UNIT-IV

PARTIAL DIFFERENTIAL EQUATIONS

Introduction

Partial differential equations are used to mathematically formulate ,and thus aid the solution of physical and other problems involving functions of several variables ,such as the propagation of heat or sound , fluid flow , elasticity , electro statistics, electro dynamics, etc.

Fluid mechanics, heat and mass transfer, and electromagnetic theory are all modeled by partial differential equations and all have plenty of real life applications.

For example,

- Fluid mechanics is used to understand how the circulatory system works, how to get rockets and planes to fly, and even to some extent how the weather behaves.
- Heat and mass transfer is used to understand how drug delivery devices work, how kidney dialysis works, and how to control heat for temperatute-sensitive things. It probably also explains why thermoses work!
- Electromagnetism is used for all electricity out there, and everything that involves light at all, from X rays to pulse oximetry and laser pointers.

Definition:

An equation which involves a dependent variable and its derivatives with respect to two or more independent variables is called partial differential equation.

Ex: $x\frac{\partial z}{\partial y} + 4y\frac{\partial z}{\partial x} = 2z + 3xy$

Notations: $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

Linear & non linear P.D.E:

If the partial derivatives of the dependent variable occur in first degree only and separately, Such a P.D.E is said to the linear P.D.E, otherwise it is said as non –linear P.D.E

Formation of partial differential equations:

Partial Differential equations can be formed by two methods

1.By the elimination of arbitrary constants

2.By the elimination of arbitrary functions

1.By elimination of arbitrary constants

Let the given function be $f(x,y,z,a,b) = 0$ ………(1) where a and b are arbitrary constants.

To eliminate a and b, differentiating (1) partially w.r.t. *'x' and 'y'*

$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.p=0$ ………………(2) and

$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.q=0$ ………………(3)

Now eliminate the constants a and b from (1), (2) and (3). We get a partial differential equation of the first order of the form. $\phi(x,y,z,p,q)=0$

Note : 1. If the number of arbitrary constants is equal to the number of variables, a partial differential equation of first order can be obtained.

2.If the number of arbitrary constants is greater than the number of variables, a partial differential equation of order higher than one can be obtained.

Solved Problems

1. **Form the partial differential equation by eliminating the arbitrary constants a and b from (i) $z = ax + by + ab$**

Sol: we have $z = ax + by + ab$ …….(1)

Differentiating (1) partially w.r.t. *'x' and 'y'*, we get

$\frac{\partial z}{\partial x}=a \Rightarrow p=a$ ……… (2) and $\frac{\partial z}{\partial y}=b \Rightarrow q=b$ ……….(3)

Putting the values of a and b from equation (2) and (3) in (1), we get

$$z = px + qy + pq$$

Which is the required partial differential equation

2. **Form the partial differential equation by eliminating the arbitrary constants a and b from (a)** $z=ax+by+a^2+b^2$ (b) $z=ax+by+\frac{a}{b}-b$

Sol: (a) we have $z=ax+by+a^2+b^2$ ……. (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\frac{\partial z}{\partial x}=a \Rightarrow p=a$ ……… (2) and $\frac{\partial z}{\partial y}=b \Rightarrow q=b$ ……….(3)

Putting the values of a and b from equation (2) and (3) in (1), we get

$$z = px + qy + p^2 + q^2$$

Which is the required partial differential equations

(b) We have $z = ax + by + \frac{a}{b} - b$ ....(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$ ..... (2) and $\frac{\partial z}{\partial y} = b \Rightarrow q = b$ .... (3)

Putting the values of a and b from equation (2) and (3) in (1), we get

$$z = px + qy + \frac{p}{q} - q$$

Which is the required partial differential equation.

3. Form the partial differential equation by eliminating the arbitrary constants from

$(x-a)^2 + (y-b)^2 + z^2 = r^2$

(OR)

Find the differential equation of all spheres of fixed radius having their centre on the xy −plane.

Sol: The equation of sphere of radius r having their centers on xy-plane is

$(x-a)^2 + (y-b)^2 + z^2 = r^2$ ....(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get.

$2(x-a) + 2z.\frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) + zp = 0 \; or \; x - a = -zp \rightarrow$ (2)

And $2(y-b) + 2z.\frac{\partial z}{2y} = 0 \; or \; (y-b) + zq = 0 \; or \; y - b = -zq \rightarrow$ (3)

Putting the values of (x-a) and (y-b) from (2) and (3) in (1), we get

$$(-zp)^2 + (-zq)^2 + z^2 = r^2$$

Which is the required partial differential equation.

4. **Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x + a)(y + b)$**

Sol: The given equation $z = (x + a)(y + b)$ −−−−−−−−−−−(1)

Differentiating (1) w.r.t., x

P = $\frac{\partial z}{\partial x} = 1.(y + b)$ −−−−−−−−−−−(2)

Differentiating (1) w.r.t., y

q = $\frac{\partial z}{\partial y} = 1.(x + a)$ −−−−−−−−−− (3)

from (2) $P = (y + b)$

$from\ (3)\ q = (x + a)$

Substituting in (1) we get

$$z = p.q$$

Which is the required partial differential equations

5. Form the partial differential by eliminating the arbitrary constants from $\log(az-1) = x + ay + b$

Sol: We have $\log(az-1) = x + ay + b$ ....(1)

Differentiating (1) partially $w.r.t.$ 'x' and 'y', we get

$$\frac{1}{(az-1)}.a.\frac{\partial z}{\partial x} = 1 \text{ or } \frac{1}{(az-1)}ap = 1 \text{ or } ap = az - 1 \text{ ....(2)}$$

and $\frac{1}{(az-1)}a.\frac{\partial z}{\partial y} = a \Rightarrow aq = (az-1)a$ ....(3)

$(3) \div (2), gives \frac{q}{p} = a \Rightarrow ap = q$ ....(4)

Putting (4) in (2), we get

$$q = \frac{q}{p}z - 1 \text{ or } pq = qz - p \text{ or } p(q+1) = q^2$$

Which is the required partial differential equation.

6.**Form the differential equation by eliminating a and b from** $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$

Sol: We have $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$ ....(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have,

$$2\frac{\partial z}{\partial x} = 2p = \frac{1}{2\sqrt{x+a}} \Rightarrow \frac{1}{\sqrt{x+a}} = 4p$$

or $\sqrt{x+a} = \frac{1}{4p}$

or $x + a = \frac{1}{16p^2} \rightarrow (2)$

And $2\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y-a}} \text{ or } 2q = \frac{1}{2\sqrt{y-a}} \text{ or } \sqrt{y-a} = \frac{1}{4q}$

$\therefore y - q = \frac{1}{16q^2} \rightarrow$ .... (3)

Adding (2) and (3), we get

$$x+y=\frac{1}{16}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)$$

$$\boxed{or\, 16(x+y)\,p^2q^2=p^2+q^2}$$

Which is the required partial differential equation.

7.Form the partial differential equation by eliminating the arbitrary constants a and b from $z=ax^3+by^3$

Sol: We have $z=ax^3+by^3 \rightarrow (1)$

Differentiating (1) partially w.r.t. 'x' *and* 'y', we get

$$\frac{\partial z}{\partial x}=3ax^2 \text{ or } p=3ax^2 \Rightarrow a=\frac{p}{3x^2}\rightarrow (2)$$

And $\dfrac{\partial z}{\partial y}=3by^2 \text{ or } q=3by^2 \Rightarrow b=\dfrac{q}{3y^2}\rightarrow (3)$

Putting the values of 'a' and 'b' from (2) and (3) in (1), we get

$$z=\frac{p}{3}x+\frac{q}{3}y$$

Or

$$\boxed{3z=px+qy}$$

8.Form the partial differential equation by eliminating the arbitrary constants a and b from $\boldsymbol{z=(x^2+a)(y^2+b)}$

Sol: The given equation $z=\ (\boldsymbol{x^2+a})\ (\boldsymbol{y^2+b})$ ------------(1)

Differentiating (1) w.r.t., x

$$p=\frac{\partial z}{\partial x}=2x(y^2+b) \text{ ------------(2)}$$

$$\therefore (y^2+b)=\frac{p}{2x}$$

Differentiating (1) w.r.t., y, we get

$$q=\frac{\partial z}{\partial y}=2y\,(x^2+a) \text{ ---------- (3)}$$

$$\therefore (x^2+a)=\frac{q}{2y}$$

Substituting in (1) we get $z=\frac{pq}{4xy}$ implies that

$$\boxed{pq-4xyz=0}$$

Which is the required partial differential equation.

9.Form the partial differential equation by eliminating the arbitrary constants from

$$(x-a)^2+(y-b)^2=z^2\cot^2\alpha$$

Sol: Given $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$……..(1)

Differentiating (1) w.r.t., x

$(x-a)=z\,\mathrm{p}\cot^2\alpha$

Differentiating (1) w.r.t., y

$(y-b)=z\,\mathrm{q}\cot^2\alpha$

Substituting (2),(3) in (1),we get

$(z\,\mathrm{p}\cot^2\alpha)^2+(z\,\mathrm{q}\cot^2\alpha)^2=z^2\cot^2\alpha$

∴The required Partial differential equation is

$$p^2+q^2=\tan^2\alpha$$

10.Form the partial differential equation by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol : Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ….. (1)

Differentiating (1) partially w.r.t. 'x' and 'y'.

$\frac{2x}{a^2}+\frac{2z}{c^2}.p=0 \text{ or } \frac{x}{a^2}+\frac{z}{c^2}.p=0\rightarrow$ (2)

And $\frac{2y}{b^2}+\frac{2z}{c^2}.q=0 \text{ or } \frac{y}{b^2}+\frac{z}{c^2}.q=0\rightarrow$ (3)

Since it is not possible to eliminate a,b,c from equation (1), (2) and (3). We require one more relation.

Differentiating (2), partially w.r.t. 'x', we get

$\frac{1}{a^2}+\frac{1}{c^2}\left(z.\frac{\partial p}{\partial x}+p.\frac{\partial z}{\partial x}\right)=0 \text{ or } \frac{1}{a^2}+\frac{1}{c^2}.z.\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}.p$

$\therefore \frac{1}{a^2}+\frac{1}{c^2}.zr+\frac{p^2}{c^2}=0\rightarrow$ (4)

Multiplying (4) by 'x' and then subtracting (2) from it, we get

$\frac{xz}{c^2}.r+\frac{xp^2}{c^2}-\frac{z}{c^2}.p=0 \text{ or } \frac{1}{c^2}\left(xzr+xp^2-zp\right)=0$

$$\therefore pz=xp^2+xzr$$

Which is the required partial differential equation.

Formation of the partial differential equation by the elimination of arbitrary functions:

Derive a p.d.e by the elimination of the arbitrary function ϕ from $\phi(u,v)=0$ where u, v are functions of x, y and z.

$\phi(u,v)=0$ …. (1)

Differentaite partially equation (1) w.r.to. x, y

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right)=0$$

i.e., $\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$……(2)

and $\frac{\partial\phi}{\partial 4}\left(\frac{\partial 4}{\partial y}+q\frac{\partial 4}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0$……(3)

Eliminating $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (2) and (3)

$$\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

i.e. $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

is the P.D.E after the elimination of ϕ from $\phi(u,v)=0$. Written in a simpler form

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

Above equation is generally written as pP+qQ=R where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\ and\ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Solved Problems

1.Form a partial differential equation by eliminating the arbitrary function

$$\boldsymbol{z = f(x^2 + y^2)}$$

Sol: We have $z = f(x^2 + y^2)$ …. . (1)

Put $u = x^2 - y^2$, we have $z = f(u) \rightarrow$ (2)

Differentiating (2) partially w.r.t. ‘x’ and ‘y’,

$$\frac{\partial z}{\partial x}=f^1(u).\frac{\partial u}{\partial x}=f^1(u).2x$$

$\therefore p = f^1(u)2x \rightarrow$ (3)

And $\frac{\partial z}{\partial y}=f^1(u).\frac{\partial u}{\partial y}=f^1(u).2y$

$\therefore q = f^1(u)2y \rightarrow$ (4)

$\therefore (3) \div (4)$, gives $\dfrac{p}{q} = \dfrac{f^1(u).2x}{f^1(u)2y} = \dfrac{x}{y}$

$$\therefore py - qx = 0$$

Which is the required partial differential equation.

2. Form a partial differential equation by eliminating the arbitrary function$\boldsymbol{\varphi(x^2 + y^2, z - xy) = 0}$

Sol: Given $\varphi(x^2 + y^2, z - xy) = 0$

This can be written as $z - xy = f(x^2 + y^2)$--------------(1)

Now we have to eliminate f from (1)

Differentiating (1) w.r.t., x

$$\frac{\partial z}{\partial x} - y = f'(x^2 + y^2)(2x)$$

$$p - y = f'(x^2 + y^2)(2x)\text{---------}(2)$$

Differentiating (2) w.r.t., y

$$q - x = f'(x^2 + y^2)(2y)\text{---------}(3)$$

Dividing (2) by (3)

$$p\,y - q\,x = y^2 - x^2$$

Which is the required partial differential equation.

3. Form a partial differential equation by eliminating the arbitrary function from $z = f\left(x^2 - y^2\right)$

Sol : We have $z = f\left(x^2 - y^2\right) \rightarrow$ (1)

Put $u = x^2 - y^2$, we have $z = f(u) \rightarrow$ (2)

Differentiating (2) partially w.r.t. 'x' and 'y',

$$\frac{\partial z}{\partial x} = f^1(u).\frac{\partial u}{\partial x} = f^1(u).2x$$

$\therefore p = f^1(u)2x \rightarrow$ (3)

Similarly we get

$q = -f^1(u)2y(4)$

$\therefore (3) \div (4)$, gives $\frac{p}{q} = \frac{x}{-y}$

$$\therefore px + qy = 0$$

Which is the required partial differential equation.

4. Form the partial differential equation by eliminating the arbitrary functions from

$$xyz = f\left(x^2+y^2+z^2\right)$$

Sol: We have $xyz = f\left(x^2+y^2+z^2\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. x and y

$$yz + xy.p = f^1\left(x^2+y^2+z^2\right).\left(2x+2z.\frac{\partial z}{\partial x}\right)$$

(or) $yz + xyp = f^1\left(x^2+y^2+z^2\right).(2x+2zp) \rightarrow$ (2)

And $xz + xy.q = f^1\left(x^2+y^2+z^2\right).(2y+2z.q) \rightarrow$ (3)

$\therefore (2)\div(3)$, gives

$$\frac{yz+xyp}{xz+xyq} = \frac{2x+2zp}{2y+2zq}$$

$$(yz+xyp)(y+zq) = (xz+xyq)(x+zp)$$

$$y^2z + z^2yq + xy^2p + xyzpq = x^2z + x^2zp + x^2yq + xyzpq$$

$$\boxed{x\left(y^2-z^2\right)p + y\left(z^2-x^2\right)q = \left(x^2-y^2\right)z}$$

Which is the required partial differential equation.

5. Form the partial differential equation by eliminating the arbitrary functions from $xyz = f(x+y+z)$

Sol: Given equations $xyz = f(x+y+z)$---------(1)

Differentiating (1) partially w.r.t. 'x'

y(xp+z) = $f'(x+y+z)(1+p)$----------(2)

Differentiating (1) partially w.r.t. 'x'

x(y q + z) = $f'(x+y+z)(1+q)$------------(3)

Dividing (2) by (3) $\frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$

$$Y(xp+z)(1+q) = x(yq+z)(1+p)$$

$$(xy-zx)p + (yz-xy)q = zx - yz$$

$$\boxed{x(y-z)p + y(z-x)q = z(x-y)}$$

Which is the required partial differential equation.

6. Form the partial differential equation by eliminating the arbitrary function

from $xy + yz + zx = f\left(\frac{z}{x+y}\right)$

Sol : We have $xy + yz + zx = f\left(\frac{z}{x+y}\right) \to (1)$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$y + y.p + z + x.p = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y).p - z\right]}{(x+y)^2} \to (2)$$

And $x + z + yq + xq = f^1\left(\frac{z}{x+y}\right)\frac{\left[(x+y)q - z\right]}{(x+y)^2} \to (3)$

Dividing (2) by (3), we get

$$\boxed{\frac{(x+y)p + y + z}{(x+y)q + x + z} = \frac{(x+y)p - z}{(x+y)q - z}}$$

is the required partial differential equation.

7.Form the partial differential equation by eliminating the arbitrary function from $z = f(x) + e^y.g(x)$

Sol: We have $z = f(x) + e^y.g(x) \to (1)$

Differentiating (1) partially w.r.t. 'x' and y, we get

$$\frac{\partial z}{\partial x} = f^1(x) + e^y.g^1(x) \text{ or } p = f^1(x) + e^y.g^1(x) \to (2)$$

And $q = e^y.g(x)$ or $\frac{\partial z}{\partial y} = e^y.g(x) \to (3)$

Differentiating (3), partially w.r.t. 'y', we get

$$\frac{\partial^2 z}{\partial y^2} = e^y.g(x) = \frac{\partial z}{\partial y} \text{ [using (3)]}$$

$$\therefore \frac{\partial^2 z}{\partial y^2} - \frac{\partial z}{\partial y} = 0$$

$$\boxed{\therefore t - q = 0}$$

Which is the required P.D.E.

8.Form a partial differential equation by eliminating the arbitrary

function $\varphi(x^2+y^2+z^2, ax+by+cz)=0$

Sol: Given function can be written as

$$x^2+y^2+z^2 = f(ax+by+cz) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiating (1) partially w.r.t. 'x' and 'y',we get

$$2x+2zp = (a+cp)f^1(ax+by+cz) \ldots. (2)$$

$$and$$

$$2y+2zq = (b+cq)f^1(ax+by+cz) \ldots. (3)$$

$\frac{(2)}{(3)}$ implies

$$\frac{x+zp}{y+zq} = \frac{(a+cp)}{(b+cq)}$$

Which is the required complete solution of given Partial differential equation.

Solution Of Partial Differential Equations :

Complete integral:

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular integral :

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral or particular solution.

Singular integral:

$Let\ f(x,y,z,p,q)=0 \rightarrow$ (1) be the partial differential equation.

$Let\ \phi(x,y,z,a,b)=0 \rightarrow$ (2)

be the complete integral of (1). Where a and b are arbitrary constants.

Now find $\frac{\partial\phi}{\partial a}=0 \rightarrow$ (3) $\frac{\partial\phi}{\partial b}=0 \rightarrow$ (4)

Eliminate a and b between the equations(2), (3) & (4) When it exists is called the singular integral of (1).

General integral : In the complete integral (2). Assume that one of the constant is a function of the other i.e. $b = f(a)$ Then (2), becomes $\phi(x,y,z,a,f(a))=0 \rightarrow$ (5)

Differentiating (5) partially w.r.t. 'a', we get $\frac{\partial\phi}{\partial a}+\frac{\partial\phi}{\partial f}.f^1(a)=0 \rightarrow$ (6)

Eliminate 'a' between (5) and (6), when it exists is called the general integral or general solution of (1).

Linear Partial Differential Equations Of The First Order:

A differential equation involving partial derivatives p and q only and no higher order derivatives is called a first order equation. If p and q occur in the first degree, it is called a linear partial differential equation of first order; otherwise it is called a non-linear partial differential equation of the first order.

For example: $px+qy^2=z$ is a linear p.d.e of first order and $p^2+q^2=1$ is non-linear

Lagrange's Linear Equation:

A linear partial differential equation of order one involving a dependent variable z and two independent variables x and y of the form $Pp+Qq=R$

Where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

Lagrange's auxiliary equations or Lagrange's subsidiary equations

The equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ are called Lagrange's auxiliary equations.

Working Rule To Solve Lagrange's Linear Equation $Pp+Qq=R$

Step 1: Write down the auxiliary equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Step 2 : Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u=a$ and $v=b$ where a, b are arbitrary constants

Step 3: Then $\emptyset(u, v) = 0$ or $u=f(v)$ is the general solution of the equation $Pp+Qq=R$

To solve $\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}$ …… (1)

(i) **Method of grouping** : In some problems, it is possible that two of the equations $\frac{dx}{P}=\frac{dy}{Q}$ or $\frac{dy}{Q}=\frac{dz}{R}$ or $\frac{dx}{P}=\frac{dz}{R}$ are directly solvable to get solutions $u(x, y)$ = constant or $v(y, z)$ =constant or $w(x, z)$ =constant. These give the complete solutions of (1)

Sometimes one of them, say $\frac{dx}{P}=\frac{dy}{Q}$ may give rise to solution $u(x, y)=c_1$

From this we may express y, as a function of x. Using this in $\frac{dy}{Q}=\frac{dz}{R}$ and integrating we get $v(y, z)=c_2$. These two relations $u= c_1, v=c_2$ give the complete solution of (1)

(ii). **Method of multipliers**: This is based on the following elementary result.

If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=......=\frac{a_n}{b_n}$ then each ratio is equal to $\frac{l_1a_1+l_2a_2+....+l_na_n}{l_1b_1+l_2b_2+....+l_nb_n}$

Consider $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

If possible identity multipliers l, m, n not necessarily constant, so that each ratio

$=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

Where $l\,P + m\,Q + n\,R = 0$ Then $l\,dx + m\,dy + n\,dz = 0$

Integrating this we get $u(x,y,z) = c_1$.

Similarly we get another solution $v(x,y,z) = c_2$ independent of the earlier one.

We have the complete solution of (1) constituted $by\ u = c_1\ and\ v = c_2$

Solved Problems

1. **Solve $p\tan x + q\tan y = \tan z$**

Sol : The given equations can be written as $\tan xp+\tan yq=\tan z \to$ (1)

Comparing with $Pp + Qq = R$, we have $P=\tan x, Q=\tan y, R=\tan z$

$\therefore$ The auxiliary equations are $\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$

Taking the first two members, we have $\frac{dx}{\tan x}=\frac{dy}{\tan y}$

Integrating $\log\sin x = log\sin y + l\,og\,c_1$

or $\log\frac{\sin x}{\sin y}=\log c_1$ *or* $\frac{\sin x}{\sin y}=c_1 \to$ (2)

Taking the last two members, we have $\frac{dy}{\tan y}=\frac{dt}{\tan z}$

Integrating, $log\ siny = log\ sinz + logc2$

or $\log\frac{\sin y}{\sin z}=\log c_2$ *or* $\frac{\sin y}{\sin z}=c_2 \to$ (3)

From (2) and (3). The general solution of (1) is

$\phi(c_1,c_2)=0$

$i.e.\phi\left(\frac{\sin x}{\sin y},\frac{\sin y}{\sin z}\right)=0$ is the required Complete Solution.

2. **Find the general solution of $y^2zp+x^2zq=y^2x$**

Sol: We have $y^2zp+x^2zq=y^2x \rightarrow$ (1)

Comparing $with\ Pp+Qq=R, we\ have$

$P=y^2z, Q=x^2z,\ R=y^2x$

$\therefore$ The auxiliary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

Taking the first two members, we have

$\frac{dx}{y^2z}=\frac{dy}{x^2z} \Rightarrow \frac{dx}{y^2}=\frac{dy}{x^2} or\ x^2dx=y^2dy$

Integrating, $\frac{x^3}{3}=y^3/3+c_1\ or\ \frac{x^3}{3}-\frac{y^3}{3}=c_1 \rightarrow$ (2)

Taking the first and last two members, we have

$\frac{dx}{y^2z}=\frac{dz}{y^2x} or\ xdx=zdz$

Integrating $\frac{x^2}{2}=\frac{z^2}{2}+c_2\ or\ \frac{x^2}{2}-\frac{z^2}{2}=c^2 \rightarrow$ (3)

From (2) and (3) The general solution of (1) is

$\phi(c_1,c_2)=0\ i.e.$

$\phi\left(\frac{x^3}{3}-\frac{y^3}{3},\frac{x^2}{2}-\frac{z^2}{2}\right)=0$ is the required Complete Solution.

3. **Solve** $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Sol: The given equation can be written as

$\sqrt{x}p+\sqrt{y}q=\sqrt{z} \rightarrow (1)$

Comparing with Pp+Qq=R, we have

$P=\sqrt{x}, Q=\sqrt{y}, R=\sqrt{z}$

$\therefore$ The auxiliary equations are $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

From the first two members, we have $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}$

Integrating, $2\sqrt{x}=2\sqrt{y}+c_1\ or\ 2\sqrt{x}-2\sqrt{y}=c_1$ or $\sqrt{x}-\sqrt{y}=a \rightarrow$ (2)

From the last two members, we have $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

Integrating, $2\sqrt{y}=2\sqrt{z}+c_2\ or\ 2\sqrt{y}-2\sqrt{z}=c_2$

$or \sqrt{y}-\sqrt{z}=b \rightarrow$ (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b)=0$ *i.e.*,

$\phi\left(\sqrt{x}-\sqrt{y},\sqrt{y}-\sqrt{z}\right)=0$ is the required Complete Solution.

4. **Solve** $x(y-z)p+y(z-x)q=z(x-y)$

Sol: We have $x(y-z)p+y(z-x)q=z(x-y)\rightarrow$ (1)

Comparing with $Pp+Qq=R$, we have

$P=x(y-z),Q=y(z-x),R=z(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$

Using $l=1,m=1,n=1$ as multipliers, we get

$\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}=\dfrac{dx+dy+dz}{0}$ $\left[\because x(y-z)+y(z-x)+z(x-y)=0\right]$

$\therefore dx+dy+dz=0$

Integrating, $x+y+z=a\rightarrow$ (2)

Again using $l=\dfrac{1}{x},m=\dfrac{1}{y},n=\dfrac{1}{z}$ as multipliers, we get

Each fraction $=\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=k$(say)

$\therefore \dfrac{1}{x}dx+\dfrac{1}{y}dy+\dfrac{1}{z}dz=0$

Integrating, $logx+logy+logz=logb.or\ xyz=b\ldots\ldots$ (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b)=0$ *i.e.*,

$\phi(x+y+z,xyz)=0$ is the required Complete Solution.

5. **Solve** $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

Sol: Given $x^2(y-z)p+y^2(z-x)q=z^2(x-y)\rightarrow$ (1)

Comparing with $Pp+Qq=R$, we have

$P=x^2(y-z),Q=y^2(z-x),R=z^2(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x^2(y-z)}=\dfrac{dy}{y^2(z-x)}=\dfrac{dz}{z^2(x-y)}$

Using $l=\frac{1}{x^2}, m=\frac{1}{y^2}, n=\frac{1}{z^2}$ as multipliers, we get

Each fraction $=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$ =k(say)

$\therefore \frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz=0$

Integrating , $-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=a \quad or \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1 \rightarrow (2)$

Again using $l=\frac{1}{x}, m=\frac{1}{y}, n=\frac{1}{z}$ as multipliers, we get

Each fraction $=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=k(\text{say})$

$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating $\log x+\log y+\log z=\log c_2$

$or\ xyz=c_2 \rightarrow$ (3)

From (2) and (3), The general solution of (1) is .

$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0$ is the required Complete Solution.

6. **Solve** $(mz-ny)p+(nx-lz)q=ly-mx$

Sol: Given eqn is $(mz-ny)p+(nx-lz)q=ly-mx \rightarrow$ (1)

Comparing with $Pp+Qq=R$, we have

$P=mz-ny, Q=nx-lz, R=ly-mx$

$\therefore$ The auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using $l=x, m=y, n=z$ as multipliers, we get

Each fraction $=\frac{xdx+ydy+zdz}{0}$

$\therefore xdx+ydy+zdz=0$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a\ or\ x^2+y^2+z^2=c_1 \rightarrow (2)$

Again using l, m, n as multipliers, we get

Each fraction $= \frac{ldx + mdy + nz}{0} = k(\text{say})$

$\therefore ldx + mdy + ndz = 0$

Integrating, $lx + my + nz = c_2 \rightarrow$ (3)

From (2) and (3), the general solution of (1) is

$\phi\left(x^2 + y^2 + z^2, lx + my + nz\right) = 0$ is the required Complete Solution.

7. Solve $xp - yq = y^2 - x^2$

Sol: Here $P = x, Q = y, R = y^2 - x^2$

$\therefore$ The auxiliary eqn's are $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$

From the first two members, $\frac{dx}{x} = \frac{dy}{-y}$

Integrating, $\log x + \log y = \log c_1 \; or \; xy = c_1 \rightarrow (1)$

Using $l = x, m = y, n = 1$ as multipliers, we get

Each fraction $= \frac{xdx + ydy + dz}{0}$

$\therefore xdx + ydy + dz = 0$

Integrating, $\frac{1}{2}x^2 + \frac{1}{2}y^2 + z = c \; or \; x^2 + y^2 + 2z = c_2 \rightarrow$ (2)

From (1) and (2), The general solution is

$\phi\left(xy, x^2 + y^2 + 2z\right) = 0$ is the required Complete Solution.

8. Find the integral surface of $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = \left(x^2 - y^2\right)z$

Which contains the straight line x+y=0, z=1

Sol: Given that $x\left(y^2 + z\right)p - y\left(x^2 + z\right)q = \left(x^2 - y^2\right)z$ ……..(1)

Comparing with $Pp + Qq = R$, we have

$P = x\left(y^2 + z\right), Q = -y\left(x^2 + z\right), R = \left(x^2 - y^2\right)z$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x\left(y^2 + z\right)} = \frac{dy}{-y\left(x^2 + z\right)} = \frac{dz}{\left(x^2 - y^2\right)z}$$

Using $l = \frac{1}{x}, m = \frac{1}{y}, n = \frac{1}{z}$ as multipliers, we get

Each fraction = $\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating, $\log x+\log y+\log z=\log a$

or $xyz=a \to$ (2)

Again using $l=x, m=y, n=-1$ as multipliers, we get

$\therefore$ Each fraction $=\dfrac{xdx+ydy-dz}{0}$ =k(say)

$\therefore xdx+ydy-dz=0$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}-z=c$ or $x^2+y^2-2z=b \to$ (3)

Given that $z=1$, using this (2) and (3), we get

$xy=a$ and $x^2+y^2-2=b$

Now b+2a= $x^2+y^2-2+2xy=(x+y)^2-2$ =0-2 $[\therefore x+y=0]$ =-2

$\therefore 2a+b+2=0$

Hence the required surface is

$x^2+y^2-2z+2xyz+2=0$ is the required Complete Solution.

9.Solve $px+qy=z$

Sol: Given $px+qy=z$ is a Lagrange's linear equation

The Auxillary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

By Consider first group, we get

$$\int\frac{dx}{x}=\int\frac{dy}{y}$$

$$\log x=\log y+\log c1$$

$c_1=\frac{x}{y}$.....(1)

By Consider second group, we get

$$\int \frac{dz}{z} = \int \frac{dy}{y}$$

$$\log y = \log z + \log c2$$

$c_2 = \frac{y}{z}$.....(2)

$\therefore f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ is the required solution.

10. Solve $(x^2 - y^2 - yz)p + (x^2 - y^2 - xz)q = z(x - y)$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2 - y^2 - yz)} = \frac{dy}{(x^2 - y^2 - xz)} = \frac{dz}{z(x - y)}$$

Taking 1,-1-1 multipliers,we get

$\frac{dx - dy - dz}{(x^2 - y^2 - yz - x^2 + y^2 + xz - xz + yz)} = \frac{dx}{(x^2 - y^2 - yz)}$

$$dx - dy - dz = 0$$

Integrating,we get

$x - y - z = c_1$.........(1)

Taking $x, -y, 0$ as multipliers,we get

$$\frac{xdx - ydy}{(x^3 - xy^2 - xyz - yx^2 + y^3 + xyz)} = \frac{dz}{z(x - y)}$$

$$\frac{xdx - ydy}{(x^2 - y^2)(x - y)} = \frac{dz}{z(x - y)}$$

$$\frac{1}{2}\log(x^2 - y^2) = \log z$$

$\frac{x^2 - y^2}{z^2} = c_2$ … … … … … … … (2)

$\therefore$Complete solution of given pde is $\varphi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$

11. Solve $x(y^2 - z^2)p - y(x^2 + z^2)q = z(x^2 + y^2)$

Sol: The auxiliary equations are

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{-y(x^2+z^2)}=\frac{dz}{z(x^2+y^2)}$$

Taking $x, y, z,$ multipliers, we get

$$\frac{xdx+ydy+zdz}{(x^2y^2-x^2z^2-y^2x^2-z^2y^2+x^2z^2+y^2z^2)}=\frac{dx}{x(y^2-z^2)}$$

$$xdx+ydy+zdz=0$$

$$x^2+y^2+z^2=c_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Taking $-\frac{1}{x},\frac{1}{y}\frac{1}{z}$, multipliers, we get

$-\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating, we get

$\frac{yz}{x}=c_2$.......(2)

From (1),(2),

Complete solution of given pde $is\ \varphi\left(\frac{yz}{x},x^2+y^2+z^2\right)=0$

12. Solve $(y^2)p-xyq=x(z-2y)$

Sol: Comparing with $Pp+Qq=R$, we have

The auxiliary equations are

$\therefore \frac{dx}{y^2}=\frac{dy}{-yx}=\frac{dz}{x(z-2y)}$

From the first two members, we have

$$\frac{dx}{y}=\frac{dy}{-x}$$

Integrating,we get

$x^2+y^2=c_1$.....(2)

From the last two members, we have

$$\frac{dy}{-y}=\frac{dz}{(z-2y)}$$

$$-ydz=zdy-2ydy$$

$$d(yz)-2ydy=0$$

$yz-y^2=c_2$ **..........(3)**

From (2) and (3). The general solution of (1) is

i.e., $\emptyset(yz - y^2, x^2 + y^2)=0$

13. Solve $(y+z)p + (z+x)q = (x+y)$

Sol: Comparing with $Pp + Qq = R$, we have

The auxiliary equations are

$\therefore \frac{dx}{(y+z)} = \frac{dy}{(z+x)} = \frac{dz}{(x+y)}$

Taking 1,1,1 and 1,-1,0 and 0,1,-1 as multipliers , we have $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{(y-x)} = \frac{dy-dz}{(z-y)}$

From the last two members, we have

$$\frac{dx-dy}{(y-x)} = \frac{dy-dz}{(z-y)}$$

Integrating,we get

$$\log\frac{(y-x)}{(z-y)} = \log C_2$$

$\frac{(y-x)}{(z-y)} = c_2$.....(1)

From the first two members, we have

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{(y-x)}$$

Integrating,we get

$$\frac{1}{2}\log(x+y+z) = \log(y-x) + \log c_1$$

$(x+y+z)(y-x)^2 = c_1$..........(2)

From (2) and (1). The general solution of given pde is

i.e., $\emptyset\left(\frac{(y-x)}{(z-y)}, (x+y+z)(y-x)^2\right)=0$

14. Solve $x^2p - y^2q = z(x-y)$

Sol: Comparing with $Pp + Qq = R$, we have

The auxiliary equations are

$\therefore \frac{dx}{x^2} = \frac{dy}{-y^2} = \frac{dz}{z(x-y)}$

From the first two members, we have

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating,we get

$$\frac{1}{x} + \frac{1}{y} = c_1 \ldots..(1)$$

Taking 1,1,0 as multipliers,we get

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)} = \frac{dz}{z}$$

Integrating,we get

$$\frac{x+y}{z} = c_2 \quad \ldots.(2)$$

From (2) and (1). The general solution is

$\emptyset(\frac{x+y}{z}, \frac{1}{x} + \frac{1}{y})=0$

15. Solve $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-xz)} = \frac{dz}{(z^2-xy)}$$

Taking 1,-1,0 and 0,-1,-1 as multipliers,we get

$\frac{dx-dy}{(x^2-yz)-(y^2-xz)}$ and also $\frac{dy-dz}{((-z^2+yx)+(y^2-xz))}$

$\therefore \frac{dx-dy}{(x^2-yz)-(y^2-xz)} = \frac{dy-dz}{((-z^2+yx)+(y^2-xz))}$

$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z))}$

solving it,we get

$\frac{(x-y)}{y-z} = c_1 \ldots..(1)$

$Taking\ x, y, z\ and$ 1,1,1 as multipliers,we get

$\frac{(xdx+ydy+zdz)}{x^3+y^3+z^3-3xyz} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$

$\frac{(xdx+ydy+zdz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$

$$(x+y+z)(dx+dy+dz) = (xdx+ydy+zdz)$$

$$(x+y+z)d(x+y+z) = (xdx+ydy+zdz)$$

Integrating,we get

$$\frac{(x+y+z)^2}{2} = \frac{x^2+y^2+z^2}{2} + c$$

$$\therefore (x+y+z)^2 = x^2+y^2+z^2+c_2$$

$xy+yz+zx = c_2$…..(2)

Complete solution of given pde is $\varphi\left(xy+yz+zx, \frac{(x-y)}{y-z}\right) = 0$

Non-Linear Partial Differential Equations Of First Order

A partial differential equation which involves first order partial derivatives $p \; and \; q$ with degree higher than one and the products of p and $q \; is$ called a non-linear partial differential equations.

Non linear PDE's can be classified in to 4 standard forms.

Standard Form I:

Equation of the form $f(p,q) = 0$ (i.e., equations containing p and q only) :

Given partial differential equation is of the form $f(p,q) = 0$ … … (1)

Procedure:

Given partial differential equation is $f(p,q) = 0$ … … (1)

Step1:Put $p = a$ in (1), then we get q value in terms of a then we can obtain 'p' value.

Step2:Substitute p, q values in $dz = p\,dx + q\,dy$

Step3:Integrating it ,we get required complete solution of (1) .

Solved Problems

1.Solve $pq = k$, where k is a constant.

Sol: Given that $pq = k$ …. (1)

Since (1) is of the form $f(p,q) = 0$

Put $p = a$ in (1),we get $q = \frac{k}{a}$

Now substitute p, q in

$dz = p\,dx + q\,dy$ then

$dz = a\,dx + \frac{k}{a}\,dy$

Integrating,we get

$z = ax + \frac{k}{a}y + c$

which contains two arbitrary constants a and c.

2.Solve $p^2 + q^2 = npq$

Sol : Given that $p^2+q^2=npq$ ……………..(1)

Since (1) is of the form $f(p,q)=0$

Put $p=a$ in (1),then we get $q=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]$

Now substitute p,q in

$dz = p\,dx+q\,dy$

$=a\,dx+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]dy$

Integrating,we get , $dz=a\int dx+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]\int dy$

$$z=ax+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]y+c$$

This is the complete integral of (1), which contains two arbitrary constants a and c.

3.Find the complete integral of $p^2+q^2=m^2$

Sol: Given that $p^2+q^2=m^2$ ………….(1)

Since (1) is of the form $f(p,q)=0$

Put $p=a\ in\ (1), we\ get\ q=\sqrt{m^2-a^2}$

Now substitute p,q in

$dz = p\,dx+q\,dy$ …….(2) then

$dz = adx+q=\sqrt{m^2-a^2}\,dy$

Now integrate on both sides

$$z=ax+\left(\sqrt{m^2-a^2}\right)y+c$$

Which is the complete integral of (1)

Standard Form II :

Equation of the form $f(p,q,z)=0$(i.e., not containing x and y)

Procedure :

Given partial differential equation is of the form $f(p,q,z)=0$ ……(1)

Step1:Put $p=aq$ in (1),then we get q value in terms of a , z then

Step2:Substitute p,q values in $dz = p\,dx+q\,dy$

Step3:Integrating it ,we get required complete solution of (1) .

Solved Problems :

Solve the following partial differential equations

1. $z=p^2+q^2$ 2. $p^2z^2+q^2=p^2q$ 3. $zpq=p+q$

Sol : 1. We have $z = p^2+q^2$…….(1)

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq$ in (1), then we get $q = \sqrt{\frac{z}{1+a^2}}$

$$\therefore p = a\sqrt{\frac{z}{1+a^2}}$$

Putting the values of p and q in $dz = p\,dx + q\,dy$, we get

$\frac{1}{\sqrt{z}}dz = \frac{1}{\sqrt{1+a^2}}(a\,dx + dy)$,

Integrating ,we get

$$\int \frac{1}{\sqrt{z}}dz = \frac{1}{\sqrt{1+a^2}}\int (a\,dx + dy)$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}(ax + y)$$

which is the required solution of (1)

2. Given that $p^2z^2 + q^2 = p^2q \rightarrow$ (1)

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq$ in (1), then we get $q = \frac{(a^2z^2+1)}{a^2}$

$$\therefore q = \frac{(a^2z^2+1)}{a}$$

Putting the values of p and q in $dz = p\,dx + q\,dy$,we get

$\frac{dz}{(a^2z^2+1)} = \frac{1}{a^2}(adx + dy)$

Integrating,we get

$\int \frac{dz}{(a^2z^2+1)} = \frac{1}{a^2}\int (adx + dy)$

$$\therefore \text{a} \tan^{-1}(az) = ax + y + c$$

which is the required complete solution of (1)

3. Given that $zpq = p + q$…..(1)

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq$ in (1), $then$ we get $q = \frac{a+1}{az}$

$$\therefore p = \frac{a+1}{z}$$

Putting the values of p and q in $dz = p\,dx + q\,dy$, we get

$zdz = \frac{a+1}{a}(a\,dx + dy)$,

Integrating ,we get

$$\int z\,dz = \frac{a+1}{a}\int (a\,dx + dy)$$

$$\frac{aZ^2}{2(a+1)} = ax + y + c$$

This is the required solution of (1)

Standard Form III :

Equation of the form $f_1(x,p) = f_2(y,q)$ i.e. Equations not involving z and the terms containing x and p can be separated from those containing y and q.

We assume that these two functions should be equal to a constant say k.

$\therefore f_1(x,p) = f_2(y,q) = k$

Solve for p and q from the resulting equations

$\therefore f_1(x,p) = k \; and \; f_2(y,q) = k$

Solve for p and q, we obtain

$p = F_1(x,k) \; and \; q = F_2(y,k)$

Since z is a function of x and y

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ [By total differentiation]

$dz = p\,dx + q\,dy$

$\therefore dz = F_1(x,k)dx + F_2(y,k)dy$

Integrating on both sides

$z = \int F_1(x,k)dx + \int F_2(y,k)dy + c$

Which is the complete solution of given equation

Solved Problems:

1. Solve $p^2 + q^2 = x + y$

Sol :. Given that $p^2 + q^2 = x + y$ ....(1)

Separating $p \; and \; x \; from \; q \; and \; y$, the given equation can be written as

$$p^2 - x = -q^2 + y$$

Let $p^2 - x = -q^2 + y = k$ (constant)

$\therefore p^2 - x = k \; and \; -q^2 + y = k$

$\Rightarrow p^2 = k + x \; and \; q^2 = y - k$

$\therefore p=\sqrt{k+x}$ and $q=\sqrt{y-k}$

Since $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore dz=\sqrt{k+x}\,dx+\sqrt{y-k}\,dy$

Integrating on both sides

$z=\int(k+x)^{\frac{1}{2}}dx+\int(y-k)^{\frac{1}{2}}dy+c$

$$\therefore z=\frac{2}{3}(k+x)^{\frac{3}{2}}+\frac{2}{3}(y-k)^{\frac{3}{2}}+c$$

Which is the complet solution of (1)

2. Solve $xp-yq=y^2-x^2$

Sol: Given that $xp-yq=y^2-x^2 \rightarrow (1)$

Separating p and x from q and y. The given equation can be written as.

$xp+x^2=yq+y^2$

Let $xp+x^2=yq+y^2=k$ (arbitrary constant)

$\therefore xp+x^2=k$ and $yq+y^2=k$

$\Rightarrow p=\frac{k-x^2}{x}$ and $q=\frac{k-y^2}{y}$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$\therefore dz=\left(\frac{k}{x}-x\right)dx+\left(\frac{k}{y}-y\right)dy$

Integrating on both sides

$z=\int\left(\frac{k}{x}-x\right)dx+\int\left(\frac{k}{y}-y\right)dy+c$

$=k\log x-\frac{x^2}{2}+k\log y-\frac{y^2}{2}+c$

$$\therefore z=k\log(xy)-\frac{1}{2}(x^2+y^2)+c$$

Which is the complete integral of (1)

3. Solve $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$

Sol: Separating p and x from q and y, the given equation can be written as.

$$\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2$$

Let $\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2=k^2$ (arbitrary constant)

$$\therefore\left(\frac{p}{2}+x\right)^2=k^2 \text{ and } 1-\left(\frac{q}{2}+y\right)^2=k^2$$

$$\Rightarrow \frac{p}{2}+x=k \text{ and } \left(\frac{q}{2}+y\right)^2=1-k^2 \text{ or } \frac{q}{2}+y=\sqrt{1-k^2}$$

$$\Rightarrow p=2(k-x) \text{ and } q=2\left[\sqrt{1-k^2}-y\right]$$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=2(k-x)dx+2\left[\sqrt{1-k^2}-y\right]dy$$

Integrating on both sides

$$z=2\int(k-x)dx+2\int\left[\sqrt{1-k^2}-y\right]dy+c$$

$$z=2(kx-\frac{x^2}{2})+2\left[\left(\sqrt{1-k^2}\right)y-\frac{y^2}{2}\right]+c$$

$$\therefore z=2kx-x^2+2\left(\sqrt{1-k^2}\right)y-y^2+c$$

This is the complete solution of (1)

4.Solve $\boldsymbol{p-x^2=q+y^2}$

Sol: Let $p-x^2=q+y^2=k^2(say)$

Then $p-x^2=k^2$ and $q+y^2=k^2$

$$\therefore p=k^2+x^2 \text{ and } q=k^2+y^2$$

But we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$$

Integrating,we get

$$\boldsymbol{z = \frac{x^3}{3} + k^2x + k^2y + \frac{y^3}{3} + c}$$

is the required complete solution.

5. Solve $q^2 - p = y - x$

Sol: Let $p - x = q^2 - y = k(say)$

Then $p = k + x \; and \; q = \sqrt{k + y}$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Integrating,we get

$$z = \frac{x^2}{2} + kx + \frac{2}{3}(k + y)^{\frac{3}{2}} + C$$

is the required complete solution.

6.Solve $q = px + p^2$

Sol: Let $q = px + p^2 = k(say)$

Then we get

$$p^2 + px - k = 0 \; and \; q = k$$

Solving,we get

$$p = \frac{-x \pm \sqrt{(x^2 + 4k}}{2} \text{ and } q = k$$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$

Integrating,we get

$$z = -\frac{x^2}{4} + \frac{1}{2}[\frac{x}{2}\sqrt{(x^2 + 4k} + 2k \sinh^{-1}(\frac{x}{2\sqrt{k}}) + ky + C$$

is the required complete solution.

STANDARD FORM IV: $Z = px + qy + f(p, q)$

An equation analogous to the clairaut's equation it is complete solution is $Z = ax + by + f(a, b)$ which is obtained by writing $a \; for \; p \; and \; b \; for \; q$ The differential equation which satisfies some specified conditions known as the boundary conditions.The differential equation together with these boundary conditions, constitute a boundary value problem

Solved Problems:

1. **Solve $z = px + qy + pq$**

Sol : The given PDE is form IV

Therefore complete solution is given by

$$z = ax + by + ab$$

2. Find the solution of $(p+q)(z-px-qy)=1$

Sol: The given equation can be written as

$$z-px-qy=\frac{1}{p+q}$$

$$\therefore z=px+qy+\frac{1}{p+q}\rightarrow(1)$$

Hence the complete solution of (1) is given by

$$\boxed{z=ax+by+\frac{1}{a+b}}$$

3. Solve $pqz=p^2\left(qx+p^2\right)+q^2\left(py+q^2\right)$

Sol: The given equation can be written as

$$pqz=p^2q\left(x+\frac{p^2}{q}\right)+q^2p\left(y+\frac{q^2}{p}\right)$$

$$\therefore z=p\left(x+\frac{p^2}{q}\right)+q\left(y+\frac{q^2}{p}\right)$$

$$\therefore z=px+qy+\left(\frac{p^3}{q}+\frac{q^3}{p}\right)\rightarrow(1)$$

Since it is in the form $z=px+qy+f\left(p,q\right)$

Hence the complete solution of (1) is given by

$$\boxed{z=ax+by+\frac{a^3}{b}+\frac{b^3}{a}}$$

4. Solve $z = px + qy + pq + q^2$

Sol: We have $z = px + qy + pq + q^2 \ldots\ldots\ldots\ldots\ldots(1)$

Since (1) is of the form $z = px + qy + f(p, q)$.

Hence the complete solution of (1) is given by

$$z = ax + by + ab + b^2 \dots\dots(2)$$

For singular solution, differentiating (2) partially w.r.t. a and b, we get

$$\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial b} = 0,$$

Implies that

$0 = x + b \dots.(3)$ and $0 = y + a + 2b \dots\dots\dots(4)$

Eliminating a, b between (2), (3) and (4), we get

$$z = x(2x - y) - xy - (2x - y)x + x^2$$

$\therefore z = x^2$

is the singular solution

Equations Reducible To Standard Forms:

Equations of the form $f(x^m p, y^n q) = 0$ $where\ m\ and\ n\ are\ constants$

The above form of the equation of the type can be transformed to an equation of the form f(p,q)=0

By substitutions given below.

Case (i):- when $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ $then\ p = \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m}\ whereP = \frac{\partial Z}{\partial X}$

$$x^m p = P(1-m) \quad and\ q = \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y} = Q(1-n)y^{-n}\ whereQ = \frac{\partial Z}{\partial Y} \rightarrow y^n q = Q(1-n)$$

Now the given equation reduces to $f[(1-m)P, (1-n)Q] = 0\ which\ is\ of\ the\ form\ f(P,Q) = 0$

Case(ii):- when $m = 1, n = 1$

Put $X = logx\ and\ Y = logy\ then$

p=$\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial X}\frac{1}{x}\ implies\ \ px = P{=}\ whereP = \frac{\partial Z}{\partial X}$

$$similarly\ \ qy = Q \quad whereQ = \frac{\partial Z}{\partial Y}$$

now the given equation reduces to the form $f(P,Q) = 0$

Equations of the form $f(x^m p, y^n q, z) = 0$ *where m and n are constants*:

This can be reduced to an equation of the form $f(P, Q, z) = 0$ by the substitutions given for the equation $F(x^m p, y^n q, z) = 0$ as above.

Solved Problems:

1. **Solve the partial differential equation** $\frac{x^2}{p} + \frac{y^2}{q} = z$

Sol: Given equation can be written as

$x^2 p^{-1} + y^2 q^{-1} = z \text{ or } \left(x^{-2}p\right)^{-1} + \left(y^{-2}q\right)^{-1} = z \rightarrow$ (1)

This is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = -2, and\ n = -2.$

Put $X = x^{1-m} = x^{1-(-2)} = x^3$ and $Y = y^{1-n} = y^{1+2} = y^3$

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2$ where $P = \frac{\partial z}{\partial X}$

$\therefore x^{-2} p = 3P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.3y^2$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^{-2} q = 3Q$

Now equation (1), becomes.

$\left(3P\right)^{-1} + \left(3Q\right)^{-1} = z \rightarrow$ (2)

Since (2) is of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ\ in\ (1), then\ we\ get\ Q = \frac{(a+1)}{3az}$

$$\therefore P = \frac{(a+1)}{3z}$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{3az}{a+1} dz = (adX + dY)$

Integrating,we get

$\int \frac{3az}{a+1} dz = (a \int dX + \int dY)$

$$\frac{3az^2}{2(a+1)} = (aX + Y) + c$$

$$\therefore 3z^2 = 2\left(\frac{a+1}{a}\right)\left(x^3 + ay^3\right) + c_1$$

,taking $c_1 = 2(\frac{a+1}{a})c$

Which is the required solution of (1)

2. **Solve the partial differential equation** $\frac{p}{x^2} + \frac{q}{y^2} = z$

Sol: The given equation can be written as

$px^{-2} + qy^{-2} = z \rightarrow (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ With $m = -2, and\ n = -2$

Put $X = x^{1-m} = x^3$, and $Y = y^{1-n} = y^3$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2$ where $P = \frac{\partial z}{\partial X}$

$\therefore x^{-2} p = 3P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q3y^2$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^{-2} q = 3Q$

Equation (1) becomes, $3P + 3Q = z \rightarrow (2)$

Since (2) is of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ\ in\ (1), then\ we\ get\ Q = \frac{z}{3(a+1)}$

$$\therefore P = \frac{az}{3(a+1)}$$

Putting the values of Pand Q in $dz = P\ dX + Q\ dY$,we get

$\frac{dz}{z} = \frac{1}{3(a+1)}(adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{3(a+1)}(a \int dX + \int dY)$

$$\log z = \frac{1}{3(a+1)}(aX + Y) + C$$

$$\Rightarrow \log z = \frac{1}{3(1+a)}\left(x^3 + ay^3\right) + c$$

This is the complete solution of (1)

3. **Solve** $q^2y^2 = z(z - px)$

Sol: Given equation can be written as

$q^2y^2 = z^2 - zpx \text{ or } (xp)z + (qy)^2 = z^2 \rightarrow (1)$

Since (1) is of the form $f(x^m p, y^n q, z) = 0$ with $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes, $Pz + Q^2 = z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $P = aQ$ in (1), then we get $Q = \frac{z}{2}[-a \pm \sqrt{a^2+4}]$

$$\therefore P = \frac{aZ}{2}[-a \pm \sqrt{a^2+4}]$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2+4}](adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2+4}](a\int dX + \int dY)$

$\log z = \frac{1}{2}[-a \pm \sqrt{a^2+4}](aX + Y) + c$

$\therefore \log z = \frac{1}{2}[-a \pm \sqrt{a^2+4}](ax^3 + y^3) + c$

Which is the complete integeral of (1)

4. **Solve the partial differential equation** $p^2x^4 + y^2zq = 2z^2$

Sol : Given that $p^2x^4 + y^2zq = 2z^2$

Then given equation can be written as

$(px^2)^2 + (qy^2)z = 2z^2 \rightarrow (1)$

Since (1) is of the form $f(x^m p, y^n q, z) = 0$ with m=2 and n=2

Put $X = x^{1-m} = x^{1-2} = x^{-1} = \frac{1}{x}$ and $Y = y^{-1} = \frac{1}{x}$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\left(\frac{-1}{x^2}\right)$, where $P = \frac{\partial z}{\partial X}$

$\therefore x^2 p = -P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\left(\frac{-1}{y^2}\right)$, where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^2 q = -Q$

Now equation (1) becomes, $P^2 - Qz = 2z^2 \; or \; P^2 - Qz = 2z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $P = aQ \; in \; (1), then \; we \; get \; Q = \frac{z}{2a^2}[1 \pm \sqrt{8a^2 + 1}]$

$$\therefore P = \frac{Z}{2a}[1 \pm \sqrt{8a^2 + 1}]$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2 + 1}](adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2 + 1}](a \int dX + \int dY)$

$\log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2 + 1}]\,(aX + Y) + c$

$\therefore \log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2 + 1}](ax^3 + y^3) + c$

Which is the complete integral of (1).

5. Solve $x^2 p^2 + xpq = z^2$

Sol : The given equation can be written as

$(xp)^2 + (xp)q = z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p.y^n q, z\right) = 0$ with m=1 and n=0

Put $X = \log x$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\frac{1}{x}$, where

$P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

Equation (1) becomes, $P^2 + Pq = z^2 \rightarrow$ (2)

Since (2) is of the form $f(P,q,z)=0$

Put $P = aq\ in\ (2), we\ get$

$q = \frac{z}{\sqrt{a(a+1)}}$, P=a$\frac{z}{\sqrt{a(a+1)}}$

But we have

$dz = P\,dX + q\,dy$

Substuting P,q ,we get

$\frac{dz}{z} = \frac{1}{\sqrt{a(a+1)}}(a\,dX + dy)$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a(a+1)}}(a\int dX + d\,y)$$

$\boxed{\sqrt{a(a+1)}\log z = (aX+y)+C}$

be the complete integral of (1)

6.Solve $z = p^2x + q^2y$

Sol : Given that $z = p^2x + q^2y$

The given equation can be written as

$$\left(p\sqrt{x}\right)^2 + \left(q\sqrt{y}\right)^2 = z\ or \left(px^{\frac{1}{2}}\right)^2 + \left(qy^{\frac{1}{2}}\right)^2 = z \to (1)$$

This is of the form $f\left(x^m p, y^n q, z\right) = 0\ with\ m = n = \frac{1}{2}$

Put $X = x^{1-m} = x^{1-\frac{1}{2}} = x^{\frac{1}{2}}\ and\ Y = y^{1-\frac{1}{2}} = y^{\frac{1}{2}}$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = P\left(\frac{1}{2}x^{\frac{-1}{2}}\right), where\ P = \frac{\partial z}{\partial X}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\cdot\frac{\partial Y}{\partial x} = Q\left(\frac{1}{2}y^{\frac{-1}{2}}\right), where\ Q = \frac{\partial z}{\partial Y}$

$\therefore px^{\frac{1}{2}} = \frac{P}{2}\ and\ qy^{\frac{1}{2}} = \frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2 + \left(\frac{Q}{2}\right)^2 = z\ i.e. P^2 + Q^2 = 4z \to (2)$

This if of the form $f\left(P,Q,z\right)=0$

Put $P = aQ\ in\ (2), we\ get$

$$a^2Q^2 + Q^2 = 4z$$

$Q = \sqrt{\frac{4z}{a^2+1}}$, P=a$\sqrt{\frac{4z}{a^2+1}}$,

But we have

$dz\ = P\ dX + Q\ dY$

Substuting P,Q ,we get

$dz\ = \sqrt{\frac{4z}{a^2+1}}(a\ dX +\ dY)$

$$\frac{dz}{\sqrt{z}} = \frac{2}{\sqrt{a^2+1}}(a\ dX +\ dY)$$

Integrating on both sides

$$\int dz/\sqrt{z} = \frac{2}{\sqrt{a^2+1}}(a\int dX + \int dY)$$

$\sqrt{(a^2+1)}\sqrt{z}$ =$(aX + Y) + C$

$\sqrt{(a^2+1)}\sqrt{z}$ =$\left(a\sqrt{x} + \sqrt{y}\right) + C$

Which is the complete integral of (1)

7.Solve $x^2p^2 + y^2q^2 = z^2$

Sol: Given $x^2p^2 + y^2q^2 = z^2 \ldots \ldots . (1)$

$$(xp)^2 + (yq)^2 = z^2$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right)=0$ with $m = 1\ and\ n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial x}=P.\frac{1}{x}$ where $P=\frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y}=Q.\frac{1}{y}$ where $Q=\frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$P^2 + Q^2 = z^2 \ldots\ldots(2)$

$$Put\ P = aQ\ in\ (2), we\ get\ Q = \frac{z}{\sqrt{a^2+1}}; P = \frac{az}{\sqrt{a^2+1}}$$

But we have

$dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$dz = \frac{z}{\sqrt{a^2+1}}(a\,dX + dY)$

$$\frac{dz}{z} = \frac{1}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a^2+1}}\left(a\int dX + \int dY\right)$$

$\sqrt{(a^2+1)}\log z = (aX + Y) + C$

$\boxed{\sqrt{(a^2+1)}\log z = (a\log x + \log y) + C}$

is the Complete solution of (1)

8. **Solve $x^2p^2 + y^2q^2 = 1$**

Sol: Given $x^2p^2 + y^2q^2 = 1 \ldots\ldots(1)$

$$(xp)^2 + (yq)^2 = 1$$

Since (1) is of the form $f(x^m p, y^n q) = 0$ with $m = 1\ and\ n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$P^2 + Q^2 = 1......(2)$

$$Put\ P = a\ in\ (2), we\ get\ Q = \sqrt{1 - a^2}$$

But we have

$dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$dz = (a\,dX + \sqrt{1 - a^2}\,dY)$

Integrating on both sides

$$\int dz = (a \int dX + \sqrt{1 - a^2} \int dY)$$

z $= \left(aX + \sqrt{1 - a^2}Y\right) + C$

z$= \left(a \log x + \sqrt{1 - a^2} \log y\right) + C$

is the Complete solution of (1)

Equations of the form $f(z^n p, z^n q) = 0$ where n is a constant:

Use the following substitution to reduce the above form to an equation of the form f(P,Q)=0

$$put \quad Z = \begin{cases} z^{n+1} & if\ n \neq -1 \\ \log z, & if\ n = -1 \end{cases}$$

Equations of the form $f(x, z^n p) = g(y, z^n q)$ where n is a constant:

An equation of the above form can be reduced to an equation of the form f(P,Q)=0

by the substitutions given for the equation $F(z^n p, z^n q) = 0$ as above

Solved Problems :

1. **Solve** $z^2\left(p^2 + q^2\right) = x^2 + y^2$

Sol: Given that $z^2\left(p^2 + q^2\right) = x^2 + y^2$

The given equation can be written as

$z^2 p^2 + z^2 q^2 = x^2 + y^2$ or $z^2 p^2 - x^2 = y^2 - z^2 q^2$

Or $(zp)^2 - x^2 = y^2 - (zq)^2 \rightarrow (1)$

Since (1) is the of the form $f\left(x, pz^n\right) = g\left(y, qz^n\right)$. with n=1

$\therefore put\ Z = z^{n+1} = z^{1+1} = z^2$

Then $\dfrac{\partial Z}{\partial x} = 2z.\dfrac{\partial z}{\partial x} \Rightarrow P = 2zp$ where $P = \dfrac{\partial z}{\partial x}$

$\therefore pz = \dfrac{P}{2}$

and $\dfrac{\partial Z}{\partial y} = 2z.\dfrac{\partial z}{\partial y} \Rightarrow Q = 2zq$ where $Q = \dfrac{\partial z}{\partial y} \therefore qz = \dfrac{Q}{2}$

$\therefore$ Equation (1) becomes, $\dfrac{P^2}{4} - x^2 = y^2 - \dfrac{Q^2}{4}$

$i.e., P^2 - 4x^2 = 4y^2 - Q^2 \rightarrow (2)$

This is of the form $f_1(x,P) = f_2(y,Q)$

Let $P^2 - 4x^2 = 4y^2 - Q^2 = 4k^2$ (say)

$\therefore P^2 - 4x^2 = 4k^2\ and\ 4y^2 - Q^2 = 4k^2$

$\Rightarrow P^2 = 4x^2 + 4k^2\ and\ Q^2 = 4y^2 - 4k^2$

$\therefore P = 2\sqrt{x^2 + k^2}\ and\ Q = 2\sqrt{y^2 - k^2}$

We have $dZ = \dfrac{\partial Z}{\partial x}.dx + \dfrac{\partial Z}{\partial y}dy$

$= Pdx + Qdy$ [By total differentiation]

$\therefore dZ = 2\sqrt{x^2 + k^2}dx + 2\sqrt{y^2 - k^2}dy$

Integrating on both sides

$Z = 2\int\sqrt{x^2 + k^2}dx + 2\int\sqrt{y^2 - k^2}dy$

$= 2\left[\dfrac{x}{2}\sqrt{x^2 + k^2} + \dfrac{k^2}{2}\sinh^{-1}\left(\dfrac{x}{k}\right)\right] + 2\left[\dfrac{x}{2}\sqrt{y^2 - k^2} - \dfrac{k^2}{2}\cosh^{-1}\left(\dfrac{y}{k}\right)\right] + c$

$= x\sqrt{x^2 + k^2} + k^2\sinh^{-1}\left(\dfrac{x}{k}\right) + x\sqrt{y^2 - k^2} + k^2\cosh^{-1}\left(\dfrac{y}{k}\right) + c$

$or\ z^2 = x\sqrt{x^2 + k^2} + y\sqrt{y^2 - k^2} + k^2\left[\sinh^{-1}\left(\dfrac{x}{k}\right) - \cosh^{-1}\left(\dfrac{y}{k}\right)\right] + c$

$$or\ z^2 = x\sqrt{x^2 + k^2} + y\sqrt{y^2 - k^2} + k^2\log\left(\dfrac{x + \sqrt{x^2 + k^2}}{y + \sqrt{y^2 - k^2}}\right) + c$$

This is the complete solution of (1)

2. **Solve the partial differential equation.** $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

Sol: Given that $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

The given equation can be written as

$(pz)^2\sin^2 x+(qz)^2\cos^2 y=1 \, or \, (pz)^2\sin^2 x=1-(qz)^2\cos^2 y\to(1)$

Since (1) is of the form $f\left(x,pz^n\right)=g\left(y,qz^n\right)$ with n=1.

Put $Z=z^{n+1}=z^2$

Now $\frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x}\Rightarrow P=2zp \, or \, pz=\frac{P}{2}$ where $P=\frac{\partial z}{\partial x};Q=\frac{\partial z}{\partial y}$

and $\frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y}\Rightarrow Q=2zq \, or \, qz=\frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2\sin^2 x=1-\left(\frac{Q}{2}\right)^2\cos^2 y$

$i.e.\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y\to(2)$

This is of the form $f_1(x,p)=f_2(y,q)$

Let $\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y=k^2$ (constant)

$\therefore\frac{P^2}{4}\sin^2 x=k^2 \, and \, 1-\frac{Q^2}{4}\cos^2 y=k^2$

$\Rightarrow P^2\sin^2 x=4k^2 \, and \, Q^2\cos^2 y=4\left(1-k^2\right)$

$\Rightarrow P=\frac{2k}{\sin x} \, and \, Q=\frac{2\sqrt{1-k^2}}{\cos y}$

We have $dZ=\frac{\partial Z}{\partial x}dx+\frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ=Pdx+Qdy$

$dZ=\frac{2k}{\sin x}dx+\frac{2\sqrt{1-k^2}}{\cos y}dy$

Integrating on both sides

$$z=2k\int\csc x\,dx+2\sqrt{1-k^2}\int\sec y\,dy$$

$= 2k\log\left(\cos ecx - \cot x\right) + 2\sqrt{1-k^2}\log\left(\sec y + \tan y\right) + c$

$\therefore z^2 = 2k\log\left(\cos ecx - \cot x\right) + 2\sqrt{1-k^2}\log\left(\sec y + \tan y\right) + c$

This is the required complete solution of (1)

3.Solve $(x+pz)^2 + (y+qz)^2 = 1$

Sol: Given $(x+pz)^2 + (y+qz)^2 = 1$…….(1)

$since\ (1)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = 1$

Put $Z = z^{n+1} = z^2$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = 2z\ implies\ that\ \frac{\partial z}{\partial Z} = \frac{1}{2z}$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x} = \frac{P}{2z}$ implies $\frac{\partial Z}{\partial x} = \frac{P}{2} = zp; Similarly\ we\ get\ qz = \frac{Q}{2}$

Substitute in (1),we get

$(x+\frac{P}{2})^2 + (y+\frac{Q}{2})^2 = 1$

Separating $P\ and\ x\ from\ Q\ and\ y$, the given equation can be written as.

$(x+\frac{P}{2})^2 = 1-(y+\frac{Q}{2})^2 = K^2$

$(x+\frac{P}{2})^2 = K^2\ AND\ 1-(y+\frac{Q}{2})^2 = K^2$

$\left(x+\frac{P}{2}\right) = K$

Implies that

$Q = 2(\sqrt{(1-K^2)} - y)$

$P = 2(K-x)$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$

$\therefore dz = 2(k-x)dx + 2\left[\sqrt{1-k^2} - y\right]dy$

Integrating on both sides

$z = 2\int(k-x)dx + 2\int\left[\sqrt{1-k^2} - y\right]dy + c$

$$z = 2(kx - \frac{x^2}{2}) + 2\left[\left(\sqrt{1-k^2}\right)y - \frac{y^2}{2}\right] + c$$

$$\therefore z = 2kx - x^2 + 2\left(\sqrt{1-k^2}\right)y - y^2 + c$$

This is the complete solution of (1).

4. Solve $\boldsymbol{z(p^2 - q^2) = x - y}$

Sol: Given

$$z(p^2 - q^2) = x - y \ldots\ldots\ldots\ldots (1)$$

$$(z^{\frac{1}{2}}p)^2 - (z^{\frac{1}{2}}q)^2 = x - y \ldots..(2)$$

$$since\ (2)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = \frac{1}{2}$$

Put $Z = z^{n+1} = z^{\frac{3}{2}}$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = \frac{3}{2} z^{\frac{1}{2}}$

$$implies\ that\ \frac{\partial z}{\partial Z} = \frac{2}{3z^{\frac{1}{2}}}$$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$=Pimplies $\frac{2}{3}P = z^{\frac{1}{2}}p;$ $Similarly\ we\ get$ $\frac{2}{3}Q = z^{\frac{1}{2}}q$ Substitute in (2),we get

$$(\frac{2}{3}P)^2 - (\frac{2}{3}Q)^2 = \boldsymbol{x - y}$$

Separating $P\ and\ x\ from\ Q\ and\ y$, the given equation can be written as.

$$(\frac{2}{3}P)^2 - \boldsymbol{x} = -\boldsymbol{y} + (\frac{2}{3}Q)^2 = \boldsymbol{k}$$

Solving, we get

$P = \frac{3}{2}\sqrt{k + x}$ and $Q = \frac{3}{2}\sqrt{k + y}$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ = Pdx + Qdy$

$dZ = \frac{3}{2}[\sqrt{k + x}dx + \sqrt{k + y}dy]$

Integrating on both sides

$$Z = \frac{3}{2}[\int \sqrt{k+x}dx + \int \sqrt{k+y}\ dy$$

$$z^{\frac{3}{2}} = (\ k+x\ )^{\frac{3}{2}} + (k+y)^{\frac{3}{2}} + c$$

This is the required complete solution of (1)

Methods Of Separation Of Variables:

This method is used to reduce one partial differential equation to two or more ordinary differential equations,each one involving one of the independent variables.This will be done by separating these variables from the beginning. This method is explained through following examples.

1. **Solve by the method of separation of variables $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$ where U(x,0)=6e^{-3x}**

Sol : Given equation is $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$----------(1)

Let U(x,t)= X(x) T(t) =XT ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant t

$$\frac{\partial U}{\partial x} = X'T \quad , \quad \frac{\partial U}{\partial t} = T'X$$

Put these values in equation (1), we have

$X'T = 2\ T'X\ + XT$ Dividing by XT

$\frac{X'}{X} = 2\frac{T'}{T} + 1$--------(3)

Since L.H.S is a function of 'x' and the R.H.S is a function of 't' where x and t are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 't' if and only if both sides are equal to a constant.

Therefore $\frac{X'}{X} = 2\frac{T'}{T} + 1$=k--------(4) where k is a constant

Now from (4) $\frac{X'}{X}$ = k----(5) and $2\frac{T'}{T} + 1$=k-------(6)

Now consider (5) $\frac{X'}{X}$ = k $\Rightarrow$ $X' - kX = 0$ $\Rightarrow X = C_1 e^{kx}$

Now consider (6) $2\frac{T'}{T} + 1$=k $\Rightarrow$ $T' - \left(\frac{k-1}{2}\right)T = 0$ $\Rightarrow T = C_2 e^{\left(\frac{k-1}{2}\right)t}$ ---(8)

Substituting the values of X and T in (2) we get

$$U(x,t) = X = C_1 e^{kx} C_2 e^{\left(\frac{k-1}{2}\right)t}$$

$$U(x,t) = X = Ae^{kx} e^{\left(\frac{k-1}{2}\right)t} \quad (where\ \text{A}=\text{C}_1\ \text{C}_2)$$

Put t=0 in the above equation ,we have U(x,0) = A e^{kx} -----(9)

but given that U(x,0)=$6e^{-3x}$--------(10)

from (9) and (10) we have A e^{kx} =$6e^{-3x}$

A=6 and k=-3 the solution of the given equation becomes

$$U(x,t) = X = 6e^{-3x} e^{(-2)t} = 6e^{-(3x+2t)}$$

2. Solve the equation by the method of separation of variables $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$

Sol: Given equation is $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$ ----------(1)

Let U(x,y)= X(x) Y(y) =X Y ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant y

$$\frac{\partial U}{\partial x} = X'Y \quad , \quad \frac{\partial U}{\partial y} = Y'X \quad \frac{\partial^2 U}{\partial x^2} = X''Y$$

Put these values in equation (1), we have

$$X''Y \;=\; Y'X + 2XY$$

Dividing by XY on both sides we have $\frac{X''}{X} = \frac{Y''}{Y}+2$

$$\frac{X''}{X} - 2 = \frac{Y''}{Y}\text{-----------------(3)}$$

Since L.H.S is a function of 'x' and the R.H.S is a function of 'y' where x and y are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 'y' if and only if both sides are equal to a constant.

$\frac{X''}{X} - 2 = \frac{Y''}{Y} = k$--------(4)

Now from (4)

$$\frac{X''}{X} - 2 = k - - - - - - -(5)$$

And $\frac{Y''}{Y} = k - - - - - - -(6)$

From (5) $X'' - 2X = kX$ $X'' - (2+k)X = 0$

Which is second order differential equation

Auxiliary equation is $m^2 - (2+k) = 0$ → m=$\pm\sqrt{(2+k)}$

Solution of the given equation (5) is $X = C_1 e^{\sqrt{(2+k)}}$

Now consider equation (6) $Y' = kY$ → $\frac{Y'}{Y} = k$

Integrating on both sides we get logy = ky + logC_3

$$\Rightarrow \log\left(\frac{Y}{C_3}\right) = ky \quad \Rightarrow Y = C_3 e^{ky} ---(8)$$

Substituting the values of X and Y in (2) we have

$$U = \left[C_1 e^{\sqrt{(2+k)}x} + C_2 e^{-\sqrt{(2+k)}x} \right] C_3 e^{ky}$$

$$U = \left[A e^{\sqrt{(2+k)}x} + B e^{-\sqrt{(2+k)}x} \right] e^{ky}$$

Where $A = C_1 C_3 \quad and \quad B = C_1 C_2$