UNIT -II

LAPLACE TRANSFORMS

Introduction

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French. Laplace Transformations is a powerful technique; it replaces operations of calculus by operations of algebra. An Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation by this method.

Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach just like the applications of transfer functions to solve ordinary differential equations.Laplace Transform methods have a key role to play in the modern approach to the analysis and design of engineering system.The purpose of the Laplace Transform is to transform ordinary differential equations (ODEs) into algebraic equations, which can then be solved by the formal rules of algebra.This makes it easier to solve ODEs.The concepts of Laplace Transforms are applied in the area of science and technology such as Electrical engineering, Communication engineering, Mechanical engineering ,Computer engineering,Control engineering,Civil engineering and Nuclear physics etc.

Laplace Transform is widely used by electronic engineers to solve quickly differential equations occurring in the analysis of electronic circuits,Digital signal processing, System modeling, Process Control. In Mechanical engineering field Laplace Transform is widely used to solve differential equations occurring in mathematical modeling of mechanical system to find transfer function of that particular system.In Computer scienceengineering it is used in Data mining (which is the analysis step of Knowledge Discovery in Databases) which focuses on the discovery of (previously) unknown properties on the data, where Laplace equation is used to determine the prediction and analyses the step of knowledge in databases.Laplace Transformations helps to find out the current and some criteria for the analyzing the circuits. It is used to build required ICs and chips for systems.So it plays a vital role in field of computer science.In order to get the true form of radioactive decay a Laplace Transform is used in Nuclear Physics.In Civil engineering it can be applied to analyze behavior of complex systems, such as bridges, skyscrapers, and so forth.In Aeronautical engineeringLaplace transform is a valuable “tool” in solving Differential equations in “feedback control” systems for example, in stability and control of aircraft systems.

Uses

- Particular Solution is obtained without first determining the general solution.
- Non-Homogeneous Equations are solved without obtaining the complementary integral.
- L.T is applicable not only to continuous functions but also to piecewise continuous functions, complicated periodic functions, step functions and impulse functions.

Applications:

- L.T is very useful in obtaining solution of linear differential equations, both ordinary and partial, solution of system of simultaneous differential equations, solution of integral equations, solution of linear difference equations and in the evaluation of definite integrals.

Definition:

Let $f(t)$ be a function of 't' defined for all positive values of t. Then Laplace transforms of $f(t)$ is denoted by $L\{f(t)\}$ is defined by

$$L\{f(t)\}=\int_0^\infty e^{-st} f(t)dt=\overline{f}(s)\rightarrow(1)$$

provided that the integral exists. Here the parameter's' is a real (or) complex number.

The relation (1) can also be written as $f(t)=L^{-1}\{\overline{f}(s)\}$

In such a case the function $f(t)$ is called the inverse Laplace transform of $\overline{f}(s)$.The symbol 'L' which transform $f(t)$ into $\overline{f}(s)$ is called the Laplace transform operator. The symbol 'L^{-1}' which transforms $\bar{f}$(s) to $f(t)$ can be called the inverse Laplace transform operator.

Conditions for Laplace Transforms

Exponential order: A function $f(t)$ is said to be of exponential order 'a' If $\underset{t\to\infty}{lt}\, e^{-st}f(t)=a$ (finite quantity).

Ex: (i). The function t^2 is of exponential order

(ii). The function e^{t^3} is not of exponential order (which is not finite quantity)

Piece – wise Continuous function: A function $f(t)$ is said to be piece-wise continuous over the closed interval $[a,b]$ if it is defined on that interval and is such that the interval can be divided into a finite number of sub intervals, in each of which $f(t)$ is continuous and has both right and left hand limits at every end point of the subinterval.

Sufficient conditions for the existence of the Laplace transform of a function:

The function $f\ (t)$ must satisfy the following conditions for the existence of the L.T.

(i).The function $f(t)$ must be piece-wise continuous (or sectionally continuous) in any limited interval $0 < a \le t \le b$.

(ii).The function $f\ (t)$ is of exponential order.

Laplace Transforms of standard functions:

1. Prove that $L\{1\} = \dfrac{1}{s}$

Proof: By definition

$$L\{1\} = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s}\ if\ s > 0$$

$$L\{1\} = \frac{1}{s}\ \left(\because e^{-\infty} = 0\right)$$

2. Prove that $L\{t\} = \frac{1}{s^2}$

Proof: By definition

$$L\{t\} = \int_0^\infty e^{-st}.tdt = \left[t.\left(\frac{e^{-st}}{-s}\right) - \int 1.\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$= \left[t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right]_0^\infty = \frac{1}{s^2}$$

3. Prove that $L\{t^n\} = \dfrac{n!}{s^{n+1}}$ **where n is a positive integer**

Proof: By definition $L\{t^n\} = \int_0^\infty e^{-st}.t^n dt = \left[t^n.\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty n.t^{n-1}.\frac{e^{-st}}{-s}dt$

$$= 0 - 0 + \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\{t^{n-1}\}$$

$$L\{t^{n-1}\} = \frac{n-1}{s}L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s}L\{t^{n-3}\}$$

By repeatedly applying this, we get

$$L\{t^n\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{2}{s}.\frac{1}{s}L\{t^{n-n}\}$$

$$=\frac{n!}{s^n}L\{1\}=\frac{n!}{s^n}.\frac{1}{s}=\frac{n!}{s^{n+1}}$$

Note: $L\{t^n\}$ can also be expressed in terms of Gamma function.

$$i.e., L\{t^n\}=\frac{n!}{s^{n+1}}=\frac{\Gamma(n+1)}{s^{n+1}}\left(\because \Gamma(n+1)=n!\right)$$

Def: If $n > 0$ then Gamma function is defined by $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$

We have $L\{t^n\}=\int_0^\infty e^{-st}.t^n dt$

Putting $x = st$ on R.H.S, we get

$$L\{t^n\}t=\int_0^\infty e^{-x}.\frac{x^n}{s^n}.\frac{1}{s}dx \qquad \begin{pmatrix} x=st \\ \frac{1}{s}dx=dt \end{pmatrix}$$

$$=\frac{1}{s^{n+1}}\int_0^\infty e^{-x}.x^n dx \qquad \begin{pmatrix} When \quad t=0, x=0 \\ When \quad t=\infty, x=\infty \end{pmatrix}$$

$$L\{t^n\}=\frac{1}{s^{n+1}}.\Gamma(n+1)$$

$If\ 'n'\text{is } a \text{ +ve integer then } \Gamma(n+1)=n!$

$\therefore L\{t^n\}=n!/s^{n+1}$

Note: The following are some important properties of the Gamma function.

1. $\Gamma(n+1)=n.\Gamma(n)\ if\ n>0$
2. $\Gamma(n+1)=n!$ if n is a positive integer
3. $\Gamma(1)=1, \Gamma(1/2)=\sqrt{\pi}$

Note: Value of $\Gamma(n)$ in terms of factorial

$\Gamma(2)=1\times\Gamma(1)=1!$

$\Gamma(3)=2\times\Gamma(2)=2!$

$\Gamma(4)=3\times\Gamma(3)=3!$ and so on.

In general $\Gamma(n+1)=n!$ provided 'n' is a positive integer.

Taking $n = 0$, it defined $0! = \Gamma(1)=1$

4. Prove that $L\{e^{at}\}=\frac{1}{s-a}$

Proof: By definition,

$$L\left\{e^{at}\right\}=\int_0^\infty e^{-st}e^{at}dt=\int_0^\infty e^{-(s-a)t}dt$$

$$=\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$=\frac{-e^{-\infty}}{s-a}+\frac{e^0}{s-a}=\frac{1}{s-a}\ if\ s>a$$

Similarly $L\left\{e^{-at}\right\}=\frac{1}{s+a}\ if\ s>-a$

5. Prove that $L\{\sinh at\}=\frac{a}{s^2-a^2}$

Proof: $L\{\sinh at\}=L\left\{\frac{e^{at}-e^{-at}}{2}\right\}=\frac{1}{2}\left[L\left\{e^{at}\right\}-L\left\{e^{-at}\right\}\right]$

$$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]=\frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]=\frac{2a}{2\left(s^2-a^2\right)}=\frac{a}{s^2-a^2}$$

6. Prove that $L\{\cosh at\}=\frac{s}{s^2-a^2}$

Proof: $L\{\cosh at\}=L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$=\frac{1}{2}\left[L\left\{e^{at}\right\}+L\left\{e^{-at}\right\}\right]=\frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}$$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]=\frac{2s}{2\left(s^2-a^2\right)}=\frac{s}{s^2-a^2}$$

7. Prove that $L\{\sin at\}=\frac{a}{s^2+a^2}$

Proof: By definition, $L\{\sin at\}=\int_0^\infty e^{-st}\sin at\,dt$

$$=\left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at-a\cos at\right)\right]_0^\infty \qquad \left[\because \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)\right]$$

$$=\frac{a}{s^2+a^2}$$

8. Prove that $L\{\cos at\}=\frac{s}{s^2+a^2}$

Proof: We know that $L\left\{e^{at}\right\}=\frac{1}{s-a}$

Replace 'a' by 'ia' we get

$$L\left\{e^{iat}\right\}=\frac{1}{s-ia}=\frac{s+ia}{(s-ia)(s+ia)}$$

$$i.e., \mathrm{L}\left\{cosat+i\sin at\right\}=\frac{s+ia}{s^2+a^2}$$

Equating the real and imaginary parts on both sides, we have

$$L\left\{\cos at\right\}=\frac{s}{s^2+a^2}\ and\ L\left\{\sin at\right\}=\frac{a}{s^2+a^2}$$

Solved Problems :

1. Find the Laplace transforms of $(t^2+1)^2$

Sol: Here $f(t)=(t^2+1)^2=t^4+2t^2+1$

$$L\{(t^2+1)^2\}=L\{t^4+2t^2+1\}=L\{t^4\}+2L\{t^2\}+L\{1\}$$

$$=\frac{4!}{s^{4+1}}+2.\frac{2!}{s^3}+\frac{1}{s}=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s}$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}(24+4s^2+s^4)$$

2. Find the Laplace transform of $L\left\{\frac{e^{-at}-1}{a}\right\}$

Sol:
$$L\left\{\frac{e^{-at}-1}{a}\right\}=\frac{1}{a}L\left\{e^{-at}-1\right\}=\frac{1}{a}\left[L\left\{e^{-at}\right\}-L\left\{1\right\}\right]$$

$$=\frac{1}{a}\left[\frac{1}{s+a}-\frac{1}{s}\right]=-\frac{1}{s(s+a)}$$

3. Find the Laplace transform of $\boldsymbol{sin2tcost}$

Sol: w.k.t $\sin 2t\cos t=\frac{1}{2}[2\sin 2t\cos t]=\frac{1}{2}[\sin 3t+\sin t]$

$$\therefore L\{\sin 2t\cos t\}=L\left\{\frac{1}{2}[\sin 3t+\sin t]\right\}=\frac{1}{2}\left[\mathrm{L}\left\{\sin 3t\right\}+L\left\{\sin t\right\}\right]$$

$$=\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]=\frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

4. Find the Laplace transform of $\boldsymbol{cosh^2 2t}$

Sol: w.k.t $\cosh^2 2t=\frac{1}{2}[1+\cosh 4t]$

$$L\{\cosh^2 2t\}=\frac{1}{2}[L(1)+L\{\cosh 4t\}]$$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s(s^2-16)}$$

5. Find the Laplace transform of $cos^3 3t$

Sol: Since cos9t=cos3(3t)

cos9t=4cos³3t-3cos3t (or) $\cos^3 3t = \frac{1}{4}[\cos 9t + 3\cos 3t]$

$$L\{\cos^3 3t\} = \frac{1}{4}L\{\cos 9t\} + \frac{3}{4}L\{\cos 3t\}$$

$$\therefore \quad = \frac{1}{4}.\frac{s}{s^2+81} + \frac{3}{4}.\frac{s}{s^2+9}$$

$$= \frac{s}{4}\left[\frac{1}{s^2+81} + \frac{3}{s^2+9}\right] = \frac{s\left(s^2+63\right)}{\left(s^2+9\right)\left(s^2+81\right)}$$

6. Find the Laplace transforms of $(\sin t + \cos t)^2$

Sol: Since $(\sin t + \cos t)^2 = \sin^2 t + \cos^2 t + 2\sin t \cos t = 1 + \sin 2t$

$$L\{(\sin t + \cos t)^2\} = L\{1 + \sin 2t\}$$

$$= L\{1\} + L\{\sin 2t\}$$

$$= \frac{1}{s} + \frac{2}{s^2+4} = \frac{s^2+2s+4}{s\left(s^2+4\right)}$$

7. Find the Laplace transforms of $cost\ cos2t\ cos3t$

Sol: $\cos t \cos 2t \cos 3t = \frac{1}{2}.\cos t\left[2.\cos 2t.\cos 3t\right]$

$$= \frac{1}{2}\cos t[\cos 5t + \cos t] = \frac{1}{2}[\cos t \cos 5t + \cos^2 t]$$

$$= \frac{1}{4}\left[2\cos t \cos 5t + 2\cos^2 t\right] = \frac{1}{4}\left[\left(\cos 6t + \cos 4t\right) + \left(1 + \cos 2t\right)\right]$$

$$= \frac{1}{4}[1 + \cos 2t + \cos 4t + \cos 6t]$$

$$\therefore L\{\cos t \cos 2t \cos 3t\} = \frac{1}{4}L\{1 + \cos 2t + \cos 4t + \cos 6t\}$$

$$= \frac{1}{4}[L\{1\} + L\{\cos 2t\} + L\{\cos 4t\} + L\{\cos 6t\}]$$

$$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4} + \frac{s}{s^2+16} + \frac{s}{s^2+36}\right]$$

8. Find L.T. of $sin^2 t$

Sol: $L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\}$

$$=\frac{1}{2}[L\{1\}-L\{\cos 2t\}]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

9. Find $L(\sqrt{t})$

Sol: $L\{\sqrt{t}\}=L\left[t^{1/2}\right]=\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ where n is not an integer

$$=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}} \quad \because \Gamma(n+1)=n.\Gamma(n)$$

10. Find $L\{sin(\omega t+\alpha)\}$, where α a constant is

Sol: $L\{sin(\omega t+\alpha)\}=L\{sin\omega t cos\alpha+cos\omega t sin\alpha\}$

$$=cos\alpha\, L\{sin\omega t\}+sin\alpha\, L\{cos\omega t\}$$

$$=cos\alpha\frac{\omega}{s^2+\omega^2}+sin\alpha\frac{\omega}{s^2+\omega^2}$$

Properties of Laplace transform:

Linearity Property:

Theorem1: The Laplace transform operator is a Linear operator.

i.e. (i). $L\{cf(t)\}=c.L\{f(t)\}$ (ii). $L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$ Where 'c' is constant

Proof: (i) By definition

$$L\{cf(t)\}=\int_0^\infty e^{-st}cf(t)\,dt=c\int_0^\infty e^{-st}f(t)\,dt=cL\{f(t)\}$$

(ii) By definition

$$L\{f(t)+g(t)\}=\int_0^\infty e^{-st}\{f(t)+g(t)\}dt$$

$$=\int_0^\infty e^{-st}f(t)dt+\int_0^\infty e^{-st}g(t)dt=L\{f(t)\}+L\{g(t)\}$$

Similarly the inverse transforms of the sum of two or more functions of 's' is the sum of the inverse transforms of the separate functions.

Thus, $L^{-1}\{\overline{f}(s)+\overline{g}(s)\}=L^{-1}\{\overline{f}(s)\}+L^{-1}\{\overline{g}(s)\}=f(t)+g(t)$

Corollary: $L\{c_1f(t)+c_2g(t)\}=c_1L\{f(t)\}+c_2L\{g(t)\}$, where c_1, c_2 are constants

Theorem2: If a, b, c be any constants and f, g, h any functions of t, then

$$L\{af(t)+bg(t)-ch(t)\}=a.L\{f(t)\}+b.L\{g(t)\}-cL\{h(t)\}$$

Proof: By the definition

$$L\{af(t)+bg(t)-ch(t)\}=\int_0^\infty e^{-st}\{af(t)+bg(t)-ch(t)\}dt$$

$$=a.\int_0^\infty e^{-st}f(t)dt+b\int_0^\infty e^{-st}g(t)dt-c\int_0^\infty e^{-st}h(t)dt$$

$$=a.L\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}$$

Change of Scale Property:

If $L\{f(t)\}=\bar{f}(s)$ *then* $L\{f(at)\}=\frac{1}{a}.\bar{f}\left(\frac{s}{a}\right)$

Proof: By the definition we have

$$L\{f(at)\}=\int_0^\infty e^{-st}f(at)dt$$

Put $at=u\Rightarrow dt=\frac{du}{a}$

when $t\to\infty$ *then* $u\to\infty$ *and* $t=0$ *then* $u=0$

$$\therefore L\{f(at)\}=\int_0^\infty e^{-\frac{su}{a}}f(u)\frac{du}{a}=\frac{1}{a}.\int_0^\infty e^{-\left(\frac{s}{a}\right).u}f(u)du=\frac{1}{a}.\bar{f}\left(s/a\right)$$

Solved Problems :

1.Find $L\{\sinh 3t\}$

Sol: $L\{\sinh t\}=\frac{1}{s^2-1}=\bar{f}(s)$

$\therefore L\{\sinh 3t\}=\frac{1}{3}\bar{f}(s/3)$(Change of scale property)

$$=\frac{1}{3}\frac{1}{\left(s/3\right)^2-1}=\frac{3}{s^2-9}$$

2. Find $L\{\cos 7t\}$

Sol: $L\{\cos t\}=\frac{s}{s^2+1}=\bar{f}(s)\ (say)$

$L\{\cos 7t\}=\frac{1}{7}\bar{f}\left(s/7\right)$ (Change of scale property)

$$L\{\cos 7t\}=\frac{1}{7}\frac{s/7}{\left(s/7\right)^2+1}=\frac{s}{s^2+49}$$

First shifting property:

If $L\{f(t)\}=\bar{f}(s)$ then $L\{e^{at}f(t)\}=\bar{f}(s-a)$

Proof: By the definition

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t) dt$$

$$= \int_0^\infty e^{-ut} f(t) dt \text{ where } u = s - a$$

$$= \overline{f}(u) = \overline{f}(s-a)$$

Note: Using the above property, we have $L\{e^{-at} f(t)\} = \overline{f}(s+a)$

Applications of this property, we obtain the following results

1. $L\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}} \left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$

2. $L\{e^{at} \sin bt\} = \dfrac{b}{(s-a)^2 + b^2} \left[\because L(\sin bt) = \dfrac{b}{s^2 + b^2}\right]$

3. $L\{e^{at} \cos bt\} = \dfrac{s-a}{(s-a)^2 + b^2} \left[\because L(\cos bt) = \dfrac{s}{s^2 + b^2}\right]$

4. $L\{e^{at} \sinh bt\} = \dfrac{b}{(s-a)^2 - b^2} \left[\because L(\sinh bt) = \dfrac{b}{s^2 - b^2}\right]$

5. $L\{e^{at} \cosh bt\} = \dfrac{s-a}{(s-a)^2 - b^2} \left[\because L(\cosh bt) = \dfrac{s}{s^2 - b^2}\right]$

Solved Problems :

1.Find the Laplace Transforms of $t^3 e^{-3t}$

Sol: Since $L\{t^3\} = \dfrac{3!}{s^4}$

Now applying first shifting theorem, we get

$$L\{t^3 e^{-3t}\} = \frac{3!}{(s+3)^4}$$

2.Find the L.T. of $e^{-t} \cos 2t$

Sol: Since $L\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying first shifting theorem, we get

$$L\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2 + 4} = \frac{s+1}{s^2 + 2s + 5}$$

3. Find L.T of $\boldsymbol{e^{2t} cos^2 t}$

Sol: - $L\,[e^{2t} cos^2 t] = L\,[e^{2t}(\frac{1+cos2t}{2})]$

$$=\frac{1}{2}\{L[e^{2t}]+L[e^{2t}cos2t]\}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\{L[cos2t]\}_{s\to s-2}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\frac{s-2}{(s-2)^2+2^2}$$

$$=\frac{1}{2}(\frac{1}{s-2})+\frac{1}{2}\frac{s-2}{(s^2-4s+8)}$$

Second translation (or) second Shifting theorem:

If $L\{f(t)\}=\overline{f}(s) and\ g(t)=\begin{cases} f(t-a), t>a \\ 0 \quad t<a\end{cases} then\ L\{g(t)\}=e^{-as}\overline{f}(s)$

Proof: By the definition

$$L\{g(t)\}=\int_0^\infty e^{-st}\,g(t)dt=\int_0^a e^{-st}\,g(t)dt+\int_a^\infty e^{-st}\,g(t)dt$$

$$=\int_0^\infty e^{-st}.odt+\int_a^\infty e^{-st}\,f(t-a)dt=\int_a^\infty e^{-st}\,f(t-a)dt$$

Let t-a = u so that dt = du And also u = 0 when t = a and u → ∞ when t → ∞

$$\therefore L\{g(t)\}=\int_0^\infty e^{-s(u+a)}\,f(u)du=e^{-as}\int_0^\infty e^{-su}\,f(u)du=e^{-as}\int_a^\infty e^{-st}\,f(t)dt$$

$$=e^{-as}L\{f(t)\}=e^{-as}\overline{f}(s)$$

Another Form of second shifting theorem:

If $L\{f(t)\}=\overline{f}(s)\ and\ a>0\ then\ L\{F(t-a)H(t-a)\}=e^{-as}\overline{f}(s)$

where H (t) = $\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$ and H(t) is called Heaviside unit step function.

Proof: By the definition

$$L\{F(t-a)H(t-a)\}=\int_0^\infty e^{-st}\,F(t-a)H(t-a)dt\to(1)$$

Put t-a=u so that dt= du and also when t=0, u=-a when t → ∞ , u→ ∞

Then $L\{F(t-a)H(t-a)\}=\int_a^\infty e^{-s(u+a)}\,F(u)H(u)du.\quad [by\ eq(1)]$

$$=\int_{-a}^{0} e^{-s(u+a)}F(u)H(u)du+\int_0^\infty e^{-s(u+a)}F(u)H(u)du$$

$$=\int_{-a}^0 e^{-s(u+a)}\,F(u).0du+\int_0^\infty e^{-s(u+a)}\,F(u).1du$$

[Since By the definition of H (t)]

$$=\int_0^\infty e^{-s(u+a)}\,F(u)du=e^{-as}\int_a^\infty e^{-su}\,F(u)du$$

$=e^{-sa}\int_0^\infty e^{-st}F(t)dt$ by property of Definite Integrals

$$=e^{-as}L\{F(t)\}=e^{-as}\overline{f}(s)$$

Note: $H(t-a)$ *is also denoted by* $u(t-a)$

Solved Problems

1.Find the L.T. of g (t) when $g(t)=\begin{cases}\cos\left(t-\pi/3\right) & if\ t>\pi/3\\ 0 & if\ t<\pi/3\end{cases}$

Sol. Let $f(t)=\cos t$

$$\therefore L\{F(t)\}=L\{cost\}=\frac{s}{s^2+1}=\overline{f}(s)$$

$$g(t)=\begin{cases}f\left(t-\pi/3\right)=cos\left(t-\pi/3\right), & if\ t>\pi/3\\ 0 & ,\ if\ t<\pi/3\end{cases}$$

Now applying second shifting theorem, then we get

$$L\{g(t)\}=e^{\frac{-\pi s}{3}}\left(\frac{s}{s^2+1}\right)=\frac{s.e^{\frac{-\pi s}{3}}}{s^2+1}$$

2.Find the L.T. of (ii)$(t-2)^3u(t-2)$ (ii) $e^{-3t}u(t-2)$

Sol: **(i).** Comparing the given function with f(t-a) u(t-a), we have a=2 and f(t)=t^3

$$\therefore L\{f(t)\}=L\{t^3\}=\frac{3!}{s^4}=\frac{6}{s^4}=\overline{f}(s)$$

Now applying second shifting theorem, then we get

$$L\{(t-2)^3u(t-2)\}=e^{-2s}\frac{6}{s^4}=\frac{6e^{-2s}}{s^4}$$

(ii). $L\{e^{-st}u(t-2)\}=L\left\{e^{-s(t-2)}.e^{-6}u(t-2)\right\}=e^{-6}L\left\{e^{-3(t-2)}u(t-2)\right\}$

$$f(t)=e^{-3t}\text{ then }\overline{f}(s)=\frac{1}{s+3}$$

Now applying second shifting theorem then, we get

$$L\{e^{-3t}u(t-2)\}=e^{-6}.e^{-2s}\frac{1}{s+3}=\frac{e^{-2(s+3)}}{s+3}$$

Multiplication by '*t*':

Theorem: If $L\{f(t)\}=\overline{f}(s)$ ***then*** $L\{tf(t)\}=\frac{-d}{ds}\overline{f}(s)$

Proof: By the definition $\overline{f}(s)=\int_0^\infty e^{-st}f(t)dt$

$$\frac{d}{ds}\{\overline{f}(s)\}=\frac{d}{ds}\int_0^\infty e^{-st}f(t)dt$$

By Leibnitz's rule for differentiating under the integral sign,

$$\frac{d}{ds}\overline{f}(s)=\int_0^\infty\frac{\partial}{\partial s}e^{-st}f(t)dt$$

$$=\int_0^\infty -te^{-st}f(t)dt$$

$$= -\int_0^{\infty} e^{-st}\{tf(t)\}dt = -L\{tf(t)\}$$

Thus $L\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

$$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} = \overline{f}(s)$$

Note: Leibnitz's Rule

If $f(x,\alpha)$ *and* $\frac{\partial}{\partial\alpha} f(x,\alpha)$ be continuous functions of x and α then

$$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)dx\right\} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

Where a, b are constants independent of α

Solved Problems:

1. Find L.T of $tcosat$

Sol: Since $L\{tcos\ at\} = \frac{s}{s^2+a^2}$

$$L\{tcos\ at\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{-s^2+a^2-s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2.Find $t^2 sin\ at$

Sol: Since $L\{sin\ at\} = \frac{a}{s^2+a^2}$

$$L\{t^2.sin\ at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2as}{\left(s^2+a^2\right)^2}\right) = \frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}$$

3.Find $L.T\ of\ te^{-t}\sin 3t$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+3^2}$

$\therefore L\{tsin\ 3t\} = \frac{-d}{ds}\left[\frac{3}{s^2+3^2}\right] = \frac{6s}{(s^2+9)^2}$ Now using the shifting property, we get

$$L\{te^{-t}sin\ 3t\} = \frac{6(s+1)}{((s+1)^2+9)^2} = \frac{6(s+1)}{(s^2+2s+10)^2}$$

4.Find $L\{te^{2t}sin\ 3t\}$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+9}$

$$\therefore \mathrm{L}\left\{e^{2t}\sin 3t\right\} = \frac{3}{(s-2)^2+9} = \frac{3}{s^2-4s+13}$$

$$L\{te^{2t}sin\ 3t\} = (-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right] = (-1)\left[\frac{0-3(2s-4)}{(s^2-4s+13)^2}\right]$$

$$=\frac{3(2s-4)}{(s^2-4s+13)^2}=\frac{6(s-2)}{(s^2-4s+13)^2}$$

5.Find the L.T. of $(1+te^{-t})^2$

Sol: Since $(1+te^{-t})^2 = 1+2te^{-t}+t^2e^{-2t}$

$$\therefore L(1+te^{-t})^2 = L\{1\}+2L\{te^{-t}\}+L\{t^2e^{-2t}\}$$

$$=\frac{1}{s}+2(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right)+(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{d}{ds}\left(\frac{-1}{(s+2)^2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{2}{(s+2)^3}$$

6.Find the L.T of t^3e^{-3t}

Sol: $L\{t^3e^{-3t}\}=(-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$

$$=-\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)=\frac{-3!(-1)^3}{(s+3)^4}$$

$$=\frac{3!}{(s+3)^4}$$

7.Find $L\{\cosh at \sin at\}$

Sol. $L\{\cosh at\sin at\}=L\left\{\frac{e^{at}+e^{-at}}{2}.\sin at\right\}$

$$=\frac{1}{2}[L\{e^{at}\sin at\}+L\{e^{-at}\sin at\}]$$

$$=\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

8.Find the L.T of the function $f(t)=(t-1)^2,\quad t>1$
$\qquad =0 \quad 0<t<1$

Sol: By the definition

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}0dt+\int_1^\infty e^{-st}(t-1)^2dt$$

$$=\int_1^\infty e^{-st}(t-1)^2dt=\left[(t-1)^2\frac{e^{-st}}{-s}\right]_1^\infty-\int_1^\infty 2(t-1)\frac{e^{-st}}{-s}dt$$

$$=0+\frac{2}{s}\int_1^\infty e^{-st}(t-1)dt$$

$$=\frac{2}{s}\left[\left\{(t-1)\left(\frac{e^{-st}}{-s}\right)\right\}_1^{\infty}-\int_1^{\infty}\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{2}{s}\left[0+\frac{1}{s}\int_1^{\infty}e^{-st}dt\right]=\frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^{\infty}=\frac{-2}{s^3}\left(e^{-st}\right)_1^{\infty}$$

$$=\frac{-2}{s^3}\left(0-e^{-s}\right)=\frac{2}{s^3}e^{-s}$$

9.Find the L.T of f (t) defined as $f(t)=3, \quad t>2$

$=0, \quad 0<t<2$

Sol: $L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt$

$$=\int_0^{2}e^{-st}f(t)dt+\int_2^{\infty}e^{-st}f(t)dt$$

$$=\int_0^{2}e^{-st}.0dt+\int_2^{\infty}e^{-st}3dt$$

$$=0+\int_2^{\infty}e^{-st}3dt=\frac{-3}{s}\left(e^{-st}\right)_2^{\infty}=\frac{-3}{s}\left(0-e^{-2s}\right)$$

$$=\frac{3}{s}e^{-2s}$$

10.Find $L\{t\cos(at+b)\}$

Sol: $L\{\cos(at+b)\}=L\{\cos at\cos b-\sin at\sin b\}$

$$=\cos b.L\{\cos at\}-\sin b\,L\{\sin at\}$$

$$=\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}$$

$$L\{t.\cos(at+b)\}=\frac{-d}{ds}\left[\cos b.\frac{s}{s^2+a^2}-\sin b.\frac{a}{s^2+a^2}\right]$$

$$=-\cos b.\left(\frac{s^2+a^2.1-s.2s}{\left(s^2+a^2\right)^2}\right)+\sin b\left(\frac{\left(s^2+a^2\right).0-a.2s}{\left(s^2+a^2\right)^2}\right)$$

$$=\frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)^2\cos b-2as\sin b\right]$$

11.Find L.T of $L\{te^t\sin t\}$

Sol: - We know that $L[\sin t]=\frac{1}{s^2+1}$

$$L[t\sin t]=(-1)\frac{d}{ds}L[\sin t]=-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=-\frac{(-1)2s}{(s^2+1)^2}$$

$$=\frac{2s}{(s^2+1)^2}$$

By First Shifting Theorem

$$L\,[te^t sint] = \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1} = \frac{2(s-1)}{((s-1)^2+1)^2} = \frac{2(s-1)}{(s^2-2s+2)^2}$$

Division by't**':**

Theorem: If $L\{f(t)\} = \overline{f}(s)$ *then* $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty \overline{f}(s)ds$

Proof: We have $\overline{f}(s) = \int_0^\infty e^{-st} f(t)dt$

Now integrating both sides w.r.t s from s to ∞, we have

$$\int_0^\infty \overline{f}(s)ds = \int_s^\infty\left[\int_0^\infty e^{-st} f(t)dt\right]ds$$

$$= \int_0^\infty \int_s^\infty f(t)e^{-st}dsdt \text{ (Change the order of integration)}$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st}ds\right]dt \text{ (} \because \text{ t is independent of's')}$$

$$= \int_0^\infty f(t)\left(\frac{e^{-st}}{-t}\right)_s^\infty dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}dt (or) L\left\{\frac{1}{t}f(t)\right\}$$

Solved Problems:

1.Find $L\left\{\frac{\sin t}{t}\right\}$

Sol: Since $L\{sint\} = \frac{1}{s^2+1} = \overline{f}(s)$

Division by't', we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= [Tan^{-1}s]_s^\infty = Tan^{-1}\infty - Tan^{-1}s$$

$$= \pi/2 - Tan^{-1}s = \cot^{-1} s$$

2.Find the L.T of $\frac{\sin at}{t}$

Sol: Since $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Division by t, we have

$$L\left\{\frac{\text{sina}\, t}{t}\right\} = \int_s^\infty \overline{f}(s)ds = \int_s^\infty \frac{a}{s^2+a^2}ds$$

$$= a.\frac{1}{a}\left[Tan^{-1}\, s/a\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}\, s/a$$

$$= \pi/2 - Tan^{-1}\left(s/a\right) = \cot^{-1} s/a$$

3.Evaluate $L\left\{\frac{1-\cos a\, t}{t}\right\}$

Sol: Since $L\{1-\cos at\} = L\{1\} - L\{\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2}$

$$L\left\{\frac{1-\cos at}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+a^2}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log\left(s^2+a^2\right)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[l\,og\left(\frac{1}{1+a^2/s^2}\right)\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log\frac{s^2}{s^2+a^2}\right]$$

$$= -\frac{1}{2}l\,og\left(\frac{s^2}{s^2+a^2}\right) = \log\left(\frac{s^2}{s^2+a^2}\right)^{\frac{-1}{2}} = \log\sqrt{\frac{s^2+a^2}{s^2}}$$

Note: $L\left\{\frac{1-\cos t}{t}\right\} = \log\sqrt{\frac{s^2+1}{s}}$ (Putting a=1 in the above problem)

4.Find $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty = \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \underset{s\to\infty}{lt}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\} - \log\left(\frac{s+a}{s+b}\right)$$

$$= \log 1 - \log(s+a) + \log(s+b) = \log\left(\frac{s+b}{s+a}\right)$$

5. Find $L\left\{\frac{1-\cos t}{t^2}\right\}$

Sol: $L\left\{\frac{1-\cos t}{t^2}\right\} = L\left\{\frac{1}{t}.\frac{1-\cos t}{t}\right\}$.....(1)

Now $L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds = \left[\log s - \frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty = \frac{-1}{2}\left[\log\frac{s^2}{s^2+1}\right] = \frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$\therefore L\left[\frac{1-\cos t}{t^2}\right]=\int_s^{\infty}\frac{1}{2}\log\frac{s^2+1}{s^2}ds$$

$$=\frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\}.s\right]_s^{\infty}-\int_s^{\infty}\frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right).sds$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\, s.\log\left(1+\frac{1}{s^2}\right)\right\}-s\log\left(\frac{s^2+1}{s^2}\right)+2\int_s^{\infty}\frac{ds}{s^2+1}\right]$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\, s\left(\frac{1}{s^2}-\frac{1}{2s^4}+\frac{1}{3s^6}+....\right)-s\log\frac{s^2+1}{s^2}\right\}+2Tan^{-1}s\right]_s^{\infty}$$

$$=\frac{1}{2}\left[\left\{0-s\log\left(1+\frac{1}{s^2}\right)+2\left(\frac{\pi}{2}-Tan^{-1}s\right)\right\}\right]\quad\because\left(\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....\right)$$

$$=\cot^{-1}s-\frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$$

6.Find L.T of $\frac{e^{-at}-e^{-bt}}{t}$

Sol: w.k.t L $[e^{-at}]=\frac{1}{s+a}$, L $[e^{-bt}]=\frac{1}{s+b}$

L$[\frac{f(t)}{t}]=\int_s^{\infty}\bar{f}(s)ds$

$$\therefore\quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^{\infty}\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=[\log(s+a)-\log(s+b)]_s^{\infty}$$

$$=\log\left(\frac{s+a}{s+b}\right)_s^{\infty}$$

$$=\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)_s^{\infty}$$

$$=\log(1)-\log\left(\frac{s+a}{s+b}\right)$$

$$=0-\log\left(\frac{s+a}{s+b}\right)=\log\left(\frac{s+b}{s+a}\right)$$

Laplace transforms of Derivatives:

If $f'(t)$be continuous and $L\{f(t)\}=\bar{f}(s)\ then\ L\{f'(t)\}=s\bar{f}(s)-f(0)$

Proof: By the definition

$$L\{f'(t)\}=\int_0^{\infty}e^{-st}f'(t)dt$$

$$=\left[e^{-st}f(t)\right]_0^{\infty}-\int_0^{\infty}(-s)e^{-st}f(t)dt \qquad \text{(Integrating by parts)}$$

$$=\left[e^{-st}f(t)\right]_0^{\infty}+s\int_0^{\infty}e^{-st}f(t)dt$$

$$= \underset{t\to\infty}{\mathrm{lt}}\, e^{-st} f(t) - f(0) + s.L\{f(t)\}$$

Since f (t) is exponential order

$$\therefore \underset{t\to\infty}{\mathrm{lt}}\, e^{-st} f(t) = 0$$

$$\therefore L\{f'(t)\} = 0 - f(0) + sL\{f(t)\}$$

$$= s\bar{f}(s) - f(0)$$

The Laplace Transform of the second derivative $f''(t)$) is similarly obtained.

$$\therefore L\{f''(t)\} = s.L\{f'(t)\} - f'(0)$$

$$= s.\left[s\bar{f}(s) - f(0)\right] - f^1(0)$$

$$= s^2\bar{f}(s) - sf(0) - f^1(0)$$

$$\therefore L\{f'''(t)\} = s.L\{f''(t)\} - f''(0)$$

$$= s[s^2L\{f(t)\} - sf(0) - f'(0)] - f''(0)$$

$$= s^3L\{f(t)\} - s^2f(0) - sf'(0) - f''(0)$$

Proceeding similarly, we have

$$L\{f^n(t)\} = s^nL\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \ldots\ldots f^{n-1}(0)$$

Note 1: $L\{f^n(t)\} = s^n\bar{f}(s)\ if\ f(0) = 0\ and\ f'(0) = 0, f''(0) = 0 \ldots\ f^{n-1}(0) = 0$

Note 2: Now $|f(t)| \le M.e^{at}\ for\ all\ t \ge 0\ and\ for\ some\ constants\ a\ amd\ M.$

We have $|e^{-st}f(t)| = e^{-st}|f(t)| \le e^{at}.Me^{at}$

$=$M. $e^{-(s-a)t} \to 0\ as\ t\ \to \infty$ if s>a

$$\therefore \underset{t\to\infty}{lt}\ e^{-st}f(t) = 0\ for\ \ s > a$$

Solved Problems:

Using the theorem on transforms of derivatives, find the Laplace Transform of the following functions.

(i). e^{at} **(ii). $cosat$** **(iii). $t\ sin\ at$**

(i). $Let\ f(t) = e^{at}\ Then\ f^1(t) = a.e^{at}\ and\ f(0) = 1$

$$Now\ L\{\,f'(t)\} = s.L\{\,f(t)\} - f(0)$$

$$i.e., L\{ae^{at}\} = s.L\{e^{at}\} - 1$$

$$i.e., L\{e^{at}\} - s.L\{e^{at}\} = -1$$

$$i.e., (a - s)L\{e^{at}\} = -1$$

$$\therefore L\{e^{at}\} = \frac{1}{s-a}$$

(ii). $Let\ f(t) = cosat\ then\ f'(t) = -asinat\ and\ f''(t) = -a^2cosat$

$\therefore L\{f''(t)\} = s^2L\{f(t)\} - s.f(0) - f'(0)$

$Now\ f(0) = \cos 0 = 1 and\ \ f^1(0) = -a\sin 0 = 0$

$Then\ L\{-a^2\cos at\} = s^2L\{\cos at\} - s.1 - 0$

$\Longrightarrow -a^2L\{\cos at\} - s^2L\{\cos at\} = -s$

$\Longrightarrow -(s^2 + a^2)L\{\cos at\} = -s \Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$

(iii). $Let\ f(t) = t\sin at\ then\ f^1(t) = \sin at + at\cos at$

$f^{11}(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$

Also $f(0) = 0$ and $f^1(0) = 0$

$Now\ L\{f''(t)\} = s^2L\{f(t)\} - sf(0) - f^1(0)$

$i.e., L\{2a\cos at - a^2t\sin at\} = s^2L\{t\sin at\} - 0 - 0$

$i.e., 2a\ L\{\cos at\} - a^2L\{t\sin at\} - s^2L\{t\sin at\} = 0$

$i.e., -(s^2 + a^2)L\{t\sin at\} = \frac{-2as}{s^2+a^2} \Rightarrow L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

Laplace Transform of Integrals:

If $L\{f(t)\} = \overline{f}(s)\ then\ L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Proof: $Let\ g(t) = \int_0^t f(x)dx$

Then $g'(t) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)\ and\ g(0) = 0$

Taking Laplace Transform on both sides

$L\{g'(t)\} = L\{f(t)\}$

But $L\{g'(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\} - 0\ \ [Since\ g(0) = 0]$

$\therefore L\{g'(t)\} = L\{f(t)\}$

$\Longrightarrow sL\{g(t)\} = L\{f(t)\}\ \Rightarrow L\{g(t)\} = \frac{1}{s}L\{f(t)\}$

$But\ g(t) = \int_0^t f(x)\,dx$

$\therefore L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Solved Problems:

1.Find the L.T of $\int_0^t \sin at dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Using the theorem of Laplace transform of the integral, we have

$$L\left\{\int_0^t f(x)\,dx\right\} = \frac{\bar{f}(s)}{s}$$

$$\therefore L\left\{\int_0^t \sin at\right\} = \frac{a}{s(s^2+a^2)}$$

2.Find the L.T of $\int_0^t \frac{\sin t}{t} dt$

Sol: $L\{\sin t\} = \frac{1}{s^2+1}$ *also* $\underset{t\to 0}{lt} \frac{\sin t}{t} = 1$ *exists*

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\} ds = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= \left[Tan^{-1} s\right]_s^\infty = Tan^{-1}\infty - Tan^{-1} s = \pi/2 - Tan^{-1} s = \cot^{-1} s\,(or)\,Tan^{-1}\left(1/s\right)$$

$$i.e., L\left\{\frac{\sin t}{t}\right\} = Tan^{-1}(1/s)(or)cot^{-1}s$$

$$\therefore\ L\left\{\int_0^t \frac{\sin t}{t} dt\right\} = \frac{1}{s} Tan^{-1}(1/s)\ (or)\ \frac{1}{s} cot^{-1}s$$

3.Find L.T of $e^{-t}\int_0^t \frac{sint}{t} dt$

Sol: $L\{e^{-t}\int_0^t \frac{sint}{t} dt\}$

We know that

$$\text{L}\ \{\text{sint}\} = \frac{1}{s^2+1} = \bar{f}(s)$$

$$\text{L}\ \{\frac{sint}{t}\} = \int_s^\infty \bar{f}(s) ds = \int_s^\infty \frac{1}{s^2+1}\ ds$$

$$=(tan^{-1}s)_s^\infty$$

$$=tan^{-1}\infty - tan^{-1}s = \frac{\pi}{2} - tan^{-1}s \quad = cot^{-1}s$$

$$\therefore\ \text{L}\ \left\{\frac{sint}{t}\right\} = \ cot^{-1}s$$

Hence $\text{L}\ \{\int_0^t \frac{sint}{t} dt\} = \frac{1}{s} cot^{-1}s$

By First Shifting Theorem

$$\text{L}\ [\ e^{-t}\int_0^t \frac{sint}{t} dt] = \bar{f}(s+1) = (\frac{cot^{-1}s}{s})_{s\to s+1}$$

$$\therefore \text{L}\left[e^{-t}\int_0^t \frac{sint}{t} dt\right] = \frac{1}{s+1} cot^{-1}(s+1)$$

Laplace transform of Periodic functions:

If $f\ (t)$ is a periodic function with period 'a'. i.e, $f(t+a) = f(t)$ then

$$\text{L}\ \{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t) dt$$

Eg: sin x is a periodic function with period 2π

i.e., $\sin x = \sin(2\pi + x) = \sin(4\pi + x)$............

Solved Problems:

1.A function $f(t)$ is periodic in (0,2b) and is defined as $f(t) = 1 \text{ if } 0 < t < b$
$= -1 \text{ if } b < t < 2b$

Find its Laplace Transform.

Sol: $$L\{f(t)\} = \frac{1}{1-e^{-2bs}}\int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\int_0^{b} e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s(1-e^{-2bs})}\left[-\left(e^{-sb}-1\right) + \left(e^{-2bs} - e^{-sb}\right)\right]$$

$$L\{f(t)\} = \frac{1}{s(1-e^{-2bs})}\left[1 - 2e^{-sb} + e^{-2bs}\right]$$

2.Find the L.T of the function $f(t) = \sin\omega t \text{ if } 0 < t < \frac{\pi}{\omega}$

$$= 0 \text{ if } \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \text{ where } f(t) \text{ has period } \frac{2\pi}{\omega}$$

Sol: Since f (t) is a periodic function with period $\frac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^{a} e^{-st} f(t)dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-s2\pi/\omega}}\int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt\right]$$

$$= \frac{1}{1-e^{-2s\pi/\omega}}\left[\frac{e^{-st}\left(-s\sin\omega t - \omega\cos\omega t\right)}{s^2+\omega^2}\right]_0^{\pi/\omega}$$

$$\because \int_a^b e^{at}\sin bt = \frac{e^{at}}{a^2+b^2}\left(a\sin bt - b\cos bt\right)$$

$$=\frac{1}{1-e^{-2\pi s/\omega}}\left[\frac{1}{s^2+\omega^2}\left(e^{-s\pi/\omega}.\omega+\omega\right)\right]$$

Laplace Transform of Some special functions:

1.The Unit step function or Heaviside's Unit functions:

It is defined as $u(t-a)=\begin{cases}0 & t<a\\ 1 & t>a\end{cases}$

Laplace Transform of unit step function:

To prove that $L\{u(t-a)\}=\frac{e^{-as}}{s}$

Proof: Unit step function is defined as $u(t-a)=\begin{cases}0 & t<a\\ 1 & t>a\end{cases}$

Then $L\{u(t-a)\}=\int_0^\infty e^{-st}u(t-a)\,dt$

$$=\int_0^a e^{-st}u(t-a)dt+\int_a^\infty e^{-st}u(t-a)dt$$

$$=\int_0^a e^{-st}.0dt+\int_a^\infty e^{-st}.1dt$$

$$=\int_a^\infty e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_a^\infty=-\frac{1}{s}.\left[e^{-\infty}-e^{-as}\right]=\frac{e^{-as}}{s}$$

$\therefore L\{u(t-a)\}=\frac{e^{-as}}{s}$

Laplace Transforms of Dirac Delta Function:

The Dirac delta function or Unit impulse function $f_\epsilon(t)=\begin{cases}1/\epsilon & 0\le t\le\epsilon\\ 0 & t>\epsilon\end{cases}$

2.Prove that $L\{f_\epsilon(t)\}=\frac{1-e^{-s\epsilon}}{s\epsilon}$ hence show that $L\{\delta(t)\}=1$

Proof: By the definition $f_\epsilon(t)=\begin{cases}1/\epsilon & 0\le t\le\epsilon\\ 0 & t>\epsilon\end{cases}$

And Hence $L\{f_\epsilon(t)\}=\int_0^\infty e^{-st}f_\epsilon(t)\,dt$

$$=\int_0^\epsilon e^{-st}f_\epsilon(t)\,dt+\int_\epsilon^\infty e^{-st}f_\epsilon(t)\,dt$$

$$=\int_0^\epsilon e^{-st}\frac{1}{\epsilon}dt+\int_\epsilon^\infty e^{-st}.0\,dt$$

$$=\frac{1}{\epsilon}\left[\frac{e^{-st}}{-s}\right]_0^\epsilon=-{}^{1}/_{\epsilon\, s}\left[e^{-s\epsilon}-e^0\right]=\frac{1-e^{-s\epsilon}}{s\epsilon}$$

$\therefore L\{f_\epsilon(t)\}=\frac{1-e^{-s\epsilon}}{s\epsilon}$

Now $L\{\delta(t)\}=\underset{\epsilon\to 0}{lt}L\{f_\epsilon(t)\}=\underset{\epsilon\to 0}{lt}\frac{1-e^{-s\epsilon}}{s\epsilon}$

$\therefore L\{\delta(t)\}=1\ using$ L-Hospital rule.

Properties of Dirac Delta Function:

1. $\int_0^\infty \delta(t)\, dt = 0$

2. $\int_0^\infty \delta(t)G(t)\, dt = G(0)$ where G(t) is some continuous function.

3. $\int_0^\infty \delta(t-a)G(t)\, dt = G(a)$ where G(t) is some continuous function.

4. $\int_0^\infty G(t)\delta^1(t-a) = -G^1(a)$

Solved Problems:

1.Prove that $L\{\delta(t-a)\} = e^{-as}$

Sol: By Translation theorem

$L\{\delta(t-a)\} = e^{-as}L\{\delta(t)\}$

$= e^{-as}$ [since $L\{\delta(t)\} = 1$]

2.Evaluate $\int_0^\infty \cos 2t\, \delta(t - \pi/3)\, dt$

Sol: By using property (3) then we get

$\int_0^\infty \delta(t-a)G(t)dt = G(a)$

Here $a = \pi/3$, $G(t) = \cos 2t$

$\therefore G(a) = G(\pi/3) = \cos 2\pi/3 = -1/2$

$\therefore \int_0^\infty \cos 2at\, \delta(t - \pi/3)\, dt = \cos 2\pi/3 = -\pi/2$

3.Evaluate $\int_0^\infty e^{-4t}\delta^1(t-2)\, dt$

Sol: By the 4th Property then we get

$\int_0^\infty \delta^1(t-a)G(t)dt = -G^1(a)$

$G(t) = e^{-4t}$ *and* $a = 2$

$G^1(t) = -4.e^{-4t}$

$\therefore G^1(a) = G^1(2) = -4.e^{-8}$

$\therefore \int_0^\infty e^{-4t}\delta^1(t-2)dt = -G^1(a) = 4.e^{-8}$

Inverse Laplace Transforms:

If $\overline{f}(s)$ is the Laplace transforms of a function of f (t) i.e. $L\{f(t)\} = \overline{f}(s)$ then f (t) is called the inverse Laplace transform of $\overline{f}(s)$ and is written as $f(t) = L^{-1}\{\overline{f}(s)\}$

$\therefore L^{-1}$ is called the inverse L.T operator.

Table of Laplace Transforms and Inverse Laplace Transforms

S.No.	$L\{f(t)\} = \overline{f}(s)$	$L^{-1}\{\overline{f}(s)\} = f(t)$
1.	$L\{1\} = {}^{1}/_{s}$	$L^{-1}\{{}^{1}/_{s}\} = 1$
2.	$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\{{}^{1}/_{s-a}\} = e^{at}$
3.	$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\{{}^{1}/_{s+a}\} = e^{at}$
4.	$L\{t^n\} = \frac{n!}{s^{n+1}} n \text{ is a } +ve \text{ integer}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$
5.	$L\{t^{n-1}\} = \frac{(n-1)!}{s^n}$	$L^{-1}\{{}^{1}/_{s^n}\} = \frac{t^{n-1}}{(n-1)!}, n = 1,2,3 \ldots$
6.	$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}.\sin at$
7.	$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
8.	$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
9.	$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
10.	$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}.e^{at}\sin bt$
11.	$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$
12.	$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}.e^{at}\sinh bt$
13.	$L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14.	$L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = \frac{1}{b}.e^{-at}\sin bt$
15.	$L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$
16.	$L\{e^{at} f(t)\} = \overline{f}(s-a)$	$L^{-1}\{\overline{f}(s-a)\} = e^{at}L^{-1}\{\overline{f}(s)\}$
17.	$L\{e^{-at} f(t)\} = \overline{f}(s+a)$	$L^{-1}\{\overline{f}(s+a)\} = e^{-at} f(t)e^{-at}L^{-1}\{\overline{f}(s)\}$

Solved Problems :

1.Find the Inverse Laplace Transform of $\frac{s^2-3s+4}{s^3}$

Sol: $L^{-1}\left\{\frac{s^3-3s+4}{s^3}\right\} = L^{-1}\left\{{}^{1}/_{s} - 3.{}^{1}/_{s^2} + {}^{4}/_{s^3}\right\}$

$$= L^{-1}\left\{{}^{1}\!/_{s}\right\} - 3L^{-1}\left\{{}^{1}\!/_{s^2}\right\} + L^{-1}\left\{{}^{4}\!/_{s^3}\right\}$$

$$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$$

2.Find the Inverse Laplace Transform of $\frac{s+2}{s^2-4s+13}$

Sol: $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\frac{s-2+4}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4.L^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

3.Find the Inverse Laplace Transform of $\frac{2s-5}{s^2-4}$

Sol: $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\frac{2s}{s^2-4} - \frac{5}{s^2-4}\right\}$

$$= 2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 2.\cosh 2t - 5.\frac{1}{2}\sinh 2t$$

4.Find $L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Sol: $L^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = e^{-t} + 1$$

5.Find $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

Sol: $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = L^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$$= \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+\left({}^{5}/_{2}\right)^2}\right\} - \frac{8}{4}L^{-1}\left\{\frac{1}{s^2+\left({}^{5}/_{2}\right)^2}\right\}$$

$$= \frac{3}{4}.\cos\frac{5}{2}t - \frac{8}{4}.\frac{2}{5}\sin\frac{5}{2}t$$

$$=\frac{3}{4}\cos\frac{5}{2}t-\frac{4}{5}\sin\frac{5}{2}t$$

6.Find the Inverse Laplace Transform of $\frac{s}{(s+a)^2}$

Sol: $L^{-1}\left\{\frac{s}{(s+a)^2}\right\}=L^{-1}\left\{\frac{s+a-a}{(s+a)^2}\right\}=e^{-at}L^{-1}\left\{\frac{s-a}{s^2}\right\}$

$$=e^{-at}L^{-1}\left\{\frac{1}{s}-\frac{a}{s^2}\right\}$$

$$=e^{-at}\left[L^{-1}\left\{\frac{1}{s}\right\}-a.L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$=e^{-at}\left[1-at\right]$$

7.Find $L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}$

Sol: $Let\ \frac{3s+7}{s^2-2s-3}=\frac{A}{s+1}+\frac{B}{s-3}$

$A(s-3)+B(s+1)=3s+7$

$put\ s=3, 4B=16\Rightarrow B=4$

$put\ s=-1, -4A=4\Rightarrow A=-1$

$\therefore\frac{3s+7}{s^2-2s-3}=\frac{-1}{s+1}+\frac{4}{s-3}$

$$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}=L^{-1}\left\{\frac{-1}{s+1}+\frac{4}{s-3}\right\}=-1L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=-e^{-t}+4.e^{3t}$$

8.Find $L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}$

Sol: $\frac{s}{(s+1)^2(s^2+1)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+1}$

$A(s+1)(s^2+1)+B(s^2+1)+(Cs+D)(s+1)^2=s$

Equating Co-efficient of s^3, A+C=0……..(1)

Equating Co-efficient of s^2, A+B+2C+D=0…….(2)

Equating Co-efficient of s, A+C+2D=1…….(3)

$put\ s=-1, 2B=-1\Rightarrow B=-\frac{1}{2}$

Substituting (1) in (3) $2D=1\Rightarrow D=\frac{1}{2}$

Substituting the values of B and D in (2)

i.e. $A-\frac{1}{2}+2C+\frac{1}{2}=0\Rightarrow A+2C=0, also\ A+C=0\Rightarrow A=0, C=0$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = \frac{\frac{-1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - te^{-t}\right]$$

9.Find $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol: Since $s^4 + 4a^4 = (s^2+2a^2)^2 - (2as)^2$

$$= (s^2+2as+2a^2)(s^2-2as+2a^2)$$

$$\therefore Let \quad \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

$$(As+B)(s^2-2as+2a^2) + (Cs+D)(s^2+2as+2a^2) = s$$

Solving we get $A=0, C=0, B=\frac{-1}{4a}, D=\frac{1}{4a}$

$$L\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{-\frac{1}{4a}}{s^2+2as+2a^2}\right\} + L^{-1}\left\{\frac{\frac{1}{4a}}{s^2-2as+2a^2}\right\}$$

$$= \frac{-1}{4}a.L^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}..L^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= \frac{-1}{4a}.\frac{1}{a}.e^{-at}\sin at + \frac{1}{4a}.\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\left(e^{at}-e^{-at}\right) = \frac{1}{4a^2}.\sin at.2\sinh at = \frac{1}{2a^2}\sin at \sinh at$$

10.Find i. $\quad L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$ $\qquad$ **ii.** $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}$

Sol:

i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$$= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3t + 4\frac{t^2}{2!} = 1 - 3t + 2t^2$$

ii. $L^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{(s^2-2)^2}{s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right\} + \frac{3}{2}\left\{L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s^5}\right\}\right\}$$

$$=\frac{3}{2}\left[1-4\frac{t^2}{2!}+\frac{4t^4}{4!}\right]=\frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]=\frac{1}{4}\left[t^4-6t^2+6\right]$$

11. Find $L^{-1}\left[\frac{s}{s^2-a^2}\right]$

Sol:

$$L^{-1}\left[\frac{s}{s^2-a^2}\right]=L^{-1}\left[\frac{2s}{2(s^2-a^2)}\right]=\frac{1}{2}L^{-1}\left[\frac{2s}{(s-a)(s+a)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[e^{at}+e^{-at}\right]=\cosh at$$

12.Find $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]$

Sol: $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]=4L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=4\,L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]=4[e^{-t}-e^{-2t}]$

13. Find $L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}$

Sol: $\frac{1}{(s+1)^2(s^2+4)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4}$

$$A=\frac{2}{25},B=\frac{1}{5},C=\frac{-2}{25},D=\frac{-3}{25}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}=\frac{2}{25}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{2}{25}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{3}{25}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{2}{25}e^{-t}\,L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{25}\cos 2t-\frac{3}{25}.\frac{1}{2}\sin 2t$$

$$=\frac{2}{25}e^{-t}+\frac{1}{5}e^{-t}.t-\frac{2}{25}\cos 2t-\frac{3}{50}\sin 2t$$

14.Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Sol: $\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$

Comparing with s^2, s, constants, we get

$$A=1/3,B=4/15,C=2/5$$

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]=L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]$$

$$= L^{-1}\left[\frac{1}{3s}\right] + L^{-1}\left[\frac{4}{15(s+3)}\right] + L^{-1}\left[\frac{2}{5(s-2)}\right]$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

15.Find $L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$

Sol: $\frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+9}$

Comparing with s^2, s, constants, we get

$$A = 31/34, B = 3/34, C = 83/34$$

$$L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right] = L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$$

$$= L^{-1}\left[\frac{31}{34(s-5)}\right] + L^{-1}\left[\frac{3}{34(s^2+9)}\right] + L^{-1}\left[\frac{83}{34(s^2+9)}\right]$$

$$= \frac{31}{34}e^{5t} + 1/34\left[3\cos 3t + \frac{83}{3}\sin 3t\right]$$

First Shifting Theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), \text{then} L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at} f(t)$

Proof: We have seen that $L\left\{e^{at} f(t)\right\} = \overline{f}(s-a) \therefore L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at} f(t) = e^{at} L^{-1}\left\{\overline{f}(s)\right\}$

Solved Problems :

1. Find $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = L^{-1}\left\{\overline{f}(s+2)\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = e^{-2t} L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$= e^{-2t}.\frac{1}{4}\sin 4t = \frac{e^{-2t}\sin 4t}{4}$$

2. Find $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

Sol: $L^{-1}\left\{\dfrac{3s-2}{s^2-4s+20}\right\} = L^{-1}\left\{\dfrac{3s-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\dfrac{3(s-2)+4}{(s-2)^2+4^2}\right\}$

$$= 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\frac{1}{4}\sin 4t$$

3. Find $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\}$

Sol: $L^{-1}\left\{\dfrac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\dfrac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\dfrac{s-5+8}{(s-5)^2+2^2}\right\}$

$$= e^{5t}\mathrm{L}^{-1}\left\{\frac{s+8}{s^2+2^2}\right\} = e^{5t}\left\{\cos 2t + 8.\frac{1}{2}\sin 2t\right\}$$

Second shifting theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), then \quad L^{-1}\{e^{-as}\overline{f}(s)\} = G(t)$, where $G(t) = \begin{cases} f\{t-a\} & if\ t>a \\ 0 & if\ t<a \end{cases}$

Proof: We have seen that $G(t) = \begin{cases} f\{t-a\} & if\ t>a \\ 0 & if\ t<a \end{cases}$

$$then \quad L\{G(t)\} = e^{-as}.\overline{f}(s)$$

$$\therefore L^{-1}\left\{e^{-as}\overline{f}(s)\right\} = G(t)$$

Solved Problems :

1. Evaluate (i) $L^{-1}\left\{\dfrac{1+e^{-\pi s}}{s^2+1}\right\}$ **(ii)** $L^{-1}\left\{\dfrac{e^{-3s}}{(s-4)^2}\right\}$

Sol: (i) $L^{-1}\left\{\dfrac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\dfrac{1}{s^2+1}\right\} + L^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$

Since $L^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t = f(t)$, say

$\therefore$ By second Shifting theorem, we have $L^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases} \sin(t-\pi) & , if\ t>\pi \\ 0 & , if\ t<\pi \end{cases}$

or $L^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}\right\}$ =sin(t-π)H(t-π)= -sint. H(t-π)

Hence $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$=sint-sint. H (t-π) =sint [1- H (t-π)]

Where H (t-π) is the Heaviside unit step function

(ii) Since $L^{-1}\left\{\frac{1}{(s-4)^2}\right\}=e^{4t}L^{-1}\left\{\frac{1}{s^2}\right\}$

$$=e^{4t}.t=f(t)\text{, say}$$

∴ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}=\begin{cases} e^{4(t-3)}.(t-3) & , if\ t>3 \\ 0 & , if\ t<3 \end{cases}$

or $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}=e^{4(t-3)}.(t-3)\,\text{H(t-3)}$

Where H (t-3) is the Heaviside unit step function

Change of scale property:

If $L\{f(t)\}=\overline{f}(s)$, Then $L^{-1}\{\overline{f}(as)\}=\frac{1}{a}f\left(\frac{t}{a}\right), a>0$

Proof: We have seen that $L\{f(t)\}=\overline{f}(s)$

$$Then\quad \overline{f}(as)=\frac{1}{a}L\left\{f\left(\frac{t}{a}\right)\right\}, a>0$$

$$\therefore L^{-1}\{\overline{f}(as)\}=\frac{1}{a}f\left(\frac{t}{a}\right), a>0$$

Solved Problems :

1. If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$**, find** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol: We have $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}=\frac{1}{2}t\sin t$,

Writing as for s,

$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\}=\frac{1}{2}.\frac{1}{a}.\frac{t}{a}\sin\frac{t}{a}=\frac{t}{2a^2}.\sin\frac{t}{a}$, by change of scale property.

Putting a=2, we get

$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\}=\frac{t}{8}\sin\frac{t}{2}$ **or** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}=\frac{1}{2}\sin\frac{t}{2}$

Inverse Laplace Transform of derivatives:

Theorem: $L^{-1}\{\overline{f}(s)\}=f(t)$, then $L^{-1}\{\overline{f}^n(s)\}=(-1)^n t^n f(t)$ where $\overline{f}^n(s)=\frac{d^n}{ds^n}\left[\overline{f}(s)\right]$

Proof: We have seen that $L\left\{t^n f(t)\right\} = (-1)^n \frac{d}{ds^n}^n \bar{f}(s)$

$$\therefore L^{-1}\left\{\bar{f}^n(s)\right\} = (-1)^n t^n f(t)$$

Solved Problems :

1. Find $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Sol: Let $L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = f(t)$

$$L\{f(t)\} = \log\frac{s+1}{s-1}$$

$$L\{tf(t)\} = \frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$$

$$L\{tf(t)\} = \frac{-1}{s+1} + \frac{1}{s-1}$$

$$tf(t) = L^{-1}\left\{\frac{-1}{s+1} + \frac{1}{s-1}\right\}$$

$$tf(t) = -1.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^{-t} + e^{t}$$

$$t\,f(t) = 2\sinh t \Rightarrow f(t) = \frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = \frac{2\sinh t}{t}$$

Note: $L^{-1}\left\{\log\frac{1+s}{s}\right\} = \frac{1-e^{-t}}{t}$

2.Find $L^{-1}\left\{\cot^{-1}(s)\right\}$

Sol: Let $L^{-1}\left\{\cot^{-1}(s)\right\} = f(t)$

$$L\{f(t)\} = \cot^{-1}(s)$$

$$L\{tf(t)\} = \frac{-d}{ds}[\cot^{-1}(s) = -\left[\frac{-1}{1+s^2}\right] = \frac{1}{1+s^2}$$

$$tf(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$\therefore L^{-1}\left\{\cot^{-1}(s)\right\} = \frac{1}{t}\sin t$$

Inverse Laplace Transform of integrals:

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, then $L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$

Proof: we have seen that $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \overline{f}(s)ds$

$$\therefore L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\} = \frac{f(t)}{t}$$

Solved Problems :

1. Find $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

Sol: **Let** $\overline{f}(s) = \frac{s+1}{(s^2+2s+2)^2}$

Then $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\int_s^\infty \frac{s+1}{(s^2+2s+2)^2}ds\right\}$

$$= L^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}, \text{ by First Shifting Theorem}$$

$$= e^{-t}\frac{t}{2}\sin t = \frac{t}{2}e^{-t}\sin t \quad \because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t}{2a}\sin at$$

Multiplication by power of's':

Theorem: $L^{-1}\left\{\overline{f}(s)\right\} = f(t)$, and f (0),then $L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$

Proof: we have seen that $L\left\{f^1(t)\right\} = s\overline{f}(s) - f(0)$

$$\therefore L\left\{f^1(t)\right\} = s\overline{f}(s) \quad [\because f(0)=0] \text{ or}$$

$$L^{-1}\left\{s\overline{f}(s)\right\} = f^1(t)$$

Note: $L^{-1}\left\{s^n\overline{f}(s)\right\} = f^n(t), \text{if } f^n(0) = 0 \text{ for } n = 1,2,3.........n-1$

Solved Problems :

1. Find (i) $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$ **(ii)** $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol: Let $\bar{f}(s)=\frac{1}{(s+2)^2}$ **Then**

$$L^{-1}\left\{\bar{f}(s)\right\}=L^{-1}\left\{\frac{1}{(s+2)^2}\right\}=e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}=e^{-2t}.t=f(t),$$

Clearly f (0) =0

$$\text{Thus } L^{-1}\left\{\frac{s}{(s+2)^2}\right\}=L^{-1}\left\{s.\frac{s}{(s+2)^2}\right\}=L^{-1}\left\{s.\bar{f}(s)\right\}=f^1(t)$$

$$=\frac{d}{dt}(te^{-2t})=\text{t}(-2e^{-2t})+e^{-2t}.1=e^{-2t}(1-2\,\text{t})$$

Note: in the above problem put 2=3, then $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}=e^{-3t}(1-3\text{t})$

Division by S:

Theorem: If $L^{-1}\left\{\bar{f}(s)\right\}=f(t)$, Then $L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(u)\,du$

Proof: We have seen that $L\left\{\int_0^t f(u)\,du\right\}=\frac{\bar{f}(s)}{s}$

$$\therefore L^{-1}\left\{\frac{\bar{f}(s)}{s}\right\}=\int_0^t f(u)\,du$$

Note: If $L^{-1}\left\{\bar{f}(s)\right\}=f(t)$, then $L^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}=\int_0^t\int_0^t f(u)\,du.du$

Solved Problems :

1. Find the inverse Laplace Transform of $\frac{1}{s^2(s^2+a^2)}$

Sol: Since $L^{-1}\left[\frac{1}{(s^2+a^2)}\right]=\frac{1}{a}\text{sinat}$, we have

$$L^{-1}\left[\frac{1}{\text{s}(s^2+a^2)}\right]=\int_0^t\frac{1}{a}\sin at dt$$

$$=\frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t=-\frac{1}{a^2}(\text{cosat}-1)=\frac{1}{a^2}(1-\text{cosat})$$

Then $L^{-1}\left[\dfrac{1}{s^2(s^2+a^2)}\right]=\int_0^t \dfrac{1}{a^2}(1-\cos at)dtdt$

$$=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)_0^t=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

Convolution Definition:

If f (t) and g (t) are two functions defined for $t\geq 0$ then the convolution of f (t) and g (t) is defined as $f(t)*g(t)=\int_0^t f(u)g(t-u)du$

$f(t)*g(t)$ *can also be written as* $(f*g)(t)$

Properties:

The convolution operation * has the following properties

1. Commutative i.e. $(f*g)(t)=(g*f)(t)$

2. Associative $[f*(g*h)](t)=[(f*g)*h](t)$

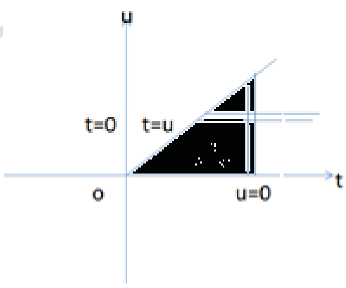
3. Distributive $[f*(g+h)](t)=(f*g)(t)+(f*h)(t)$ *for* $t\geq 0$

Convolution Theorem: If $f(t)$ **and** $g(t)$ **are functions defined for** $t\geq 0$ **then**

$\mathrm{L}\{f(t)*g(t)\}=\mathrm{L}\{f(t)\}.\mathrm{L}\{g(t)\}=\overline{f}(s).\overline{g}(s)$

i.e., The L.T of convolution of f(t) and g(t) is equal to the product of the L.T of f(t) and g(t)

Proof: WKT $\mathrm{L}\{\phi(t)\}=\int_0^\infty e^{-st}\left\{\int_0^t f(u)g(t-u)du\right\}dt$



$$=\int_0^\infty\int_0^t e^{-st}f(u)g(t-u)du\,dt$$

The double integral is considered within the region enclosed by the line u=0 and u=t

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_u^\infty e^{-s(t-u)} g(t-u)\,dt\right\} du$$

$$= \int_0^\infty e^{-su} f(u)\left\{\int_0^\infty e^{-sv} g(v)\,dv\right\} du \quad put\ t-u=v$$

$$= \int_0^\infty e^{-su} f(u)\{\bar{g}(s)\} du = \bar{g}(s)\int_0^\infty e^{-su} f(u)\,du = \bar{g}(s).\bar{f}(s)$$

$$L\{f(t)*g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s).\bar{g}(s)$$

Solved Problems :

1.Using the convolution theorem find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

$Let\ \bar{f}(s) = \frac{s}{s^2+a^2}\ and\ \bar{g}(s) = \frac{1}{s^2+a^2}$

So that $L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t) - say$

$L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \frac{1}{a}\sin at = g(t) \to say$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\frac{1}{a}.\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t [\sin(au+at-au) - \sin(au-at+au)]du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}.\cos(2au-at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos(2at-at) - \frac{1}{2a}\cos(-at)\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos at\right]$$

$$= \frac{t}{2a} \sin at$$

2.Use convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\}$

$$Let\ \overline{f}(s) = \frac{s}{s^2+a^2}\ and\ \overline{g}(s) = \frac{s}{s^2+b^2}$$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f(t) \rightarrow say$

$$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+b^2)}\right\} = \cos bt = g(t) \rightarrow say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\} = \int_0^t \cos au.\cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au-bu+bt)+\cos(au+bu-bt)\right]du$$

$$= \frac{1}{2}\left[\frac{\sin(au-bu+bt)}{a-b} + \frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}$$

3.Use convolution theorem to evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Sol: $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\} = L^{-1}\left\{\frac{1}{s^2}.\frac{s}{(s^2+4)^2}\right\}$

$$Let\ \overline{f}(s) = \frac{1}{s^2}\ and\ \overline{g}(s) = \frac{s}{(s^2+4)^2}$$

So that $L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t) \rightarrow say$

$$L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t.\sin 2t}{4} = f(t) - say\left[\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{t\sin 2t}{2a}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}=\int_0^t \frac{u}{4}\sin 2u(t-u)du$$

$$= t/4\int_0^t u\sin 2udu-\frac{1}{4}\int_0^t u^2\sin 2udu$$

$$= t/4\left(-\frac{u}{2}\cos 2u+\frac{1}{4}\sin 2u\right)_0^t$$

$$=-\frac{1}{4}\left[\frac{-u^2}{2}\cos 2u+\frac{u}{2}\sin 2u+\frac{1}{4}\cos 2u\right]_0^t$$

$$=\frac{1}{16}[1-t\sin 2t-\cos 2t]$$

4.Find $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Sol: $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]=L^{-1}\left[\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right]$

$Let\ \bar{f}(s)=\frac{1}{s-2}\ and\ \bar{g}(s)=\frac{1}{s^2+1}$

So that $L^{-1}\{\bar{f}(s)\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=f(t)\to say$

$L^{-1}\{\bar{g}(s)\}=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t=g(t)\to say$

$\therefore L^{-1}\left\{\frac{1}{s-2}\cdot\frac{1}{s^2+1}\right\}=\int_0^t f(u).\mathrm{g}(\mathrm{t}-u)\,\mathrm{du}$ (By Convolution theorem)

$$=\int_0^t e^{2u}\sin(\mathrm{t}-\mathrm{u})\,\mathrm{du}\ \ (\mathrm{or})\ \int_0^t \sin\mathrm{u}.e^{2(t-u)}\,\mathrm{du}$$

$$=e^{2t}\int_0^t \sin ue^{-2u}du$$

$$=e^{2t}\left[\frac{e^{-2u}}{2^2+1}[-2\sin u-\cos u]\right]_0^t$$

$$=e^{2t}\left[\frac{1}{5}e^{-2t}(-2\sin t-\cos t)-1/5(-1)\right]$$

$$=\frac{1}{5}\left(e^{2t}-2\sin t-\cos t\right)$$

5.Find $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}=L^{-1}\left\{\frac{1}{s+1}.\frac{1}{s-2}\right\}$

$Let\ \overline{f}(s)=\frac{1}{s+1}\ and\ \overline{g}(s)=\frac{1}{s-2}$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}=f(t)\rightarrow say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=g(t)\rightarrow say$

$\therefore$ By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}=\int_0^t e^{-u}e^{2(t-u)}du$$

$$=\int_0^t e^{2t}e^{-3u}du=e^{2t}\int_0^t e^{-3u}du=e^{2t}\left[\frac{e^{-3u}}{-3}\right]_0^t=\frac{1}{3}\left[e^{2t}-e^{-t}\right]$$

6.Find $\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Sol: $\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}=\mathrm{L}^{-1}\left\{\frac{1}{s^2}.\frac{1}{s^2-a^2}\right\}$

$Let\ \overline{f}(s)=\frac{1}{s^2}\ and\ \overline{g}(s)=\frac{1}{s^2-a^2}$

So that $\mathrm{L}^{-1}\left\{\overline{f}(s)\right\}=\mathrm{L}^{-1}\left\{\frac{1}{s^2}\right\}=t=f(t)-say$

$\mathrm{L}^{-1}\left\{\overline{g}(s)\right\}=\mathrm{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{1}{a}\sinh at=g(t)-say$

By using convolution theorem, we have

$$\mathrm{L}^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}=\int_0^t u.\frac{1}{a}\sinh a(t-u)du$$

$$=\frac{1}{a}\int_0^t u\sinh(at-au)du$$

$$=\frac{1}{a}\left[\frac{-u}{a}\cosh(at-au)-\frac{\sin(at-au)}{a^2}\right]_0^t$$

$$=\frac{1}{a}\left[\frac{-t}{a}\cosh(at-at)-0-\frac{1}{a^2}[0-\sinh at]\right]$$

$$=\frac{1}{a}\left[\frac{-t}{a}+\frac{1}{a^2}\sinh at\right]$$

$$=\frac{1}{a^3}[-at+\sinh at]$$

7.Using Convolution theorem, evaluate $L^{-1}\{\frac{s}{(S+2)(s^2+9)}\}$

Sol: $L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\}=L^{-1}\{\frac{1}{s+2}.\frac{s}{s^2+3^2}\}=L^{-1}\{\bar{f}(s).\bar{g}(s)\}$

$\bar{f}(s)=\frac{1}{s+2}=L\{f(t)\}\Rightarrow f(t)=L^{-1}\left\{\frac{1}{s+2}\right\}=e^{-2t}$----------------- (1)

$\bar{g}(s)=\frac{s}{s^2+3^2}=L\{g(t)\}\Rightarrow g(t)=L^{-1}\left\{\frac{s}{s^2+3^2}\right\}=cos3t$----------------- (2)

By Convolution theorem we have

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\}=f(t)*g(t)$$

Where $f(t)*g(t)=\int_0^t g(u)f(t-u)du$

$\therefore \quad L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\}=\int_0^t e^{-2(t-u)}cos3udu$

$$=e^{-2t}\int_0^t e^{2u}cos3udu$$

$$=e^{-2t}.\frac{1}{2^2+3^2}[2cos3u-3sin3u]_0^t$$

$$=\frac{e^{-2t}}{13}[2cos3t-2-3sin3t]$$

$$=\frac{1}{13}[e^{-2t}(2cos3t-3sin3t)]-\frac{2e^{-2t}}{13}$$

Application of L.T to ordinary differential equations:

Solutions of ordinary DE with constant coefficient:

1. **Step1:** Take the Laplace Transform on both the sides of the DE and then by using the formula

   $L\{f^n(t)\}=s^nL\{f(t)\}-s^{n-1}f(0)-s^{n-1}f^1(0)-s^{n-2}f^2(0)-............f^{n-1}(0)$ and apply given initial conditions. This gives an algebraic equation.

2. **Step2:** replace f (0), $f^1(0)$, $f^2(0)$,……… $f^{n-1}(0)$ with the given initial conditions.

   Where $f^1(0)=s\overline{f}(0)-f(0)$

   $f^2(0)=s^2\overline{f}(s)-sf(0)-f^1(0)$, and so on

3. **Step3:** solve the algebraic equation to get derivatives in terms of s.

4. **Step4:** take the inverse Laplace transform on both sides this gives f as a function of t which gives the solution of the given DE

Solved Problems :

1.Solve $y^{111}+2y^{11}-y^{1}-2y=0$ **using Laplace Transformation given that**

$y(0)=y^{1}(0)=0$ **and** $y^{11}(0)=6$

Sol: Given that $y^{111}+2y^{11}-y^{1}-2y=0$

Taking the Laplace transform on both sides, we get

$$\mathrm{L}\left\{y^{111}(t)\right\}+2\mathrm{L}\left\{y^{11}(t)\right\}-\mathrm{L}\left\{y^{1}\right\}-2\mathrm{L}\left\{y\right\}=0$$

$$\Rightarrow s^{3}\mathrm{L}\left\{y(t)\right\}-s^{2}y(0)-sy^{1}(0)-y^{11}(0)+2\left\{s^{2}\mathrm{L}\left\{y(t)\right\}-sy(0)-y^{1}(0)\right\}-$$
$$\left\{s\mathrm{L}\left\{y(t)\right\}-y(0)\right\}-2\mathrm{L}\left\{y(t)\right\}=0$$

$$\Rightarrow\left\{s^{3}+2s^{2}-s-2\right\}\mathrm{L}\left\{y(t)\right\}=s^{2}y(0)+sy^{1}(0)+y^{11}(0)+2sy(0)+2y^{1}(0)-y(0)$$

$$=0+0+6+2.0+2.0-0$$

$$\Rightarrow\left\{s^{3}+2s^{2}-s-2\right\}\mathrm{L}\left\{y(t)\right\}=6$$

$$\mathrm{L}\left\{y(t)\right\}=\frac{6}{s^{3}+2s^{2}-s-2}=\frac{6}{(s-1)(s+1)(s+2)}$$

$$=\frac{A}{s-1}+\frac{B}{s+1}+\frac{C}{s+2}$$

$$\Rightarrow A(s+1)(s+2)+B(s-1)(s+2)+C(s-1)(s+1)=6$$

$$\Rightarrow A(s^{2}+3s+2)+B(s^{2}-s-2)+C(s^{2}-1)=6$$

Comparing both sides s^2,s,constants,we have

$$\Rightarrow A+B+C=0,3A-B=0,2A-2B-C=6$$

$$\mathrm{A+B+C=0}$$
$$\mathrm{2A-2B-C=6}$$

$$3A-B=6$$
$$3A+B=0$$

$$6A=6\Rightarrow A=1$$
$$3A+B=0\Rightarrow B=-3A\Rightarrow B=-3$$

$$\therefore A+B+C=0\Rightarrow C=-A-B=-1+3=2$$

$$\therefore \mathrm{L}\left\{y(t)\right\}=\frac{1}{s-1}-\frac{3}{s+1}+\frac{2}{s+2}$$

$$y(t) = L^{-1}\left\{\frac{1}{s-1}\right\} - 3.L^{-1}\left\{\frac{1}{s+1}\right\} + 2.L^{-1}\left\{\frac{1}{s+2}\right\} = e^{t} - 3e^{-t} + 2.e^{-2t}$$

Which is the required solution

2.Solve $y^{11} - 3y^{1} + 2y = 4t + e^{3t}$ **using Laplace Transformation given that** $y(0) = 1 and\ y^{1}(0) = -1$

Sol: Given that $y^{11} - 3y^{1} + 2y = 4t + e^{3t}$

Taking the Laplace transform on both sides, we get

$$L\left\{y^{11}(t)\right\} - 3L\left\{y^{1}(t)\right\} + 2L\left\{y(t)\right\} = 4L\left\{t\right\} + L\left\{e^{3t}\right\}$$

$$\Rightarrow s^{2}L\left\{y(t)\right\} - sy(0) - y^{1}(0) - 3\left[sL\left\{y(t)\right\} - y(0)\right] + 2L\left\{y(t)\right\} = \frac{4}{s^{2}} + \frac{1}{s-3}$$

$$\Rightarrow (s^{2} - 3s + 2)L\left\{y(t)\right\} = \frac{4}{s^{2}} + \frac{1}{s-3} + s - 4$$

$$\Rightarrow (s^{2} - 3s + 2)L\left\{y(t)\right\} = \frac{4s - 12 + s^{4} + s^{2} - 3s^{3} - 4s^{3} + 12s^{2}}{s^{2}(s-3)}$$

$$\Rightarrow L\left\{y(t)\right\} = \frac{s^{4} - 7s^{3} + 13s^{2} + 4s - 12}{s^{2}(s-3)(s^{2} - 3s + 2)}$$

$$\Rightarrow L\left\{y(t)\right\} = \frac{s^{4} - 7s^{3} + 13s^{2} + 4s - 12}{s^{2}(s-3)(s-1)(s-2)}$$

$$\Rightarrow \frac{s^{4} - 7s^{3} + 13s^{2} + 4s - 12}{s^{2}(s-3)(s-1)(s-2)} = \frac{As + B}{s^{2}} + \frac{C}{s-3} + \frac{D}{s-1} + \frac{E}{s-2}$$

$$= \frac{(As+B)(s-1)(s-2)(s-3) + C(s^{2})(s-1)(s-2) + D(s^{2})(s-2)(s-3) + E(s^{2})(s-1)(s-3)}{s^{2}(s-3)(s-1)(s-2)}$$

$$\Rightarrow s^{4} - 7s^{3} + 13s^{2} + 4s - 12 = (As + B)(s^{3} - 6s^{2} + 11s - 6) +$$
$$C(s^{2})(s^{2} - 3s + 2) + D(s^{2})(s^{2} - 5s + 6) + E.s^{2}(s^{2} - 4s + 3)$$

Comparing both sides s^4,s^3,we have

$A + C + D + E = 1$......................(1)

$-6A + B - 3C - 5D - 4E = -7$......................(2)

$put\ s = 1, 2D = -1 \Rightarrow D = \frac{-1}{2}$

$put\ s = 2, -4E = 8 \Rightarrow E = -2$

$put\ \ s = 3, 18C = 9 \Rightarrow C = \frac{1}{2}$

$$from\, eq.(1)\ A = 1 - \frac{1}{2} + \frac{1}{2} + 2 \Rightarrow A = 3$$

from eq.(2) B= -7+18+$\frac{3}{2} - \frac{5}{2} - 8 = 3 - 1 = 2$

$$y(t) = L^{-1}\left\{\frac{3}{s} + \frac{2}{s^2} + \frac{1}{2(s-3)} - \frac{1}{2(s-1)} - \frac{2}{s-2}\right\}$$

$$y(t) = 3 + 2t + \frac{1}{2}e^{3t} - \frac{1}{2}e^{t} - 2.e^{2t}$$

3. Using Laplace Transform Solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t,$ **given that** $y = \frac{dy}{dt} = 0$ **when t=0**

Sol: Given equation is $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t.$

$$L\{y^{11}(t)\} + 2L\{y^{1}(t)\} - 3L\{y(t)\} = L\{\sin t\}$$

$$s^2L\{y(t)\} - sy(0) - y^1(0) + 2\left[sL\{y(t)\} - y(0)\right] - 3.L\{y(t)\} = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2 + 2s - 3)L\{y(t)\} = \frac{1}{s^2+1}$$

$$\Rightarrow L\{y(t)\} = \left(\frac{1}{(s^2+1)(s^2+2s-3)}\right)$$

$$\Rightarrow y(t) = L^{-1}\left(\frac{1}{(s-1)(s+3)(s^2+1)}\right)$$

Now consider

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1$$

Comparing both sides s^3,we have

$$put\ s = 1, 8A = 1 \Rightarrow A = \frac{1}{8}$$

$$put\ s = -3, -40B = 1 \Rightarrow B = \frac{-1}{40}$$

$$A + B + C = 0 \Rightarrow C = 0 - \frac{1}{8} + \frac{1}{40}$$

$$C=\frac{-5+1}{40}=\frac{-4}{40}=\frac{-1}{10}$$

$$3A-B+2C+D=0\Rightarrow D=-\frac{3}{8}-\frac{1}{40}+\frac{1}{5}$$

$$D=\frac{-15-1+8}{40}=\frac{-8}{40}=\frac{-1}{5}$$

$$\therefore y(t)=\mathrm{L}^{-1}\left\{\frac{\frac{1}{8}}{s-1}+\frac{\frac{-1}{40}}{s+3}+\frac{\frac{-1}{10}s-\frac{1}{5}}{s^2+1}\right\}$$

$$=\frac{1}{8}\mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\}-\frac{1}{40}\mathrm{L}^{-1}\left\{\frac{1}{s+3}\right\}-\frac{1}{10}\mathrm{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-\frac{1}{5}\mathrm{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore y(t)=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\cos t-\frac{1}{5}\sin t$$

4.Solve $\frac{dx}{dt}+x=\sin\omega\mathrm{t},x(0)=2$

Sol: Given equation is $\frac{dx}{dt}+x=\sin\omega\mathrm{t}$

$$\mathrm{L}\{x^{1}(t)\}+\mathrm{L}\{x(t)\}=\mathrm{L}\{\sin\omega\mathrm{t}\}$$

$$\Rightarrow \mathrm{s.L}\{x(t)\}-x(0)+\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow \mathrm{s.L}\{x(t)\}-2+\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow (s+1)\mathrm{L}\{x(t)\}=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow x(t)=\mathrm{L}^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}+\frac{2}{s+1}\right\}$$

$$=2\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathrm{L}^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}\right\}$$ (By using partial fractions)

$$=2e^{-t}+\mathrm{L}^{-1}\left\{\frac{\frac{\omega}{\omega^2+1}}{s+1}-\frac{\frac{s\omega}{1+\omega^2}}{s^2+\omega^2}+\frac{\frac{\omega}{1+\omega^2}}{s^2+\omega^2}\right\}$$

$$=2e^{-t}+\frac{\omega}{\omega^2+1}e^{-t}-\frac{\omega}{1+\omega^2}\cos\omega t+\frac{\omega}{1+\omega^2}.\frac{1}{\omega}\sin\omega t$$

5.Solve $\left(D^2+n^2\right)x = a\sin\left(nt+\alpha\right)$ **given that $x = Dx = 0$, when $t = 0$.**

Sol: Given equation is $\left(D^2+n^2\right)x = a\sin\left(nt+\alpha\right)$

$$x^{11}(t)+n^2x(t)=a\sin(nt+\alpha)$$

$$\mathrm{L}\left\{x^{11}(t)\right\}+n^2\mathrm{L}\left\{x(t)\right\}=\mathrm{L}\left\{a\sin nt\cos\alpha+a\cos nt\sin\alpha\right\}$$

$$\Rightarrow s^2\mathrm{L}\left\{x(t)\right\}-sx(0)-x^1(0)+n^2\mathrm{L}\left\{x(t)\right\}=a\cos\alpha\mathrm{L}\left\{\sin nt\right\}+a\sin\alpha\mathrm{L}\left\{\cos nt\right\}$$

$$\Rightarrow\left(s^2+n^2\right)\mathrm{L}\left\{x(t)\right\}=a\cos\alpha\frac{n}{s^2+n^2}+a\sin\alpha.\frac{s}{s^2+n^2}$$

$$\Rightarrow\mathrm{L}\left\{x(t)\right\}=a\cos\alpha\frac{n}{\left(s^2+n^2\right)^2}+a\sin\alpha\frac{s}{\left(s^2+n^2\right)^2}$$

(By using convolution theorem I –part, partial fraction in II-part)

$$=na\cos\alpha\int_0^t\frac{1}{n}.\sin nx.\frac{1}{n}\sin n(t-x)\,dx-\frac{a\sin\alpha}{2}\mathrm{L}^{-1}\left\{\frac{d}{ds}\frac{1}{\left(s^2+n^2\right)}\right\}$$

$$=\frac{a\cos\alpha}{2n}\int_0^t\left\{\cos(nt-2nx)-\cos nt\right\}dx+\frac{a\sin\alpha}{2}t\frac{1}{n}\sin nt$$

$$=\frac{a\cos\alpha}{2n}\left[\int_0^t\left\{\cos n(t-2x)-\cos nt\right\}dx+\frac{a}{2n}\sin\alpha t\sin nt\right]$$

$$=\frac{a\cos\alpha}{2n}\left[\frac{-1}{2n}.\sin n(t-2x)-x\cos nt\right]_0^t+\frac{at\sin\alpha}{2n}\sin nt$$

$$=\frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n}-t\cos nt\right]+\frac{at\sin\alpha}{2n}\sin nt$$

$$=\frac{a\cos\alpha\sin nt}{2n^2}-\frac{at}{2n}\left[\cos\alpha\cos nt-\sin\alpha\sin nt\right]$$

$$=\frac{a\cos\alpha\sin nt}{2n^2}-\frac{at}{2n}\cos(\alpha+nt)$$

6. Solve $y''-4y'+3y=e^{-t}$ **using L.T given that $y(0) = y'(0) = 1$.**

Sol: Given equation is $y''-4y'+3y=e^{-t}$

Applying L.T on both sides we get $L(y'')-4L(y')+3L(y)=L(e^{-t})$

$$\Rightarrow\{s^2L[y]-s\,y(0)-y^1(0)\}-4\{s\,L[y]-y(0)\}+3L\{y\}=\frac{1}{s+1}$$

$$\Rightarrow(s^2+4s+3)\,L\{y\}-s-1-4=\frac{1}{s+1}$$

$$\Rightarrow(s^2+4s+3)\,L\{y\}=\frac{1}{s+1}+s+5$$

$$\Rightarrow(s^2+4s+3)\,L\{y\}=\frac{1}{s+1}+s+5$$

$$L\{y\} = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{(s^2+4s+3)}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] + L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]$$

Let us consider

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right]$$

$$\frac{1}{(s+1)(s^2+4s+3)} = \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{(S+1)^2} + \frac{C}{S+3}$$

$$= \frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}$$

$$= L^{-1}\left[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}\right]$$

$$= L^{-1}\left[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}\right]$$

$$= -\frac{1}{4}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{(S+1)^2}\right] + \frac{1}{4}L^{-1}\left[\frac{1}{S+3}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] = -\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t} ---\rightarrow (1)$$

$$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right] = L^{-1}\left[\frac{s+2}{((s+2)^2-1)}\right] + L^{-1}\left[\frac{3}{((s+2)^2-1)}\right]$$

$$= \boldsymbol{e^{-2t}}L^{-1}\left[\frac{s}{(s^2-1)}\right] + L^{-1} + \boldsymbol{3e^{-2t}}L^{-1}\left[\frac{1}{(s^2-1)}\right]$$

$$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right] = \boldsymbol{cost} + \boldsymbol{3e^{-2t}sint} ---\rightarrow (2)$$

From (1) & (2)

$$\therefore y = -\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} + \frac{1}{4}e^{-3t} + \boldsymbol{e^{-2t}cost} + \boldsymbol{3e^{-2t}sint}$$

7.Solve $\frac{d^2x}{d^2t} + 9x = cos2t$ using L.T. given $x(0) = 1, x\left(\frac{\pi}{2}\right) = -1$.

Sol: Given $x'' + 9x = cos2t$

$L\,[x''] + 9[x] = L[cos2t]$

$\Rightarrow\ s^2L[x] - sx(0) - x^1(0) + 9L[x\} = \frac{s}{s^2+4}$

$\Rightarrow (s^2 + 9)L[x] - s - a = \frac{s}{s^2+4}$

$\Rightarrow (s^2 + 9)L[x] = \frac{s}{s^2+4} + (s + a)$

$L[x] = \frac{s}{(s^2+4((s^2+9)} + \frac{s}{(s^2+9)} + \frac{a}{(s^2+9)}$

$X = L^{-1}\left[\frac{s}{(s^2+4((s^2+9)}\right] + L^{-1}\left[\frac{s}{(s^2+9)}\right] + L^{-1}\left[\frac{a}{(s^2+9)}\right]$

$= \frac{1}{5} L^{-1}[\frac{s}{s^2+4} - \frac{s}{s^2+9}] + cos3t + \frac{a}{3} sin3t$

$= \frac{1}{5} L^{-1}[\frac{s}{s^2+4}] - \frac{1}{5} L^{-1}[\frac{s}{s^2+9}] + cos3t + \frac{a}{3} sin3t$

$= \frac{1}{5} cos2t - \frac{1}{5} cos3t + cos3t + \frac{a}{3} sin3t$ ------------------→ (1)

Given x $(\frac{\pi}{2})$ = -1.

$\therefore\ -1 = \frac{1}{5} cos2\left(\frac{\pi}{2}\right) - \frac{1}{5} cos\frac{3\pi}{2} + cos\frac{3\pi}{2} + cos\frac{3\pi}{2} + \frac{a}{3} sin\frac{3\pi}{2}$

$\Rightarrow\ -1 = -\frac{1}{5} - 0 + 0 - \frac{a}{3}$

$\frac{a}{3} = -\frac{1}{5} + 1$

$\frac{a}{3} = \frac{4}{5}$

$\therefore$ x = $\frac{1}{5} cos2t + \frac{4}{5} cos3t + \frac{4}{5} sin3t$ From (1)

8.Solve$(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ **Using L.T given** $y(0) = 1, y' = 0, y''(0) = -2$

Sol: Given $y''' - 3y'' + 3y' - y = t^2e^t$

$L[y'''] - 3L[y''] + 3L[y'] - L[y] = L[t^2e^t]$

$\Rightarrow \{s^3L[y] - s^2y(0) - sy'(0) - y''(0)\} - 3\{s^2L[y] - sy'(0) - y(0)\} + 3\{sL[y] - y(0)\} - L[y] = L[t^2e^t]$

$\Rightarrow (s^3 - 3s^2 + 3s - 1)L[y] - s^2 - 0 + 2 + 0 + 3(1) - 3(1) = (-1)^2 \frac{d^2}{ds^2} L[e^t]$

$\Rightarrow (s-1)^3 L[y] - s^2 + 2 = \frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$

$= \frac{2}{(s-1)^3}$

$\Rightarrow (s-1)^3 L[y] = \frac{2}{(s-1)^3} + s^2 - 2$

$$L[y] = \frac{2}{(s-1)^6} + \frac{s^2}{(s-1)^3} - \frac{2}{(s-1)^3}$$

$y = L^{-1}[\frac{2}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - L^{-1}[\frac{2}{(s-1)^3}]$

$= 2L^{-1}[\frac{1}{(s-1)^6}] + L^{-1}[\frac{s^2}{(s-1)^3}] - 2L^{-1}[\frac{1}{(s-1)^3}]$

$= 2e^t L^{-1}\left[\frac{1}{(s)^6}\right] + L^{-1}\frac{s^2}{(s-1)^3} - 2e^t L^{-1}\left[\frac{1}{s^3}\right]$

$= 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} + L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

Consider $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

W.K.T $L^{-1}\left[\frac{1}{(s-1)^3}\right] = e^t L^{-1}\left[\frac{1}{s^3}\right] = e^t\frac{t^2}{2!} = \frac{e^t t^2}{2}$

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right] = \frac{d^2}{ds^2}\left(\frac{e^t t^2}{2}\right) = \frac{1}{2}\frac{d}{dt}(2te^t + t^2e^t) = \frac{1}{2}(2e^t + 2te^t + 2te^t + t^2e^t)$$

$$= \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$

$$\therefore \quad y = 2e^t\frac{t^5}{5!} - 2e^t\frac{t^2}{2!} - \frac{1}{2}(2e^t + 4te^t + t^2e^t)$$