Unit-2

Laplace transforms

Definition

Let $f(t)$ be a given function and defined for all positive values of 't'
Then Laplace transform of $f(t)$ is denoted by $L[f(t)]$ (or) $F(s)$ defined by

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt = F(s)$$

where 's' is a real (or) complex number.

Sufficient condition for existance of Laplace transform

① $f(t)$ must be piecewise continuous function
② The function $f(t)$ is of exponential order 'a'
ie $\lim_{t\to\infty} e^{-at} f(t) =$ finite quantity.

Ex The function $f(t) = t^2$ is of exponential order '3'

$$\lim_{t\to\infty} e^{-3t} f(t) = \lim_{t\to\infty} e^{-3t} t^2 = \lim_{t\to\infty} \frac{t^2}{e^{3t}} = \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{t\to\infty} \frac{2t}{3e^{3t}} \quad \text{by L' Hospital Rule}$$

$$= \lim_{t\to\infty} \frac{2}{9e^{3t}} = \frac{2}{\infty} = 0 = \text{finite quant.}$$

Hence $f(t) = t^2$ is of exponential order '3'.

Note $f(t) = t^2 \sin at$, e^{at} etc are all of exponential order and also continuous, But $f(t) = e^{t^3}$, e^{t^2} is not exponential order and such that it's Laplace transforms does not exist.

Laplace transforms of some standard functions

① $L[c] = c/s$
② $L[1] = 1/s$
③ $L[e^{at}] = 1/(s-a)$
④ $L[e^{-at}] = \frac{1}{s+a}$
⑤ $L[\sin at] = a/(s^2+a^2)$
⑥ $L[\cos at] = s/(s^2+a^2)$
⑦ $L[\sinh at] = a/(s^2-a^2)$
⑧ $L[\cosh at] = \frac{s}{s^2-a^2}$
⑨ $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}$
⑩ $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx$
⑪ $\Gamma(1) = 1$, $\Gamma(1/2) = \sqrt{\pi}$, $\Gamma(n+1) = n!$
$\Gamma(n+1) = n\Gamma(n)$
⑫ $L[c_1 f_1(t) + c_2 f_2(t)] = c_1 L[f_1(t)] + c_2 L[f_2(t)]$

① Find Laplace transforms of $e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9$

Sol $L\left[e^{3t} - 2e^{-2t} + \sin 2t + \cos 3t + \sinh 3t - 2\cosh 4t + 9\right]$

$= L[e^{3t}] - 2L[e^{-2t}] + L[\sin 2t] + L[\cos 3t] + L[\sinh 3t] - 2L[\cosh 4t] + L(9)$

$= \frac{1}{s-3} - 2\frac{1}{s+2} + \frac{2}{s^2+4} + \frac{s}{s^2+9} + \frac{3}{s^2-3^2} - 2\frac{s}{s^2-4^2} + \frac{9}{s} = F(s)$

② $f(t) = \begin{cases} 1 & ; \text{If } 0<t<1 \\ 0 & ; \text{If } 1<t<2 \\ 5 & \text{If } 2<t<3 \\ 0 & \text{If } t>3 \end{cases}$ Find $L[f(t)]$

Sol $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^1 e^{-st} \cdot 1\,dt + \int_1^2 e^{-st} \cdot 0\,dt + \int_2^3 e^{-st} \cdot 5\,dt + \int_3^\infty e^{-st} \cdot 0\,dt$

$= \left(\frac{e^{-st}}{-s}\right)_0^1 + 0 + 5\left(\frac{e^{-st}}{-s}\right)_2^3 + 0$

$= \frac{e^{-s}}{-s} - \frac{1}{-s} + 5\left(\frac{e^{-3s}}{-s} - \frac{e^{-2s}}{-s}\right)$

$= \frac{1-e^{-s}}{s} - \frac{5(e^{-3s} - e^{-2s})}{s}$

$L(f(t)) = \frac{1}{s}(1 - e^{-s} + 5e^{-2s} - 5e^{-3s})$

③ Find $L[\cos 3t \sin 5t] = L\left[\frac{1}{2} 2\sin 5t \cos 3t\right]$

$= \frac{1}{2} L[2\sin 5t \cos 3t]$

$= \frac{1}{2} L[\sin(5t+3t) + \sin(5t-3t)]$

$= \frac{1}{2}\left[L[\sin 8t] + L[\sin 2t]\right]$

$L[\cos 3t \sin 5t] = \frac{1}{2}\left[\frac{8}{s^2+8^2} + \frac{2}{s^2+2^2}\right]$

④ $L[(\sin t - \cos t)^3]$

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$= L[(\sin^3 t - \cos^3 t) - 3(1-\cos^2 t)\cos t + 3\sin t(1-\sin^2 t)$

$= L[3(\sin t - \cos t) - 2(\sin^3 t - \cos^3 t)]$

$= L[3(\sin t - \cos t) - 2[\frac{1}{4}(\sin 3t + 3\sin t) - \frac{1}{4}(\cos 3t + 3\cos t)]]$

$= L[\frac{1}{2}(3\sin t - 3\cos t + \sin 3t + \cos 3t)]$

$= \frac{1}{2}\left[\frac{3}{s^2+1} - \frac{3s}{s^2+1} + \frac{3}{s^2+9} + \frac{s}{s^2+9}\right] = L[(\sin t - \cos t)^3]$

(5) find $L[\cos t \cos 2t \cos 3t] = L[\frac{1}{2}\cos t \, 2\cos 2t \cos 3t)]$

$= \frac{1}{2} L[\cos t(\cos 5t + \cos t)]$

$= \frac{1}{2} L[\cos t \cos 5t + \cos^2 t)]$

$= \frac{1}{4} L[\cos 6t + \cos 4t + 1 + \cos 2t]$

$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+4} + \frac{s}{s^2+16} + \frac{s}{s^2+36}\right]$

(6) $L[\sin(\omega t + \alpha)] = L[\sin\omega t \cos\alpha + \cos\omega t \sin\alpha)$

$L[\sin(\omega t + \alpha)] = \cos\alpha \cdot \frac{\omega}{s^2+\omega^2} + \sin\alpha \cdot \frac{s}{s^2+\omega^2}$

(7) $L[t^2 + at + b] = L[t^2] + aL[t] + L[b]$

$= \frac{2!}{s^3} + a \cdot \frac{1!}{s^2} + b/s$

$L[t^2 + at + b] = \frac{2}{s^3} + \frac{a}{s^2} + b/s$

First Translation theorem (or First shifting theorem)

Statement

If $L[f(t)] = F(s)$ then $L[e^{at} f(t)] = F(s-a)$

Ex find $L[e^{-t}(3\sin 2t - 5\cosh 2t]$

$= L[3e^{-t}\sin 2t] - L[5e^{-t}\cosh 2t]$

$= 3L[e^{-t}\sin 2t] - 5L[e^{-t}\cosh 2t]$

$= 3\frac{2}{(s+1)^2+2^2} - 5 \cdot \frac{s}{(s+1)^2-2^2}$

$= \frac{6}{(s^2+2s+5)} - \frac{5(s+1)}{s^2+2s-3}$

Ex Find $L[e^{3t}\sin^2 t]$

Sol $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right]$

$= \frac{1}{2} L[1-\cos 2t]$

$= \frac{1}{2}\left[L[1] - L[\cos 2t]\right]$

$L[\sin^2 t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L[e^{3t}\sin^2 t] = \frac{1}{2}\left[\frac{1}{s-3} - \frac{s-3}{(s-3)^2+4}\right]$

Ex $L[\cosh at \sin bt] = L\left[\frac{e^{at}+e^{-at}}{2}\sin bt\right]$

$= \frac{1}{2} L[e^{at}\sin bt + e^{-at}\sin bt]$

$L[\cosh at \sin bt] = \frac{1}{2}\left[\frac{b}{(s-a)^2+b^2} + \frac{b}{(s+a)^2+b^2}\right]$

Ex Find $L[e^{4t}\sin t\cos 2t] = L\left[\frac{1}{2}e^{4t}\, 2\cos 2t \sin t\right]$

$= \frac{1}{2} L[e^{4t}(\sin 3t - \sin t)]$

$= \frac{1}{2}\left[L[e^{4t}\sin 3t] - L[e^{4t}\sin t]\right]$

$L[e^{4t}\sin t\cos 2t] = \frac{1}{2}\left[\frac{3}{(s-4)^2+3^2} - \frac{1}{(s-4)^2+1^2}\right]$

Second translation theorem (or) second shifting theorem

Statement

If $L[f(t)] = F(s)$ and $g(t) = \begin{cases} f(t-a); & t>a \\ 0; & t<a \end{cases}$

Then $L[g(t)] = e^{-as}F(s)$

Proof LHS $L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt$

$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$

$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$

put $t-a = x$

$\Rightarrow dt = dx$

when $t=a \Rightarrow x=0$

when $t=\infty \Rightarrow x=\infty$

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$$L(g(t)) = \int_{x=0}^{\infty} e^{-s(x+a)} f(x)\,dx$$

$$= e^{-as} \int_{0}^{\infty} e^{-sx} f(x)\,dx$$

$$L[g(t)] = e^{-as} F(s)$$

Another form of Second Shifting theorem

Statement

If $L[f(t)] = F(s)$ then

$L[f(t-a)H(t-a)] = e^{-as}F(s)$ where

$H(t) = \begin{cases} 1 & \text{If } t>0 \\ 0 & \text{If } t<0 \end{cases}$ and $H(t)$ is called Heaviside unit step function.

Ex Find Laplace transform of $g(t)$

where $g(t) = \begin{cases} \cos(t-2\pi/3), & \text{If } t>2\pi/3 \\ 0 & ; \ t<2\pi/3 \end{cases}$

Sol Let $f(t) = \cos t$

$L[f(t)] = L[\cos t] = \frac{s}{s^2+1} = F(s)$

Now $g(t) = \begin{cases} f(t-2\pi/3) & ; \ t>2\pi/3 \\ 0 & ; \ t<2\pi/3 \end{cases}$

Applying second shifting theorem

$L[g(t)] = e^{-as}F(s)$

$L[g(t)] = e^{-\frac{2\pi}{3}s} \frac{s}{s^2+1} = \frac{s e^{-2\pi s/3}}{s^2+1}$

Ex Find Laplace transform of $3\cos 4(t-2)u(t-2)$

Sol Let $f(t) = 3\cos 4t$

Then $L[f(t)] = L[3\cos 4t] = 3L[\cos 4t] = \frac{3s}{s^2+4^2}$

By Second shifting theorem $L[f(t-2)u(t-2)] = e^{-2s}F(s)$

$L[3\cos 4(t-2)u(t-2)] = e^{-2s}\frac{3s}{s^2+16} = \frac{3se^{-2s}}{s^2+16}$.

$L[3\cos 4(t-2)u(t-2)]$

Change of scale property

If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F(s/a)$

Ex If $L[f(t)] = \frac{9s^2 - 12s + 15}{(s-1)^3}$ find $L[f(3t)]$ using change of scale property.

Sol Given $L[f(t)] = \frac{9s^2 - 12s + 15}{(s-1)^3} = F(s)$

By change of scale property

$$L[f(3t)] = \frac{1}{3} F(s/3) = \frac{1}{3} \frac{9(s/3)^2 - 12(s/3) + 15}{(s/3 - 1)^3}$$

$$\therefore L[f(3t)] = \frac{9(s^2 - 4s + 15)}{(s-3)^3}$$

Laplace Transform of derivatives :-

If $L[f(t)] = F(s)$ then

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$$

$$L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$$

$$L[f^{IV}(t)] = s^4 L[f(t)] - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

Laplace transform of integrals

Statement

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Ex If $L[f(t)] = \frac{9s^2-12s+15}{(s-1)^3}$ find $L[f(3t)]$ using change of scale property.

Sol Given $L[f(t)] = \frac{9s^2-12s+15}{(s-1)^3} = F(s)$

By change of scale property

$$L[f(3t)] = \frac{1}{3}F(s/3) = \frac{1}{3}\,\frac{9(s/3)^2-12(s/3)+15}{(s/3-1)^3}$$

$$\therefore L[f(3t)] = \frac{9(s^2-4s+15)}{(s-3)^3}$$

Laplace Transform of derivatives :-

If $L[f(t)] = F(s)$ then

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f''(t)] = s^2L[f(t)] - sf(0) - f'(0)$

$L[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0)$

$L[f^{iv}(t)] = s^4L[f(t)] - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$

Laplace transform of integrals

Statement

If $L[f(t)] = F(s)$ then $L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Ex Find $L\left[\int_0^t e^{-t}\cos t\,dt\right]$

Sol Let $f(t) = e^{-t}\cos t$

$L[f(t)] = L[e^{-t}\cos t]$

$= \frac{s+1}{(s+1)^2+1^2} = \frac{s+1}{s^2+2s+2}$ $\left(\because L[\cos t] = \frac{s}{s^2+1}\right)$

$\therefore L\left[\int_0^t e^{-t}\cos t\,dt\right] = \frac{F(s)}{s} = \frac{1}{s}\,\frac{s+1}{s^2+2s+2} = \frac{s+1}{s(s^2+2s+2)}$

Multiplication by t.

If $L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} L[f(t)]$

Ex Find $L[t e^{2t}\cos 2t]$

Sol $L[\cos 2t] = s/(s^2+4)$

$L[e^{2t}\cos 2t] = (s-2)/((s-2)^2+4) = (s-2)/(s^2-4s+8)$

$L[t e^{2t}\cos 2t] =$

$$L[t e^{2t}\cos 2t] = (-1)^1 \frac{d^1}{ds^1}\left(\frac{s-2}{s^2-4s+8}\right)$$

$$= -\left(\frac{1(s^2-4s+8)-(s-2)(2s-4)}{(s^2-4s+8)^2}\right)$$

$$= -\frac{\left(s^2-4s+8-(2s^2-4s-4s+8)\right)}{(s^2-4s+8)^2}$$

$$\therefore L[t e^{2t}\cos 2t] = -\frac{(-s^2+4s)}{(s^2-4s+8)^2} = \frac{s(s-4)}{(s^2-4s+8)^2}$$

Ex find $L[t^3 e^{2t}\sin t]$

Sol $L\{\sin t\} = \frac{1}{s^2+1}$

$$L[e^{2t}\sin t] = \frac{1}{(s-2)^2+1} = \frac{1}{s^2-4s+5}$$

$$L[t^3 e^{2t}\sin t] = (-1)^3\frac{d^3}{ds^3}\frac{1}{s^2-4s+5} = -\frac{d^2}{ds^2}\,\frac{-(2s-4)}{(s^2-4s+5)^2}$$

$$= 2\frac{d}{ds}\,\frac{1(s^2-4s+5)^2-(s-2)\cdot 2(s^2-4s+5)(2s-4)}{(s^2-4s+5)^4}$$

$$= 2\frac{d}{ds}\,\frac{(s^2-4s+5)\left[(s^2-4s+5)-4(s-2)^2\right]}{(s^2-4s+5)^4}$$

$$= 2\frac{d}{ds}\left(\frac{(s^2-4s+5)-4(s^2-4s+4)}{(s^2-4s+5)^3}\right)$$

$$= 2\frac{d}{ds}\left(\frac{-3s^2+12s-11}{(s^2-4s+5)^3}\right)$$

$$= 2\left(\frac{(-6s+12)(s^2-4s+5)^3-(-3s^2+12s-11)\cdot 3(s^2-4s+5)^2(2s-4)}{(s^2-4s+5)^6}\right)$$

$$= 2\,\frac{(s^2-4s+5)^2\left[-3(2s-4)(s^2-4s+5)-3(2s-4)(-3s^2+12s-11)\right]}{(s^2-4s+5)^6}$$

$$= 3\cdot 2(s^2-4s+5)^2(2s-4)\,\frac{\left[(s^2-4s+5)+(-3s^2+12s-11)\right]}{\ldots}$$

$$\therefore L[t^3 e^{2t}\sin t] = \frac{-(12s+24)(-2s^2+8s-6)}{(s^2-4s+5)^4}$$

$$\frac{12(s-2)(-2s^2+8s-6)}{(s^2-4s+5)^4}$$

~~$= -12(s-2)$ [illegible] $/(s^2-4s+5)^5$~~

Ex $L[t^2 e^{-2t}] = \frac{2}{(s+2)^3}$

Ex $L[t^2 \sin 2t] = \frac{4(3s^2-4)}{(s^2+4)^3}$

Ex Find $L\left[\int_0^t t e^{-t} \sin 2t \, dt\right]$

Sol $L[\sin 2t] = \frac{2}{s^2+2^2}$

$L[e^{-t}\sin 2t] = \frac{2}{(s+1)^2+2^2} = \frac{2}{s^2+2s+5}$

$L[t e^{-t} \sin 2t] = (-1)^1 \frac{d^1}{ds^1} L\left[\frac{2}{s^2+2s+5}\right]$

$= -\frac{-2}{(s^2+2s+5)^2}(2s+2)$

$L[t e^{-t}\sin 2t] = \frac{4(s+1)}{(s^2+2s+5)^2}$

Hence $L\left[\int_0^t t e^{-t}\sin 2t \, dt\right] = L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

$L\left[\int_0^t t e^{-t}\sin 2t \, dt\right] = \frac{1}{s}\frac{4(s+1)}{(s^2+2s+5)^2}$

Ex Division by t

If $L[f(t)] = F(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\,ds$

Sol Find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = L\left[\frac{f(t)}{t}\right]$

where $f(t) = \frac{e^{-at}-e^{-bt}}{t}$

$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[e^{-at}-e^{-bt}]\,ds$

$= \int_s^\infty L[e^{-at}] - L[e^{-bt}]\,ds$

$= \int_s^\infty \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$

$L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \left[\log\frac{(s+a)}{s+b}\right]_s^\infty = \log\left(\frac{s+b}{s+a}\right)$

Find $L\left(\frac{\cos at - \cos bt}{t}\right) = L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]\, ds$

where $f(t) = \cos at - \cos bt$

$= \int_s^\infty \left(L[\cos at] - L[\cos bt]\right) ds$

$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$

$= \frac{1}{2}\left(\log \frac{(s^2+a^2)}{s^2+b^2}\right)_s^\infty$

$L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log \frac{(s^2+b^2)}{(s^2+a^2)}$

Ex find $L\left[\int_0^t \frac{e^t \sin t}{t} dt\right]$

Sol $L[\sin t] = 1/s^2+1$

$L\left[\frac{\sin t}{t}\right] = \int_s^\infty L[f(t)]\, ds = \int_s^\infty L[\sin t]\, ds = \int_s^\infty \frac{1}{s^2+1}\, ds$

$= \left(\tan^{-1} s\right)_s^\infty = \tan^{-1}\infty - \tan^{-1} s = \pi/2 - \tan^{-1} s = \cot^{-1} s.$

$L\left[e^t \frac{\sin t}{t}\right] = \cot^{-1}(s-1)$

$L\left[\int_0^t e^t \frac{\sin t}{t} dt\right] = \frac{\cot^{-1}(s-1)}{s}$

Ex Find $L\left[\frac{1-\cos t}{t^2}\right]$

Sol $L\left[\frac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]\, ds$

$= \int_s^\infty L[1] - L[\cos t]\, ds$

$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$

$= \left(\log s - \frac{1}{2}\log(s^2+1)\right)_s^\infty$

$= \left(\log s - \log\sqrt{s^2+1}\right)_s^\infty$

$L\left[\frac{1-\cos t}{t}\right] = \left(\log \frac{s}{\sqrt{s^2+1}}\right)_s^\infty = \log \frac{\sqrt{s^2+1}}{s}$

$$L\left[\frac{1-\cos t}{t^2}\right] = L\left(\frac{1}{t}\frac{(1-\cos t)}{t}\right)$$

$$= \int_s^\infty L\left[\frac{1-\cos t}{t}\right] ds$$

$$= \int_s^\infty \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\log(s^2+1) - \log s^2\right) ds$$

$$= \frac{1}{2}\left[\left(s\log\frac{(s^2+1)}{s^2}\right)_s^\infty + \int_s^\infty \frac{1}{s^2+1} ds\right.$$

$$= \left(\frac{s}{2}\log(1+1/s^2)\right)_s^\infty + \left(\tan^{-1}s\right)_s^\infty$$

$$= \left(\tan^{-1}\infty - \tan^{-1}s\right) - \frac{s}{2}\log(1+1/s^2)$$

$$= \left(\pi/2 - \tan^{-1}s\right) - \frac{s}{2}\log(1+1/s^2)$$

$$L\left[\frac{1-\cos t}{t^2}\right] = \cot^{-1}s - s/2\log(1+1/s^2) \quad \left(\because \pi/2 - \tan^{-1}s = \cot^{-1}s\right)$$

① Use laplace transform of $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$

② Using Laplace transform, Evaluate $\int_0^\infty \frac{e^{-at}\sin^2 t}{t}dt$

put a = 1 in the above result
then we have

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4} \log \frac{5}{1} = \frac{1}{4} \log 5.$$

Ex) Evaluate $\int_0^\infty t e^{-t} \sin t \, dt = \int_0^\infty t e^{-t} \sin t \, dt = 1/2$

Ex) " $\int_0^\infty t e^{-4t} \sin 2t \, dt = 11/500$

Laplace transform of some special functions

(1) Unit step function (Heavisides unit function)

This function denoted by $u(t-a)$ (or) $H(t-a)$ and is defined as

$$H(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases}$$

Laplace transform of unit step function

P.T $L[H(t-a)] = L[u(t-a)] = \frac{e^{-as}}{s}$

Proof

LHS $L[H(t-a)] = \int_0^\infty e^{-st} H(t-a) dt$

$$= \int_0^a e^{-st} H(t-a) dt + \int_a^\infty e^{-st} H(t-a) dt$$

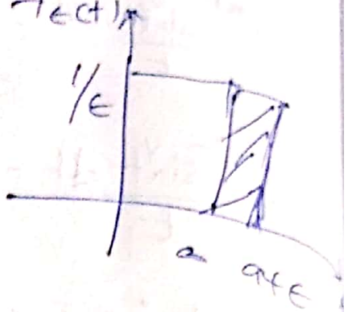
$$= 0 + \int_a^\infty e^{-st} \cdot 1 \, dt = \left(\frac{e^{-st}}{-s} \right)_a^\infty$$

$$L[H(t-a)] = 0 - \frac{e^{-as}}{-s} = \frac{e^{-as}}{s}$$

$$\therefore L(H(t-a)) = \frac{e^{-as}}{s}$$

Consider a function

$$f_\epsilon(t-a) = \begin{cases} 1/\epsilon & \text{for } a \le t \le a+\epsilon \\ 0 & ; \text{otherwise} \end{cases}$$



Laplace transform

$$L[f_\epsilon(t-a)] = \int_0^\infty e^{-st} f_\epsilon(t-a)\,dt$$

$$= \int_a^{a+\epsilon} e^{-st} \cdot \frac{1}{\epsilon}\,dt + 0$$

$$= \frac{1}{\epsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\epsilon}$$

$$= \frac{1}{\epsilon}\left(\frac{e^{-as} - e^{-as-s\epsilon}}{s}\right)$$

$$L[f_\epsilon(t-a)] = e^{-as}\frac{(1-e^{-s\epsilon})}{s\epsilon}$$

The limit of $f_\epsilon(t-a)$ as $\epsilon \to 0$ is denoted by $\delta(t-a)$ and is called the Dirac Delta function (or unit impulse function)

Laplace transform of Dirac Delta function

$$L[\delta(t-a)] = \lim_{\epsilon \to 0} L[f_\epsilon(t-a)] = \lim_{\epsilon \to 0} e^{-as}\frac{(1-e^{-s\epsilon})}{s\epsilon} = \left(\frac{0}{0}\right)$$

$$= e^{-as} \lim_{\epsilon \to 0} \frac{s e^{-s\epsilon}}{s} \quad \text{using L'Hospital rule}$$

$$\therefore L[\delta(t-a)] = e^{-as} \cdot 1 = e^{-as}$$

Hence Laplace transform of Dirac Delta function is e^{-as}.

Note $\int_0^\infty \delta(t-a)\,dt = 1$

Ex Find $L[e^{t-3}u(t-3)]$

Sol Since $L[u(t-3)] = \frac{e^{-as}}{s} = \frac{e^{-3s}}{s}$

$$\therefore \; L\left[e^{t-3}u(t-3)\right] = e^{-3}L\left[e^{t}u(t-3)\right]$$

$$= e^{-3} \cdot \frac{e^{-3(s-1)}}{(s-1)}$$

$$= \frac{e^{-3} \cdot e^{-3s} \cdot e^{3}}{(s-1)} = \frac{e^{-3s}}{(s-1)}$$

$$\therefore L[e^{t-3}u(t-3)] = e^{-3}L[e^{t}u(t-3)]$$

$$= e^{-3}\cdot \frac{e^{-3(s-1)}}{(s-1)}$$

$$= \frac{e^{-3}\cdot e^{-3s}\cdot e^{3}}{(s-1)} = \frac{e^{-3s}}{(s-1)}$$

Laplace transform of periodic functions

A function $f(t)$ is said to be periodic if and only if $f(t+T) = f(t)$ for some value of T and for every value of 't'. (where T is non-zero least +ve integer)

If $f(t)$ is a periodic function with period T

Then $L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$

Ex Find the Laplace transformation of the rectified semi-wave function defined by

$$f(t) = \sin\omega t, \quad 0 < t < \pi/\omega$$
$$= 0, \quad \pi/\omega < t < \frac{2\pi}{\omega}$$

Sol Since $f(t)$ is a periodic function with period $\frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st}f(t)\,dt + \int_{\pi/\omega}^{\frac{2\pi}{\omega}} e^{-st}f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\int_0^{\pi/\omega} e^{-st}\sin\omega t\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi}{\omega}s}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-\frac{\pi s}{\omega}}}{\omega^2+s^2}\left(-\sin\omega\cdot\frac{\pi}{\omega} - \omega\cos\frac{\pi}{\omega}\cdot\omega\right) - \frac{e^{0}}{s^2+\omega^2}(0-\omega\cdot 1)\right]$$

$$= \frac{\omega}{1-e^{-\frac{2\pi s}{\omega}}}\;\frac{1+e^{-\pi s/\omega}}{s^2+\omega^2} \quad \left(\because \begin{array}{l}\sin 0 = 0\\ \cos 0 = 1\end{array}\right)$$

Ex

If $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1; & 1 \le t < 2 \end{cases}$ is a periodic function

with period '2' find it's Laplace transform.

Inverse Laplace transforms

Definition

If $L[f(t)] = F(s)$

Then $f(t) = L^{-1}[F(s)]$

ie If $F(s)$ is the Laplace transform of a function $f(t)$,

Then $f(t)$ is called the inverse Laplace transform of $F(s)$.

Ex ① If $L[e^{at}] = \frac{1}{s-a}$

$\Rightarrow e^{at} = L^{-1}\left[\frac{1}{s-a}\right]$

② If $L[\cos at] = \frac{s}{s^2+a^2}$

$\Rightarrow \cos at = L^{-1}\left[\frac{s}{s^2+a^2}\right]$

properties of inverse Laplace transform

If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$

respectively. Then

$$L^{-1}[c_1F_1(s) + c_2F_2(s)] = c_1L^{-1}[F_1(s)] + c_2L^{-1}[F_2(s)]$$

where c_1 & c_2 are constants.

Ex find $L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = L^{-1}\left[\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 4L^{-1}\left[\frac{1}{s^3}\right]$

$= 1 - 3t + 4\frac{t^2}{2!}$

$\therefore L^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = 1 - 3t + 2t^2$

Ex Find $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$

Sol $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}L^{-1}\left[\frac{(s^2-2)^2}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4-4s^2+4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4}{s^5} - \frac{4s^2}{s^5} + \frac{4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{1}{s} - 4\frac{1}{s^3} + \frac{4}{s^5}\right]$

$= 3/2 \cdot 1 - 6 \cdot \frac{t^2}{2!} + \frac{3}{2} \cdot 4 \cdot \frac{t^4}{4!}$

$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2} - 3t^2 + \frac{1}{4}t^4$

First shifting theorem

F

Ex find $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right]$

Sol $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2}L^{-1}\left[\frac{(s^2-2)^2}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4-4s^2+4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{s^4}{s^5} - \frac{4s^2}{s^5} + \frac{4}{s^5}\right]$

$= \frac{3}{2}L^{-1}\left[\frac{1}{s} - 4\frac{1}{s^3} + \frac{4}{s^5}\right]$

$= 3/2 \cdot 1 - 6 \cdot \frac{t^2}{2!} + \frac{3}{2} \cdot 4 \cdot \frac{t^4}{4!}$

$L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = \frac{3}{2} - 3t^2 + \frac{1}{4}t^4$

First shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}(F(s-a)) = e^{at}f(t)$

Ex find $L^{-1}\left[\frac{3s-2}{s^2-4s+20}\right] = L^{-1}\left[\frac{3(s-2)+4}{(s-2)^2+4^2}\right]$

$= 3L^{-1}\left[\frac{(s-2)}{(s-2)^2+4^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+4^2}\right]$

$= 3e^{2t}L^{-1}\left[\frac{s}{s^2+4^2}\right] + 4L^{-1}\left[\frac{1}{s^2+4^2}\right]e^{2t}$

$= 3e^{2t}\cdot\cos 4t + \frac{4}{4}L^{-1}\left[\frac{4}{s^2+4^2}\right]e^{2t}$

$\therefore L^{-1}\left[\frac{3s-2}{s^2-4s+20}\right] = 3e^{2t}\cos 4t + e^{2t}\sin 4t.$

Ex find $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{(s+3)-3}{(s+3)^2}\right]$

$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2}\right]$

$= e^{-3t}\left[L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{3}{s^2}\right)\right]$

$L^{-1}\left[\frac{s}{(s+3)^2}\right] = e^{-3t}(1-3t)$

Ex ① find $L^{-1}\left[\frac{1}{s^2+2s+5}\right]$ ④ find $L^{-1}\left[\frac{s+3}{s^2-10s+29}\right]$

② find $L^{-1}\left[\frac{2s+12}{s^2+6s+13}\right]$

③ find $L^{-1}\left[\frac{1}{(s+1)^3}\right]$

Use of partial fractions to find inverse Laplace transform

Ex find $L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = L^{-1}[F(s)]$

Sol where $F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{Cs+D}{s^2+4}$

$\Rightarrow \frac{1}{(s+1)^2(s^2+4)} = \frac{A(s+1)(s^2+4)+B(s^2+4)+(Cs+D)(s+1)^2}{(s+1)^2(s^2+4)}$

$\Rightarrow 1 = A(s+1)(s^2+4)+B(s^2+4)+(Cs+D)(s+1)^2$

put $s=-1$

$1 = 0 + B(1+4) + 0$

$\Rightarrow 5B = 1 \Rightarrow B = 1/5$

Comparing s^3 coeff. on both sides

$0 = A + C \Rightarrow A = -C$ — ①

Comparing s^2 coeff on both sides

$A + B + 2C + D = 0$ — ②

Comparing s coeff on both sides

$4A + C + 2D = 0$ — ③

Comparing s^0 i.e constants on both sides

$4A + 4B + D = 1$ — ④

sub eq ① in ② we get

$-C + B + 2C + D = 0 \Rightarrow B + D + C = 0$

Since $4A + 4B + D = 1$

$-4C + 4B + D = 1$

$B + D + C = 0$

$4B + D - 4C = 1$

$(-)\ (-)\ (+)\ (-)$

$-3B + 5C = -1$

$-\frac{3}{5} + 5C = -1$

$\Rightarrow 5C = -1 + \frac{3}{5} = \frac{-2}{5}$

$\Rightarrow C = -2/25$

Since $A = -C = -(-2/25) = \frac{2}{25}$.

Since $4A + C + 2D = 0$

$4 \cdot \frac{2}{25} - \frac{2}{25} + 2D = 0$

$\Rightarrow +\frac{8}{25} - \frac{2}{25} + 2D = 0 \Rightarrow 2D = -6/25 \Rightarrow D = \frac{-3}{25}$

$\Rightarrow 2D = -\frac{6}{25} = \frac{-3}{25} \Rightarrow D = \frac{-3}{25} = \frac{-3}{25}$

$\therefore A = 2/25,\ B = 1/5,\ C = -2/25,\ D = -3/5.$

$$\therefore F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{\frac{2}{25}}{s+1} + \frac{\frac{1}{5}}{(s+1)^2} + \frac{-\frac{2}{25}s - \frac{3}{25}}{s^2+4}$$

Taking L^{-1} Operator on both sides we get—

$$L^{-1}[F(s)] = L^{-1}\left(\frac{\frac{2}{25}}{s+1} + \frac{\frac{1}{5}}{(s+1)^2} + \frac{\frac{1}{25}(2s - 3)}{s^2+4}\right)$$

$$= \frac{2}{25}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{5}L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{2}{25}L^{-1}\left[\frac{s}{s^2+4}\right] - \frac{3}{25}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = e^{-t}\cdot\frac{2}{25} + \frac{e^{-t}t}{5} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t$$

$$L^{-1}\left[\frac{1}{(s+1)^2(s^2+4)}\right] = \frac{e^{-t}(2+5t)}{25} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t.$$

Ex $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+25)}\right] = \frac{1}{21}(5\sin 5t - 2\sin 2t]$

Ex $L^{-1}\left[\frac{s}{(s^2+1)(s^2+9)(s^2+25)}\right] = \frac{1}{3092}(16\cos t - 24\cos 3t + 8\cos 5t)$

Ex $L^{-1}\left[\frac{2s^2 - 6s + 5}{\ldots 11s - 6}\right]$

Ex Find $L^{-1}\left[\frac{2s+3}{s^3-6s^2+11s-6}\right] = \frac{5}{2}e^{t} - 7e^{2t} + \frac{9}{2}e^{3t}$

Ex Find $L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sin at \sinh at$

Sol

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = L^{-1}\left[\frac{s}{(s^2)^2+(2a^2)^2}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{1}{4a}\left[L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right]$$

$$= \frac{1}{4a}\left[e^{at}\frac{1}{a}\sin at - e^{-at}\frac{1}{a}\sin at\right]$$

$$= \frac{1}{4a^2}\left(e^{at}-e^{-at}\right)\sin at$$

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right] = \frac{1}{2a^2}\sinh at \sin at$$

Second shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[e^{-as}F(s)] = G(t)$

where $G(t) = \begin{cases} f(t-a); & t>a \\ 0; & t<a \end{cases}$

Ex Find $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right]$

Sol $L^{-1}\left[\frac{1}{s^2+4s+5}\right] = L^{-1}\left[\frac{1}{(s+2)^2+1}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$

$\therefore L^{-1}\left[\frac{1}{s^2+4s+5}\right] = e^{-2t}\sin t = f(t)$ say

By second shifting theorem

$$L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = \begin{cases} e^{-2(t-2)}\sin(t-2); & t>2 \\ 0; & t<2 \end{cases}$$

(or) $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = e^{-2(t-2)}\sin(t-2)H(t-2)$

where $H(t-2)$ is the Heaviside's unit step function.

Ex Find inverse Laplace transform of $\frac{e^{-\pi(s+2)}}{s+2}$

Ans $e^{-2t} L^{-1}\left[\frac{e^{-\pi s}}{s}\right]$ (using FST)

$= e^{-2t} u(t-\pi)$ $\left(L^{-1}\left[\frac{1}{s}\right] = 1 \right]$ unit step function)

Ex Find $L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] = \sin t - \sin t\, H(t-\pi)$ $(\because \sin(t-\pi) = -\sin t)$

Change of scale property

If $L[f(t)] = F(s)$ then $L^{-1}[F(as)] = \frac{1}{a} f(t/a)$

Inverse Laplace transform of derivatives

Theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[F^{(n)}(s)] = (-1)^n t^n f(t)$

where $F^{(n)}(s) = \frac{d^n}{ds^n} F(s)$

Ex Find $L^{-1}\left[\log\frac{(s+1)}{s-1}\right]$

Sol Given $L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = L^{-1}[F(s)]$

where $F(s) = \log\frac{(s+1)}{(s-1)} = \log(s+1) - \log(s-1)$

$F'(s) = \frac{1}{s+1} - \frac{1}{s-1}$

Taking L^{-1} on both sides

$L^{-1}[F'(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$(-1)^1 t^1 f(t) = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s-1}\right]$

$\Rightarrow -t L^{-1}[F(s)] = e^{-t} - e^{t}$

$\Rightarrow L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = \frac{e^{t}-e^{-t}}{t} = 2\frac{e^{t}-e^{-t}}{2t} = \frac{2}{t}\sinh t.$

$\therefore L^{-1}\left[\log\frac{(s+1)}{(s-1)}\right] = \frac{2}{t}\sinh t.$

Ex $\log(1+1/s^2)$ Find inverse Laplace transform of

Ex $L^{-1}\left[\cot^{-1}\left(\frac{s+2}{3}\right)\right]$ Ex: $L^{-1}[\cot^{-1}s]$, $L^{-1}\left[\frac{s}{(s^2-25)^2}\right]$

$L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]$

Ex Find $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

Sol

$$L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = L^{-1}\left[F(s)\right]$$

where $F(s) = \frac{1}{(s^2+4)}$

$\Rightarrow$ Diff. w.r.t 's'

$$F'(s) = \frac{-1}{(s^2+4)^2}\,2s = \frac{-2s}{(s^2+4)^2}$$

$$\Rightarrow F'(s) = \frac{-2s}{(s^2+4)^2}$$

Taking L^{-1} on both sides

$$L^{-1}[F'(s)] = L^{-1}\left[\frac{-2s}{(s^2+4)^2}\right]$$

$$(-1)^1 t^1 f(t) = -2\,L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\Rightarrow \frac{t}{2} f(t) = L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$$

$$\Rightarrow L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{2} f(t) = \frac{t}{2} L^{-1}[F(s)] = \frac{t}{2} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{t}{4}\sin 2t.$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{t}{4}\sin 2t.$$

Ex Evaluate $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$

Inverse laplace transform of integrals

Theorem If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

Ex Find $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$

Sol

$$L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = L^{-1}[F(s)]$$

where $F(s) = \frac{s+1}{((s+1)^2+1)^2}$

Taking L^{-1} on both sides

$$L^{-1}[F(s)] = L^{-1}\left[\frac{s+1}{((s+1)^2+1)^2}\right] = e^{-t} L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = e^{-t}\cdot\frac{t\sin t}{2\cdot 1}$$

Ex $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \frac{t}{4}e^{-3t}\sin 2t$

$L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = \frac{t}{2}e^{-t}\sin t.$

Multiplication by powers of s

Theorem

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$ then $L^{-1}[sF(s)] = f'(t)$

Ex Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

Sol $L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2}\right] = L^{-1}[s \cdot F(s)] = f'(t)$

where $F(s) = \frac{1}{(s+2)^2} \Rightarrow L^{-1}[F(s)] = f(t) = L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$$= \frac{d}{dt}\left(e^{-2t} \cdot t\right) \quad \left(\because L^{-1}\left[\frac{1}{s^2}\right] = t\right)$$

$$= 1 \cdot e^{-2t} - 2e^{-2t} \cdot t$$

$$\therefore L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2t}(1-2t)$$

Find

Ex $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}(\sin at + at\cos at)$

Division by s

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

Ex Find $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]$

Sol $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$

$$= \int_0^t \frac{1}{a}\sin at\,dt$$

$$= \frac{-1}{a^2}(\cos at)_0^t$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s^2+a^2)}\right] = \int_0^t \frac{1}{a^2}(1-\cos at)\,dt$$

$$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t = \frac{1}{a^2}\left(t - \frac{\sin at}{a}\right)$$

Convolution

Definition

Let $f(t)$ and $g(t)$ be two functions defined for $t > 0$

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

Convolution Theorem

Statement

If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$

Then $L[f(t) \times g(t)] = F(s)G(s)$

(or) $f(t) * g(t) = L^{-1}[F(s)G(s)]$

ie Convolution of two functions $f(t)$ and $g(t)$ is the inverse Laplace transform of their product.

Ex find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \dfrac{a\sin at - b\sin bt}{a^2-b^2}$

Ex $L^{-1}\left[\dfrac{s^2}{(s^2+4)(s^2+9)}\right] = \dfrac{1}{5}\left[2\sin 2t - 3\sin 3t\right]$

Ex Solve $\dfrac{dy}{dt} + 3y + 2\int_0^t y(t)\,dt = t$ by using Laplace transform method

Ex find $L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right]$

Ex Find $L^{-1}\left[\dfrac{1}{(s+2)^2(s^2+4)}\right]$

Laplace transform of integrals

Theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t}$

Ex Find $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right]$. $L^{-1}\left[\int_s^\infty \frac{1}{s}\left(\frac{s}{(s^2+1)}\right)ds\right]$

sol $L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = L^{-1}\left[\frac{s+1}{((s+1)^2+1)^2}\right]$

$= e^{-t} L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$ ($\because$ F.S.T)

$L^{-1}\left[\frac{s+1}{(s^2+2s+2)^2}\right] = e^{-t}\frac{t}{2}\sin t$

$\left(\because L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right) = \frac{t}{2a}\sin at\right)$.

$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{t}{2 \cdot 1}\sin t$

multiplication by powers of s

Theorem

If $L^{-1}[F(s)] = f(t)$ and $f(0)=0$

Then $L^{-1}[s F(s)] = f'(t) = \frac{d}{dt} f(t)$

$L[f'(t)] = sL[f(t)] - f(0)$

$L[f'(t)] = s F(s)$

$f'(t) = L^{-1}[s F(s)]$

Ex $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

sol $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[s \cdot \frac{s}{(s^2+a^2)^2}\right]$

$= L^{-1}[s\, F(s)]$

$= f'(t) = \frac{d}{dt} f(t)$

$= \frac{d}{dt} L^{-1}[F(s)]$

$= \frac{d}{dt}\left(L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]\right)$

$= \frac{d}{dt}\left(\frac{t}{2a}\sin at\right)$

$L^{-1}\left(\frac{s^2}{(s^2+a^2)^2}\right) = \frac{1}{2a}[1.\sin at + ta\cos at]$

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Division by s

Statement :- If $L^{-1}[F(s)] = f(t)$

Then $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(t)\,dt$

$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

Ex $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = L^{-1}\left[\frac{\frac{1}{(s^2+a^2)}}{s}\right]$ $\quad \int_0^t f(t)\,dt = L^{-1}\left[\frac{F(s)}{s}\right]$

$= L^{-1}\left[\frac{F(s)}{s}\right]$

$= \int_0^t L^{-1}[F(s)]\,dt \quad (f(t) = L^{-1}[F(s)])$

$= \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right]dt$

$= \int_0^t \frac{1}{a} L^{-1}\left[\frac{a}{s^2+a^2}\right] dt$

$= \frac{1}{a}\int_0^t \sin at\,dt$

$= \frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t$

$= \frac{-1}{a^2}(\cos at - \cos 0)$

$= \frac{+1}{a^2}(-\cos at + 1)$

$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}(1-\cos at)$

Ex $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = L^{-1}\left[\frac{\frac{1}{s(s^2+a^2)}}{s}\right] = L^{-1}\left[\frac{F(s)}{s}\right]$

$= \int_0^t L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]dt$

$= \int_0^t \frac{1}{a^2}(1-\cos at)\,dt$

$= \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]_0^t$

$L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right] = \frac{1}{a^2}\left[t - \frac{\sin at}{a}\right]$

Second shifting theorem

If $L^{-1}[F(s)] = f(t)$ then $L^{-1}[e^{-as}F(s)] = G(t)$

where $G(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$

Ex Find $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right]$

Sol $L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = L^{-1}\left[e^{-2s} \cdot \frac{1}{s^2+4s+5}\right]$

$= L^{-1}\left[e^{-as} F(s)\right]$, $a = 2$

$f(t) = L^{-1}[F(s)]$

$= L^{-1}\left[\frac{1}{s^2+4s+5}\right]$

$= L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{s^2+1}\right]$

$= e^{-2t}\sin t$

$= f(t)$

By second shifting th

$L^{-1}\left[\frac{e^{-2s}}{s^2+4s+5}\right] = G(t) = \begin{cases} e^{-2(t-2)} \sin(t-2) & ; t > 2 \\ 0 & ; t < 2 \end{cases}$

Ex Find $L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right]$

Sol. $L^{-1}\left[\frac{1+e^{-\pi s}}{s^2+1}\right] = L^{-1}[F(s)]$

$= L^{-1}\left[\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\right]$

$= L^{-1}\left[\frac{1}{s^2+1}\right] + L^{-1}\left[\frac{e^{-\pi s}}{s^2+1}\right]$

$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s^2+1}\right]$

$a = \pi$

$f(t) = \sin t$

$= \sin t + \begin{cases} \sin(t-\pi) & ; t > \pi \\ 0 & ; t < \pi \end{cases}$

$\sin(-\theta) = -\sin\theta$

$\sin(t-\pi) = \sin-(\pi-t)$

$= -\sin(\pi-t)$

$= -\sin t$

$\sin(t-\pi) = -\sin t$

$= \sin t + (-\sin t) H(t-\pi)$

$= \sin t - \sin t \, H(t-\pi)$

$L^{-1}\left[\frac{1+e^{-\pi s}}{\ldots}\right] = \sin t - \sin t \, H(t-\pi)$

Convolution theorem

Def Let $f(t)$, $g(t)$ be two functions defined, $t>0$

$$f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Convolution theorem

St If $L\{f(t)\} = F(s)$, $L[g(t)] = G(s)$

Then $L\{f(t) * g(t)\} = F(s)G(s)$

i.e. convolution of Laplace transform of $f(t)$ & $g(t)$ is the product of their Laplace transforms

$$L[f(t) * g(t)] = F(s)G(s)$$

$$\Rightarrow \boxed{f(t) * g(t) = L^{-1}[F(s)G(s)]}$$

Ex Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ by using convolution theorem.

Sol

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}[F(s)G(s)]$$

By convolution theorem

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

By using convolution Def

$$= \int_0^t f(u)\,g(t-u)\,du$$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos \underset{A}{au}\,\cos \underset{B}{b(t-u)}\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu) + \cos(au-bt+bu)]\,du$$

$(\because \cos(A+B)+\cos(A-B) = 2\cos A\cos B)$

$$= \frac{1}{2}\left[\frac{\sin((a-b)u+bt)}{a-b} + \frac{\sin((a+b)u-bt)}{}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)t + bt)}{a-b} + \frac{\sin((a+b)t - bt)}{a+b} - \left(\frac{\sin(bt)}{a-b} + \frac{\sin(-bt)}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b}\right) - \left(\frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{a+b+a-b}{a^2-b^2}\right)\sin at - \left(\frac{a+b-(a-b)}{a^2-b^2}\right)\sin bt\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at}{a^2-b^2} - \frac{2b\sin bt}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \left(\frac{a\sin at - b\sin bt}{a^2-b^2}\right)$$

Ex $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right] = \frac{2\sin 2t - 3\sin 3t}{4-9} = \frac{3\sin 3t - 2\sin 2t}{5}$

($4 = 2^2$, $9 = 3^2$)

Ex Find $L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$ by using convolution th

sol

$$L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = L^{-1}\left[\frac{s}{(s^2+1)}\cdot\frac{1}{(s^2+1)}\right]$$

$$= L^{-1}\left[F(s)G(s)\right] = f(t) * g(t)$$

$$= L^{-1}\left[\frac{s}{s^2+1}\right] * L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \cos t * \sin t = f(t) * g(t)$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t \cos u \sin(t-u)\, du$$

$$= \int_0^t \cos u \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos u \sin(t-u)\,du$$

$(\because \sin(A+B) - \sin(A-B) = 2\cos A \sin B)$

$$= \frac{1}{2}\int_0^t \left[\sin(u+t-u) - \sin(u-(t-u))\right] du$$

$$= \frac{1}{2}\int_0^t \left[\sin t - \sin(2u-t)\right] du$$

$$= \frac{1}{2}\left[\sin t\,(u)_{u=0}^{t} + \left(\frac{\cos(2u-t)}{2}\right)_{u=0}^{t}\right]$$

$$= \frac{1}{2}\left[\sin t\,(t-0) + \frac{\cos t}{2} - \frac{\cos(-t)}{2}\right]$$

$$= \frac{1}{2}\left[t\sin t + \frac{\cos t}{2} - \frac{\cos t}{2}\right]$$

$$\Rightarrow \frac{t}{2}\sin t = L^{-1}\left[\frac{s}{(s^2+1)^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

$$L^{-1}\left[\frac{s}{(s^2+5)^2}\right] = \frac{t}{2\cdot 5}\sin 5t$$

Ex Solve $\frac{dy}{dt} + 3y + 2\int_0^t y(t)\,dt = t$, using Laplace transform.

Sol Given

$$y'(t) + 3y(t) + 2\int_0^t y(t)\,dt = t$$

Taking Laplace transform on both sides

$$L\left[y'(t) + 3y(t) + 2\int_0^t y(t)\,dt\right] = L[t]$$

$$\Rightarrow L[y'(t)] + 3L[y(t)] + 2L\left[\int_0^t y(t)\,dt\right] = L[t]$$

$$\Rightarrow L[y(t)] = \frac{1}{s(s^2+3s+2)} + \frac{as}{s^2+3s+2}$$

Taking inverse Laplace transform

$$y(t) = L^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] + aL^{-1}\left[\frac{s}{s^2+3s+2}\right] \quad (1)$$

(A) (B)

Consider $L^{-1}\left[\frac{1}{s(s^2+3s+2)}\right] = L^{-1}[F(s)]$

$$\therefore F(s) = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+2)} = \frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1)$$

put $s=0$

$$1 = 2A \Rightarrow A = 1/2$$

put $s=-1$

$$1 = B(-1)1 \Rightarrow -B = 1 \Rightarrow B = -1$$

put $s=-2$

$$1 = C(-2)(-1) \Rightarrow 2C = 1 \Rightarrow C = 1/2$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} \quad (A)$$

Consider $L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}[F(s)]$

$$\therefore F(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$s = A(s+2) + B(s+1)$$

put $s=-1$

$$-1 = A \Rightarrow A = -1$$

put $s=-2$

$$-2 = -B \Rightarrow B = 2$$

Taking L^{-1} on both side

$$L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s+2}\right]$$

$$= \ldots -2t \quad (B)$$

put $s=-1$

$1 = B(-1)1 \Rightarrow -B=1 \Rightarrow B=-1$

put $s=-2$

$1 = C(-2)(-1) \Rightarrow 2C=1 \Rightarrow C=\frac{1}{2}$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{2}}{s} - \frac{1}{s+1} + \frac{\frac{1}{2}}{s+2}\right]$$

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} \quad \text{— (A)}$$

Consider

$$L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}[F(s)]$$

$$\therefore F(s) = \frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$s = A(s+2) + B(s+1)$$

put $s=-1$

$-1 = A \Rightarrow A=-1$

put $s=-2$

$-2 = -B \Rightarrow B=2$

Taking L^{-1} on both side

$$L^{-1}\left[\frac{s}{s^2+3s+2}\right] = L^{-1}\left[\frac{-1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s+2}\right]$$

$$= -e^{-t} + 2e^{-2t} \quad \text{— (B)}$$

Sub (A) & (B) in eq (1)

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} + a\left(2e^{-2t} - e^{-t}\right)$$

Application of Laplace transform to solve ordinary diff. equations

Procedure

(1) Take the Laplace transform on both sides

(2) Use formulas, apply initial conditions

(3) Rearrangement of terms

(4) Take L^{-1} operator on both sides and get the sol of O.D.E

Ex: Using Laplace transform, solve the DE

$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, Given that $y(0)=0$, $y'(0)=1$

Sol: $y''(t) + 2y'(t) + 5y(t) = e^{-t}\sin t$

Step-1 $L[y'' + 2y' + 5y] = L[e^{-t}\sin t]$

$L[y''(t)] + 2L[y'(t)] + 5L[y(t)] = L[e^{-t}\sin t]$

Step-2 $(s^2 L[y(t)] - sy(0) - y'(0)) + 2[sL(y(t)] - y(0)] + 5L(y(t)] = \frac{1}{(s+1)^2+1}$

Given $y(0)=0$, $y'(0)=1$

$(s^2 L[y(t)] - s.0 - 1) + 2(sL(y(t)] - 0) + 5L[y(t)] = \frac{1}{s^2+2s+2}$

$L[y(t)][s^2+2s+5] - 1 = \frac{1}{s^2+2s+2}$

$\Rightarrow L[y(t)](s^2+2s+5) = \frac{1}{s^2+2s+2} + 1$

$\Rightarrow L[y(t)] = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$ — (1)

Consider

$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$

$\Rightarrow 1 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$

C.C of s^3 on both sides

$A + C = 0$ — (2)

C.C of s^2 on both sides

$2A + B + 2C + D = 0$ — (3)

C.C of s on both sides

$5A + 2B + 2C + 2D = 0$ — (4)

$5B+2D=1$ (5)

Application of Laplace transform to solve ordinary diff. equations

Workrule

1. Take the Laplace transform on both sides
2. Use formulas, apply initial conditions
3. Rearrangement of terms.
4. Take L^{-1} operator on both sides and get the sol of O.D.E

Ex Using Laplace transform, solve the DE

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t,$$ Given that $y(0)=0$, $y'(0)=1$

Sol $y''(t) + 2y'(t) + 5y(t) = e^{-t}\sin t$

Step-1 $L[y'' + 2y' + 5y] = L[e^{-t}\sin t]$

$$L[y''(t)] + 2L[y'(t)] + 5L[y(t)] = L[e^{-t}\sin t]$$

Step-2 $(s^2L[y(t)] - sy(0) - y'(0)) + 2[sL(y(t)] - y(0)]$

$$+ 5L[y(t)] = \frac{1}{(s+1)^2+1}$$

Given $y(0)=0$, $y'(0)=1$

$$(s^2L[y(t)] - s\cdot 0 - 1) + 2(sL[y(t)] - 0)$$

$$+ 5L[y(t)] = \frac{1}{s^2+2s+2}$$

$$L[y(t)][s^2+2s+5] - 1 = \frac{1}{s^2+2s+2}$$

$$\Rightarrow L[y(t)](s^2+2s+5) = \frac{1}{s^2+2s+2} + 1$$

$$\Rightarrow L[y(t)] = \frac{1}{(s^2+2s+2)(s^2+2s+5)} + \frac{1}{s^2+2s+5} \quad \text{--- (1)}$$

Consider

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\Rightarrow 1 = (As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)$$

C.C of s^3 on both sides

$$A + C = 0 \quad \text{--- (2)}$$

C.C of s^2 on both sides

$$2A + B + 2C + D = 0 \quad \text{--- (3)}$$

C.C of s on both sides

$$5A + 2B + 2C + 2D = 0 \quad \text{(4)}$$

$5B + 2D = 1$ (5)

$A+C=0$ — (2)

$2A+B+2C+D=0$ — (3)

$5A+2B+2C+2D=0$ — (4)

$5B+2D=1$ — (5)

From (2) $A+C=0 \Rightarrow A=-C$

Sub in (3)

$2A+B-2A+D=0$

$\Rightarrow B+D=0$ — (6)

Solving (5) & (6)

$5B+2D=1$

$2B+2D=0$

$(-) \quad (-) \quad (-)$

$3B=1$

$\Rightarrow \boxed{B=1/3}$

Since (6) eq.

$B+D=0$

$\Rightarrow D+1/3=0 \Rightarrow \boxed{D=-1/3}$

From (4)

$5A+2B+2C+2D=0$

$5A+2/3-2A+2/3=0$

$5A - 2/3 - 2A$

$\Rightarrow 3A=0 \Rightarrow A=0$

From (2) $A=-C \Rightarrow C=-A=0$

$\therefore \boxed{A=C=0}$, $B=+1/3$, $D=-1/3$

$\therefore A=C=0$, $B=+1/3$, $D=-1/3$

Sub. all these in eq (1) we get

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$\frac{1}{(s^2+2s+2)(s^2+2s+5)} = \frac{+1/3}{s^2+2s+2} - \frac{\frac{1}{3}}{s^2+2s+5}$$

$$L[y(t)] = \frac{+\frac{1}{3}}{s^2+2s+2} + \frac{-\frac{1}{3}}{s^2+2s+5} + \frac{1}{s^2+2s+5}$$

$+1/3$, 2 1

$$y(t) = L^{-1}\left[\frac{-1}{3} \frac{1}{s^2+2s+2} + \frac{2}{3} \frac{1}{s^2+2s+5} \right]$$

$$= \frac{-1}{3} L^{-1}\left[\frac{1}{s^2+2s+2} \right] + \frac{2}{3} L^{-1}\left[\frac{1}{s^2+2s+5} \right] \text{(by lp)}$$

$$= \frac{-1}{3} L^{-1}\left[\frac{1}{(s+1)^2+1} \right] + \frac{2}{3} L^{-1}\left[\frac{1}{(s+1)^2+2^2} \right]$$

$$= \frac{-1}{3} e^{-t} L^{-1}\left[\frac{1}{s^2+1} \right] + \frac{2}{3}\cdot\frac{1}{2} e^{-t} L^{-1}\left[\frac{2(1)}{s^2+(2)^2} \right]$$

$$= \frac{-1}{3} e^{-t} \sin t + \frac{1}{3} e^{-t} \sin 2t$$

$$\therefore y(t) = \frac{e^{-t}}{3} (\sin 2t - \sin t).$$

1. solve
   $y'' - 8y' + 15y = 9te^{2t}$, $y(0) = 5$, and $y'(0) = 10$ using laplace transform
2. solve $(D^2 + 1)x = t\cos 2t$ given $x = 0$, $\frac{dx}{dt} = 0$ at $t = 0$
3. solve $\frac{d^2x}{dt^2} + 9x = \sin t$ using L.T, Given that $x(0) = 1$, $x(\pi/2) = 1$
4. solve $y'' = t\cos 2t$, $y(0) = 0$ and $y'(0) = 0$
5. solve $(D^2 - 2D + 2)x = 0$ given that $x = Dx = 1$ at $t = 0$
6. solve $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0) = 1$, $y'(0) = 1$

Solutions

① $y(t) = 4e^{2t} + 3te^{2t} + 3e^{3t} - 2e^{5t}$

② $x(t) = \frac{4}{9}\sin 2t - \frac{5}{9}\sin t - \frac{t\cos 2t}{3}$

③ $x(t) = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right) + \cos 3t - \frac{5}{6}\sin 3t$

④ $y(t) = \frac{1}{4}\left(\sin 2t - t - t\cos 2t\right)$

⑤ $x = e^{t}\cos t$

⑥ $y = 3 + 2t - \frac{5}{2}e^{t} + \frac{1}{2}e^{3t}$.

Convolution theorem

① Using Convolution theorem find $L^{-1}\left[\frac{1}{(s^2+9)(s+1)^2}\right]$

Ans $-\frac{1}{50}\left[\cos 3t + \frac{4}{3}\sin 3t\right] + \frac{e^{-t}}{50} + \frac{te^{-t}}{10}$

② Find $L^{-1}\left[\frac{1}{(s+2)^2(s^2+4)}\right]$

Ans

③ Solve integro differential equation

$$y(t) = 1 - e^{-t} + \int_0^t y(t-u)\sin u\,du$$

Ans $1 - e^{t} + L^{-1}\left[\frac{1}{s^3(s+1)}\right]$

④ Find $L^{-1}\left[\frac{1}{(s^2+4)(s^2+25)}\right] = \frac{1}{21}\left[5\sin 5t - 2\sin 2t\right]$

⑤ Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Ans $\frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$

⑥ $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$

Ans $t(e^{-t}+1) + 2(e^{-t}-1)$