic form of eqn (1) is

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n) z = Q(x,y)$$

$$\text{ie,} \quad f(D, D') z = Q(x,y)$$

Its soln consists of two parts, the complementary soln (C.F) and the particular integral (P.I)

The complementary function is the general soln of $f(D, D') = 0$. it must ~~Const~~ contain 'n' arbitrary constants where n is the order of the P.D.E.

The particular integral is a particular soln (free from arbitrary constants) of $f(D, D') = Q(x,y)$.

$\therefore$ The complete soln of eqn (1) is $\boxed{z = C.F + P.I}$

Rules to find complementary function.

putting $D = m$ and $D' = 1$ in $f(D, D') = 0$, we get the auxillary eqn (A.E) $f(m, 1) = 0$, solve it for m

(i) If the roots of A.E are $m_1, m_2, \ldots, m_n$ all are distinct (real or imaginary) then

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \cdots + f_n(y + m_n x)$$

(ii) If two roots are equal and remaining roots are different ie $m_1 = m_2 = m$, $m_3, m_4, \ldots, m_n$ then

$$C.F = f_1(y + mx) + x f_2(y + mx) + f_3(y + m_3 x) + f_4(y + m_4 x) + \cdots + f_n(y + m_n x)$$

(3) (iii) If three roots are equal and remaining roots are different i.e $m_1 = m_2 = m_3 = m$, $m_4, m_5, \ldots, m_n$

Then $C.F = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx) + f_4(y+m_4x)$
$$+ f_5(y+m_5x) + \cdots + f_n(y+m_nx)$$

where $f_1, f_2, f_3, \ldots, f_n$ are arbitrary functions.

Problems.

(1) Solve $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Sol:- Since in the given P.D.E each term is same order so this is Homogeneous P.D.E.

Symbolic form of the given eqn is

$$(2D^2 + 5DD' + 2D'^2)z = 0 \qquad (\text{put } \tfrac{\partial}{\partial x} = D, \tfrac{\partial}{\partial y} = D')$$

Clearly this is of the form $f(D, D')z = 0$.

where $f(D, D') = 2D^2 + 5DD' + 2D'^2$

To get the auxillary eqn (A.E) put $D = m$, $D' = 1$

we have $f(m, 1) = 0$.

$$2m^2 + 5m + 2 = 0.$$

$$\Rightarrow (2m+1)(m+2) = 0$$

$\Rightarrow m_1 = -\frac{1}{2}, m_2 = -2$ are different roots

$\therefore$ General soln $z = C.F$

$$= f_1(y+m_1x) + f_2(y+m_2x)$$

$$z = f_1(y - \tfrac{1}{2}x) + f_2(y-2x).$$

(2) Solve $4r + 12s + 9t = 0.$

W.K.T $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

$\therefore$ Given P.D.E is $4\frac{\partial^2 z}{\partial x^2} + 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0.$

symbolic form is $(4D^2 + 12DD' + 9D'^2)z = 0.$

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(4)

Here $f(D,D') = 4D^2 + 12DD' + 9D'^2$

A.E is $f(m,1) = 0$

$\Rightarrow 4m^2 + 12m + 9 = 0$

$\Rightarrow m = \frac{-12 \pm \sqrt{12^2 - 4\times4\times9}}{2\times4}$ $\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right.$

$\left. a=4, b=12, c=9\right)$

$\Rightarrow m = \frac{-12 \pm \sqrt{144-144}}{8}$

$\Rightarrow m = \frac{-3}{2}$

$\therefore m = -\frac{3}{2}, -\frac{3}{2}$ are equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$= f_1(y - \frac{3}{2}x) + x f_2(y - \frac{3}{2}x)$ or

$C.F = f_1(2y-3x) + x f_2(2y-3x)$

Since R.H.S of the given eq'n is zero, The complete soln is $z = C.F$

$\Rightarrow z = f_1(2y-3x) + x f_2(2y-3x)$

(3) Solve $(D^3 - 4D^2D' + 4DD'^2) z = 0.$

Sol:- Here $f(D,D') = D^3 - 4D^2D' + 4DD'^2$

A.E is $f(m,1) = 0$ (ie put $D = m$, $D' = 1$)

$m^3 - 4m^2 + 4m = 0$

$\Rightarrow m(m^2 - 4m + 4) = 0$

$\Rightarrow m(m-2)^2 = 0$

$\Rightarrow m = 0, 2, 2$ ie $m_1 = 0$, $m_2 = m_3 = 2$

Here one root is different and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x) + x f_3(y+m_2x)$

$\Rightarrow C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x)$

General soln is $z = C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x).$

(5)

(4) Solve $\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = 0.$

Sol:- Symbolic form of the given Homo P.D.E is

$$(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = 0.$$

Here $f(D,D') = D^3 + 2D^2D' - DD'^2 - 2D'^3$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 + 2m^2 - m - 2 = 0$

$m_1 = 1,\ m_2 = -1,\ m_3 = -2$

are different roots

1	1	2	-1	-2
	0	1	3	2
-1	1	3	2	0
	0	-1	-2	
-2	1	2	0	
	0	-2		
	1	0		

$\therefore$ G.S is $z = C.F$

$$z = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$$

$$\Rightarrow z = f_1(y+x) + f_2(y-x) + f_3(y-2x).$$

(5) Solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$

Sol:- Symbolic form is $(D^4 - D'^4)z = 0$

Here $f(D,D') = D^4 - D'^4$

A.E is $f(m,1) = 0$

$\Rightarrow m^4 - 1 = 0$

$\Rightarrow (m^2+1)(m^2-1) = 0$

$\Rightarrow m^2+1 = 0,\ m^2-1 = 0$

$\Rightarrow m^2 = -1,\ m^2 = 1$

$\Rightarrow m = \pm i,\ m^2 = \pm 1$

$\therefore$ roots are $m_1 = -i,\ m_2 = i,\ m_3 = -1,\ m_4 = 1$ are different roots

$z = C.F$

General soln is $z = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + f_4(y+m_4x)$

$$\Rightarrow z = f_1(y-ix) + f_2(y+ix) + f_3(y-x) + f_4(y+x)$$

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(6) $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 3\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$

Sol:- symbolic form of the given Homo P.D.E is

$(D^3 - 4D^2D' + 3DD'^2)z = 0.$

A.E is $f(m, 1) = 0.$

$\Rightarrow m^3 - 4m^2 + 3m = 0.$

$\Rightarrow m(m^2 - 4m + 3) = 0$

$\Rightarrow m(m-1)(m-3) = 0$

$\Rightarrow m_1 = 0,\ m_2 = 1,\ m_3 = 3$ are different roots

$\therefore$ G.S is $z = C.F = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x)$

$\Rightarrow z = f_1(y) + f_2(y + x) + f_3(y + 3x).$

H.W

(1) $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x \partial y^2} + 6\dfrac{\partial^3 z}{\partial y^3} = 0$, roots are $m = 1, 2, -3$

(2) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$, roots are $m = 1, 2, 3.$

(3) $(D + 2D')(D - 3D')^2 z = 0$ roots are $m = -2, 3, 3.$

(7) $\dfrac{\partial^4 z}{\partial x^4} - 2\dfrac{\partial^4 z}{\partial x^3 \partial y} + 2\dfrac{\partial^4 z}{\partial x \partial y^3} - \dfrac{\partial^4 z}{\partial y^4} = 0$

Sol:- symbolic form of given Homo P.D.E is

$(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0.$ (put $\frac{\partial}{\partial x} = D,\ \frac{\partial}{\partial y} = D'$)

Here $f(D, D') = D^4 - 2D^3D' + 2DD'^3 - D'^4$

A.E is $f(m, 1) = 0$ (Put $D = m,\ D' = 1$)

$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$

1	1	−2	0	2	−1
	0	1	−1	−1	1
1	1	−1	−1	1	0
	0	1	0	−1	
1	1	0	−1	0	
	0	1	1		
−1	1	1	0		
	0	−1			
	1	0			

1	1	−2	2	−1
	0	1	−1	1
	1	−1	1	0

G.S is $z = C.F$

$\Rightarrow z = f_1(y + m_1x) + f_2(y + m_2x) + x f_3(y + m_2x) + x^2 f_4(y + m_2x)$

$\Rightarrow z = f_1(y - x) + f_2(y + x) + x f_3(y + x) + x^2 f_4(y + x)$

roots are $m_1 = -1,\ m_2 = m_3 = m_4 = 1$

ie one root is different, remaining 3 roots are equal

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(7)

Rules for finding P.I

particular integral of the equ'n $f(D,D')z = Q(x,y)$ is

$$z_p = \frac{1}{f(D,D')} Q(x,y)$$

where $Q(x,y)$ is a function of $ax+by$

i.e, e^{ax+by}, $\sin(ax+by)$, $\cos(ax+by)$, $\tan(ax+by)$,

$(ax+by)^n$, $\log(ax+by)$ etc then

Case (i)

$$P.I = \frac{1}{f(D,D')} Q(ax+by)$$

$$= \frac{1}{f(a,b)} \underbrace{\iiint\cdots\int}_{(n\ times)} Q(u)\, \underbrace{du\,du\cdots du}_{(n\ times)}$$

provided $f(a,b) \neq 0$

ie ① replace D by a, D' by b where a,b are cofficients of x and y respectively in $ax+by$.

This method is applicable only when $f(a,b) \neq 0$.

If $f(a,b)=0$. Then this method fails

② If $f(a,b) \neq 0$, Put $ax+by=u$; in $Q(ax+by)$ to get $Q(u)$, integrate $Q(u)$ ntimes ie 'n' times

ie as many times as the degree of the P.D.E (Order of P.D.E)

③ Now replace u by $ax+by$ to get the required P.I

Case (ii) Case of failure ie $f(a,b)=0$

Then $$P.I = x.\frac{1}{\frac{\partial}{\partial D}[f(D,D')]} Q(ax+by)$$

ie multiply by x and differentiate $f(D,D')$ partially w.r.to D

If the method fails again, multiply again by x and differentiate the denominator partially w.r.to D.

(8) Problems

① solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$

clearly given P.D.E is Homogeneous since each term order is same.

Symbolic form of the given P.D.E is

$$(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$$

Clearly This is of the form $f(D,D')z = \phi(x,y)$

$f(D,D') = D^3 - 3D^2D' + 4D'^3$, $\phi(x,y) = e^{x+2y}$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 3m^2 + 4 = 0$

-1	1	-3	0	4
	0	-1	$+4$	-4
	1	-4	$+4$	0
2	0	2	-4	
	1	-2	0	
2	0	2		
	1	0		

$m_1 = -1$, $m_2 = m_3 = 2 = m$

Here one root is different root and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$\Rightarrow C.F = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$

$$P.I = \frac{1}{f(D,D')}\phi(x,y)$$

$$= \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

Here $\phi(x,y) = e^{x+2y}$ is of the form e^{ax+by}

so ~~first~~ put $D = a$, $D' = b$ in $f(D,D')$ we have

ie put $D = 1$, $D' = 2$ in

$f(D,D') = D^3 - 3D^2D' + 4D'^3$

$\Rightarrow f(a,b) = f(1,2) = 1 - 3\times 2 + 4\times 8 = 27 \neq 0$.

$\therefore P.I = \frac{1}{f(a,b)}\iiint \phi(u)\,du\,du\,du$ where $u = x+2y$.

$= \frac{1}{27}\iiint e^u\,du\,du\,du$

order of P.D.E is 3 so. $n = 3$ Integrate 3 times

$$P.I = \frac{1}{27} e^{u} = \frac{1}{27} e^{x+2y}$$

$\therefore$ Complete soln $z = C.F + P.I$

$$\Rightarrow z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

(2) solve $(D^3 - 4D^2D' + 4DD'^2)z = 6\sin(3x+2y)$

Sol:- Here $f(D,D') = D^3 - 4D^2D' + 4DD'^2$, $Q(x,y) = 6\sin(3x+2y)$

A.E is $f(m,1) = 0$

$$\Rightarrow m^3 - 4m^2 + 4m = 0$$

$$\Rightarrow m(m^2 - 4m + 4) = 0$$

$$\Rightarrow m(m-2)^2 = 0$$

$$\Rightarrow m = 0, 2, 2$$

i.e $m_1 = 0$, $m_2 = m_3 = 2 = m$

Here one root is different root and remaining two roots are equal roots

$$C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$$

$$C.F = f_1(y) + f_2(y+2x) + x f_3(y+2x)$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^3 - 4D^2D' + 4DD'^2} \cdot 6\sin(3x+2y) \quad \text{(Compare } \sin(ax+by))$$

Here $a=3$, $b=2$, $n = \text{order} = 3$

put $D = a$, $D' = b$

$$f(D,D') = f(a,b) = 3^3 - 4\times 3^2 \times 2 + 4\times 3\times 2^2 = 3 \neq 0$$

$$P.I = 6 \times \frac{1}{f(a,b)} \iiint \sin u \, du\, du\, du, \quad u = ax+by$$

$$= \frac{6}{3} \iiint \sin u \, du\, du\, du$$

$$= 2 \iint (-\cos u)\, du\, du$$

$$= 2 \int (-\sin u)\, du$$

$$P.I = 2\cos u = 2\cos(3x+2y)$$

$\therefore$ Complete soln $z = C.F + P.I$

$$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + 2\cos(3x+2y)$$

3. Solve $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$

Sol:- Symbolic form of the given Homo P.D.E is

$$(D^2 - 2DD' + D'^2)z = \sin(x+0y)$$

Here $f(D,D') = D^2 - 2DD' + D'^2$, $Q(x,y) = \sin(x+0y)$

A.E is $f(m,1)=0 \Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2 = 0$

$\Rightarrow m = 1, 1$ are two equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+x) + x f_2(y+x)$.

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} \sin(x+0y) \quad \left(\text{It is of the form } \sin(ax+by)\right)$$

put $D = a = 1$, $D' = b = 0$ in $f(D,D')$ we have, n = order $= 2$

$f(D,D') = f(a,b) = f(1,0) = 1 - 0 + 0 = 1 \neq 0$

$$P.I = \frac{1}{f(a,b)} \iint \sin u \, du\, du, \quad u = x+0y$$

$$P.I = \frac{1}{1} \iint \sin u \, du\, du$$

$P.I = -\sin u = -\sin x$.

G.S is $z = C.F + P.I = f_1(y+x) + x f_2(y+x) - \sin x$.

4. Solve $r + s - 2t = \sqrt{2x+y}$

Sol:- Given Homo P.D.E is $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = \sqrt{2x+y}$

Symbolic form of given P.D.E is $(D^2 + DD' - 2D'^2)z = \sqrt{2x+y}$

Here $f(D,D') = D^2 + DD' - 2D'^2$

$Q(x,y) = \sqrt{2x+y}$

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(11) A.E is $f(m,1)=0$

$\Rightarrow m^2+m-2=0$

$\Rightarrow (m+2)(m-1)=0$

$\Rightarrow m_1=-2, m_2=1$ are different roots

$C.F = f_1(y+m_1x)+f_2(y+m_2x)$

$\Rightarrow C.F = f_1(y-2x)+f_2(y+x).$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{D^2+DD'-2D'^2} \sqrt{2x+y}$

Here $a=2, b=1, n=\frac{1}{2}$, put $D=a, D'=b$ in $f(D,D')$

Now $f(D,D')=f(a,b)=f(2,1)=4+2-2=4\neq 0.$

$P.I = \frac{1}{f(a,b)} \iint \sqrt{u}\, du\, du, \quad u=2x+y$

$= \frac{1}{4}\iint u^{1/2}\, du\, du$

$= \frac{1}{4}\int \frac{u^{3/2}}{3/2}\, du$

$= \frac{1}{4}\times\frac{2}{3}\times\frac{u^{5/2}}{5/2}$

$P.I = \frac{1}{4}\times\frac{2}{3}\times\frac{2}{5}\, u^{5/2}$ (replace u)

$\Rightarrow P.I = \frac{1}{15}(2x+y)^{5/2}$

$\therefore$ Complete sol'n $z = C.F + P.I$

$z = f_1(y-2x)+f_2(y+x)+\frac{1}{15}(2x+y)^{5/2}$

(5) Solve $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}=\sin x\cos 2y$

Sol: Symbolic form of given Homo P.D.E is

$(D^2-DD')z = \sin x\cos 2y$

$f(D,D')=D^2-DD' \qquad Q(x,y)=\sin x\cos 2y$

$=\frac{1}{2}(2\sin x\cos 2y)$

$=\frac{1}{2}\left[\sin(x+2y)+\sin(x-2y)\right]$

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(12)

A.E is $f(m,1)=0$

$\Rightarrow m^2-m=0$

$\Rightarrow m(m-1)=0$

$\Rightarrow m_1=0,\ m_2=1$ are different roots

$C.F = f_1(y+m_1x) + f_2(y+m_2x)$

$C.F = f_1(y)+f_2(y+x)$.

$$P.I = \frac{1}{f(D,D')}\phi(x,y)$$

$$= \frac{1}{f(D,D')}\frac{1}{2}\left[\sin(x+2y)+\sin(x-2y)\right]$$

$$= \frac{1}{2}\left[\frac{1}{f(D,D')}\sin(x+2y)+\frac{1}{f(D,D')}\sin(x-2y)\right] \leftarrow (1)$$

Now $\frac{1}{f(D,D')}\sin(x+2y) = \frac{1}{D^2-DD'}\sin(x+2y)$

Here $a=1$, $b=2$, $n=2$ (order)

put $D=a=1$, $D'=b=2$ in $f(D,D')$ we have

$f(D,D')=f(a,b)=f(1,2)=1-1\times 2=-1\neq 0$

$$\frac{1}{f(D,D')}\sin(x+2y)=\frac{1}{-1}\iint \sin u\,du\,du,\quad u=x+2y$$

$$= \sin u = \sin(x+2y)$$

Illy $\frac{1}{f(D,D')}\sin(x-2y) = \frac{1}{D^2-DD'}\sin(x-2y)$

Here $a=1$, $b=-2$, $n=2$

$f(D,D')=f(a,b)=f(1,-2)= 1^2-1(-2)=3\neq 0$

$$\frac{1}{f(D,D')}\sin(x-2y) = \frac{1}{f(a,b)}\iint \sin u\,du\,dy,\ u=x-2y$$

$$= \frac{1}{3}\iint \sin u\,du\,dy = -\frac{1}{3}\sin u$$

$$= \frac{1}{3}\sin(x-2y)$$

Sub these in (1), we have

$$P.I = \frac{1}{2}\left[\sin(x+2y)-\frac{1}{3}\sin(x-2y)\right]$$

G.S is $z = C.F + P.I = f_1(y)+f_2(y+x)+\frac{1}{2}\sin(x+2y)-\frac{1}{6}\sin(x-2y)$

(13) H.W. Solve $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

A) $z = f_1(y-x) + f_2(y+3x) + f_3(y-2x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$

(6) Solve $(D^2 - 2DD' + D'^2)z = e^{x+y}$

Sol:- $f(D,D') = D^2 - 2DD' + D'^2$, $Q(x,y) = e^{x+y}$

A.E is $f(m,1) = 0$

$\Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$ are equal roots

$\therefore C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+x) + x f_2(y+x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - 2DD' + D'^2} e^{x+y} \quad (\text{Compare with } e^{ax+by})$$

Here $a=1$, $b=1$, $n=2$

put $D=a$, $D'=b$ in $f(D,D')$ we have.

$f(D,D') = f(a,b) = f(1,1) = 1^2 - 2\times1\times1 + 1^2 = 0$

It is a case of failure,

in this case $P.I = x \cdot \frac{1}{\frac{\partial}{\partial D}[D^2 - 2DD' + D'^2]} e^{x+y}$

$$= x \cdot \frac{1}{2D - 2D'} e^{x+y}$$

Here $a=1$, $b=1$, $n=1$

put $D=a$, $D'=b$ in $2D-2D'$ we have.

$2D - 2D' = 2\times1 - 2\times1 = 0$

again it is case of failure.

$$P.I = x^2 \cdot \frac{1}{\frac{\partial}{\partial D}(2D - 2D')} e^{x+y}$$

$= x^2 \cdot \frac{1}{2} \cdot e^{x+y}$ (partial derivative w.r.to D, D' is constant)

$$P.I = \frac{x^2}{2} e^{x+y}$$

$\therefore$ G.S is $z = C.F + P.I$.

(7) Solve $r + s - 6t = \cos(2x+y)$

Sol:- Given $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x+y)$

Symbolic form is $(D^2 + DD' - 6D'^2) z = \cos(2x+y)$

Here $f(D,D') = D^2 + DD' - 6D'^2$, $Q(x,y) = \cos(2x+y)$

A.E is $f(m,1) = 0 \Rightarrow m^2 + m - 6 = 0 \Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m_1 = -3, m_2 = 2$ are different roots

$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

$\Rightarrow C.F = f_1(y - 3x) + f_2(y + 2x)$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$$

Here $a = 2,\ b = 1,\ n = 2$

put $D = a = 2,\ D' = b = 1$ in $f(D,D')$ we have

$f(a,b) = f(2,1) = 2^2 + 2\times 1 - 6\times 1^2 = 0$

It is a case of failure, in this case

$$P.I = x \cdot \frac{1}{\frac{\partial}{\partial D}[D^2 + DD' - 6D'^2]} \cos(2x+y)$$

$$= x \cdot \frac{1}{2D + D'} \cos(2x+y)$$

put $D = a = 2,\ D' = b = 1,\ n = 1$ (order of $2D+D'$ is 1)

in the above, we have $f(2,1) = 5 \neq 0$

$$P.I = x \cdot \frac{1}{u+1} \int \cos u \, du \quad , \quad u = 2x+y$$

$$= \frac{x}{5} \sin u$$

$$\Rightarrow P.I = \frac{x}{5} \sin(2x+y)$$

G.S is $z = C.F + P.I$.

8. Solve $(D^2+5DD'+6D'^2)z = \frac{1}{y-2x}$

Sol:- Here $f(D,D') = D^2+5DD'+6D'^2$, $Q(x,y) = \frac{1}{y-2x}$

A.E is $f(m,1)=0$

$\Rightarrow m^2+5m+6=0 \Rightarrow (m+2)(m+3)=0$

$\Rightarrow m_1=-2,\ m_2=-3$ are different roots

$C.F = f_1(y+m_1x)+f_2(y+m_2x)$

$\Rightarrow C.F = f_1(y-2x)+f_2(y-3x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{D^2+5DD'+6D'^2}\left(\frac{1}{y-2x}\right)$ (Compare with $(ax+by)^{-1}$)

Here $a=-2,\ b=1,\ n=2$

put $D=a,\ D'=b$ in $f(D,D')$ we have

$f(D,D') = f(a,b) = f(-2,1) = 4-10+6 = 0$.

It is a case of failure, in this case

$P.I = x.\frac{1}{\frac{\partial}{\partial D}[D^2+5DD'+6D'^2]}(y-2x)^{-1}$

$= x.\frac{1}{2D+5D'}(y-2x)^{-1}$

Here $a=-2,\ b=1,\ n=1$

put $D=a,\ D'=b$ in $2D+5D'$ we have

$2D+5D' = 2(-2)+5(1) = 1 \neq 0$

$\therefore P.I = x.\frac{1}{(1)}\int u^{-1}\,du,\quad u=y-2x$

$= x\int\frac{1}{u}\,du$

$P.I = x\log u = x\log(y-2x)$

$\therefore$ G.S is $z = C.F+P.I$

$z = f_1(y-2x)+f_2(y-3x)+x\log(y-2x)$.

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(16)

9. Solve $4r-4s+t = 16\log(x+2y)$

Sol:- Given P.D.E $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\partial y} + \frac{\partial^2 z}{\partial y^2} = 16\log(x+2y)$

Symbolic form of given P.D.E is

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

Here $f(D,D') = 4D^2 - 4DD' + D'^2$, $Q(x,y) = 16\log(x+2y)$

A.E is $f(m,1) = 0 \Rightarrow 4m^2 - 4m + 1 = 0$

$\Rightarrow (2m-1)^2 = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$ are 2 equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y+\frac{1}{2}x) + x f_2(y+\frac{1}{2}x)$ (or)

$C.F = f_1(2y+x) + x f_2(2y+x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{4D^2-4DD'+D'^2} 16\log(x+2y)$ (Compare with $\log(ax+by)$)

Here $a=1, b=2, n=2$

Sub $D=a=1$, $D'=b=2$ in $f(D,D')$, we have.

$f(1,2) = 4\times1^2 - 4\times1\times2 + 2^2 = 8-8=0$

It is a Case of failure, in this Case

$P.I = 16x \cdot \frac{1}{\frac{\partial}{\partial D}[4D^2-4DD'+D'^2]} \log(x+2y)$

$= 16x \cdot \frac{1}{8D-4D'} \log(x+2y)$

put $D=a=1$, $D'=b=2$ in $8D-4D'$, $n=1$

we have $8D-4D' = 8\times1 - 4\times2 = 0$, Case of failure

$P.I = 16x^2 \cdot \frac{1}{\frac{\partial}{\partial D}(8D-4D')} \log(x+2y)$

$= 16x^2 \cdot \frac{1}{8} \log(x+2y)$

$P.I = 2x^2\log(x+2y)$

$\therefore z = C.F + P.I$

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(17)

(10) Solve $(D-2D')(D-D')^2 z = e^{x+y}$

Sol:- Here $f(D,D') = (D-2D')(D-D')^2$, $Q(x,y) = e^{x+y}$

A.E is $f(m,1)=0$

$\Rightarrow (m-2)(m-1)^2=0 \Rightarrow m_1=2,\ m_2=m_3=1,=m$

Here one root is different root and remaining two roots are equal roots

$\therefore C.F = f_1(y+m_1x) + f_2(y+mx) + x f_3(y+mx)$

$\Rightarrow C.F = f_1(y+2x) + f_2(y+x) + x f_3(y+x)$.

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-D')^2(D-2D')} e^{x+y}$$

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{D-2D'} e^{x+y}\right\}$$

$$= \frac{1}{(D-D')^2}\left\{\frac{1}{1-2}\int e^u\,du\right\} \quad (\because a=1, b=1, n=1)$$

$u = ax+by$

$$= \frac{1}{(D-D')^2}\left\{-e^u\right\}$$

$$= \frac{-1}{(D-D')^2} e^{x+y}$$

$$= \frac{-1}{0} e^{x+y}, \text{ Case of failure put } D=1, D'=1$$

$$= -x.\frac{1}{\frac{\partial}{\partial D}[D-D']^2} e^{x+y}$$

$$= -x.\frac{1}{2(D-D')} e^{x+y}$$

$$= \frac{-x}{2}.\frac{1}{(1-1)} e^{x+y} \quad \text{again Case of failure}$$

$\rightarrow 0$

$$P.I = -\frac{x}{2}\left\{x.\frac{1}{\frac{\partial}{\partial D}(D-D')} e^{x+y}\right\} = -\frac{x^2}{2} e^{x+y}$$

$$\therefore z = C.F + P.I = f_1(y+2x) + f_2(y+x) + x f_3(y+x) - \frac{x^2}{2} e^{x+y}$$

P.I of $f(D,D')z = Q(x,y)$ when $Q(x,y) = x^m y^n$

when $Q(x,y) = x^m y^n$ then

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')} x^m y^n$$

$$P.I = [f(D,D')]^{-1} x^m y^n$$

If $m<n$, expand $[f(D,D')]^{-1}$ in powers of $\frac{D}{D'}$

ie take common D' from $f(D,D')$

If $n<m$, expand $[f(D,D')]^{-1}$ in powers of $\frac{D'}{D}$

ie take common D from $f(D,D')$

Also we have $\frac{1}{D} Q(x,y) = \int\limits_{y \text{ constant}} Q(x,y)\, dx$

$$\frac{1}{D'} Q(x,y) = \int\limits_{x \text{ constant}} Q(x,y)\, dy$$

Note (i) $Q(x,y) = x^m y^n$, (i) If x power is highest then take common D from $f(D,D')$

(ii). If y power is highest then take common D' from $f(D,D')$.

Problems ① solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$

Sol:- symbolic form of the given eqn is put $D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$

$$(D^3 - 2D^2 D')z = 2e^{2x} + 3x^2 y.$$

Here $f(D,D') = D^3 - 2D^2 D'$, $Q(x,y) = 2e^{2x} + 3x^2 y$

A.E is $f(m,1) = 0$

$\Rightarrow m^3 - 2m^2 = 0$

$\Rightarrow m^2(m-2) = 0$

$\Rightarrow m_1 = 2, m_2 = m_3 = 0$

Here one root is different and remaining two roots are equal.

$$C.F = f_1(y+m_1 x) + f_2(y+m_2 x) + x f_3(y+m_3 x).$$

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$C \cdot F = f_1(y+2x) + f_2(y) + x f_3(y)$.

$$P \cdot I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')} \left(2e^{2x} + 3x^2 y\right)$$

$$= \left[2 \cdot \frac{1}{f(D,D')} e^{2x} + 3 \frac{1}{f(D,D')} x^2 y\right]$$

$$P.I = 2P_1 + 3P_2 \quad \text{— } ①$$

Now $2P_1 = 2 \cdot \frac{1}{f(D,D')} e^{2x}$

$$= 2 \cdot \frac{1}{D^3 - 2D^2D'} e^{2x+0y} \quad \left(\text{Compare with } e^{ax+by}\right)$$

Here $a = 2$, $b = 0$, $n = 3$ (order)

put $D = a = 2$, $D' = b = 0$ in $f(D,D')$, we have

$f(D,D') = f(2,0) = 2^3 - 2 \times 2^2 \times 0 = 8 \neq 0$

$\therefore 2P_1 = 2 \cdot \frac{1}{8} \iiint e^u \, du\,du\,du$, $u = 2x + 0y$

$= \frac{1}{4} e^u \Rightarrow 2P_1 = \frac{1}{4} e^{2x}$.

$$3P_2 = 3 \frac{1}{f(D,D')} x^2 y$$

$$= 3 \cdot \frac{1}{D^3 - 2D^2D'} x^2 y \quad \left(\text{Compare with } x^m y^n\right)$$

$m = 2$, $n = 1$, $m > n$ (ie x power is high).

So take common highest power of D from $f(D,D')$, we have

$$= 3 \cdot \frac{1}{D^3\left[1 - \frac{2D'}{D}\right]} x^2 y.$$

$$= 3 \cdot \frac{1}{D^3} \left[1 - \frac{2D'}{D}\right]^{-1} x^2 y.$$

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(20)

$$zP_2 = \frac{3}{D^3}\left[1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 + \cdots\right] x^2 y$$

$(\because (1-x)^{-1} = 1 + x^2 + x^2 + \cdots)$

$$= \frac{3}{D^3}\left[1 + \frac{2D'}{D}\right] x^2 y$$ (neglect ~~higher power~~ from D'^2 and higher power since y power is one so consider upto D' power is one)

$$= \frac{3}{D^3}\left[x^2 y + \frac{2D'}{D}(x^2 y)\right]$$

$$= \frac{3}{D^3}\left[x^2 y + \frac{2}{D} x^2 D'(y)\right]$$

$dy = 1,\ D''y = D'''y = \cdots = 0$

$\rightarrow$ w.r.to D', D is constant

$$= \frac{3}{D^3}\left[x^2 y + \frac{2}{D} x^2\right]$$

$$= \frac{3}{D^3}\left[x^2 y + 2 \cdot \frac{x^3}{3}\right]$$ $\left[\frac{1}{D} \rightarrow \int dx.\right]$

$$= 3y \iiint x^2\, dx\,dx\,dx + \cancel{3} \cdot 2 \iiint x^3\, dx\,dx\,dx$$

$$= 3y \cdot \left(\frac{x^5}{60}\right) + 2 \cdot \left(\frac{x^6}{120}\right)$$

$$zP_2 = \frac{y x^5}{20} + \frac{x^6}{60}$$

Sub zP_1, zP_2 in (1), we have

$$P.I = \frac{1}{4} e^{2x} + \frac{y x^5}{20} + \frac{x^6}{60}$$

$\therefore$ G.S is $z = C.F + P.I$

$$z = f_1(y+2x) + f_2(y) + x f_3(y) + \frac{1}{4} e^{2x} + \frac{y x^5}{20} + \frac{x^6}{60}.$$

Note

$\frac{1}{D^3} \rightarrow \iiint dx\,dx\,dx.$ lly $\frac{1}{D'^3} = \iiint dy\,dy\,dy$

$\frac{1}{D^2} \rightarrow \iint dx\,dx$ $\frac{1}{D'^2} = \iint dy\,dy.$

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(2) solve $(D^2-6DD'+9D'^2)z = 12x^2+36xy$.

Sol:- Here $f(D,D') = D^2-6DD'+9D'^2$, $\phi(x,y) = 12x^2+36xy$

A.E is $f(m,1) = 0$ (put $D=m, D'=1$)

$\Rightarrow m^2-6m+9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3$ are two equal roots

$\therefore$ C.F $= f_1(y+mx) + x f_2(y+mx)$.

$\Rightarrow$ C.F $= f_1(y+3x) + x f_2(y+3x)$

$P.I = \frac{1}{f(D,D')}\phi(x,y)$

$= \frac{1}{f(D,D')}[12x^2+36xy] = 12.\frac{1}{f(D,D')}x^2 + 36.\frac{1}{f(D,D')}xy$

$= z_{P_1} + z_{P_2}$ —— (1)

$z_{P_1} = 12.\frac{1}{f(D,D')}x^2$

$= 12.\frac{1}{D^2-6DD'+9D'^2}(x+0y)^2$ (compare with $(ax+by)$)

$a=1, b=0$, $n=$ order of D.E $= 2$

put $D=a=1$, $D'=b=0$ in $f(D,D')$, we have

$f(D,D') = f(1,0) = 1 \neq 0$.

$z_{P_1} = 12.\frac{1}{1}\iint u^2\,du\,du$ $(u = x)$.

$= 12.\frac{u^4}{12}$ $\Rightarrow z_{P_1} = (x)^4 = x^4$

$z_{P_2} = 36.\frac{1}{f(D,D')}xy$.

$= 36.\frac{1}{D^2-6DD'+9D'^2}(xy)$ Compare $(x^m y^n)$

$= 36\frac{1}{(D-3D')^2}(xy)$

Here $m=1$, $n=1$ so $m=n$.

Hence take Higher power of D or higher power of D' Common from $f(D,D')$

Here I take highest power of D Common from $f(D,D')$

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(22)

$$zP_2 = 36 \cdot \frac{1}{D^2\left[1-\frac{3D'}{D}\right]^2} xy$$

$$= \frac{36}{D^2}\left[1-\frac{3D'}{D}\right]^{-2}(xy)$$

$$= \frac{36}{D^2}\left[1+2\left(\frac{3D'}{D}\right)+\cdots\right](xy) \qquad \left((1-x)^{-2} = 1+2x+3x^2+\cdots\right)$$

y power is one so
Consider upto D' power is one

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}D'(xy)\right]$$

$$= \frac{36}{D^2}\left[xy + \frac{6}{D}x\right]$$

$$= 36y\,\frac{1}{D^2}(x) + 36\times 6\,\frac{1}{D^3}(x)$$

$$= 36y\iint x\,dx\,dx + 36\times 6\iiint x\,dx\,dx\,dx.$$

$$= 36y\left(\frac{x^3}{6}\right) + 36\times\cancel{6}\left(\frac{x^4}{\cancel{24}_{4}}\right)$$

$$zP_2 = 6x^3y + 9x^4.$$

sub zP_1, zP_2 in (1), we have

$$P.I = x^4 + 6x^3y + 9x^4 = 6x^3y + 10x^4$$

$\therefore$ G.S is $z = C.F + P.I$

$$\Rightarrow z = f_1(y+3x) + x f_2(y+3x) + 6x^3y + 10x^4.$$

(13) $$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x^2+y^2$$

Sol:- Symbolic form of the given D.E is $(D^2-D'^2)z = x^2+y^2$

Here $f(D,D') = D^2 - D'^2$, $Q(x,y) = x^2+y^2$

A.E is $f(m,1) = 0 \Rightarrow m^2-1 = 0$

$\Rightarrow m = \pm 1$ ie $m_1 = -1$, $m_2 = 1$
are different roots

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$C.F = f_1(y+m_1 x) + f_2(y+m_2 x)$

$\Rightarrow C.F = f_1(y-x) + f_2(y+x)$.

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{D^2-D'^2}(x^2+y^2)$

$= \frac{1}{D^2-D'^2}(x^2) + \frac{1}{D^2-D'^2}(y^2) = P_1 + P_2$ — (1)

$P_1 = \frac{1}{D^2-D'^2}(x^2) = \frac{1}{D^2-D'^2}(x+0y)^2$ compare $(ax+by)$

$a=1,\ b=0,\ n=2$.

put $D=a=1,\ D'=b=0$ in $f(D,D')$

$f(D,D') = f(1,0) = 1 \neq 0$

$P_1 = \frac{1}{1}\iint u^2\,du\,du$ where $u = x+0y$

$= \frac{u^4}{12} = \frac{x^4}{12}$

$P_2 = \frac{1}{D^2-D'^2}(y^2) = \frac{1}{D^2-D'^2}(0x+y)^2$

$a=0,\ b=1,\ n=2$ (order of D.E)

$f(D,D') = f(0,1) = 0^2-1^2 = -1 \neq 0$

$P_2 = \frac{1}{-1}\iint u^2\,du\,du$, where $u = 0x+y$

$= -\frac{u^4}{12} = -\frac{y^4}{12}$

$\iint u^2\,du\,du = \int\left[\int u^2\,du\right]du$

$= \int\left(\frac{u^3}{3}\right)du$

$= \frac{u^4}{12}$

sub P_1, P_2 in (1), we have

$P.I = \frac{x^4}{12} - \frac{y^4}{12}$

$\therefore$ G.S is $z = C.F + P.I$

$\Rightarrow z = f_1(y-x) + f_2(y+x) + \frac{1}{12}(x^4-y^4)$.

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(24) (4) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Sol:- Symbolic form of the given DE is

$$(D^2 + 2DD' + D'^2) z = x^2 + xy + y^2$$

Here $f(D,D') = D^2 + 2DD' + D'^2$, $Q(x,y) = x^2 + xy + y^2$

A.E is $f(m,1) = 0 \Rightarrow m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0$

$\Rightarrow m = -1, -1$ are two equal roots

$C.F = f_1(y+mx) + x f_2(y+mx)$

$\Rightarrow C.F = f_1(y-x) + x f_2(y-x)$.

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{f(D,D')}(x^2 + xy + y^2)$$

$$= \frac{1}{f(D,D')} x^2 + \frac{1}{f(D,D')}(xy) + \frac{1}{f(D,D')}(y^2)$$

$$= zP_1 + zP_2 + zP_3 \quad (1)$$

$$zP_1 = \frac{1}{f(D,D')} x^2 = \frac{1}{(D+D')^2} x^2$$

$= \frac{1}{(D+D')^2}(x+0y)^2$ (Compare with $ax+by$)

$a=1, b=0, n=2$

$f(D,D') = f(a,b) = f(1,0) = (1+0)^2 = 1 \neq 0$

$zP_1 = \frac{1}{1}\iint u^2\, du\, du$, where $u = x+0y$

$$= \frac{u^4}{12} = \frac{x^4}{12}$$

$$zP_3 = \frac{1}{f(D,D')} y^2 = \frac{1}{(D+D')^2}(0x+y)^2$$

$a=0, b=1, n=2$

$f(D,D') = f(a,b) = (0+1)^2 = 1 \neq 0$

$zP_3 = \frac{1}{1}\iint u^2\, du\, du$, $u = 0x+y$

$$= \frac{u^4}{12} = \frac{y^4}{12}$$

$$z P_2 = \frac{1}{f(D,D')}(xy)$$

$$= \frac{1}{(D+D')^2}(xy)$$ (Compare with $x^m y^n$, here $m=n=1$ so take highest power of D or take highest power of D' common from $f(D,D')$)

$$= \frac{1}{D^2\left[1+\frac{D'}{D}\right]^2}(xy)$$

$$= \frac{1}{D^2}\left[1+\frac{D'}{D}\right]^{-2}(xy)$$

$$= \frac{1}{D^2}\left[1-\frac{2D'}{D}+\cdots\right](xy)$$

$$= \frac{1}{D^2}\left[xy - 2\frac{D'}{D}(xy)\right]$$

$$= \frac{1}{D^2}\left[xy - \frac{2}{D}(x)D'(y)\right]$$

$$= y\frac{1}{D^2}(x) - \frac{2}{D^3}(x)$$

$$= y\iint x\,dx\,dx - 2\iiint x\,dx\,dx\,dx$$

$$z P_2 = y\frac{x^3}{6} - 2\left(\frac{x^4}{24}\right)$$

$$z P_2 = \frac{yx^3}{6} - \frac{x^4}{12}$$

sub zP_1, zP_2, zP_3 values in ①, we have

$$P.I = \cancel{\frac{x^4}{12}} + \frac{yx^3}{6} - \cancel{\frac{x^4}{12}} + \frac{y^4}{12}$$

$$P.I = \frac{yx^3}{6} + \frac{y^4}{12}$$

G.S is $z = C.F + P.I$

$$= f_1(y-x) + x f_2(y-x) + \frac{yx^3}{6} + \frac{y^4}{12}$$

H.W

① $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$ A) $f_1(y-x) + f_2(y-2x) + 2x^3y - \frac{3}{2}x^4$

② $(D^2 - 2DD')z = e^{2x} + x^3y$ A) $f_1(y) + f_2(y+2x) + \frac{1}{4}e^{2x} + \frac{x^5 y}{20} + \frac{x^6}{60}$.

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(5) $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y$ ✓

Sol: ~~Here~~ symbolic form of given D.E is $(D^3 - D'^3) z = x^3 y$ ✓

Here $f(D, D') = D^3 - D'^3$, $Q(x, y) = x^3 y$ ✓

A.E is $f(m, 1) = 0$

$\Rightarrow m^3 - 1 = 0$

$\Rightarrow$ ~~(m+1)~~ $(m-1)(m^2+m+1) = 0 \Rightarrow (m-1) = 0, m^2+m+1 = 0$

$a = 1, b = 1, c = 1$

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\Rightarrow m_1 = 1, \quad m = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2}$

$m = \frac{-1 \pm \sqrt{-3}}{2}$

$\Rightarrow m_2 = \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore m_1 = 1, \; m_2 = \frac{-1 + i\sqrt{3}}{2}, \; m_3 = \frac{-1 - i\sqrt{3}}{2}$ are different roots

Let us take $m_1 = 1, \; m_2 = \frac{-1+i\sqrt{3}}{2} = \omega, \; m_3 = \frac{-1-i\sqrt{3}}{2} = \omega^2$ ✓

Since w.r.t the given eqn $m^3 - 1 = 0 \Rightarrow m^3 = 1$ means cube roots of unity which are $1, \omega, \omega^2$ ✓

$\therefore C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$

$\Rightarrow C.F = f_1(y + x) + f_2(y + \omega x) + f_3(y + \omega^2 x)$

$P.I = \frac{1}{f(D, D')} Q(x, y)$

$= \frac{1}{(D^3 - D'^3)} x^3 y$ ✓ $\quad (x^m y^n, \; m = 3, n = 2)$

$m > n$ so take common highest power of D from $f(D, D')$ we have

$= \frac{1}{D^3 \left[1 - \frac{D'^3}{D^3}\right]} x^3 y$

$= \frac{1}{D^3} \left[1 - \frac{D'^3}{D^3}\right]^{-1} x^3 y$ ✓

$= \frac{1}{D^3} \left[1 + \cancel{\frac{D'^3}{D^3}} + \cancel{\left(\frac{D'^3}{D^3}\right)^2} + \cdots \right] x^3 y$ ✓

$$P.I = \frac{1}{D^3}(x^3y^2) = y^2 \iiint x^3\,dx\,dx\,dx.$$

$$P.I = \frac{y^2 \cdot x^6}{120} \qquad \int^{2} \frac{x^4}{4} \quad \text{II} \quad \frac{x^5}{20} \quad \text{III} \quad \frac{x^6}{120}.$$

$\therefore$ G.S is $y = C.F + P.I$

$$\Rightarrow z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{y^2 x^6}{120}.$$

General method for finding P.I

If $Q(x,y)$ is of a form different from the forms already discussed, Then use.

$$\frac{1}{D-mD'}Q(x,y) = \int Q(x, c-mx)\,dx$$

ie replace y by $c-mx$, and after integration replace c by $y+mx$.

procedure factorise $f(D,D')$ into Linear factors, Thus

$$P.I = \frac{1}{f(D,D')}Q(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')\cdots}Q(x,y)$$

$$= \frac{1}{D-m_1D'} \cdot \frac{1}{D-m_2D'} \cdots Q(x,y)$$

Repeated application of the above rule gives the P.I

problems ① Solve $(D^2 - DD' - 2D'^2)z = (y-1)e^x$.

Sol: $f(D,D') = D^2 - DD' - 2D'^2$, $Q(x,y) = (y-1)e^x$

A.E is $f(m,1) = 0 \Rightarrow m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$\Rightarrow m_1 = 2, m_2 = -1$ are different roots

$$C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$= f_1(y+2x) + f_2(y-x)$$

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$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' - 2D'^2} (y-1)e^x$$

$$= \frac{1}{(D-2D')(D+D')} (y-1)e^x$$

$$= \frac{1}{D-2D'} \cdot \frac{1}{D+D'} (y-1)e^x$$

(Comparing $D+D'$ with $D-mD'$, $m=-1$

replace $y = c - mx$

$y = c + x$

$\Rightarrow \boxed{c = y - x}$)

$$= \frac{1}{D-2D'} \int \underset{v}{e^x} \underset{u}{(c+x-1)}\, dx$$

$$= \frac{1}{D-2D'} \left\{ u \int v\,dx - \int \left(\frac{du}{dx} \int v\,dx \right) dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1) \int e^x\,dx - \int \left[\frac{d}{dx}(c+x-1) \int e^x\,dx \right] dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1)e^x - \int 1 \times e^x\,dx \right\}$$

$$= \frac{1}{D-2D'} \left\{ (c+x-1)e^x - e^x \right\}$$

replace $c = y - x$, we have

$$= \frac{1}{D-2D'} \left\{ (y - \cancel{x} + \cancel{x} - 1)e^x - e^x \right\}$$

$$= \frac{1}{D-2D'} \left[(y-1)e^x - e^x \right]$$

$$= \frac{1}{D-2D'} \left[(y-2)e^x \right]$$

Compare $D-2D'$ with $D-mD'$, here $m=2$

replace y by $c - mx = c - 2x$.

after integration replace $c = y + 2x$.

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$$P.I = \int \underset{u}{(c-2x-2)}\, \underset{v}{e^x}\, dx$$

$$= u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

$$= (c-2x-2)\int e^x dx - \int\left[\frac{d}{dx}(c-2x-2)\int e^x dx\right]dx.$$

$$= (c-2x-2)\,e^x - \int (-2)\,e^x\,dx$$

$$= (c-2x-2)\,e^x + 2e^x$$

$$P.I = (c-2x-2+2)\,e^x$$

$$= (c-2x)\,e^x$$

$$P.I = (y+2x-2x)\,e^x \quad (\because c = y+2x)$$

$$\Rightarrow P.I = y e^x$$

$$\therefore G.S \; z = C.F + P.I$$

$$z = f_1(y+2x) + f_2(y-x) + y e^x$$

(2) Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x.$

Sol: Symbolic form of the given eqn is

$$(D^2 + DD' - 6D'^2)\, z = y\cos x.$$

Here $f(D, D') = D^2 + DD' - 6D'^2$, $Q(x) = y\cos x$.

A.E is $f(m,1) = 0 \Rightarrow m^2 + m - 6 = 0$

$$\Rightarrow (m+3)(m-2) = 0$$

$\Rightarrow m_1 = -3,\ m_2 = 2$ are diff roots

$$\therefore C.F = f_1(y+m_1x) + f_2(y+m_2x)$$

$$\Rightarrow C.F = f_1(y-3x) + f_2(y+2x).$$

$$P.I = \frac{1}{f(D,D')}\, Q(x,y)$$

$$= \frac{1}{D^2 + DD' - 6D'^2}(y\cos x)$$

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$$P.I = \frac{1}{(D+3D')(D-2D')}(y\cos x)$$

$$= \frac{1}{D+3D'}\left\{\frac{1}{D-2D'}(y\cos x)\right\}$$

Compare $D-2D'$ with $D-mD'$.

Here $m=2$

put $y = c-mx$.

$\Rightarrow y = c-2x$.

replace after integration

$\boxed{c = y+2x}$

$$= \frac{1}{D+3D'}\int \underset{u}{(c-2x)}\underset{v}{\cos x}\,dx.$$

$$= \frac{1}{D+3D'}\left\{u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx\right\}$$

$$= \frac{1}{D+3D'}\left\{(c-2x)\int\cos x\,dx - \int\left[\frac{d}{dx}(c-2x)\int\cos x\,dx\right]dx\right\}$$

$$= \frac{1}{D+3D'}\left\{(c-2x)\sin x - \int(-2)\sin x\,dx\right\}$$

$$= \frac{1}{D+3D'}\left\{(c-2x)\sin x - 2\cos x\right\}$$

$$= \frac{1}{D+3D'}\left\{(y+\cancel{2x}-\cancel{2x})\sin x - 2\cos x\right\}$$

$$= \frac{1}{D-(-3D')}(y\sin x - 2\cos x)$$

Here $m=-3$,

put $y = c-mx$

$\Rightarrow y = c+3x$

$\Rightarrow \boxed{c = y-3x}$

$$= \int\left[(c+3x)\sin x - 2\cos x\right]dx$$

$$= \int \underset{u}{(c+3x)}\underset{v}{\sin x}\,dx - 2\int\cos x\,dx.$$

$$= u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx - 2\sin x.$$

$$= (c+3x)\int\sin x\,dx - \int\left[\frac{d}{dx}(c+3x)\int\sin x\,dx\right]dx - 2\sin x$$

$$= -(c+3x)\cos x - \int 3(-\cos x)\,dx - 2\sin x$$

$$= -(c+3x)\cos x + 3\sin x - 2\sin x$$

$$= -[y-\cancel{3x}+\cancel{3x}]\cos x + \sin x$$

$$P.I = -y\cos x + \sin x$$

$$z = C.F + P.I$$

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③ Solve $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$

Sol: Here $f(D,D') = D^2 + DD' - 6D'^2$, $\phi(x,y) = x^2 \sin(x+y)$

A.E is $f(m,1) = 0 \Rightarrow m^2 + m - 6 = 0$

$\Rightarrow (m+3)(m-2) = 0$

$\Rightarrow m_1 = -3,\ m_2 = 2$ are diff roots

$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$.

$\Rightarrow C.F = f_1(y - 3x) + f_2(y + 2x)$.

$$P.I = \frac{1}{f(D,D')}\phi(x,y)$$

$$= \frac{1}{(D+3D')(D-2D')} x^2 \sin(x+y)$$

$$= \frac{1}{D+3D'}\left\{\frac{1}{D-2D'} x^2 \sin(x+y)\right\}$$

Here $m = 2$, put $y = c - mx$

$\Rightarrow y = c - 2x$

$\therefore \boxed{c = y + 2x}$

$$= \frac{1}{D+3D'} \int x^2 \sin(x + c - 2x)\, dx.$$

$$= \frac{1}{D+3D'} \int \underset{u}{x^2}\ \underset{v}{\sin(c-x)}\, dx.$$

$$= \frac{1}{D+3D'}\left[u v_1 - u' v_2 + u'' v_3\right]$$ (Bernoulli's rule $\int uv\,dx$)

$u = x^2$ $\qquad v = \sin(c-x)$

$u' = 2x$ $\qquad v_1 = \int v\,dx = -\frac{\cos(c-x)}{-1} = \cos(c-x)$

$u'' = 2$ $\qquad v_2 = \int v_1\, dx = \frac{\sin(c-x)}{-1} = -\sin(c-x)$

$v_3 = \int v_2\, dx = -\left[\frac{-\cos(c-x)}{-1}\right] = -\cos(c-x)$

$$P.I = \frac{1}{D+3D'}\left\{x^2\cos(c-x) - 2x[-\sin(c-x)] + 2[-\cos(c-x)]\right\}$$

$$= \frac{1}{D+3D'}\left\{x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)\right\}$$

$$= \frac{1}{D+3D'}\left\{x^2\cos(y+2x-x) + 2x\sin(y+2x-x) - 2\cos(y+2x-x)\right.$$

$(\because c = y + 2x)$

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$$P.I = \frac{1}{D+3D'}\left\{x^2\cos(y+x) + 2x\sin(y+x) - 2\cos(y+x)\right\}$$

Compare $D+3D'$ with $D-mD'$. Here $m=-3$, Put $y = c - mx$

$\Rightarrow y = c+3x$

$\Rightarrow \boxed{c = y-3x}$

~~$P.I = \int x^2 \cos(y-3x+x) + 2x\sin($~~

$$P.I = \int\left[x^2\cos(c+3x+x) + 2x\sin(c+3x+x) - 2\cos(c+3x+x)\right]dx$$

$$P.I = \int x^2\cos(c+4x)\,dx + 2\int x\sin(c+4x)\,dx - 2\int\cos(c+4x)\,dx \quad \text{--- (1)}$$

$$\int \underset{u}{x^2}\underset{v}{\cos(4x+c)}\,dx = uv_1 - u'v_2 + u''v_3$$

$u = x^2$ $\quad v = \cos(4x+c)$

$v_1 = \int v\,dx = \frac{\sin(4x+c)}{4}$

$u' = 2x$

$u'' = 2$ $\quad v_2 = \int v_1\,dx = -\frac{\cos(4x+c)}{4^2} = -\frac{\cos(4x+c)}{16}$

$v_3 = \int v_2\,dx = -\frac{\sin(4x+c)}{4^3} = -\frac{\sin(4x+c)}{64}$

$$\int x^2\cos(4x+c)\,dx = x^2\frac{\sin(4x+c)}{4} - 2x\left[-\frac{\cos(4x+c)}{16}\right] + 2\left[-\frac{\sin(4x+c)}{64}\right]$$

$$= \frac{x^2}{4}\sin(4x+y-3x) + \frac{x}{8}\cos(4x+y-3x) - \frac{1}{32}\sin(4x+y-3x)$$

$$\int x^2\cos(4x+c)\,dx = \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y)$$

$$\int\cos(c+4x)\,dx = \frac{\sin(4x+c)}{4} = \frac{1}{4}\sin(4x+y-3x) = \frac{1}{4}\sin(x+y)$$

$$\int \underset{u}{x}\,\underset{v}{\sin(4x+c)}\,dx = uv_1 - u'v_2$$

$u = x$, $\quad v = \sin(4x+c)$

$u' = 1$, $\quad v_1 = \int v\,dx = -\frac{\cos(4x+c)}{4}$

$v_2 = \int v_1\,dx = -\frac{\sin(4x+c)}{16}$

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$$\int x \sin(4x+c)\,dx = x\left[\frac{-\cos(4x+c)}{4}\right] - \left[\frac{-\sin(4x+c)}{16}\right]$$

$$= \frac{-x}{4}\cos(4x+y-3x) + \frac{1}{16}\sin(4x+y-3x)$$

$$\int x\sin(4x+c)\,dx. = \frac{-x}{4}\cos(x+y) + \frac{1}{16}\sin(x+y)$$

sub all these values in (1), we have

$$P.I = \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y) + 2\left[\frac{-x}{4}\cos(x+y) + \frac{1}{16}\sin(x+y)\right] - 2\left[\frac{1}{4}\sin(x+y)\right]$$

$$= \frac{x^2}{4}\sin(x+y) + \frac{x}{8}\cos(x+y) - \frac{1}{32}\sin(x+y) - \frac{x}{2}\cos(x+y) + \frac{1}{8}\sin(x+y) - \frac{1}{2}\sin(x+y)$$

$$= \sin(x+y)\left[\frac{x^2}{4} - \frac{1}{32} + \frac{1}{8} - \frac{1}{2}\right] + \cos(x+y)\left[\frac{x}{8} - \frac{x}{2}\right]$$

$$= \sin(x+y)\left[\frac{x^2}{4} + \frac{-1+4-16}{32}\right] + x\cos(x+y)\left[\frac{1-4}{8}\right]$$

$$P.I = \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y)$$

$\therefore$ G.S is $z = C.F + P.I$

$$= f_1(y-3x) + f_2(y+2x) + \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y).$$

(4) ~~solve r-s-2t~~ H.W problems

(1) $(D^2 + DD' - 6D'^2)z = y\sin x$ A) $f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x$

(2) $(D-D')(D+2D')z = (y+1)e^x$
A) $f_1(y+x) + f_2(y-2x) + ye^x$.

(3) $(D^2 + 2DD' + D'^2)z = 2\sin y - x\cos y$

Non- Homogeneous Linear partial differential equn's

In the equn $f(D,D')z = Q(x,y)$, if the polynomial $f(D,D')$ in D, D' is not homogeneous.(ie having different ~~powers~~ orders) Then it is called a non homogeneous linear Partial differential equn's

Its complete soln is $z = C.F + P.I$

Method for finding C.F, when $f(D,D')$ can be factorised into linear factors :-

Resolve $f(D,D')$ into linear factors of the form $(D - mD' - \alpha)$. The C.F Corresponding to the factor $D - mD' - \alpha$ is $e^{\alpha x}\phi(y+mx)$.

if $f(D,D') = (D - m_1D' - \alpha_1)(D - m_2D' - \alpha_2) \cdots (D - m_nD' - \alpha_n)$

$$C.F = e^{\alpha_1 x} f_1(y+m_1x) + e^{\alpha_2 x} f_2(y+m_2x) + \cdots + e^{\alpha_n x} f_n(y+m_nx)$$

In case of repeated factors

If $f(D,D') = (D - mD' - \alpha)^2$ then

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx)$$

If $f(D,D') = (D - mD' - \alpha)^3$ then

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) + x^2 e^{\alpha x} f_3(y+mx).$$

Methods for finding P.I

Let the given equn be $f(D,D')z = Q(x,y)$.

Then $P.I = \dfrac{1}{f(D,D')} Q(x,y)$

Case(i) when $Q(x,y) = e^{ax+by}$ and $f(a,b) \neq 0$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}$$

ie replace D by a & D' by b.

Note If $f(a,b)=0$ then it is a case of failure

in this case $P.I = x.\dfrac{1}{\frac{\partial}{\partial D}[f(D,D')]}e^{ax+by}$ (or) $y\dfrac{1}{\frac{\partial}{\partial D'}[f(D,D')]}e^{ax+by}$

Case (ii) when $Q(x,y)=\sin(ax+by)$ or $\cos(ax+by)$

Then $P.I = \dfrac{1}{f(D,D')}\sin(ax+by)$

put $D^2=-(a^2)$, $D'^2=-(b^2)$, $DD'=-(ab)$ providing the denominator is non-zero

Note If the denominator is zero then

$$P.I = x.\frac{1}{\frac{\partial}{\partial D}[f(D,D')]}\sin(ax+by) \text{ or } y.\frac{1}{\frac{\partial}{\partial D'}[f(D,D')]}\sin(ax+by)$$

Use the above conditions providing denominator is non zero, if the denominator is zero then

$$P.I = x^2.\frac{1}{\frac{\partial^2}{\partial D^2}[f(D,D')]}\sin(ax+by) \text{ or } y^2.\frac{1}{\frac{\partial^2}{\partial D'^2}[f(D,D')]}\sin(ax+by)$$

Case (iii) when $Q(x,y)=x^m y^n$ where m and n are positive integers.

$$P.I = \frac{1}{f(D,D')}x^m y^n = [f(D,D')]^{-1}x^m y^n$$

which can be evaluated after expanding $[f(D,D')]^{-1}$ in ascending powers of $\frac{D'}{D}$ when $m>n$ or $\frac{D}{D'}$ when $m<n$

Case (iv) when $Q(x,y)=e^{ax+by}V$ where V is a function of x and y.

$$P.I = \frac{1}{f(D,D')}e^{ax+by}V = e^{ax+by}\frac{1}{f(D+a,D'+b)}V$$

which can be evaluated by using the previous cases.

problems

① Solve $(D-D'-1)(D-D'-2)z = e^{2x-y}$

Here $f(D,D') = (D-D'-1)(D-D'-2)$, $Q(x,y) = e^{2x-y}$

Compare each factor of $f(D,D')$ with $D-mD'-\alpha$, we have

$m_1 = 1, \alpha_1 = 1$, $m_2 = 1, \alpha_2 = 2$

$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$

$= e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

$P.I = \frac{1}{f(D,D')} Q(x,y)$

$= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y}$

put $D=2$, $D'=-1$, we have

$= \frac{1}{(2+1-1)(2+1-2)} e^{2x-y} = \frac{1}{2} e^{2x-y}$

$\therefore$ Complete soln $z = C.F + P.I$

$\Rightarrow z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2} e^{2x-y}$.

② Solve $(D^2 - DD' + D' - 1) z = \cos(x+2y)$

Clearly this is non Homogeneous D.E since

Here $f(D,D')$ have different order terms (D^2 DD' D' 1)

(order $\downarrow$ 2, $\downarrow$ 2, $\downarrow$ 1, $\downarrow$ 0)

Here $f(D,D') = D^2 - DD' + D' - 1$, $Q(x,y) = \cos(x+2y)$

$\Rightarrow f(D,D') = D^2 - 1 - DD' + D'$

$= (D+1)(D-1) - D'(D-1)$

$= (D-1)(D+1-D')$

$= (D-0.D'-1)(D-D'+1)$ are product of two different linear factors Compare with $D-mD'-\alpha$, we have.

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$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

Here $m_1 = 0,\ \alpha_1 = 1,\quad m_2 = 1,\ \alpha_2 = -1$

$$\boxed{C.F = e^{x} f_1(y) + e^{-x} f_2(y+x).}$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' + D' - 1} \cos(x+2y) \quad \left[\cos(ax+by) \text{ Model } (2)\right]$$

$a=1,\ b=2$

put $D^2 = -(a^2) = -1,\quad DD' = -(ab) = -2$

$$P.I = \frac{1}{-1+2+D'-1} \cos(x+2y) \qquad \frac{1}{D'} \to \int dy$$

$$= \frac{1}{D'} \cos(x+2y) = \int \cos(x+2y)\, dy$$

(x constant)

$$P.I = \frac{\sin(x+2y)}{2}$$

Gs. is $z = C.F + P.I$

$$z = e^{x} f_1(y) + e^{-x} f_2(y+x) + \frac{1}{2}\sin(x+2y)$$

(3) Solve $[D^2 - D'^2 - 3D + 3D']z = xy$

Sol: Here $f(D,D') = D^2 - D'^2 - 3D + 3D'$, $Q(x,y) = xy$

Here $f(D,D')$ is a non homogeneous D.E in D, D'.

$$f(D,D') = (D+D')(D-D') - 3(D-D')$$

$$= (D-D')[D+D'-3]$$ are product of two

different linear factors Compare with $D - mD' - \alpha$

Here $m_1 = 1,\ \alpha_1 = 0;\ m_2 = -1,\ \alpha_2 = 3.$

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$= e^{0x} f_1(y+x) + e^{3x} f_2(y-x)$$

$$\boxed{C.F = f_1(y+x) + e^{3x} f_2(y-x)}$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-D')(D+D'-3)} xy$$

$$= \frac{1}{D\left[1-\frac{D'}{D}\right]\left\{-3\left(1-\frac{D+D'}{3}\right)\right\}} xy$$

(Here $x^m y^n$ $m=1=n$, both are same so express $\frac{D'}{D}$ or $\frac{D}{D'}$)

$$= \frac{-1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left[1-\left(\frac{D+D'}{3}\right)\right]^{-1} xy$$

$$= \frac{-1}{3D}\left(1+\frac{D'}{D}+\cdots\right)\left[1+\frac{D+D'}{3}+\left(\frac{D+D'}{3}\right)^2+\cdots\right] xy$$

$$= \frac{-1}{3D}\left[1+\frac{D'}{D}+\cdots\right]\left[1+\frac{D+D'}{3}+\cancel{\frac{D^2}{9}}+\cancel{\frac{D'^2}{9}}+\frac{2DD'}{9}+\cdots\right] xy$$

$$= \frac{-1}{3D}\left[1+\frac{D+D'}{3}+\frac{2DD'}{3}+\frac{D'}{D}+\frac{D'}{D}\left(\frac{D+D'}{3}\right)+\frac{D'}{D}\left(\frac{2DD'}{3}\right)\right] xy$$

($\because D^2(x)=0,\ D'^2(y)=0$ so neglecting second and higher order derivatives of D, D')

$$= \frac{-1}{3D}\left[1+\frac{D}{3}+\frac{D'}{3}+\frac{2DD'}{9}+\frac{D'}{D}+\frac{D'}{3}+\cancel{\frac{D'^2}{3D}}+\cancel{\frac{2D'^2}{3}}\right] xy.$$

$$= \frac{-1}{3D}\left[1+\frac{D}{3}+\frac{2D'}{3}+\frac{D'}{D}+\frac{2DD'}{9}\right] xy$$

$$= \frac{-1}{3D}\left[xy+\frac{1}{3}y+\frac{2}{3}x+\frac{x^2}{2}+\frac{2}{9}\right]$$

$DD'(xy)=D(x)\,D'(y) = 1$

$$= \frac{-1}{3}\left[\frac{x^2}{2}y+\frac{1}{3}yx+\frac{2}{3}\cdot\frac{x^2}{2}+\frac{x^3}{6}+\frac{2}{9}x\right]$$

$\frac{D'}{D}(xy) = \frac{1}{D}(x)\,D'(y) = \int x\,dx \times 1 = \frac{x^2}{2}$

$$P.I = \frac{-1}{3}\left[\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2}{9}x\right]$$

Complete soln is $z = C.F + P.I$

$$\Rightarrow z = f_1(y+x)+e^{3x}f_2(y-x)-\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{x^3}{6}+\frac{2}{9}x\right).$$

(39) (4) Solve $(D-3D^1-2)^2 z = 2e^{2x}\tan(y+3x)$

Sol:- Here $f(D,D^1) = (D-3D^1-2)^2$ are product of two repeated linear factors. Compare with $(D-mD^1-\alpha)^2$

Here $m=3,\ \alpha=2$

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx)$$
$$= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x).$$

Given $Q(x,y) = 2e^{2x}\tan(y+3x)$

$$P.I = \frac{1}{f(D,D^1)} Q(x,y)$$
$$= \frac{1}{(D-3D^1-2)^2}\, 2e^{2x}\tan(y+3x)$$

put $D = D+2,\ D^1 = D^1+0$ (Using Model (4) $e^{ax+by}Q(x,y)$)

$$= 2e^{2x} \frac{1}{(D+\cancel{2}-3D^1-\cancel{2})^2}\tan(y+3x)$$
$$= 2e^{2x}\left[\frac{1}{(D-3D^1)^2}\tan(y+3x)\right]$$ (Compare with $\tan(ax+by)$)

Now this eqn becomes to Homogeneous D.E

Here $f_1(D,D^1) = (D-3D^1)^2$,

put $D=a=3,\ D^1=b=1,\ n=\text{order}=2$

$f_1(D,D^1) = f(3,1) = (3-3)^2 = 0$, Case of failure

$$P.I = 2e^{2x}\,x.\frac{1}{\frac{\partial}{\partial D}(D-3D^1)^2}\tan(y+3x)$$
$$= \cancel{2}\,xe^{2x}\frac{1}{\cancel{2}(D-3D^1)}\tan(y+3x)$$

put $D=a=3,\ D^1=b=1,\ D-3D^1=3-3=0$

again Case of failure.

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$$P.I = x\, e^{2x} \cdot x \cdot \frac{1}{\frac{\partial}{\partial D}(D-3D')} \tan(y+3x)$$

$$P.I = x^2 e^{2x} \cdot \tan(y+3x)$$

G.S is $z = C.F + P.I$

$$\Rightarrow z = e^{2x} f_1(y+3x) + x\, e^{2x} f_2(y+3x) + x^2 e^{2x} \tan(y+3x)$$

(5) $(D^2 - DD' - 2D)\, z = \sin(3x+4y)$

Sol: $f(D,D') = D^2 - DD' - 2D$, $\quad Q(x,y) = \sin(3x+4y)$

$$\cancel{= D^2 - 2DD' + DD' - 2D = D^2 + DD' - 2DD' - 2D}$$

$$\cancel{= D^2 - 2D - 2DD' + DD'} = \cancel{D(D+D') - 2}$$

$$\cancel{= D(D-2) -}$$

$$f(D,D') = D^2 - DD' - 2D$$

$$= D\left[D - D' - 2\right]$$

$$= (D - 0\cdot D' - 0)(D - D' - 2) \quad \text{Compare with } D - mD' - \alpha$$

Here $m_1 = 0,\ \alpha_1 = 0,\quad m_2 = 1,\ \alpha_2 = 2$

$$C.F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$= e^{0x} f_1(y + 0\times x) + e^{2x} f_2(y + x)$$

$$C.F = f_1(y) + e^{2x} f_2(y+x).$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2 - DD' - 2D} \sin(3x+4y) \qquad (\sin(ax+by),\ a=3, b=4)$$

put $D^2 = -(3^2) = -9,\quad DD' = -(ab) = -12$

$$P.I = \frac{1}{-9+12-2D} \sin(3x+4y)$$

$$= \frac{1}{3-2D} \cdot \frac{(3+2D)}{(3+2D)} \sin(3x+4y)$$

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(41)

$$P.I = \frac{(3+2D)}{9-4D^2} \sin(3x+4y)$$

put $D^2 = -(a^2) = -(3^2) = -9$

$$= \frac{(3+2D)}{9-4(-9)} \sin(3x+4y)$$

$$\Rightarrow P.I = \frac{3\sin(3x+4y) + 2D[\sin(3x+4y)]}{45}$$

$$= \frac{3\sin(3x+4y) + 6 \cancel{\sin} \cos(3x+4y)}{45}$$

$$P.I = \frac{1}{15}[\sin(3x+4y) + 2\cos(3x+4y)]$$

$$\left[D(\sin(3x+4y)) = \frac{\partial}{\partial x}\sin(3x+4y) = 3\cos(3x+4y) \right]$$

G.S is $z = C.F + P.I$

$$\Rightarrow z = f_1(y) + e^{2x} f_2(y+x) + \frac{1}{15}[\sin(3x+4y) + 2\cos(3x+4y)].$$

---

(6) $r - s + p = 1$

Sol:- W.K.T $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $p = \frac{\partial z}{\partial x}$.

Symbolic form of the given D.E is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} = 1$$

$$\Rightarrow (D^2 - DD' + D)z = 1$$

Here $f(D, D') = D^2 - DD' + D$, $Q(x,y) = 1 = e^{0x+0y}$

$$\Rightarrow f(D,D') = D[D - D' + 1]$$

$= (D - 0D' + 0)(D - D' + 1)$ are two different linear factors Compare with $(D - mD' - \alpha)$, we have

$m_1 = 0, \alpha_1 = 0$; $m_2 = 1$, $\alpha_2 = -1$

$$C.F = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

$$\Rightarrow C.F = e^{0x} f_1(y) + e^{-x} f_2(y+x)$$

$$C.F = f_1(y) + e^{-x} f_2(y+x).$$

(42)

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{D^2-DD'+D} e^{0x+0y} \quad \left(e^{ax+by} \text{ Model } (1) \right)$$

put $D=0, D'=0$, we have

$$P.I = \frac{1}{0} e^{0x+0y}, \quad \text{it is case of failure}$$

$$= x \cdot \frac{1}{\frac{\partial}{\partial D}(D^2-DD'+D)} e^{0x+0y}$$

$$= x \cdot \frac{1}{(2D-D'+1)} e^{0x+0y}$$

put $D=0, D'=0$, we have

$$P.I = x \cdot e^{0x+0y} = x,$$

$$G.S \; z = C.F + P.I$$

$$= f_1(y) + e^{-x} f_2(y+x) + x.$$

---

(7) $(D+D'-1)(D+2D'-3)z = 4+3x+6y$

Sol:- $f(D,D') = (D+D'-1)(D+2D'-3), \quad Q(x,y) = 4+3x+6y$

$m_1 = -1, \alpha_1 = 1 \quad m_2 = -2, \alpha_2 = 3$ different

$$C.F = e^{\alpha_1 x} f_1(y+m_1 x) + e^{\alpha_2 x} f_2(y+m_2 x)$$

$$C.F = e^{x} f_1(y-x) + e^{3x} f_2(y-2x)$$

(43)

Now $\frac{1}{f(D,D')} e^{0x+0y} = \frac{1}{(D+D'-1)(D+2D'-3)} e^{0x+0y}$

put $D=0$, $D'=0$, we have,

$= \frac{1}{3} e^{0x+0y} = \frac{1}{3}$.

$\frac{1}{f(D,D')} x = \frac{1}{(D+D'-1)(D+2D'-3)} x$

$= \frac{1}{(-1)\left[1-(D+D')\right](-3)\left[1-\left(\frac{D+2D'}{3}\right)\right]} x$

$= \frac{1}{3} \left[1-(D+D')\right]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} x.$

$= \frac{1}{3} \left[1+(D+D')+(D+D')^2+\cdots\right] \left[1+\frac{1}{3}(D+2D')+\frac{1}{9}(D+2D')^2+\cdots\right] x$

$= \frac{1}{3} \left[1+D+\cancel{D'}\right]\left[1+\frac{1}{3}D+\cancel{\frac{2}{3}D'}\right] x.$

$= \frac{1}{3} \left[1+D\right]\left[1+\frac{1}{3}D\right] x \qquad \left(\because D'(x)=0 \text{ i.e. } \frac{\partial}{\partial y}(x)=0\right)$

$= \frac{1}{3} \left[1+\frac{1}{3}D+D+\cancel{\frac{1}{3}D^2}\right] x \qquad \left(D^2(x)=0\right)$

$= \frac{1}{3} \left(1+\frac{4}{3}D\right) x = \underline{\frac{1}{3}\left(x+\frac{4}{3}\right)}$

$\frac{1}{f(D,D')} y = \frac{1}{(D+D'-1)(D+2D'-3)} (y)$

$= \frac{1}{(-1)\left[1-(D+D')\right](-3)\left[1-\left(\frac{D+2D'}{3}\right)\right]} (y)$

$= \frac{1}{3} \left[1-(D+D')\right]^{-1} \left[1-\left(\frac{D+2D'}{3}\right)\right]^{-1} (y)$

$= \frac{1}{3} \left[1+\cancel{D}+D'\right]\left[1+\cancel{\frac{D}{3}}+\frac{2}{3}D'\right] y \qquad \left[D(y)=0.\right]$

$= \frac{1}{3} (1+D')\left(1+\frac{2}{3}D'\right) y$

$= \frac{1}{3} \left(1+\frac{2}{3}D'+D'+\frac{2}{3}\cancel{D'^2}\right) y \qquad D'^2(y)=0$

(44)

$$\frac{1}{f(D,D')}(y) = \frac{1}{3}\left(1+\frac{5}{3}D'\right)y = \frac{1}{3}\left(y+\frac{5}{3}\right)$$

sub these in ①, we have

$$P.I = 4\left(\frac{1}{3}\right) + \cancel{3}\cdot\frac{1}{\cancel{3}}\left(x+\frac{4}{3}\right) + \overset{2}{\cancel{6}}\cdot\frac{1}{\cancel{3}}\left(y+\frac{5}{3}\right)$$

$$= \frac{4}{3} + x + \frac{4}{3} + 2y + \frac{10}{3}$$

$$P.I = x+2y+\frac{18}{3} = x+2y+6$$

$\therefore$ G.S is $z = C.F + P.I$

$$= e^{x} f_1(y-x) + e^{3x} f_2(y+2x) + x+2y+6.$$

(8) $(D-3D'-2)^3 z = 6e^{2x}\sin(3x+y)$

Sol: Here $f(D,D') = (D-3D'-2)^3$, repeated 3 times

$m=3,\ \alpha=2$

$$C.F = e^{\alpha x} f_1(y+mx) + x e^{\alpha x} f_2(y+mx) + x^2 e^{\alpha x} f_3(y+mx)$$

$$\Rightarrow C.F = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x).$$

$$P.I = \frac{1}{f(D,D')} Q(x,y)$$

$$= \frac{1}{(D-3D'-2)^3}\, 6e^{2x}\sin(3x+y) \qquad \left(e^{ax+by}\, V\right)$$

put $D = D+a$, $D' = D'+b$, $a=2, b=0$

$$= 6e^{2x}\cdot\frac{1}{(D+\cancel{2}-3D'-\cancel{2})^3}\sin(3x+y)$$

$$= 6e^{2x}\frac{1}{(D-3D')^3}\sin(3x+y),$$ This is Homogeneous D.E of the form $\frac{1}{f(D,D')}\sin(ax+by)$. ie $ax+by$ Combination fcn

Here $a=3,\ b=1,\ n=\text{order}=3$

$f_1(D,D') = f_1(3,1) = (3-3)^3 = 0$

Case of failure.

$$P.I = 6e^{2x}\cdot x\cdot\frac{1}{\frac{\partial}{\partial D}(D-3D')^3}\sin(3x+y)$$

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$$P.I = \cancel{6}^{2} x e^{2x} \frac{1}{\cancel{3}(D-3D')^2} \sin(3x+y)$$

put $D=3,\ D'=1,\ n=2$

$(D-3D')^2 = (3-3)^2 = 0$, Case of failure.

$$P.I = 2x e^{2x} \left\{ x \frac{1}{\frac{\partial}{\partial D}(D-3D')^2} \sin(3x+y) \right\}$$

$$= \cancel{2} x^2 e^{2x} \frac{1}{\cancel{2}(D-3D')} \sin(3x+y)$$

put $D=a=3,\ D'=b=1,\ n=1$

$D-3D' = 3-3=0$, again Case of failure.

$$P.I = x^2 e^{2x} \left\{ x \cdot \frac{1}{\frac{\partial}{\partial D}(D-3D')} \sin(3x+y) \right\}$$

$$P.I = x^3 e^{2x} \sin(3x+y)$$

G.S $z = C.F + P.I$

$$= e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x) + x^3 e^{2x} \sin(3x+y).$$

H.W ① $r - t + p - q = 0$ A) $z = f_1(y+x) + e^{-x} f_2(y-x)$

② $(D+D'-1)(D+2D'-2)z = 0$ A) $z = e^{x} f_1(y-x) + e^{2x} f_2(y-2x)$

③ $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$

A) $z = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{12} e^{x-2y}$

④ $(D^2 - DD' + D' - 1)z = \sin(x+2y)$

A) $z = e^{x} f_1(y) + e^{-x} f_2(y+x) - \frac{1}{2}\cos(x+2y)$

⑤ $(D-D'-1)(D-D'-2)z = e^{3x-y} + x$

A) $z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{6} e^{3x-y} + \frac{1}{2}(x + 3/2)$

Classification of second order P.D.E

The second order linear P.D.E

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + F(u) = 0$$

where A, B, C, D, E are real constants is said to be

(i) Hyperbolic if $B^2 - 4AC > 0$

(ii) Parabolic if $B^2 - 4AC = 0$

(iii) Elliptic if $B^2 - 4AC < 0$

for Example.

(i) The eqn $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ (Special case of wave eqn)

is hyperbolic since $A = 1$, $B = 0$, $C = -1$,

$$B^2 - 4AC = 0 - 4(1)(-1) = 4 > 0$$

(ii) The eqn $\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial t^2}$ (heat eqn) is parabolic

since $A = c$, $B = 0$, $C = 0$ and $B^2 - 4AC = 0$

(iii) The eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Two dimensional Laplace eqn) is elliptic, since $A = 1, B = 0, C = 1$

$$B^2 - 4AC = -4 < 0.$$

Applications of PDE.

Method of Separation of variables :-

Where we have a partial differential eqn involving two independent variables say x and y, we seek a solution in the form $X(x)Y(y)$ and write down various types of solutions. The following examples will explain the method.

(47) <u>problems on method of separation of variables</u>

(1) solve $y^3\frac{\partial z}{\partial x}+x^2\frac{\partial z}{\partial y}=0$ — (1)

<u>Sol:</u> let $z=X(x)Y(y)$ be the soln of (1)

$$\frac{\partial z}{\partial x}=X'(x)Y(y),\quad \frac{\partial z}{\partial y}=X(x)Y'(y)$$

sub These in (1), we have

$$y^3X'(x)Y(y)+x^2X(x)Y'(y)=0$$

$$\Rightarrow y^3X'(x)Y(y)=-x^2X(x)Y'(y)=0$$

$$\Rightarrow \frac{X'(x)}{x^2X(x)}=-\frac{Y'(y)}{y^3Y(y)}$$

In the above L.H.S is a function of x and R.H.S is a function of y and these are equal for all values of x and y. This is possible if and only if each is equal to the same constant (say) λ. This λ is called separation constant

$\therefore$ we have $\frac{X'(x)}{x^2X(x)}=\frac{-Y'(y)}{y^3Y(y)}=\lambda$ — (3)

from (3), we get The two ordinary differential equations

$$X'(x)=\lambda x^2X(x)$$

$$\Rightarrow \frac{X'(x)}{\cancel{X(x)}}=\lambda x^2X(x)$$

$$\Rightarrow \frac{dX}{dx}=\lambda x^2X$$

$$\Rightarrow \frac{dX}{X}=\lambda x^2dx$$

Integrate on B-S

$$\int\frac{dX}{X}=\lambda\int x^2dx$$

$$\Rightarrow \log X=\lambda\frac{x^3}{3}+\log C_1$$

$$\Rightarrow \log\left(\frac{X}{C_1}\right)=\lambda\frac{x^3}{3}$$

&

$$Y'(y)=-\lambda y^3Y(y)$$

$$\frac{dY}{dy}=-\lambda y^3Y$$

$$\Rightarrow \frac{dY}{Y}=-\lambda y^3dy,$$ Integrate on B-S

$$\Rightarrow \log Y=-\lambda\frac{y^4}{4}+\log C_2$$

$$\Rightarrow \log Y-\log C_2=-\lambda\frac{y^4}{4}$$

$$\Rightarrow \log\left(\frac{Y}{C_2}\right)=-\lambda\frac{y^4}{4}$$

$$\Rightarrow \frac{Y}{C_2}=e^{-\lambda\frac{y^4}{4}}$$

$\Rightarrow \frac{X}{c_1} = e^{\lambda \frac{x^3}{3}}$, $\frac{Y}{c_2} = e^{-\lambda \frac{y^4}{4}}$

$\Rightarrow X = c_1 e^{\lambda \frac{x^3}{3}}$, $Y = c_2 e^{-\lambda \frac{y^4}{4}}$

$\therefore$ soln of ① is given by $z = X(x)Y(y)$

$\Rightarrow z = c_1 e^{\lambda \frac{x^3}{3}} \cdot c_2 e^{-\lambda \frac{y^4}{4}}$

$z = c\, e^{\lambda \left(\frac{x^3}{3} - \frac{y^4}{4}\right)}$, $c = c_1 c_2$ is an arbitrary constant

(2) Solve by the method of separation of variables $u_x = 2u_t + u$ where $u(x,0) = 6e^{-3x}$

Sol: Given P.D.E $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ —— ①

Let the soln of eqn ① using method of seperation of variables is

$u = X(x)\,T(t)$

$\Rightarrow \frac{\partial u}{\partial x} = X'T$, $\frac{\partial u}{\partial t} = XT'$

sub these in ①, we have

$X'T = 2XT' + XT$

$\Rightarrow X'T = X[2T' + T]$

$\Rightarrow \frac{X'}{X} = \frac{2T' + T}{T} = \lambda$ (separation constant)

Now $\frac{X'}{X} = \lambda$

$\Rightarrow X' = \lambda X$

$\Rightarrow \frac{dX}{dx} = \lambda X$

$\Rightarrow \frac{dX}{X} = \lambda dx$

Variable separable, Integrate on B.S

$\frac{2T' + T}{T} = \lambda$

$2T' + T = \lambda T$

$\Rightarrow 2T' = (\lambda - 1)T$

$\Rightarrow 2\frac{dT}{dt} = (\lambda - 1)T$

$\Rightarrow \frac{dT}{T} = \frac{(\lambda - 1)}{2} dt$

V.S, Integrate on B.S

(49)

$$\int \frac{dX}{X} = \lambda \int dx.$$

$$\Rightarrow \log X = \lambda x + \log c_1$$

$$\Rightarrow \log\left(\frac{X}{c_1}\right) = \lambda x$$

$$\Rightarrow \frac{X}{c_1} = e^{\lambda x}$$

$$\Rightarrow \boxed{X(x) = c_1 e^{\lambda x}}$$

$$\int \frac{dT}{T} = \frac{(\lambda-1)}{2} \int dt$$

$$\Rightarrow \log T = \frac{(\lambda-1)}{2} t + \log c_2$$

$$\Rightarrow \log T - \log c_2 = \frac{(\lambda-1)}{2} t$$

$$\Rightarrow \log\left(\frac{T}{c_2}\right) = \frac{(\lambda-1)}{2} t$$

$$\Rightarrow \frac{T}{c_2} = e^{\frac{(\lambda-1)}{2} t}$$

$$\Rightarrow \boxed{T(t) = c_2 \, e^{\left(\frac{\lambda-1}{2}\right)t}}.$$

$\therefore u(x,t) = X(x)T(t)$

$$\Rightarrow u(x,t) = c_1 e^{\lambda x} \cdot c_2 \, e^{\left(\frac{\lambda-1}{2}\right)t}$$

$$\Rightarrow u(x,t) = c \, e^{\lambda x + \left(\frac{\lambda-1}{2}\right)t} \quad (2) \quad (\because c_1 \cdot c_2 = c)$$

but given $u(x,0) = 6e^{-3x}$

$$\therefore u(x,0) = c\, e^{\lambda x}.$$

$$\Rightarrow 6e^{-3x} = c\, e^{\lambda x}$$

Comparing like coefficients on B.S, we have.

$$\boxed{c = 6, \ \lambda = -3.}$$

Sub c, λ values in (2) we get The sol'n of (1)

$$\boxed{u(x,t) = 6\, e^{-3x-2t}}$$

(13) Using method of separation of variables solve $u_{xt} = e^{-t}\cos x$ with $u(x,0)=0$ and $u(0,t)=0$

Sol: Given P.D.E is $\frac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$ — (1)

Let the sol'n of eq'n (1) using method of separation of variables is $u = X(x)T(t)$

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$\Rightarrow \dfrac{\partial^2 u}{\partial x \partial t} = X' T'$.

sub this in ①, we have

$X'T' = e^{-t} \cos x$. where $X' = \dfrac{dX}{dx}$, $T' = \dfrac{dT}{dt}$

$\Rightarrow \dfrac{X'}{\cos x} = \dfrac{e^{-t}}{T'} = \lambda$ (separation constant)

$\dfrac{X'}{\cos x} = \lambda$

$\Rightarrow X' = \lambda \cos x$

$\Rightarrow \dfrac{dX}{dx} = \lambda \cos x$

$\Rightarrow dX = \lambda \cos x \, dx$

Integrate on B.S

$\Rightarrow \underline{X = \lambda \sin x + c_1}$

$\dfrac{e^{-t}}{T'} = \lambda$

$\Rightarrow T'\lambda = e^{-t}$

$\Rightarrow T' = \dfrac{1}{\lambda} e^{-t}$ (v-s, $\int$ on B.S)

$\Rightarrow \dfrac{dT}{dt} = \dfrac{1}{\lambda} e^{-t}$

$\Rightarrow dT = \dfrac{1}{\lambda} e^{-t} dt$

$\Rightarrow \boxed{T = -\dfrac{1}{\lambda} e^{-t} + c_2}$

sub $X(x)$ & $T(t)$ values in ②, we have.

$u(x,t) = [\lambda \sin x + c_1]\left[-\dfrac{1}{\lambda} e^{-t} + c_2\right]$ —— ③

but given $u(x,0) = 0$ & $u(0,t) = 0$.

Now $u(x,0) = 0$, from ③

$\Rightarrow u(x,0) = (\lambda \sin x + c_1)\left(\dfrac{-1}{\lambda} + c_2\right)$

$\Rightarrow 0 = (\lambda \sin x + c_1)\left(\dfrac{-1}{\lambda} + c_2\right)$

$\Rightarrow \dfrac{-1}{\lambda} + c_2 = 0 \Rightarrow \boxed{c_2 = \dfrac{1}{\lambda}}$ $(\because \lambda \sin x + c_1 \neq 0)$

Now $u(0,t) = 0$, from ③

$u(0,t) = c_1\left[\dfrac{-1}{\lambda} e^{-t} + c_2\right]$

$\Rightarrow 0 = c_1\left[\dfrac{-1}{\lambda} e^{-t} + c_2\right] \Rightarrow \boxed{c_1 = 0}$

sub c_1, c_2 values in ③ we get the soln of ①

$u(x,t) = \lambda \sin x\left[\dfrac{-1}{\lambda} e^{-t} + \dfrac{1}{\lambda}\right] = \underline{\sin x(1 - e^{-t})}$

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