UNIT-IV

Partial Differential Equations of Ist order

Partial d.E :- An equation containing 1 dependent variable and its partial derivatives w.r.t 2 or more independent variables is called a partial d.E

Ex :- 1) $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = 6z$ 3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial z^2} = 0$ 4) $\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial z}\right)^3$.

In the case of 1 dependent variable two independent variables, usually we consider z as the dependent variable x and y are independent variables.

The first and second order partial derivatives of z with respect to x and y are $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are respectively denoted by p, q, r, s, t

i.e. $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

Order: The order of a p.d.E is defined as the order of the highest derivative occurred in the given equation.

Degree : The degree of a p.d.E is defined as the degree of the highest order derivative occured in the given equation

Ex : $\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + \frac{\partial^2 u}{\partial y^2} = 2u$ ∴ Or is a partial d.E of order 2 and degree - 3

Formation of a p.d.E:

A partial d.E can be formed by

(i) Elimination of arbitary constants

(ii) " " " functions

Elimination of arbitary constants:-

* If the number of arbitary constants to be eliminated is equal to the number of independent variables, then we get a p.d.E of 1^{st} order

* If the number of arbitary constants to be eliminated is greater than the number of independent variables then we get a p.d.E of 2^{nd} and higher order.

Sums:-

1) Form the p.d.E by eliminating the arbitary constants a and b from $z = ax + by + a^2 + b^2$

Sol Given $z = ax + by + a^2 + b^2$ —① Diff ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a, \quad \frac{\partial z}{\partial y} = b$$

$p = a$ —② $q = b$ —③

sub ② and ③ in eq-①

$z = px + qy + p^2 + q^2$ is the required p.d.E

2) Form the p.d.E by eliminating the arbitary constants a and b from $2z = \frac{x^2}{a^2} + y^2/b^2$

Sol Given $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ —① Diff ① partially w.r.t x & y, we get

$$\cancel{2}\cdot\frac{\partial z}{\partial x} = \frac{\cancel{2}x}{a^2} \Rightarrow a^2 = x/p \quad —②$$

$$\left[\because \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q\right]$$

$$\cancel{2}\cdot\frac{\partial z}{\partial y} = \frac{\cancel{2}y}{b^2} \Rightarrow b^2 = y/q \quad —③$$

Sub ② & ③ in ① $2z = \frac{x^2}{x/p} + \frac{y^2}{y/p}$

$2z = px + qy$ is the required p.d.E

3) Find the p.d.E of all spheres whose centeres lies on z-axis with radius r

Sol

with center (a,b,c) and radius 'r'

Equation of spheres : $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Since center lies on z-axis $a=0, b=0$

$$x^2+y^2+(z-c)^2=r^2 \quad -(1)$$

where 'c' and 'r' are orbitary constants

Differentiating eq (1) partially w.r.t x and y

$$\cancel{2}x+\cancel{2}(z-c)\frac{\partial z}{\partial x}=0$$

$$(z-c)\frac{\partial z}{\partial x}=-x \Rightarrow (z-c)p=-x$$

$$z-c=-x/p \quad -(2)$$

$$\cancel{2}y+\cancel{2}(z-c)\frac{\partial z}{\partial y}=0$$

$$(z-c)q=-y \Rightarrow z-c=-y/q \quad -(3)$$

$\therefore$ from (2) and (3)

$$\frac{-x}{p}=\frac{-y}{q} \Rightarrow xq-yp=0$$

(3) $z = (x^2+a)(y^2+b)$ —— (1)

The no: of independent variables x, y = The no: of arbitrary constants.

Then we get first order PDE.

Scanned with CamScanner

differentiate eq ① w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\Rightarrow y^2+b = \frac{p}{2x} \quad \text{—— (2)}$$

differentiate eq ① w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = 2y(x^2+a)$$

$$\Rightarrow x^2+a = \frac{q}{2y} \quad \text{—— (3)}$$

Substitute (2) (3) in (1) we get

$$z = \frac{q}{2y} \cdot \frac{p}{2x}$$

$$\Rightarrow 4xyz = pq.$$

5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\frac{z^2}{c^2} = 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

$$\Rightarrow z^2 = c^2\left(1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}\right) \quad \text{—— (1)}$$

Here the no: of independent variables x, y $\neq$ The

Here the no: of independent variables x, y & The no: of arbitrary constants a, b, c.

i.e The no: of arbitrary constants > The no: of Independent variables.

differentiate eq (1) w.r.t 'x' partially

$$2z \frac{\partial z}{\partial x} = c^2\left(-\frac{2x}{a^2}\right)$$

$$\Rightarrow z \frac{\partial z}{\partial x} = \frac{-c^2 x}{a^2} \quad \text{——— (2)}$$

differentiate w.r.t 'y' partially

$$2z \frac{\partial z}{\partial y} = c^2\left(-\frac{2y}{b^2}\right)$$

$$\Rightarrow z \frac{\partial z}{\partial y} = \frac{-c^2 y}{b^2} \quad \text{——— (3)}$$

again differentiate eq (2) w.r.t 'x' partially

$$z \frac{\partial^2 z}{\partial x^2} + (1) \frac{\partial z}{\partial x} = \frac{-c^2}{a^2} \quad \text{——— (4)}$$

Substitute (4) in (2) we get

$$z \frac{\partial z}{\partial x} = \left(z \frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\right) x$$

$$\Rightarrow pz = zrx + px$$

$$\Rightarrow zxr + px - pz = 0$$

7. $z = xy + y\sqrt{x^2-a^2} + b$ —— (1)

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = y + y\frac{1}{2\sqrt{x^2-a^2}}(2x) \quad \text{—— (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = x + \sqrt{x^2-a^2} \quad \text{—— (3)}$$

$$q = x + \sqrt{x^2-a^2}$$

$$\Rightarrow \sqrt{x^2-a^2} = q - x.$$

from (2) $$P = y + \frac{y}{\cancel{2}(q-x)}(\cancel{2}x)$$

$$\Rightarrow P = y + \frac{xy}{(q-x)}$$

Scanned with CamScanner

$$P(q-x) = y(q-x) + xy$$

$$\Rightarrow pq - px = qy - \cancel{xy} + \cancel{xy}$$

$$\Rightarrow px + qy = pq$$

$=$

8. $(x-a)^2 + (y-b)^2 + z^2 = c^2$ where c is fixed.

—— (1)

differentiate eq (1) w.r.t. 'x' partially

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) + zp = 0 \quad \text{—— (2)} \quad \Rightarrow (x-a) = -zp$$

Scanned with CamScanner

Scanned with CamScanner

$\Rightarrow \quad (x-a)+zp=0 \quad$ ——— (2) $\quad \Rightarrow (x-a)=-zp$

differentiate eq (1) w.r.t 'y' partially

$2(y-b)+2z\frac{\partial z}{\partial y}=0$

$\Rightarrow \quad (y-b)+zq=0 \quad$ ——— (3) $\quad \Rightarrow (y-b)=-zq$

from (1) $\quad (-zp)^2+(-zq)^2+z^2=c^2$

$\Rightarrow \quad z^2p^2+z^2q^2+z^2=c^2$

$\Rightarrow \quad z^2(p^2+q^2+1)=c^2$

9. $\quad z=axe^y+\frac{1}{2}a^2e^{2y}+b.$ ——— (1)

differentiate eq (1) w.r.t. 'x' partially

$p=\frac{\partial z}{\partial x}=ae^y(1)$

$\Rightarrow \quad p=ae^y$ ——— (2)

differentiate eq (1) w.r.t 'y' partially

$q=\frac{\partial z}{\partial y}=axe^y+\frac{1}{2}a^2e^{2y}(2)$

$q=axe^y+a^2e^{2y}$

$q=(ae^y)x+(ae^y)^2$

$\Rightarrow \quad q=px+p^2 \quad [\because$ from (2)$]$

11. $\log(az-1) = x + ay + b$ —— (1)

differentiate eq (1) w.r.t 'x' partially

$\frac{1}{az-1}\left(a\frac{\partial z}{\partial x}\right) = 1.$

$\Rightarrow \quad ap = (az-1).$ —— (2)

differentiate eq (1) w.r.t 'y' partially

$\frac{1}{az-1} \cdot a\frac{\partial z}{\partial y} = a$

$\Rightarrow \quad q = (az-1)$ —— (3)

from (2) & (3) $\quad ap = q$

$\Rightarrow a = \frac{q}{p}.$

from (3) $\quad q = \left(\frac{q}{p}\right)z - 1$

$\Rightarrow \quad pq = qz - p$

$\Rightarrow \quad p + pq = qz.$

$\Rightarrow \quad p(1+q) = qz.$

Scanned with CamScanner

13. $z = a \log \left[\frac{b(y-1)}{1-x}\right]$

$$z = a\left[\log [b(y-1)] - \log(1-x)\right]$$

$$z = a\left[\log b + \log(y-1) - \log(1-x)\right] \text{ —— } (1)$$

differentiate eq (1) w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = p = a\left[0 - \frac{1}{1-x}(-1)\right]$$

$$\Rightarrow p = a\left[\frac{1}{1-x}\right]$$

$$\Rightarrow p = \frac{a}{1-x} \text{ —— } (2) \quad \Rightarrow a = p(1-x)$$

differentiate w.r.t 'y' partially to eq (1)

$$\frac{\partial z}{\partial y} = q = a\left[\frac{1}{(y-1)}\right]$$

$$q = \frac{a}{y-1} \text{ —— } (3) \quad \Rightarrow q(y-1) = a$$

$$(y-1)q = p(1-x)$$

$$\Rightarrow qy - q = p - px$$

$$\Rightarrow px + qy = p + q$$

14. find the differential eq of all spheres of radius '8' and having their centres in the yz - plane.

$$x^2 + (y-b)^2 + (z-c)^2 = 64 \text{ —— } (1)$$

differentiate eq (1) w.r.t 'x' partially

$$2x + 0 + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x + (z-c)p = 0 \text{ —— } (2) \quad \Rightarrow (z-c) = \frac{-x}{p}$$

Scanned with CamScanner

Scanned with CamScanner

17. Find the differential eq of all spheres of radius '8' and having their centres in the yz- plane.

$$x^2+(y-b)^2+(z-c)^2=64 \quad \text{——} (1)$$

differentiate eq (1) w.r.t 'x' partially

$$2x+0+2(z-c)\frac{\partial z}{\partial x}=0$$

$$\Rightarrow x+(z-c)p=0 \quad \text{——}(2) \quad \Rightarrow (z-c)=\frac{-x}{p}$$

differentiate eq (1) w.r.t 'y' partially

$$2(y-b)+2(z-c)\frac{\partial z}{\partial y}=0$$

$$\Rightarrow (y-b)+(z-c)q=0 \quad \text{——} (3) \quad \Rightarrow (y-b)=-(z-c)q$$

$$=-\left(\frac{-x}{p}\right)q$$

$$(y-b)=\frac{x}{p}q$$

Substitute $(y-b)$ and $(z-c)$ in (1)

$$x^2+\left(\frac{xq}{p}\right)^2+\left(\frac{-x}{p}\right)^2=64$$

$$\Rightarrow x^2+\frac{x^2q^2}{p^2}+\frac{x^2}{p^2}=64$$

$$\Rightarrow x^2\left(1+\frac{q}{p^2}+\frac{1}{p^2}\right)=64$$

$$\Rightarrow x^2(p^2+q+1)=64p^2.$$

Formation of P.D.E by Eliminating arbitrary functions.

1. $z = f(x^2+y^2)$
2. $z = f(x) + e^y g(x)$
3. $z = f\left(\frac{y}{x}\right)$
4. $z = \phi_1(x+iy) + \phi_2(x-iy)$
5. $lx+my+nz = \phi(x^2+y^2+z^2)$
6. $\phi(x+y+z,\ x^2+y^2+z^2) = 0$
7. $f(x^2+y^2,\ z-xy) = 0$.
8. $z = yf(x) + xg(y)$
9. $xyz = f(x^2+y^2+z^2)$
10. $z = (x+y)f(x^2-y^2)$
11. $z = e^{ax+by} f(ax-by)$
12. $z = xy + f(x^2+y^2)$

13. $\phi(x^2+y^2+z^2,\ z^2-2xy) = 0$ 10
14. $z = f(x+ct) + g(x-ct)$
15. $z = x f(ax+by) + g(ax+by)$
16. $z = f(x-it) + g(x-it)$
17. $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$
18. $z = \phi_1(x)\,\phi_2(y)$
19. $z = x f_1(x+t) + f_2(x+t)$
20. $z = x^n f\left(\frac{y}{x}\right)$

<u>Note</u> Similar form of the equations

$u = f(v)$ (or) $v = f(u)$ (or) $f(u, v) = 0$

2. $z = f\left(\frac{y}{x}\right)$ —— ①

differentiate eq ① w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right) \text{ —— (2)}$$

differentiate eq ① w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) \text{ —— (3)}$$

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{\cancel{f'\left(\frac{y}{x}\right)} \cdot \left(\frac{-y}{x^{\cancel{2}}}\right)}{\cancel{f'\left(\frac{y}{x}\right)}\left(\frac{1}{\cancel{x}}\right)}$$

$$\frac{p}{q} = \frac{-y}{x}$$

$$\Rightarrow px + qy = 0$$

Scanned with CamScanner

7. $f(x^2+y^2, z-xy) = 0$

$z - xy = f(x^2+y^2)$ (or) $x^2+y^2 = f(z-xy)$

consider $x^2+y^2 = f(z-xy)$ —— ①

differentiate w.r.t 'x' partially

$$2x = f'(z-xy)\left(\frac{\partial z}{\partial x} - y\right) \text{ —— (2)}$$

differentiate w.r.t 'y' partially

Scanned with CamScanner

2. $z = f(x) + e^{y} g(x)$ —— (1).

differentiate eq (1) w.r.t 'x' partially

$p = \frac{\partial z}{\partial x} = f'(x) + e^{y} g'(x)$ —— (2)

differentiate eq (1) w.r.t 'y' partially

$q = \frac{\partial z}{\partial y} = e^{y} g(x)$ —— (3)

differentiate eq (2) w.r.t 'x' partially

$r = \frac{\partial^2 z}{\partial x^2} = f''(x) + e^{y} g''(x)$ —— (4)

differentiate eq (2) w.r.t 'y' partially

$s = \frac{\partial^2 z}{\partial x \partial y} = e^{y} g'(x)$ —— (5)

differentiate eq (3) w.r.t 'y' partially

$t = \frac{\partial^2 z}{\partial y^2} = e^{y} g(x)$ —— (6)

from (3) & (6) $q = e^{y} g(x) = t$

$\Rightarrow q = t$

$\Rightarrow q - t = 0$

$\frac{\partial z}{\partial y} - \frac{\partial^2 z}{\partial y^2} = 0$

Scanned with CamScanner

$$2y = f'(z-xy)\left(\frac{\partial z}{\partial y}-x\right) \text{ ——— } (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{2x}{2y} = \frac{\cancel{f'(z-xy)}(p-y)}{\cancel{f'(z-xy)}(q-x)}$$

$$\Rightarrow (q-x)x = (p-y)y$$

$$\Rightarrow qx - x^2 = py - y^2$$

$$\Rightarrow -py + qx = x^2 - y^2$$

9. $$xyz = f(x^2+y^2+z^2) \text{ ——— } (1)$$

differentiate eq (1) w.r.t 'x' partially

$$y\left[(1)z + x\frac{\partial z}{\partial x}\right] = f'(x^2+y^2+z^2)\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow y(z+xp) = f'\cdot(2x+2zp) \longrightarrow (2)$$

differentiate eq (1) w.r.t 'y' partially

$$x\left[(1)z + y\frac{\partial z}{\partial y}\right] = f'(x^2+y^2+z^2)\ 2y+2z\left(\frac{\partial z}{\partial y}\right)$$

$$x(z+yq) = f'(2y+2zq) \text{ ——— } (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{y(z+xp)}{x(z+yq)} = \frac{\cancel{f'}\cdot(2x+2zp)}{\cancel{f'}\cdot(2y+2zq)}$$

$$\Rightarrow (yz+xyp)\,\cancel{2}(y+zq) = (xz+xyq)\cancel{2}(x+zp)$$

$$\Rightarrow y^2z + xy^2p + yz^2q + xy\cancel{z}pq = x^2z + x^2yq + xz^2p + xy\cancel{zpq}$$

$$\Rightarrow px(y^2-z^2)+qy(z^2-x^2)+z(y^2-x^2)=0.$$

11 $z = e^{ax+by} f(ax-by)$ —

$$z = e^{ax} \cdot e^{by} f(ax-by) \quad \text{——} (1)$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = e^{by}\left[e^{ax} . a \; f(ax-by) + e^{ax} f'(ax-by) \cdot a\right]$$

$$p = e^{ax+by} . a \cdot f(ax-by) + a \, e^{ax+by} f'(ax-by)$$

$$\Rightarrow p = az + ae^{ax+by} f'(ax-by) \quad \text{——} (2)$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = e^{ax}\left[e^{by} . b \; f(ax-by) + e^{by} f'(ax-by)(-b)\right]$$

$$q = e^{ax+by} . b \, f(ax-by) - b \, e^{ax+by} f'(ax-by)$$

$$q = bz - be^{ax+by} f'(ax-by) \quad \text{——} (3)$$

from (2) $\quad p-az = a e^{ax+by} f'(ax-by) \quad$ —— (4)

and from (3) $\quad q-bz = -b e^{ax+by} f'(ax-by) \quad$ —— (5)

13

$$\frac{(4)}{(5)} \Rightarrow \frac{p-az}{q-bz} = \frac{a\, e^{ax+by} . f'}{-b\, e^{ax+by} . f'}$$

$$z = e^{ax} \cdot e^{by} f(ax - by) \quad \text{———} (1)$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = e^{by}\left[e^{ax}.a\, f(ax-by) + e^{ax} f'(ax-by) \cdot a\right]$$

$$p = e^{ax+by}.a \cdot f(ax-by) + a\, e^{ax+by} f'(ax-by)$$

$$\Rightarrow p = az + a\, e^{ax+by} f'(ax-by) \quad \text{——} (2)$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = e^{ax}\left[e^{by}.b\, f(ax-by) + e^{by} f'(ax-by)(-b)\right]$$

$$q = e^{ax+by}.b\, f(ax-by) - b\, e^{ax+by} f'(ax-by)$$

$$q = bz - b\, e^{ax+by} f'(ax-by) \quad \text{——} (3)$$

from (2) $\quad p - az = a\, e^{ax+by} f'(ax-by) \quad$ —— (4)

and from (3) $\quad q - bz = -b\, e^{ax+by} f'(ax-by) \quad$ —— (5)

Scanned with CamScanner

$$\frac{(4)}{(5)} \Rightarrow \frac{p-az}{q-bz} = \frac{a\, \cancel{e^{ax+by}} . \cancel{f'}}{-b\, \cancel{e^{ax+by}} . \cancel{f'}}$$

$$\Rightarrow -b(p-az) = a(q-bz)$$

$$\Rightarrow -bp + abz = aq - abz$$

$$\Rightarrow aq + bp = 2abz$$

$$z = \phi_1(x+iy) + \phi_2(x-iy) \quad \text{--- (1)}$$

differentiate eq (1) w.r.t 'x' partially

$$p = \frac{\partial z}{\partial x} = \phi_1'(x+iy) + \phi_2'(x-iy) \quad \text{--- (2)}$$

differentiate eq (1) w.r.t 'y' partially

$$q = \frac{\partial z}{\partial y} = \phi_1'(x+iy)\,i + \phi_2'(x-iy)(-i)$$

$$q = \phi_1'(x+iy)\,i - \phi_2'(x-iy)\,i \quad \text{--- (3)}$$

differentiate eq (2) w.r.t 'x' partially

$$r = \frac{\partial^2 z}{\partial x^2} = \phi_1''(x+iy) + \phi_2''(x-iy) \quad \text{--- (4)}$$

differentiate eq (2) w.r.t 'y' partially

$$s = \frac{\partial^2 z}{\partial x \partial y} = \phi_1''(x+iy)\,i + \phi_2''(x-iy)(-i)$$

$$s = \phi_1''(x+iy)\,i - \phi_2''(x-iy) \quad \text{--- (5)}$$

differentiate eq (3) w.r.t 'y' partially

$$t = \frac{\partial^2 z}{\partial y^2} = \phi_1''(x+iy)\,i^2 - \phi_2''(x-iy)\,i(-i)$$

$$t = -\phi_1''(x+iy) - \phi_2''(x-iy) \quad \text{--- (6)}$$

$$= -\left[\phi_1''(x+iy) + \phi_2''(x-iy)\right]$$

$$t = -r$$

$$\Rightarrow \quad r + t = 0$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Scanned with CamScanner

Scanned with CamScanner

8. $z = y f(x) + x g(y)$ —— ①

differentiate ① w.r.t 'x' partially

$p = \frac{\partial z}{\partial x} = y f'(x) + g(y)$ —— (2)

differentiate ① w.r.t 'y' partially

$q = \frac{\partial z}{\partial y} = f(x) + x g'(y)$ —— (3)

differentiate (2) w.r.t 'x' partially

$r = \frac{\partial^2 z}{\partial x^2} = y f''(x)$ —— (4)

differentiate (2) w.r.t 'y' partially

$s = \frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$ —— (5)

differentiate (3) w.r.t 'y' partially

$t = \frac{\partial^2 z}{\partial y^2} = x g''(y)$ —— (6)

from (2) $px = xy f'(x) + x g(y)$

from (3) $qy = y f(x) + xy g'(y)$

$px + qy = xy f'(x) + x g(y) + y f(x) + xy g'(y)$

$\Rightarrow px + qy = xy(f'(x) + g'(y)) + z$

$\Rightarrow px + qy = xys + z$

Scanned with CamScanner
Scanned with CamScanner

Types of p.d.E :

There are two types of p.d.E

1) Linear p.d.E
2) Non-linear p.d.E

Linear p.d.E :- A partial d.E is said to be linear if the derivaties of the dependent variable appear here in the 1st degree only. and ~~doesnot~~ the derivatives cannot be multiplyed together.

Example :- 1. $x^2 \frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} = z$ is a linear p.d.E of order 2

2. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2$ is a linear p.d.E of order 1

Non linear p.d.E :- A partial d.E which is not linear is called a non-linear p.d.E

Ex :- $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = 0$

Solution of a p.d.E :-

A solution of a p.d.E is a relation b/w dependent and independent variables

Types : There are 3 types of solutions

1) Complete solution
2) particular "
3) General "

Complete solution :- Let $f(x, y, z, p, q) = 0$ — ① be the p.d.E of 1st order. A realtion of the form $\phi(x, y, z, a, b) = 0$ — ②

Clearly eq-② is a complete solution of eq-①

($\because$ no. of arbitary constants = no. of independent variables)

Particular solution :

A solution is obtained by giving particular values to the arbitary constants in the complete integral is called a particular solution

General Solution: A solution which contains arbitrary functions is called a General solution.

Ex:- $z = f(x+y)$ is a general solution of $p-q=0$ where 'f' is arbitrary function.

Linear Partial d.E:

Lagranges Linear equation:

A first order linear p.d.E is of the form $Pp+Qq=R$ — (1), where P, Q, R are functions of x, y, z is called a Lagranges linear equation.

Let $f(u,v)=0$ — (2) be the general solution of eq-(1)

If (1) is obtained from (2) by eleminating the arbitary function 'f' where u and v are independent functions of x, y, z

Working Rule to solve $Pp+Qq=R$: — (1)

Step-1: write the Lagranges Auxilary equations (subsidary equations)

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{— (4)}$$

step-2: Find any two independent solutions of subsidary equations

Let the solutions be $u=a$, $v=b$

step-3: The general solution of eq-(1) is given by $f(u,v)=0$

Method of grouping:-

In some problems, It is possible that the subsidary equations are directly solvable to get the solutions. Some times the subsidary equations are not convinient to solve using method of grouping. In such cases we choose the method of multiplyers.

Each ratio is equal to the $\frac{ldx+mdy+ndz}{lP+mQ+nR}$

where l, m, n are called as Multiplyers

If l, m, n are so choosen that $lP+mQ+nR=0$ then

$ldx+mdy+ndz=0$

Integrating this, we get one independent solution (say $u=a$)

Similarly we get another independent solution (say $v=b$)

Sums:

1) solve $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

sol Given $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$ — ①

eq-① is of the form $Pp+Qq=R$

Here $P=\sqrt{x}, Q=\sqrt{y}, R=\sqrt{z}$

The lagrangel Auxilary equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$$

Taking 1st two Ratios

$$\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}$$

I.o.b.s

$$\int x^{-1/2}dx=\int y^{-1/2}dy$$

$$2\sqrt{x}=2\sqrt{y}+c$$

$$\sqrt{x}-\sqrt{y}=\cancel{2}c/2 \Rightarrow \boxed{\sqrt{x}-\sqrt{y}=k_1}$$

Taking last two ratios $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}} \Rightarrow$ I.o.b.s

$$\int y^{-1/2}dy=\int z^{-1/2}dz$$

$$2\sqrt{y} = 2\sqrt{z} + c_1$$

$$\sqrt{y} - \sqrt{z} = c_1/2 \quad \therefore$$

$$\sqrt{y} - \sqrt{z} = k_2 \; (\because k_2 = c_1/2)$$

$\therefore$ General solution is $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$

Q) solve $\tan x \cdot p + \tan y \cdot q = \tan z$ (10)

Sol Given $\tan x \cdot p + \tan y \cdot q = \tan z$ —(1)

It is of the form ~~$P = \sqrt{x}, Q$~~

$$P = \tan x, \; Q = \tan y, \; R = \tan z$$

The Lagranges Auxilary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Taking 1st two equations

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} \Rightarrow \text{I.o.b.s}$$

$$\int \frac{\cos x}{\sin x} dx = \int \frac{\cos y}{\sin y} dy$$

$$\log(\sin x) = \log(\sin y) + \log c$$

$$\frac{\sin x}{\sin y} = k_1$$

Taking last two

$$\frac{dy}{\tan y} = \frac{dz}{\tan z} \Rightarrow \text{I.o.b.s} \Rightarrow \int \frac{\cos y}{\sin y} dy = \int \frac{\cos z}{\sin z} dz$$

$$\log(\sin y) = \log(\sin z) + \log c$$

$$\frac{\sin y}{\sin z} = k_2$$

$\therefore$ The g.s is $f\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

3) Solve $\frac{y^2 z}{x} \cdot p + xzq = y^2$

$y^2z \cdot p + x^2zq = y^2x$ — ①

eq-① is of the form $Pp + Qq = R$

$P = y^2z, Q = x^2z, R = xy^2$

The Auxilary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

Comparing 1st two co-efficients

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}} \Rightarrow x^2dx = y^2dy$$

I.o.b.s

$$\int x^2dx = \int y^2dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c$$

$$x^3 - y^3 = 3c = K_1$$

$$\frac{dx}{y^2z} = \frac{dz}{xy^2}$$

$xdx = zdx$ I.o.b.s

$$\frac{x^2}{2} - \frac{z^2}{2} = c \Rightarrow x^2 - z^2 = K_2$$

$\therefore$ The g.s is $f(x^3 - y^3, x^2 - z^2) = 0$.

4) Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Sol $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

eq-① is of the form $Pp + Qq = R$

$P = x^2(y-z), Q = y^2(z-x), R = z^2(x-y)$

The Auxilary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)}$$

Here the auxilary equations are not convinient to solve by the method of grouping. so we choose Method of Multiplyers

Taking $l = 1/x, m = 1/y, n = 1/z$ as multiplyers are so choosen that

$lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

I.o.bs

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log c$$

$$\boxed{xyz = c}$$

Again let $l = 1/x^2, m = 1/y^2, n = 1/z^2$ such that

$lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$

$$\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

I.o.b.s

$$\int \frac{1}{x^2}dx + \int \frac{1}{y^2}dy + \int \frac{1}{z^2}dz = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = K \Rightarrow \boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -K}$$

$\therefore$ The general solution of eq ① is

$$f\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$$

5) Solve $xp - yq = y^2 - x^2$

sol Given $xp - yq = y^2 - x^2$ is o.f the form $Pp + Qq = R$

Here $P = x, Q = -y, R = y^2 - x^2$

The Auxilary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{y^2 - x^2}$$

From the first two terms $\frac{dx}{x} = \frac{dy}{-y}$

I.o.bs

$\int \frac{1}{x} dx = -\int \frac{1}{y} dy \Rightarrow \log x = -\log y + \log c$

$\log(xy) = \log c$

$\boxed{xy = c}$

let $l = x, m = y, n = 1$ so choosen that

$lP + mQ + nR = 0$

$\therefore l dx + m dy + n dz = 0$

$x dx + y dy + dz = 0$

I.o.bs

$\int x dx + \int y dy + \int dz = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + z = K$

$\Rightarrow \boxed{x^2 + y^2 + 2z = 2K}$

The general solution of given equation is $f(xy, x^2+y^2+2z) = 0$

6) Solve $(y^2+z^2)p - xy\, q + zx = 0$

Sol) Given $(y^2+z^2)p - xyq = -zx$

It is of the form $Pp + Qq = R$

$P = y^2+z^2$, $Q = -xy$, $R = -zx$

$\therefore$ The Auxilary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx}$$

$\therefore$ from last two $\frac{dy}{-xy} = \frac{dz}{-zx}$

I.o.b.s

$\int \frac{1}{y} dy = \int \frac{1}{z} dz \Rightarrow \log y = \log z + \log c$

$y = zc \Rightarrow \boxed{y/z = c}$

Taking $l = x, m = y, n = z$ are so choosen that $lP + mQ + nR = 0$

$l dx + m dy + n dz = 0$ and

$x dx + y dy + z dz = 0$ I.o.bs

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = K \Rightarrow \boxed{x^2+y^2+z^2 = 2K}$

$\therefore$ The given auxilary equation is $f\left(y/z,\ x^2+y^2+z^2\right)=0$

7) Solve $(x^2-y^2-z^2)p + 2xyq = 2xz$

sol Given $(x^2-y^2-z^2)p + 2xyq = 2xz$

It is of the form $Pp + Qq = R$

$P = x^2-y^2-z^2,\ Q = 2xy,\ R = 2xz$

$\therefore \frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$ — I

$\therefore$ from last two $\frac{dy}{\cancel{2x}y} = \frac{dz}{\cancel{2x}z} \Rightarrow$ I.o.b.s

$\int \frac{1}{y}\,dy = \int \frac{1}{z}\,dz \Rightarrow \log y = \log z + \log c$

$\Rightarrow \boxed{y/z = c}$

let $l = x,\ m = y,\ n = z$ and each ratio is equal to $\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$

$l\,dx + m\,dy + n\,dz = x\,dx + y\,dy + z\,dz$, $lP + mQ + nR$

$= x^3 \cancel{- xy^2 - xz^2 + 2xy^2 + 2xz^2}$

$= x^3 \cancel{+ xy^2 + xz^2}$

$= x\cancel{(x^2+y^2+z^2)}$

$= \frac{x\,dx + y\,dy + z\,dz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$

$= \frac{x\,dx + y\,dy + dz}{x^3 + xy^2 + xz^2}$

$= \frac{x\,dx + y\,dy + dz}{x(x^2+y^2+z^2)}$

Each ratio $= \frac{x\,dx + y\,dy + z\,dz}{\cancel{x}(x^2+y^2+z^2)} = \frac{dz}{2\cancel{x}z}$

$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \frac{1}{z}\,dz$

I.o.b.s

$\log(x^2+y^2+z^2) = \log z + \log K$

$x^2+y^2+z^2 = zK \Rightarrow \frac{x^2+y^2+z^2}{z} = K.$

$\therefore$ General solution is $F\left(y/z,\ \frac{x^2+y^2+z^2}{z}\right) = 0$

Non Linear Partial d.E:

There are four types of non-linear p.d.E's.

1) $f(p,q)=0$

2) $f(z,p,q)=0$

3) $f(x,p)=g(y,q)$

4) $z=px+qy+f(p,q)$

Standard form-1:

Equations of the form $f(p,q)=0$ — ①

i.e, equations containing p and q only

Working Rule:-

Let $z=ax+by+c$ be a trail solution of equation ①
— ②

then $\frac{\partial z}{\partial x}=a$ i.e, $p=a$

similarly $\frac{\partial z}{\partial y}=b$ i.e, $q=b$

substituting these values in eq-① , we get

$f(a,b)=0$

To find the complete solution of eq-①, we have to eliminate any one of the constant from ②

∴ from ②, we can express b in terms of 'a'

let $b=\phi(a)$

Then the required complete solution is

$$\boxed{z=ax+\phi(a)\cdot y+c}$$

Sums:

1) Solve $p^2+q^2=1$

Sol Given $p^2+q^2=1$ — ①

eq-① is of the form $f(p,q)=0$

let $z=ax+by+c$ be a trail solution of eq-①
— ②

$\frac{\partial z}{\partial x} = a$, i.e, $p = a$ and $\frac{\partial z}{\partial y} = b$ i.e, $q = b$

Substituting these values in ①, we get

$$a^2 + b^2 = 1 \quad — ③$$

To find ~~the~~ complete solution, we have to eliminate any one of the constant from ②

∴ from ③ $b^2 = 1 - a^2 \Rightarrow b = \pm\sqrt{1-a^2}$

Sub 'b' value in ②, we get

$\boxed{z = ax \pm \sqrt{1-a^2}\, y + c}$ is the required complete solution.

2) Solve $pq + p + q = 0$

Sol Given $pq + p + q = 0$ — ①

It is of the form $f(p,q) = 0$

let $z = ax + by + c$ is the trail solution of eq-① — ②

$\frac{\partial z}{\partial x} = a$ i.e, $p = a$, $\frac{\partial z}{\partial y} = b$ i.e, $q = b$

Sub these values in ①. we get

$$ab + a + b = 0 \quad — ③$$

To find the complete solution we have to eliminate any one of the constant from ②

∴ from ③ $b(a+1) + a = 0 \Rightarrow b = \frac{-a}{a+1}$

∴ sub 'b' value in eq-①, we get

$\boxed{z = ax - \frac{a}{a+1} y + c}$ is the required complete solution.

(3) ~~4(a)~~) solve $p^2+q^2=npq$

Given $p^2+q^2=npq$ — ①

eq ① is of the ~~the~~ standard form - ①

let $z=ax+by+c$ be the trail solution — ②

Then $\frac{\partial z}{\partial x}=a \Rightarrow p=a$

$\frac{\partial z}{\partial y}=b(1) \Rightarrow q=b$

sub these values in ①

$a^2+b^2=nab$

$a^2-nab+b^2=0$

$\therefore a=\frac{nb\pm\sqrt{n^2b^2-4b^2}}{2}$

$=\frac{b}{2}\left[n\pm\sqrt{n^2-4}\right]$ — ③

sub ③ in ②, we get

$z=bx+\frac{b}{2}y\left[n\pm\sqrt{n^2-4}\right]+c.$

Equations reducible to standard forms

Equations of the form $f(x^m p, y^n q) = 0$ — (1)

eq-(1) can be reduced to an equation of the form

$f(P, Q) = 0$

By the following transformations

Case (i):- When $m=1, n=1$ then the eq-① becomes.

$f(px, qy)=0$.

For the combination px, we use the transformation

$$X = \log x$$

For the combination qy, we use the transformation

$$Y = \log y$$

W.K.T

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1/x)$$

$$px = \frac{\partial z}{\partial X} = P \text{ (say)}$$

||ly $$qy = \frac{\partial z}{\partial Y} = Q \text{ (say)}$$

Now eq-① will be reduced to the form

$$f(P, Q) = 0$$

Case (ii):- When $m \neq 1$ and $n \neq 1$

let $X = x^{1-m}, Y = y^{1-n}$

then $\frac{\partial X}{\partial x} = (1-m)x^{-m}$ and $\frac{\partial Y}{\partial y} = (1-n)y^{-n}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)x^{-m}$$

$$\Rightarrow x^m p = (1-m)\frac{\partial z}{\partial X} = (1-m)P$$

similarly $y^n q = (1-n)\frac{\partial z}{\partial Y} = (1-n)Q$

Now eq-① will be transformed to $f((1-m)P, (1-n)Q) = 0$

Sums:-

1) Solve $x^2p^2 + y^2q^2 = 1$

Sol: Given equation can be written as $(px)^2 + (qy)^2 = 1$ —①

eq-① will be transformed in the form of $f(P, Q) = 0$

By $X = \log x$ & $Y = \log y$

$$\frac{\partial X}{\partial x} = \frac{1}{x} \qquad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

W.K.T $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{1}{x}$$

$\Rightarrow px = \partial z/\partial X = P$

Scanned with CamScanner

III'ly $qy = \frac{\partial z}{\partial Y} = Q$

Sub these values in ①, we get

$$P^2 + Q^2 = 1 \quad \text{—⑤}$$

let $z = ax + bY \pm c$ ③ be a solution of ②

then $\frac{\partial z}{\partial x} = a$ | $\frac{\partial z}{\partial Y} = b$

$P = a$ | $Q = b$

$\therefore$ from ② $a^2 + b^2 = 1 \Rightarrow b^2 = 1 - a^2 \Rightarrow b = \pm\sqrt{1-a^2}$

sub 'b' values in ③, we get

$$z = ax \pm \sqrt{1-a^2}\,y + c$$

$\boxed{z = a\log x \pm \sqrt{1-a^2}\log y + c}$ is the r.c.s

Standard form-2:

equations of the form $f(z, p, q) = 0$ ① i.e. equations not containing x and y.

Working Rule:-

Let $z = f(u)$ where $u = x + ay$ be a tentative solution of eq-①. Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}(1) = \frac{\partial z}{\partial u}$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = \frac{\partial z}{\partial u}(a) = a\cdot\frac{\partial z}{\partial u}$$

Substituting these values in eq-①, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\cdot\frac{\partial z}{\partial u}\right) = 0 \quad (\text{or}) \quad f\left(z, \frac{dz}{\partial u}, a\cdot\frac{dz}{du}\right) = 0$$

which is an ordinary d.E of 1^{st} order

The solution of above d.E will give the complete solution of eq-①

Sums

1) Solve $z = p^2 + q^2$

Sol Given $z = p^2 + q^2$ — ①

eq-① is of the form $f(z, p, q) = 0$

let $z = f(u) \Rightarrow u = x + ay$ be a trail solution of eq-①

then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}(1) = \frac{\partial z}{\partial u}$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a\frac{\partial z}{\partial u}$

Sub these values in ①, we get

$$z = \left(\frac{\partial z}{\partial u}\right)^2 + \left(a \cdot \frac{\partial z}{\partial u}\right)^2 \quad (\text{or}) \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z$$

$$\Rightarrow z = \left(\frac{dz}{du}\right)^2(1+a^2)$$

$$\frac{z}{1+a^2} = \left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \frac{dz}{du} = \frac{\sqrt{z}}{\sqrt{1+a^2}} \Rightarrow \int z^{-1/2}dz = \frac{1}{\sqrt{1+a^2}}\int du + c$$

$$\frac{z^{1/2}}{1/2} = \frac{1}{\sqrt{a^2+1}}(u) + c$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{1+a^2}} + c$$

2) solve $q^2 = z^2p^2(1-p^2)$

Sol Given $q^2 = z^2p^2(1-p^2)$ — ①

eq-① is of the form $f(z,p,q)=0$

let $z = f(u) \Rightarrow u = ay + x$ be a trail solution of eq-①

then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}(1)$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} = a \cdot \frac{\partial z}{\partial u}$

sub these values in ①, we get

$$\left(a \cdot \frac{\partial z}{\partial u}\right)^2 = z^2\left(\frac{\partial z}{\partial u}\right)^2\left(1-\left(\frac{\partial z}{\partial u}\right)^2\right)$$

(or)

$$a^2\left(\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left(1-\left(\frac{dz}{du}\right)^2\right)$$

$$1-\left(\frac{dz}{du}\right)^2 = \frac{a^2}{z^2}$$

$$1-\frac{a^2}{z^2} = \left(\frac{dz}{du}\right)^2$$

$$\frac{dz}{du} = \sqrt{\frac{z^2-a^2}{z^2}} = \pm\frac{\sqrt{z^2-a^2}}{z}$$

$$\frac{dz}{du} = \frac{z\,dz}{\sqrt{z^2-a^2}} = \pm du$$

I.o.b.s

$$\frac{1}{2}\int \frac{2z\,dz}{\sqrt{z^2-a^2}} = \pm\int du$$

$$\frac{1}{2}\cdot 2\sqrt{z^2-a^2} = \pm u + c$$

$$\boxed{z^2-a^2 = (u+c)^2 = (x+ay+c)^2}$$ is the required c.s

3) Solve $x^2p^2+y^2q^2=z^2$

Sol Given equation can be written as $(px)^2+(qy)^2=z^2$ — ①

let $X=\log x$, $Y=\log y$ then

$$\frac{\partial X}{\partial x} = \frac{1}{x}, \quad \frac{\partial Y}{\partial y} = 1/y$$

W.K.T $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\cdot\frac{1}{x}$

$px = \frac{\partial z}{\partial X}$ and similarly $qy = \frac{\partial z}{\partial Y}$

Sub these values in ①, we have

$$\left(\frac{\partial z}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial Y}\right)^2 = z^2$$

$$P^2+Q^2=z^2 \text{ — ②}$$

② is of form $f(P,Q,z)=0$

let $z=f(u)$, where $u=X+aY$ then

$$P = \frac{\partial z}{\partial X} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial X} = \frac{\partial z}{\partial u}(1)$$

$$Q = \frac{\partial z}{\partial Y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial Y} = \frac{\partial z}{\partial u}(a)$$

Sub P, Q values in ②, we get

$$\left(\frac{\partial z}{\partial u}\right)^2 + \left(a\frac{\partial z}{\partial u}\right)^2 = z^2 \text{ (or) } \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2\left[1+a^2\right] = z^2$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

Scanned with CamScanner

$$\frac{1}{z}dz = \frac{1}{\sqrt{1+a^2}}\cdot du$$

I.o.b.s

$$\int \frac{1}{z}\cdot dz = \frac{1}{\sqrt{1+a^2}}\int 1\cdot du + c$$

$$\log z = \frac{u}{\sqrt{1+a^2}} + c \Rightarrow \log z = \frac{x+ay}{\sqrt{1+a^2}} + c$$

$$\boxed{\log z = \frac{\log x + a\log y}{\sqrt{1+a^2}} + c}$$ is the required complete solution

Standard form-3

Equations of the form $f(x,p)=g(y,q)$ i.e, equations not containing z and the equations containg x and p can be seperated from those containing y and q

Working Rule :-

* consider $f(x,p)=g(y,q)$ — ① as a trail solution, we assume that each term of eq-① is equal to an arbitary constant 'a'

i.e, $f(x,p)=g(y,q)=a$

Solving for p and q we have

$$p=F_1(x,a), \quad q=F_2(y,a)$$

$\because$ z is a function of x and y then the total derivative of z

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}.dy = p\,dx + q\,dy \quad \text{—②}$$

Substituting the values of p, q in eq-② we have

$$dz = F_1(x,a)dx + F_2(y,a)dy$$

$$\text{I.o.b.s} \Rightarrow \int dz = \int F_1(x,a)dx + \int F_2(y,a)dy$$

$$\therefore \boxed{z = \int F_1(x,a)dx + \int F_2(y,a)dy}$$ is the r.c.s

Sums:-

1) Solve $yp + xq + pq = 0$

Sol Given $yp + xq + pq = 0$ — ①

Dividing ① with pq $\Rightarrow \frac{y}{q} + \frac{x}{p} + 1 = 0$

$$\frac{x}{p} = -\left(1+\frac{y}{q}\right)$$

Assume that each term is equal to an arbitary constant

$$\frac{x}{p} = -\left(1+\frac{y}{q}\right) = a$$

$\frac{x}{p} = a \Rightarrow p = \frac{x}{a}$ || $-\left(1+\frac{y}{q}\right) = a \Rightarrow -\frac{y}{q} = 1+a \Rightarrow q = -\frac{y}{1+a}$

sub p, q values in $dz = pdx + qdy$

$$dz = \frac{x}{a}dx - \frac{y}{1+a}dy \Rightarrow \text{I.o.b.s}$$

$$\int dz = \frac{1}{a}\int x\,dx - \frac{1}{1+a}\int y\,dy + c$$

$$\boxed{z = \frac{x^2}{2a} - \frac{y^2}{2(1+a)} + c}$$ is the r.c.s

2) Solve $p^2+q^2 = x^2+y^2$

sol Given equation can be written as $p^2-x^2 = y^2-q^2$ — ①

Assume that each term is equal to arbitary Constant

$$p^2-x^2 = y^2-q^2 = a^2 \text{ (say)}$$

$p^2-x^2 = a^2$ || $y^2-q^2 = a^2$

$p = \sqrt{x^2+a^2}$ || $\Rightarrow q = \sqrt{y^2-a^2}$

Sub these values in $dz = pdx + qdy$

$$dz = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy \Rightarrow \text{I.o.b.s}$$

$$\int dz = \int\sqrt{a^2+x^2}\,dx + \int\sqrt{y^2-a^2}\,dy + c$$

$$z = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c$$

3) Solve $\sqrt{p}+\sqrt{q} = x^2$

Given equation can be written as $\sqrt{p}-x^2 = -\sqrt{q} = a$

Solving for p, q

$\sqrt{p}-x^2 = a$ | $-\sqrt{q} = a$

$\sqrt{p} = a+x^2$ | $q = a^2$

$p = (a+x^2)^2$

Sub p, q values in $dz = pdx + qdy$

$$= (a+x^2)^2dx + a^2dy$$

$$\int dz = \int(a+x^2)^2dx + a^2\int dy + c$$

Sub these values in $dz = Pdx + Qdy$

$$dz = (\sqrt{a} - x)dx + (\sqrt{1-a} - y)dy$$

I.O.b.S

$$\int dz = \int(\sqrt{a} - x)\,dx + \int(\sqrt{1-a} - y)dy + c$$

$$z = \sqrt{a}x - \frac{x^2}{2} + \sqrt{1-a}\cdot y - \frac{y^2}{2} + c$$

$$\frac{z^2}{2} = \sqrt{a}x + \sqrt{1-a}\cdot y - \left(\frac{x^2+y^2}{2}\right) + c$$ is the r.c.s

Standard form-IV:

Equations of the form $z = px + qy + f(p,q)$

This equation is known as Clairauit's equations

Consider $z = px + qy + f(p,q)$ —(1)

A relation of the form $z = ax + by + f(a,b)$ —(2)

differentiating eq-(2) partially w.r.t 'x' and y

$$\frac{\partial z}{\partial x} = a \qquad \frac{\partial z}{\partial y} = b$$

$$p = a \qquad q = b$$

eleminating a, b from (2), we get eq-(1)

$\therefore z = ax + by + f(a,b)$ is the complete solution of eq-(1)

Sums:

1) Solve $z = px + qy \mp 2\sqrt{pq}$

Given $z = px + qy \mp 2\sqrt{pq}$ —(1)

eq-(1) is of the form $px + qy + f(p,q) = z$

The complete solution of eq-(1) is obtained by writing $p = a$ and $q = b$

$$\therefore z = ax + by \mp 2\sqrt{ab}$$

2) Solve $(p+q)(z - px - qy) = 1$

Sol: Given equation can be written as

$$z - px - qy = \frac{1}{p+q}$$

$$z = px + qy + \frac{1}{p+q} \quad —(1)$$

equation ① is of the form $px+qy+f(p,q)=z$

The complete solution of eq-① can be obtained by writing $p=a, q=b$

$$\therefore z = ax + by + \frac{1}{a+b}$$

3) Solve $pqz = p^2(qx+p^2) + q^2(py+q^2)$

Sol Given equation can be written as Given $pqz = p^2(qx+p^2) + q^2(py+q^2)$

Dividing ① with pq

$$z = px + \frac{p^3}{q} + qy + \frac{q^3}{p}$$

$$= px + qy + \frac{p^3}{q} + \frac{q^3}{p} \quad \text{—①}$$

eq-① is of the form $z = px + qy + f(p,q)$

The complete solution of eq-① can be obtained by writing $p=a$ and $q=b$

$$z = ax + by + \frac{a^3}{b} + \frac{b^3}{a}$$

4) Solve $z = px + qy + x\log x \cdot p + y\log y \cdot q$

Sol Given equation can be written as $z = px + qy + \log x\,(px) + \log y\,(qy)$ —①

Eq-① can be reduced to the standard form ④ by the transformation $X = \log x$ and $Y = \log y$

$$\frac{\partial X}{\partial x} = \frac{1}{x}, \quad \frac{\partial Y}{\partial y} = \frac{1}{y}$$

W.K.T $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\left(\frac{1}{x}\right)$

$$px = \frac{\partial z}{\partial X} = P$$

Similarly $qy = \frac{\partial z}{\partial y} = Q$

Sub these values in ①, we get

$$z = P + Q + XP + YQ$$

$$z = PX + QY + P + Q \quad ①$$

Scanned with CamScanner

eq-② is of the form standard form ④

The complete solution of the eq-② is obtained by replacing

$P = a, Q = b$

$Z = aX + bY + a + b$

$= a\log x + b\log y + a + b$

---

1) Solve $\frac{x^2}{p} + \frac{y^2}{q} = z$

Sol:- Given equation can be written as $(x^{-2}p)^{-1} + (y^{-2}q)^{-1} = z$ —①

This is of the form $f(x^m p, y^n q, z) = 0$ with $m = -2$ and $n = -2$

$\therefore$ put $X = x^{1-m} = x^{1-(-2)} = x^3$ and $Y = y^{1-n} = y^3$.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P \cdot 3x^2$ where $P = \frac{\partial z}{\partial X} \Rightarrow x^{-2}p = 3P$

$||^{ly}$ $y^{-2}q = 3Q$ where $Q = \frac{\partial z}{\partial Y}$.

Now ① becomes $(3P)^{-1} + (3Q)^{-1} = z$

This is of the form $f(z, P, Q) = 0$ —②

Let $z = g(X + aY)$ be the solution of ②

put $u = X + aY$ then $z = g(u)$

$\therefore P = \frac{dz}{du}$ and $Q = a\frac{dz}{du}$

Now ② becomes $\left(3\frac{dz}{du}\right)^{-1} + \left(3a\frac{dz}{du}\right)^{-1} = z$

ie. $\frac{1}{3}\frac{du}{dz} + \frac{1}{3a}\frac{du}{dz} = z$.

(or) $\frac{1}{3}\frac{du}{dz}\left[1 + \frac{1}{a}\right] = z$, Separating the variables,

$z\,dz = \left(\frac{a+1}{3a}\right)du$, Integrating, we get

$\frac{z^2}{2} = \left(\frac{a+1}{3a}\right)u + c$

$\Rightarrow \frac{z^2}{2} = \frac{a+1}{3a}(X + aY) + c$

$\Rightarrow \frac{z^2}{2} = \left(\frac{a+1}{3a}\right)\left[x^3 + ay^3\right] + c$, is the required solution.

② . Solve $z^2(p^2+q^2) = x^2+y^2$

Sol - Given equation can be written as $(pz)^2 - x^2 = y^2 - (qz)^2$

This is of the form $f(x, pz^n) = g(y, qz^n)$ with $n=1$

$\therefore$ put $Z = z^{n+1} = z^{1+1} = z^2$ then

$\frac{\partial Z}{\partial x} = 2z\frac{\partial z}{\partial x} \Rightarrow P = 2zp$ (or) $pz = \frac{P}{2}$ and $\frac{\partial Z}{\partial y} = 2z\frac{\partial z}{\partial y}$

$\Rightarrow Q = 2zq$ (or) $qz = \frac{Q}{2}$.

substituting the values of pz and qz in ①, we have

$$\left(\frac{P}{2}\right)^2 - x^2 = y^2 - \left(\frac{Q}{2}\right)^2$$

i.e. $P^2 - 4x^2 = 4y^2 - Q^2$

which is of the form $f_1(x, P) = f_2(y, Q)$

i.e. $P^2 - 4x^2 = 4y^2 - Q^2 = 4a^2$ (say)

$\therefore P = \sqrt{4a^2+4x^2} = 2\sqrt{x^2+a^2}$

$Q = \sqrt{4y^2-4a^2} = 2\sqrt{y^2-a^2}$.

$\therefore dZ = P\,dx + Q\,dy = 2\sqrt{x^2+a^2}\,dx + 2\sqrt{y^2-a^2}\,dy$

I.o.b.s.

$$Z = 2\int\sqrt{x^2+a^2}\,dx + 2\int\sqrt{y^2-a^2}\,dy$$

$$= 2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)\right] + 2\left[\frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)\right] + c$$

(or) $z^2 = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a^2} - a^2\cosh^{-1}\left(\frac{y}{a}\right) + c$

$$= x\sqrt{x^2+a^2} + y\sqrt{y^2-a^2} + a^2\left[\sinh^{-1}\left(\frac{x}{a}\right) - \cosh^{-1}(y/a)\right] + c.$$

Scanned with CamScanner