Vector Differentiation

Definition

The vector differential operator ∇ (read as del)

is defined as

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Gradient

Let $\phi(x,y,z)$ be a scalar point function of position

defined then the vector function $\frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$

is known as the gradient of ϕ and is denoted by grad ϕ (or) $\nabla\phi$

$$\therefore \nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

Example

prove that $\nabla r^n = n.r^{n-2}\vec{r}$

Sol Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

and let $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

Diff. w.r.t 'x' partially.

$$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\text{lly} \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

LHS $\nabla r^n = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) r^n$

$$= \sum \frac{\partial}{\partial x} r^n \vec{i}$$

$$= \sum \vec{i}\, n.r^{n-1}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}.n.r^{n-1}\frac{x}{r} \quad \left(\because \frac{\partial r}{\partial x} = x/r\right)$$

=

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$= \sum \vec{i}\, n r^{n-2} \cdot x$

$= n \cdot r^{n-2} \sum x \vec{i}$

$= n \cdot r^{n-2} (x\vec{i} + y\vec{j} + z\vec{k})$

$= n \cdot r^{n-2} \vec{r}$

$=$ RHS

$\therefore$ LHS $=$ RHS

ie $\nabla r^n = n \cdot r^{n-2} \vec{r}$

Ex prove that $\nabla f(r) = \frac{f'(r)}{r} \vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Ex prove that $\nabla \log r = \frac{1}{r^2} \vec{r}$

Ex Find a unit normal vector to the surface $x^2 + y^2 + 2z^2 = 26$ at the point $(2, 2, 3)$

Sol Let the given surface be

$f(x,y,z) = x^2 + y^2 + 2z^2 - 26 = 0$

$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 4z$

$\therefore$ gradϕ (or) grad$f$ $= \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\nabla f = 2x\vec{i} + 2y\vec{j} + 4z\vec{k}$

$(\nabla f)_{at(2,2,3)} = 4\vec{i} + 4\vec{j} + 12\vec{k}$

$\therefore$ unit normal vector $= \frac{\nabla f}{|\nabla f|} = \frac{4\vec{i} + 4\vec{j} + 12\vec{k}}{\sqrt{(4)^2 + (4)^2 + (12)^2}}$

$= \frac{4\vec{i} + 4\vec{j} + 12\vec{k}}{\sqrt{176}}$

$= \frac{4\vec{i} + 4\vec{j} + 12\vec{k}}{\sqrt{16 \times 11}}$

$\therefore$ unit normal vector $= \frac{\vec{i} + \vec{j} + 3\vec{k}}{\sqrt{11}}$

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Ex

Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$

Sol

Let $\phi_1(x,y,z) = x^2+y^2+z^2-9$

and $\phi_2(x,y,z) = x^2+y^2-3-z$ be the two given surfaces

Then $\nabla\phi_1 = \frac{\partial\phi_1}{\partial x}\vec{i} + \frac{\partial\phi_1}{\partial y}\vec{j} + \frac{\partial\phi_1}{\partial z}\vec{k}$ | $\nabla\phi_2 = 2x\vec{i}+2y\vec{j}-\vec{k}$

$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$

$\vec{n_1} = (\nabla\phi_1)_{at\,(2,-1,2)} = 4\vec{i}-2\vec{j}+4\vec{k}$

$\vec{n_2} = (\nabla\phi_2)_{at\,(2,-1,2)} = 4\vec{i}-2\vec{j}-\vec{k}$

Since the angle between the two surfaces is the angle between their normals.

$$\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$$

$$= \frac{(4\vec{i}+4\vec{k}-2\vec{j})\cdot(4\vec{i}-2\vec{j}-\vec{k})}{\sqrt{16+16+4}\,\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Directional derivative

The directional derivative of ϕ at a point p in the direction of a vector $\vec{a}$ is given by

$$D.D = (\text{grad}\,\phi)_{at\,p}\cdot\frac{\vec{a}}{|\vec{a}|}$$

Ex Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point $P=(1,2,3)$ in the direction of the line PQ where $Q=(5,0,4)$

Sol $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= (5-1)\vec{i} + (0-2)\vec{j} + (4-3)\vec{k}$

$\overrightarrow{PQ} = 4\vec{i} - 2\vec{j} + \vec{k} = \vec{r}$

$|\vec{r}| = \sqrt{16+4+1}$

$|\vec{r}| = \sqrt{21}$

grad ϕ (or) grad $f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$= 2x\vec{i} - 2y\vec{j} + 4z\vec{k}$

directional derivative of f at P(1, 2, 3) in the direction of $\overrightarrow{PQ}$

$(\text{grad} f)_{\text{at } P} = 2\vec{i} - 4\vec{j} + 12\vec{k}$

is $(\text{grad} f)_{\text{at } P} \cdot \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$D.D = (2\vec{i} - 4\vec{j} + 12\vec{k}) \cdot \frac{(4\vec{i} - 2\vec{j} + \vec{k})}{\sqrt{21}}$

$= \frac{8+8+12}{\sqrt{21}}$

$D.D = \frac{28}{\sqrt{21}}$

Ex Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x=t, y=t^2, z=t^3$ at the point (1, 1, 1)

Sol Here $f(x,y,z) = xy^2 + yz^2 + zx^2$

$\nabla f = \text{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\nabla f = (y^2 + 2xz)\vec{i} + (z^2 + 2xy)\vec{j} + (x^2 + 2yz)\vec{k}$

$(\nabla f)_{\text{at }(1,1,1)} = 3\vec{i} + 3\vec{j} + 3\vec{k}$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = t\vec{i} + t^2\vec{j} \ldots$

$$\frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{1+4+9}} = \frac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}}$$

Directional derivative along the tangent $= \text{grad} f \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{1}{\sqrt{14}}(\vec{i}+2\vec{j}+3\vec{k}) \cdot (3\vec{i}+3\vec{j}+3\vec{k})$$

$$= \frac{3+6+9}{\sqrt{14}}$$

$$D.D = \frac{18}{\sqrt{14}}$$

Ex Find the directional derivative of xyz^2+xz at $(1,1,1)$ in a direction of the normal to the surface $3xy^2+y=z$ at $(0,1,1)$

Ans $4/\sqrt{11}$

Ex Find the constants a & b so that the surface $ax^2-byz=(a+2)x$ will be orthogonal to the surface $4x^2y+z^3=4$ at the point $(-1,1,2)$.

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$$D.D = \frac{18}{\sqrt{14}}$$

Ex Find the directional derivative of $xyz^2 + xz$ at $(1,1,1)$ in a direction of the normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

Ans $4/\sqrt{11}$

Ex Find the constants a & b so that the surface $ax^2 - byz = (a+2)x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point $(-1,1,2)$.

Sol

Let $f(x,y,z) = ax^2 - byz - (a+2)x$ — ①

$g(x,y,z) = 4x^2y + z^3 - 4$ — ②

Given two surfaces meet at the point $(1,-1,2)$

sub point $(1,-1,2)$ in ① we get

$$a + 2b - (a+2) = 0$$

$$\Rightarrow 2b = 2 \Rightarrow b = 1$$

$$\frac{\partial f}{\partial x} = 2ax - (a+2), \quad \frac{\partial f}{\partial y} = -bz \text{ and } \frac{\partial f}{\partial z} = -by$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\nabla f = [2ax - (a+2)]\vec{i} - bz\vec{j} - by\vec{k}$$

$$(\nabla f)_{at\ (1,-1,2)} = (a-2)\vec{i} - 2b\vec{j} + b\vec{k} = \vec{n}_1$$

Since $g = 4x^2y + z^3 - 4$

$$\frac{\partial g}{\partial x} = 8xy, \quad \frac{\partial g}{\partial y} = 4x^2, \quad \frac{\partial g}{\partial z} = 3z^2$$

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$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k}$$

$$\nabla g = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

$$(\nabla g)_{at\,(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k} = \vec{n_2}$$

Since the two surfaces are orthogonal at the point $(1,-1,2)$

$$\nabla f \cdot \nabla g = 0$$

$$\Rightarrow \left((a-2)\vec{i} - 2\vec{j} + \vec{k}\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\right) = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 = 0$$

$$\Rightarrow -8a + 20 = 0 \Rightarrow 8a = 20 \Rightarrow a = \frac{5}{2}$$

Hence $a = \frac{5}{2}$, $b = 1$

Divergence of a vector

Let $\vec{F}$ be any continuously differentiable vector point function

Then

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

Then $\operatorname{div}\vec{F} = \nabla \cdot \vec{F}$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla \cdot \vec{F} = \text{divergence of } \vec{F}$$

Then $\operatorname{div}\vec{F}$ is called divergence of $\vec{F}$ which is a scalar point function.

Note A vector point function $\vec{F}$ is said to be Solenoidal

If $\operatorname{div}\vec{F} = 0$

This $\operatorname{div}\vec{F} = 0$ is also called equation of Continuity (or) Conservation of mass.

Ex Find $\operatorname{div}\vec{F}$ where $\vec{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Sol Let $\phi = x^3 + y^3 + z^3 - 3xyz$

Then $\frac{\partial \phi}{\partial x} = 3x^2 - 3yz$, $\frac{\partial \phi}{\partial y} = 3y^2 - 3xz$, $\frac{\partial \phi}{\partial z} = 3z^2 - 3xy$

$\therefore \vec{F} = \text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$

$$\vec{F} = \underbrace{(3x^2 - 3yz)}_{F_1}\vec{i} + \underbrace{(3y^2 - 3zx)}_{F_2}\vec{j} + \underbrace{(3z^2 - 3xy)}_{F_3}\vec{k}$$

Hence $\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

$$= 3(2x) + 3(2y) + 3(2z)$$

$$\text{div}\,\vec{F} = 6(x+y+z)$$

Ex Find $\text{div}\,\vec{F}$ where $\vec{F} = r^n\vec{r}$

Find 'n' if it is solenoidal.

(or) prove that $r^n\vec{r}$ is solenoidal if $n = -3$.

proof Given $\vec{F} = r^n\vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

and $r = |\vec{r}|$

$\Rightarrow r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

Diff partially w.r.t 'x' we get

$\cancel{2}r\frac{\partial r}{\partial x} = \cancel{2}x \Rightarrow \frac{\partial r}{\partial x} = x/r$

|||ly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = z/r$

Given $\vec{F} = r^n\vec{r}$

$\therefore \text{div}\,\vec{F} = \nabla \cdot r^n\vec{r}$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot r^n(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \Sigma \frac{\partial}{\partial x} x r^n \quad (\because \vec{i}\cdot\vec{i} = 1)$$

$$= \Sigma\left[n.r^{n-1}\frac{\partial r}{\partial x}.x + r^n.1\right]$$

$$= \Sigma n.r^{n-1}\frac{x}{r}.x + \Sigma r^n \quad (\because \Sigma(x+y) = \Sigma x + \Sigma y)$$

$$= n.r^{n-2}\Sigma x^2 + 3r^n \quad (\because \Sigma r^n = r^n + r^n + r^n = 3r^n)$$

$$= n.r^{n-2}(x^2+y^2+z^2) + 3r^n$$

$$= n.r^{n-2}r^2 + 3r^n$$

$\therefore \text{div}\,\vec{F} = n.r^n + 3r^n = r^n(n+3)$

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$\therefore \operatorname{div}\vec{F} = r^n(n+3)$

Let $\vec{F} = r^n\vec{r}$ be solenoidal

Then $\operatorname{div}\vec{F} = 0$

$\Rightarrow \operatorname{div}\vec{F} = r^n(n+3) = 0$

$\Rightarrow n+3 = 0 \quad (\because r^n \neq 0)$

$\Rightarrow n = -3$

ie $\vec{F} = r^n\vec{r} = \vec{r}\,r^{-3} = \dfrac{\vec{r}}{r^3}$ is solenoidal.

Ex Show that $\dfrac{\vec{r}}{r^3}$ is solenoidal

Sol we have to prove that $\dfrac{\vec{r}}{r^3}$ is solenoidal

ie $\operatorname{div}\dfrac{\vec{r}}{r^3} = 0$

LHS Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\operatorname{div}\frac{\vec{r}}{r^3} = \nabla\cdot\frac{\vec{r}}{r^3}$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{r^3}\right)$$

$$= \sum\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) \quad (\because \vec{i}\cdot\vec{i} = 1)$$

$$= \sum\frac{1.r^3 - x.3r^2\frac{\partial r}{\partial x}}{(r^3)^2}$$

$$= \sum\frac{r^3 - x.3r^{\cancel{2}}.\frac{x}{\cancel{r}}}{r^6}$$

$$= \sum\frac{r^3 - 3x^2.r}{r^6}$$

$$= \frac{r^3 - 3x^2r + r^3 - 3y^2r + r^3 - 3z^2r}{r^6}$$

$$= \frac{3r^3 - 3r(x^2+y^2+z^2)}{r^6}$$

$$= 3r^3 - 3r(r^2)/r^6 = 3r^3 - 3r^3/r^6 = 0/r^6 = 0$$

$\therefore \operatorname{div}\vec{F} = 0 \Rightarrow \vec{F} = \vec{r}/r^3$ is solenoidal.

Ex If $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$ is solenoidal, find p

Sol $\operatorname{div} \vec{F} = 0$

$\Rightarrow p+2=0 \Rightarrow p=-2$

Ex S.T $3y^4z^2\vec{i} + z^3x^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal

ie $\operatorname{div} \vec{F} = 0$.

Curl of a vector

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Ex If $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+pz)\vec{k}$ is solenoidal, find p

Sol $\operatorname{div}\vec{F} = 0$

$\Rightarrow p+2=0 \Rightarrow p=-2$

Ex S.T $3y^4z^2\vec{i} + 2x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal

ie $\operatorname{div}\vec{F} = 0$.

Curl of a vector

Let $\vec{F}$ be any differentiable vector point function then Curl of $\vec{F}$ ie Curl$\vec{F}$ is defined as

$$\text{Curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

where $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$$\text{Curl}\vec{F} = \nabla\times\vec{F} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

Note

A vector $\vec{F}$ is said to be irrotational, If Curl$\vec{F} = \vec{0}$

Ex Find curl$\vec{F}$ where $\vec{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol Let $\phi(x,y,z) = x^3+y^3+z^3-3xyz$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\text{grad}\,\phi = (3x^2-3yz)\vec{i} + (3y^2-3xz)\vec{j} + (3z^2-3xy)\vec{k}$$

$$\text{Curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$\operatorname{Curl}\vec{F} = 3\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= 3\Big[\vec{i}\Big(\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-zx)\Big)$$

$$-\vec{j}\Big(\frac{\partial}{\partial x}(z^2-xy) - \frac{\partial}{\partial z}(x^2-yz)\Big) + \vec{k}\Big(\frac{\partial}{\partial x}(y^2-zx) - \frac{\partial}{\partial y}(x^2-yz)\Big)$$

$$= 3\Big[\vec{i}\,(-x-(-x)) - \vec{j}\,(-y-(-y)) + \vec{k}\,(-z-(-z))\Big]$$

$$= 3\,(0\vec{i} + 0\vec{j} + 0\vec{k})$$

$\therefore \operatorname{Curl}\vec{F} = \vec{0} = \nabla\times\vec{F}$

Since $\operatorname{curl}\vec{F}$ is $\vec{0}$ ie $\operatorname{Curl}\vec{F} = \vec{0}$

Hence $\vec{F}$ is irrotational.

Ex Find the constants a, b, c. If $\vec{F} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$ is irrotational.

Sol Given $\vec{F} = (2x+3y+az)\vec{i} + (bx+2y+3z)\vec{j} + (2x+cy+3z)\vec{k}$

Since $\vec{F}$ is irrotational.

Hence $\operatorname{Curl}\vec{F} = \vec{0}$

ie $\operatorname{Curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix} = \vec{0}$

$$= \vec{i}\,(c-3) - \vec{j}\,(2-a) + \vec{k}\,(b-3)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$\Rightarrow c-3=0 \Rightarrow c=3$

$\Rightarrow a-2=0 \Rightarrow a=2$

$\Rightarrow b-3=0 \Rightarrow b=3$

$\therefore a=2,\ b=3,\ c=3$

Ex If $\vec{A} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$

then show that $\vec{A} \cdot \text{Curl}\,\vec{A} = 0$

Ex If $\vec{A}$ is irrotational, Evaluate $\text{div}(\vec{A} \times \vec{r})$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Sol $\text{div}(\vec{A} \times \vec{r}) = 0$

Scalar potential function

For any given vector point function $\vec{F}$ there exist a scalar point function ϕ such that $\vec{F} = \nabla\phi$

then ϕ is called scalar potential function.

In this case $\text{Curl}\,\vec{F} = \vec{0}$

Ex show that the vector $(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ is irrotational and find its scalar potential.

Sol Given that

$$\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} +\vec{i} & -\vec{j} & +\vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

where $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

$$= \vec{i}\left(\frac{\partial}{\partial y}(z^2 - xy) - \frac{\partial}{\partial z}(y^2 - zx)\right)$$
$$- \vec{j}\left(\frac{\partial}{\partial x}(z^2 - xy) - \frac{\partial}{\partial z}(x^2 - yz)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(y^2 - zx) - \frac{\partial}{\partial y}(x^2 - yz)\right)$$

$$= \vec{i}\left(-x - (-x)\right) - \vec{j}\left(-y - (-y)\right) + \vec{k}\left(-z - (-z)\right)$$

$$\text{Curl}\,\vec{F} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Since $\text{Curl}\,\vec{F} = \vec{0}$

Hence $\vec{F}$ is said to be irrotational.

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Then there exist ϕ such that $\vec{F} = \nabla\phi$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k})$$

$$(x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial\phi}{\partial x} &= x^2 - yz \\ \frac{\partial\phi}{\partial y} &= y^2 - zx \\ \frac{\partial\phi}{\partial z} &= z^2 - xy \end{aligned}\right\} \text{Comparing components.}$$

Integrating

$$\int \frac{\partial\phi}{\partial x}dx = \int (x^2 - yz)dx \quad \Bigg| \quad \int \frac{\partial\phi}{\partial y}dy = \int (y^2 - zx)dy$$

$$\int \frac{\partial\phi}{\partial z}dz = \int (z^2 - xy)dz$$

$$\Rightarrow \phi = \frac{x^3}{3} - xyz + f_1(y, z)$$

Ill^{ly}

$$\phi = y^3/3 - xyz + f_2(x, z)$$

$$\phi = z^3/3 - xyz + f_3(x, y)$$

$$\therefore \phi = \frac{x^3}{3} - xyz + \frac{y^3}{3} - xyz + \frac{z^3}{3} - xyz + \text{constant}.$$

$$\Rightarrow \phi = 1/3(x^3 + y^3 + z^3) - xyz + \text{constant}.$$

which is the required scalar potential function.

Ex Find constants a, b, c so that the vector $\vec{A} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational

Also find ϕ such that $\vec{A} = \nabla\phi$.

Sol Since $\vec{A}$ is irrotational

Hence $\text{Curl}\,\vec{A} = \vec{0}$

ie $\text{Curl}\,\vec{A} = \nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} =$

$\Rightarrow a - 4 = 0 \Rightarrow a = 4$

$b - 2 = 0 \Rightarrow b = 2$

$c + 1 = 0 \Rightarrow c = -1$

$\therefore a = 4, b = 2, c = -1$

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Let ϕ be a scalar point function such that
$$\vec{A} = \nabla\phi$$

$$(x+2y+4z)\vec{i} + (2x-3y-z)\vec{j} + (4x-y+2z)\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\frac{\partial\phi}{\partial x} = x+2y+4z$$
$$\frac{\partial\phi}{\partial y} = 2x-3y-z$$
$$\frac{\partial\phi}{\partial z} = 4x-y+2z$$

integrating

$$\int\frac{\partial\phi}{\partial x}dx = \int(x+2y+4z)dx = \frac{x^2}{2}+2xy+4xz + f_1(y,z)$$

$$\int\frac{\partial\phi}{\partial y}dy = \int(2x-3y-z)dy = 2xy-\frac{3y^2}{2}-yz + f_1(x,z)$$

$$\int\frac{\partial\phi}{\partial z}dz = \int(4x-y-2z)dz = 4xz-yz+2z^2/2 + f_1(x,y)$$

(or) $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + C$

Ex Show that the vector field $\vec{F} = 2xyz^2\vec{i} + (x^2z^2+z\cos yz)\vec{j} + (2x^2yz+y\cos yz)\vec{k}$ is irrotational. Find the potential function.

$\Rightarrow$ we have to prove that $\operatorname{curl}\vec{F} = \vec{0}$

sol we have to $\phi = x^2yz^2 + \sin yz + C$
and

Ex If ω is a constant vector Evaluate curl V where $V = \omega \times \vec{r}$ where $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$

Ex If $\vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$ find curl curl $\vec{F}$

Ex If $\vec{A}$ is irrotational vector Evaluate $\operatorname{div}(\vec{A}\times\vec{r})$

Ans 0.

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List of formulae

(1) $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$

(2) $\nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$ = normal to ϕ

(3) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ (vector)

(4) $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ (scalar)

(5) $r^n = \left((x^2+y^2+z^2)^{1/2}\right)^n = (x^2+y^2+z^2)^{n/2}$

(6) $\vec{i}\cdot\vec{i} = 1$, $\vec{j}\cdot\vec{j} = 1$, $\vec{i}\cdot\vec{k} = 0$

(7) $r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow r^2 = x^2+y^2+z^2$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

(8) $\nabla f(r) = \frac{f'(r)}{r}\vec{r}$

(9) unit normal vector $= \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$

(10) $\cos\theta = \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|} = \frac{\text{grad}\,\phi_1 \cdot \text{grad}\,\phi_2}{|\text{grad}\,\phi_1|\,|\text{grad}\,\phi_2|}$

(11) Directional derivative of f $= \text{grad} f \cdot \frac{\vec{a}}{|\vec{a}|}$

(12) If $\vec{a}, \vec{b}$ are orthogonal then $\vec{a}\cdot\vec{b} = 0$

(13) $\text{div}\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$

(14) If $\text{div}\,\vec{A} = 0$ then $\vec{A}$ is said to be solenoidal

(15) If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ then $\text{div}\,\vec{r} = 3$

(16) If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

then $\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \overset{+}{\vec{i}} & \overset{-}{\vec{j}} & \overset{+}{\vec{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$

$\nabla\times\vec{F} = \sum \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)$

(17) If $\vec{F}$ is irrotational then $\text{curl}\,\vec{F} = \vec{0}$

(18) $\text{div curl}\,\vec{F} = \nabla\cdot(\nabla\times\vec{F})$ (19) $\text{curl grad}\,\phi = \nabla\times\nabla\phi = \vec{0}$

(19) If $\vec{F}$ given $\exists\,\phi$ such that $\vec{F} = \nabla\phi$ then ϕ is scalar potential

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Ex Show that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

proof $\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

$$= \nabla \cdot \frac{f'(r)}{r}\vec{r} \quad \left(\because \nabla f(r) = \frac{f'(r)}{r}\vec{r}\right)$$

$$= \sum \left[\frac{\partial}{\partial x}\vec{i} \cdot \frac{f'(r)}{r} x\vec{i}\right]$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{f'(r)}{r} \cdot x\right)$$

$$= \sum \frac{\partial}{\partial x}\left(f'(r) \cdot \frac{1}{r} \cdot x\right)$$

$$= \sum \left[f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r} + f'(r) \frac{-1}{r^2}\frac{\partial r}{\partial x} \cdot x + \frac{f'(r)}{r} \cdot 1\right]$$

$$= \sum \left[f''(r)\frac{x}{r} \cdot \frac{x}{r}\right] + \sum f'(r)\frac{-1}{r^2}\frac{x}{r} \cdot x + \sum \frac{f'(r)}{r}$$

$$= \sum \frac{f''(r)}{r^2}x^2 - \frac{f'(r)}{r^3}\sum x^2 + \frac{\sum f'(r)}{r}$$

$$= \frac{f''(r)}{r^2}\sum x^2 - \frac{f'(r)}{r^3}(x^2+y^2+z^2) + \frac{3f'(r)}{r}$$

$$= \frac{f''(r)(x^2+y^2+z^2)}{r^2} - \frac{f'(r)}{r^3}r^2 + \frac{3f'(r)}{r}$$

$$= \frac{f''(r) \cdot r^2}{r^2} - \frac{f'(r)}{r} + \frac{3f'(r)}{r}$$

$$\therefore \text{LHS} = \nabla^2 f(r) = f''(r) + \frac{2f'(r)}{r} = \text{RHS}$$

Workdone by a force

If $\vec{F}$ represents the force vector acting on a particle moving along an arc AB then the workdone during a small displacement $\delta\vec{r}$ is $\vec{F} \cdot \delta\vec{r}$

Hence the total workdone by $\vec{F}$ during displacement from A to B is given by the line integral $\int_A^B \vec{F} \cdot d\vec{r}$

Note If $\vec{F}$ is conservative force If $\nabla \times \vec{F} = \vec{0}$ ie $\text{curl}\,\vec{F} = \vec{0}$ ie $\vec{F}$ is irrotational.

Ex If $\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ along the curve 'C'

in xy-plane $y = x^3$ from $(1,1)$ to $(2,8)$

Sol Given

$$\vec{F} = (5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}$$

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$$\vec{F} \cdot d\vec{r} = \left[(5xy - 6x^2)\vec{i} + (2y - 4x)\vec{j}\right] \cdot \left[dx\vec{i} + dy\vec{j} + dz\vec{k}\right]$$

$$\vec{F} \cdot d\vec{r} = (5xy - 6x^2)dx + (2y - 4x)dy$$

Given curve is

$$y = x^3$$

Diff. w.r.t 'x'

$$\Rightarrow dy = 3x^2 dx$$

$$\therefore \vec{F} \cdot d\vec{r} = (5x^4 - 6x^2)dx + (2x^3 - 4x)(3x^2)dx$$

$$\vec{F} \cdot d\vec{r} = (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

Line integral $= \int_C \vec{F} \cdot d\vec{r}$

$$= \int_{\text{along } AB} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

Let $A = (1,1)$ and $B = (2,8)$

$$= \int_{(1,1)}^{(2,8)} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \int_{x=1}^{2} (6x^5 + 5x^4 - 12x^3 - 6x^2)\,dx$$

$$= \left(\frac{6x^6}{6} + \frac{5x^5}{5} - \frac{12x^4}{4} - \frac{6x^3}{3}\right)_1^2$$

$$= (64 + 32 - 48 - 16) - (1 + 1 - 3 - 2)$$

$\therefore \int_C \vec{F} \cdot d\vec{r} = 35 =$ workdone $=$ line integral.

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Ex II $\vec{F} = (x^2 - 27)\vec{i} - 6yz\vec{j} + 8xz^2\vec{k}$ Evaluate $\int_C \vec{F}\cdot d\vec{r}$ from the point $(0,0,0)$ to the point $(1,1,1)$ along the straight line from $(0,0,0)$ to $(1,0,0)$, $(1,1,0)$ and $(1,1,0)$ to $(1,1,1)$

Sol Given $\vec{F} = (x^2 - 27)\vec{i} - 6yz\vec{j} + 8xz^2\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}\cdot d\vec{r} = (x^2 - 27)dx - 6yz\,dy + 8xz^2 dz$

① Along the st line from $O = (0,0,0)$ to $A = (1,0,0)$

Here $y = 0,\ z = 0$

$\Rightarrow dy = 0,\ dz = 0$

$$\int_{\text{along } OA} \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} (x^2 - 27)dx$$

$$= \left(\frac{x^3}{3} - 27x\right)_0^1$$

$$= \frac{1}{3} - 27$$

$$\therefore \int_{\text{along } OA} \vec{F}\cdot d\vec{r} = -\frac{80}{3}$$

② Along the st. line from $A = (1,0,0)$ to $B = (1,1,0)$

$$\int_{\text{along } AB} \vec{F}\cdot d\vec{r} = \int_{(1,0,0)}^{(1,1,0)} (x^2 - 27)dx - 6yz\,dy + 8xz^2 dz$$

Here $x = 1 \Rightarrow dx = 0$

$z = 0 \Rightarrow dz = 0$

y varies from 0 to 1

$$= \int_0^1 -6yz\,dy$$

$$\int_{\text{along } AB} \vec{F}\cdot d\vec{r} = 0 \quad (\because z = 0)$$

③ Along the st. Line from $B = (1,1,0)$ to $C = (1,1,1)$

Here $x = 1,\ y = 1$

$\Rightarrow dx = 0,\ dy = 0$

z varies from 0 to 1

$$\int_{\text{along } BC} \vec{F}\cdot d\vec{r} = \int_{(1,1,0)}^{(1,1,1)} (x^2 - 27)dx - 6yz\,dy + 8xz^2 dz$$

$$= \int_0^1 8(1)z^2\,dz = \left(8\frac{z^3}{3}\right)_0^1 = 8/3$$

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$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{\text{along OA}} \vec{F}\cdot d\vec{r} + \int_{\text{along AB}} \vec{F}\cdot d\vec{r} + \int_{\text{along BC}} \vec{F}\cdot d\vec{r}$$

$$= \frac{-80}{3} + 0 + 8/3 = -72/3$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -72/3.$$

Ex Find the workdone in moving a particle in the force field

(a) $\vec{F} = 3x^2\vec{i} + \vec{j} + 2\vec{k}$

(b) $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + 2\vec{k}$

along the straight line from $(0,0,0)$ to $(2,1,3)$.

Sol we have eq. of St line is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} = t$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{\text{along }OA} \vec{F}\cdot d\vec{r} + \int_{\text{along }AB} \vec{F}\cdot d\vec{r} + \int_{\text{along }BC} \vec{F}\cdot d\vec{r}$$

$$= \frac{-80}{3} + 0 + 8/3 = -72/3$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -72/3.$$

Ex Find the workdone in moving a particle in the force field

(a) $\vec{F} = 3x^2\vec{i} + \vec{j} + 2\vec{k}$ ———— $27/2$

(b) $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + 2\vec{k}$ Ans 16 unit

along the straight line from $(0,0,0)$ to $(2,1,3)$.

Sol we have eq. of St line is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} = t$$

① $\vec{F} = 3x^2\vec{i} + \vec{j} + 2\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}\cdot d\vec{r} = 3x^2dx + dy + 2dz$, Let $A = (0,0,0)$ and $B = (2,1,3)$

Equation of St. line AB is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} = t$$

$$\frac{x - 0}{0 - 2} = \frac{y - 0}{0 - 1} = \frac{z - 0}{0 - 3} = t$$

$\Rightarrow x = 2t \quad y = t \quad z = 3t$

$\Rightarrow dx = 2dt \quad dy = dt \quad dz = 3dt$

$$\text{workdone} = \int_{\text{along }AB} \vec{F}\cdot d\vec{r} = \int_{(0,0,0)}^{(2,1,3)} 3x^2dx + dy + 2dz$$

$$= \int_{t=0}^{1} 3(2t)^2 2dt + dt + 3t\,(3dt) \quad (\because y = t)$$

$$= \left(24\frac{t^3}{3} + t + 9t^2/2\right)_0^1$$

$$= 8 + 1 + 9/2$$

$$\therefore \int_{\text{along }AB} \vec{F}\cdot d\vec{r} = \frac{27}{2}.$$

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Ex Find the work done by $\vec{F} = (2x-y-z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}$ along a curve 'c' in the xy-plane given by

① $x^2+y^2=9,\ z=0$

② $x^2+y^2=4,\ z=0$

Sol Given

$\vec{F} = (2x-y-z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

in the xy-plane $z=0$

$\Rightarrow dz = 0$

$\therefore \vec{F}\cdot d\vec{r} = (dx\vec{i} + dy\vec{j} + dz\vec{k}) \cdot (2x-y-z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-5z)\vec{k}$

$\vec{F}\cdot d\vec{r} = (2x-y)\,dx + (x+y)\,dy$

Give c: $x^2+y^2=9$

Take $x = 3\cos\theta \quad y = 3\sin\theta$

$\Rightarrow dx = -3\sin\theta\, d\theta \quad dy = 3\cos\theta\, d\theta$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\theta=0}^{2\pi} (6\cos\theta - 3\sin\theta)(-3\sin\theta)\,d\theta + (3\cos\theta + 3\sin\theta)(3\cos\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[9(\sin^2\theta + \cos^2\theta) - 9\sin\theta\cos\theta\right] d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 18 - 9(2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{1}{2}\left[18\theta + \frac{9\cos 2\theta}{2}\right]_0^{2\pi}$$

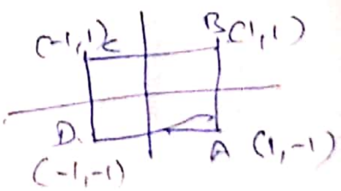
$$= \frac{1}{2}\left[\left(36\pi + \frac{9}{2}\right) - \left(0 + \frac{9}{2}\right)\right] = 18\pi$$

$\therefore \int_C \vec{F}\cdot d\vec{r} = 18\pi.$

Ex Compute the line integral $\int (y^2dx - x^2dy)$ round the ~~rect~~ triangle whose vertices are $(1,0)$ $(0,1)$ $(-1,0)$

Ans $= -2/3$.

Ex Evaluate the line integral $\oint (x^2+xy)\,dx + (x^2+y^2)\,dy$ where 'c' is the square formed by the lines $x = \pm 1$, and $y = \pm 1$



Ans $\int_C \vec{F}\cdot d\vec{r} = 0$

Surface integrals

Surface integrals

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ where F_1, F_2, F_3 are continuous and differentiable function of x, y, z

Then $\int_S \vec{F}\cdot\hat{n}\, ds$ is the integral of a normal component of $\vec{F}$ taken over the surface 'S'.

Let R_1 be the projection of S on xy plane

Then $\int_S \vec{F}\cdot\hat{n}\, ds = \iint_{R_1} \vec{F}\cdot\hat{n} \dfrac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$

IIIly $\int_S \vec{F}\cdot\hat{n}\, ds = \iint_{R_2} \vec{F}\cdot\hat{n} \dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|} = \iint_{R_3} \vec{F}\cdot\hat{n} \dfrac{dx\,dz}{|\hat{n}\cdot\vec{j}|}$

where R_2, R_3 are the projections of S on yz, zx planes respectively

Ex Evaluate $\iint_S \vec{F}\cdot ds$ If $\vec{F} = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$ and S is the surface of the cylinder $x^2+y^2=9$ contained in the first octant between the planes $z=0$ and $z=2$.

Sol Given $\vec{F} = yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k}$

Let $\phi(x,y,z) = x^2+y^2-9$

$\text{grad}\,\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\vec{i} + \dfrac{\partial\phi}{\partial y}\vec{j} + \dfrac{\partial\phi}{\partial z}\vec{k}$

$= 2x\vec{i} + 2y\vec{j} + 0\vec{k}$

$\nabla\phi = 2x\vec{i} + 2y\vec{j} = 2(x\vec{i} + y\vec{j})$

$|\text{grad}\,\phi| = |\nabla\phi| = \sqrt{4x^2+4y^2} = \sqrt{4(x^2+y^2)} = \sqrt{4.9} = 6$

Now unit normal vector $= \hat{n} = \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$= \dfrac{2(x\vec{i}+y\vec{j})}{6}$

$\hat{n} = \dfrac{x\vec{i}+y\vec{j}}{3}$

$\vec{F}\cdot\hat{n} = (yz\vec{i} + 2y^2\vec{j} + xz^2\vec{k})\cdot\left(\dfrac{x\vec{i}+y\vec{j}}{3}\right)$

$\vec{F}\cdot\hat{n} = \dfrac{xyz + 2y^3}{3} = \frac{1}{3}(xyz+2y^3)$, $\hat{n}\cdot\vec{i} = \dfrac{x\vec{i}+y\vec{j}}{3}\cdot\vec{i} = x/3$

$\int_S \vec{F}\cdot\hat{n}\, ds = \int_S \vec{F}\cdot ds = \iint_R \vec{F}\cdot\hat{n} \dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|} = \int_{y=0}\int_{z=0}^{2} \dfrac{\left(\frac{xyz+2y^3}{3}\right)}{x/3} dy\,dz$

$= \int_{y=0}^{3}\int_{z=0}^{2}\left(yz + \dfrac{2y^3}{\sqrt{9-y^2}}\right) dy\,dz$ $\left(\because x=0, y=3 \atop z=0 \text{ to } z=2\right)$

$$= \int_{y=0}^{3}\left(2y + \frac{4y^3}{\sqrt{9-y^2}}\right)dy$$

$$= \left(\cancel{2}\, y^2/\cancel{2}\right)_0^3 + 4\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy$$

$$= 9 + 4(18) = 81$$

(Let $y = 3\sin\theta \Rightarrow dy = 3\cos\theta\, d\theta$, $y=0 \Rightarrow \theta = 0$
$y = 3 \Rightarrow \theta = \pi/2$

$$\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy = \int_0^{\pi/2} \frac{(3\sin\theta)^3}{\cancel{3.\cos\theta}}\, 3\cancel{\cos\theta}\, d\theta$$

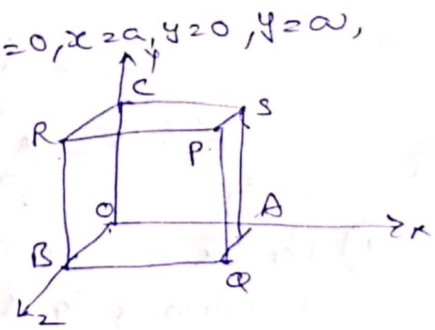
$$= 27\int_0^{\pi/2} \sin^3\theta\, d\theta$$

$$= \frac{\overset{9}{\cancel{81}}}{\cancel{3}} \cdot \frac{2}{3} \times 1$$

$$\int_0^3 \frac{y^3}{\sqrt{9-y^2}}\,dy = 18$$)

$$\therefore \int_S \vec{F}\cdot \hat{n}\, ds = 81$$

Ex If $\vec{F} = 4xz\,\bar{i} - y^2\bar{j} + yz\,\bar{k}$ Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where 's' is the Surface of the Cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$



Sol Given $\vec{F} = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$

(i) ~~Along~~, For the Surface PQAS

equation of PQAS is $x = a$

$ds = \dfrac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$, where $\hat{n} = \vec{i}$ (unit normal vector)

$$\Rightarrow ds = \frac{dy\,dz}{|\vec{i}\cdot\vec{i}|} = \frac{dy\,dz}{|1|} = dy\,dz$$

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k})\cdot\vec{i}\; dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} 4xz\; dy\,dz \qquad (\because x = a)$$

$$= 4a\int_{y=0}^{a} \left(z^2/2\right)_0^a dy$$

$$= 4a\cdot\frac{a^2}{2}\,(y)_0^a$$

$$\therefore \iint \vec{F}\cdot\hat{n}\,ds = 2a^4$$

② For the Surface S_2 (OCRB), equation of S_2 surface is $x=0$

$$ds = \frac{dy\,dz}{|-\vec{i}\cdot\vec{i}|} = \frac{dy\,dz}{1} = dy\,dz, \text{ Here } \hat{n} = -\vec{i}$$

$$\vec{F}\cdot\hat{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot -\vec{i}$$

$$= -4xz$$

$$\vec{F}\cdot\hat{n} = 0 \;(\because x=0) \Rightarrow \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = 0$$

③ For the Surface S_3 (RBQP)

Equation of S_3 surface is $z = a$, Here $\hat{n} = \vec{k}$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = \frac{dx\,dy}{|1|} = dx\,dy$$

$$\vec{F}\cdot\hat{n} = yz \quad (\because \vec{k}\cdot\vec{k} = 1)$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} yz\,dx\,dy$$

$$= a\int_{x=0}^{a}\int_{y=0}^{a} y\,dx\,dy$$

$$= a\int_{x=0}^{a}\left(\frac{y^2}{2}\right)_0^a dx$$

$$= \frac{a^3}{2}(x)_0^a = \frac{a^4}{2}$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \frac{a^4}{2}$$

④ For the Surface S_4 (OASC)

Equation of S_4 is $z=0$, Here $\hat{n} = -\vec{k}$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|-\vec{k}\cdot\vec{k}|} = \frac{dx\,dy}{|-1|} = dx\,dy$$

$$\vec{F}\cdot\hat{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot -\vec{k}$$

$$\vec{F}\cdot\hat{n} = -yz = 0 \quad (\because z=0)$$

$$\therefore \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \iint_{S_4} 0\,ds = 0$$

⑤ For RPSC, $y = a$, $ds = dx\,dz$, $\hat{n} = \vec{j}$

$$\vec{F}\cdot\hat{n} = -y^2 = -a^2$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} -a^2\,dx\,dz = -a^4$$

⑥ For OBQA, $y=0$, $ds = dx\,dz$, $\hat{n} = -\vec{j}$

$$\vec{F}\cdot\hat{n} = y^2 = 0 \quad (\because y=0)$$

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \iint_{S_6} 0\,ds = 0$$

Hence $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} = 2a^4 + 0 + \frac{a^4}{2} + 0 - a^4 + 0$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3a^4}{2}$$

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Evaluate

$\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 12x^2y\,\hat{i} - 3yz\,\hat{j} + 2z\,\hat{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans $-\frac{55}{24}$

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Ex Evaluate

$\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 12x^2y\,\vec{i} - 3yz\,\vec{j} + 2z\,\vec{k}$ and S is the portion of the plane $x+y+z=1$ included in the first octant.

Ans $-\frac{55}{24}$

Ex Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 18z\,\vec{i} - 12\,\vec{j} + 3y\,\vec{k}$

and S is the part of the surface of the plane $2x+3y+6z=12$ located in the first octant.

Ans 24.

Ex If $\vec{F} = y\,\vec{i} + (x-2xz)\,\vec{j} - xy\,\vec{k}$

Evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$ where 's' is the surface of the sphere $x^2+y^2+z^2=a^2$

Ans 0

volume integral

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ where F_1, F_2, F_3 are function of x, y, z

we have $dv = dx\,dy\,dz$

Then the volume integral is given by

$$\int_V \vec{F}\,dv = \iiint (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})\,dx\,dy\,dz$$

Ex If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

Then Evaluate (1) $\int_V \operatorname{div}\vec{F}\,dv$ (2) $\int_V \nabla\times\vec{F}\,dv$ where V is the closed region bounded by $x=0, y=0, z=0, 2x+2y+z=4$.

Sol Given $\vec{F} = (2x^2-3z)\vec{i} - 2xy\,\vec{j} - 4x\vec{k}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 4x - 2x$$

$$\therefore \operatorname{div}\vec{F} = 2x$$

$$\int_V \operatorname{div}\vec{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x\,[z]_0^{4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} 2x(4-2x-2y)\,dy\,dx$$

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$$= \int_{x=0}^{2} \int_{y=0}^{2-x} (8x - 4x^2 - 4xy)\, dy\, dx$$

$$= \int_{x=0}^{2} \left(8xy - 4x^2y - 4xy^2/2\right)_0^{2-x} dx$$

$$= \int_{x=0}^{2} \left[8x(2-x) - 4x^2(2-x) - 2x(2-x)^2\right] dx$$

$$= \int_{x=0}^{2} (16x - 8x^2 - 8x^2 + 4x^3 - 8x - 2x^3 + 8x^2)\, dx$$

$$= \int_{x=0}^{2} (2x^3 - 8x^2 + 8x)\, dx$$

$$= \left(\frac{2x^4}{4} - \frac{8x^3}{3} + 8x^2/2\right)_0^2$$

$$= 2 \cdot \frac{16}{4} - 8 \cdot \frac{8}{3} + 8 \cdot \frac{4}{2}$$

$$= 8 - \frac{64}{3} + 16$$

$$= 24 - \frac{64}{3}$$

$$= \frac{72-64}{3}$$

$$\therefore \quad = 8/3 = \iiint_V \operatorname{div} \vec{F}\, dv$$

$$\iiint_V \operatorname{div} \vec{F}\, dv = 8/3.$$

Given that $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\operatorname{Curl}(\vec{F}) = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}\,[0 - 0] - \vec{j}\,[-4 + 3] + \vec{k}\,[-2y - 0]$$

$$\operatorname{Curl} \vec{F} = +\vec{j} - 2y\vec{k}$$

$$\iiint_V \operatorname{Curl} \vec{F}\, dv = \iiint_V (\vec{j} - 2y\vec{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} (\vec{i} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \vec{i} \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} dz\, dy\, dx - 2\vec{k} \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2y\, dz\, dy\, dx$$

$$= \frac{8}{3}\vec{i} - \frac{8}{3}\vec{k}$$

$$\iiint_V \text{curl}\,\vec{F}\,dv = \frac{8}{3}(\vec{i} - \vec{k}) = \iiint_V \text{curl}\,\vec{F}\,dv = \int_V \nabla \times \vec{F}\,dv$$

Ex $\iiint (2x+y)\,dv$ where V is the closed region bounded by the cylinder $z = 4 - x^2$, and planes $x=0, y=0, y=2$ and $z=0$

Ans $\frac{80}{3}$.

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$$= \frac{8}{3}\vec{i} - \frac{8}{3}\vec{k}$$

$\iiint \text{curl}\,\vec{F}\,dv =$ $\frac{8}{3}(\vec{i}-\vec{k}) = \iiint_V \text{Curl}\,\vec{F}\,dv = \int_V \nabla\times\vec{F}\,dv$

Ex $\iiint (2x+y)\,dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0, y=0, y=2$ and $z=0$

Ans $\frac{80}{3}$.

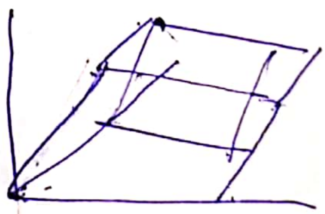
Ex If $\phi = 45x^2y$ Evaluate $\iiint_V \phi\,dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$

Gauss divergence theorem (Vector integral theorems)

Statement :- Let S be a closed surface enclosing a volume. If $\vec{F}$ is a continuously differentiable vector point function

Then
$$\iiint_V \text{div}\,\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$
where $\hat{n}$ is the outward drawn normal vector at any point of S.

Ex verify Divergence theorem for $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ taken over a rectangular parallelopiped $0\le x\le a, 0\le y\le b, 0\le z\le c$



Sol Given $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$

we have to prove that
$$\iiint_V \text{div}\,\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Sol $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left((x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$\text{div}\,\vec{F} = 2x+2y+2z = 2(x+y+z)$$

$$\therefore \iiint_V \operatorname{div} \vec{F}\, dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} 2(x+y+z)\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \left(xz + yz + \frac{z^2}{2}\right)_0^c dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b} \left(cx + yc + c^2/2\right) dy\, dx$$

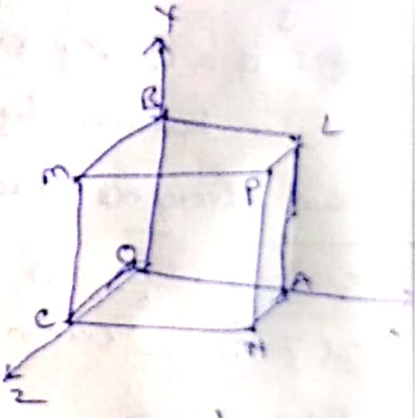
$$= 2\int_{x=0}^{a} \left(cxy + \frac{y^2}{2}c + \frac{c^2}{2}y\right)_0^b dx$$

$$= 2\int_{x=0}^{a} \left(cx\cdot b + \frac{b^2}{2}c + \frac{c^2}{2}\cdot b\right) dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{b^2 c}{2}x + \frac{c^2}{2}bx\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2} + \frac{b^2ca}{2} + \frac{c^2ab}{2}\right]$$

$$\therefore \iiint_V \operatorname{div} \vec{F}\, dv = abc(a+b+c) \quad \text{—— (1)}$$



To find $\int_S \vec{F}\cdot\hat{n}\, ds$

(1) on the face ANPL (S_1)

Equation S_1 surface is $x = a$, $\hat{n} = \vec{i}$

Here $ds = dy\, dz$ (plane parallel to yz plane)

$$\int_{S_1} \vec{F}\cdot\hat{n}\, ds = \iint_R (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k} \cdot \vec{i}\, dy\, dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (x^2 - yz)\, dy\, dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (a^2 - yz)\, dy\, dz \quad (\because x = a)$$

$$= \int_{z=0}^{c} \left(a^2 y - \frac{y^2}{2}z\right)_0^b dz = \int_{z=0}^{c} \left(ba^2 - \frac{b^2}{2}z\right) dz$$

$$= \left(a^2bz - \frac{b^2}{2}\frac{z^2}{2}\right)_0^c = a^2bc - \frac{b^2c^2}{4}$$

$$\therefore \int_{S_1} \vec{F}\cdot\hat{n}\, ds = a^2bc - \frac{b^2c^2}{4}$$

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(2) For the surface S_2 (OCMB), equation of OCMB (S_2) is
$x=0$ (Equation of yz plane)

Here $\hat{n} = -\vec{i}$, $ds = dy\,dz$.

$$\int_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{y=0}^{b} -(x^2 - yz)\,dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} -(0^2 - yz)\,dy\,dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} +(yz)\,dy\,dz = \int_{z=0}^{c}\left(\frac{y^2}{2}z\right)_0^b dz$$

$$= \frac{b^2}{2}\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

$$\therefore \int_{S_2} \vec{F}\cdot\hat{n}\,ds = \frac{b^2c^2}{4}$$

(3) For the surface S_3 (MPLB), equation of S_3 is $y=b$ (eq. of plane parllel to xz plane), $\hat{n} = \vec{j}$

Here $ds = dx\,dz$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a}(y^2 - zx)\,dx\,dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a}(b^2 - xz)\,dx\,dz = \int_{z=0}^{c}\left(b^2x - \frac{x^2}{2}z\right)_0^a dz$$

$$= \int_{z=0}^{c}\left(b^2a - \frac{a^2}{2}z\right) dz$$

$$= \left(ab^2z - \frac{a^2}{2}\frac{z^2}{2}\right)_0^c = ab^2c - \frac{a^2c^2}{4}$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds = ab^2c - \frac{a^2c^2}{4}$$

(4) For the surface S_4(OACM), equation of S_4 is $y=0$ (equation of plane xz), $\hat{n} = -\vec{j}$

Here $ds = dx\,dz$

$$\int_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} -(y^2 - zx)\,dx\,dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} -(0 - zx)\,dz\,dx = +\int_{z=0}^{c}\left(z\frac{x^2}{2}\right)_0^a dz$$

$$= \left(\frac{z^2}{2}\frac{a^2}{2}\right)_0^c = \frac{a^2c^2}{4}$$

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(5) For the surface S_5 (CNPM), equation of S_5 surface is $z=c$

$\hat{n} = \vec{k}$, $ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = \frac{dx\,dy}{|1|} = dx\,dy$

$\Rightarrow ds = dx\,dy$

$z=c$ is (equation of plane parllel to xy-plane)

$$\int_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} (z^2 - xy)\,dx\,dy$$

$$= \int_{y=0}^{b}\int_{x=0}^{a} (c^2 - xy)\,dx\,dy$$

$$= \int_{y=0}^{b} \left(c^2 x - x^2 y/2\right)_0^a dy$$

$$= \int_0^b \left(ac^2 - a^2 y/2\right)dy = \left(ac^2 \cdot y - \frac{a^2 y^2}{2\cdot 2}\right)_0^b$$

$$= ac^2 \cdot b - \frac{a^2 b^2}{2\cdot 2}$$

$$\int_{S_5} \vec{F}\cdot\hat{n}\,ds = abc^2 - \frac{a^2 b^2}{4}$$

(6) For the surface S_6 (OALB), equation of S_6 is $z=0$ (equation of xy-plane), $\hat{n} = -\vec{k}$

$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|-\vec{k}\cdot\vec{k}|} = \frac{dx\,dy}{|-1|} = \frac{dx\,dy}{1} = dx\,dy.$

$\Rightarrow ds = dx\,dy$

$$\int_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{x=0}^{a} -(z^2 - xy)\,dx\,dy$$

$$= \int_{y=0}^{b}\int_{x=0}^{a} (xy - z^2)\,dx\,dy$$

$$= \int_{y=0}^{b} \left(\frac{x^2 y}{2} - z^2 x\right)_0^a dy$$

$$= \int_{y=0}^{b} \left(\frac{a^2 y}{2} - z^2 \cdot a\right)dy$$

$$= \left(\frac{a^2 y^2}{4}\right)_0^b - 0 = \frac{a^2 b^2}{4}$$

$$\therefore \int_{S_6} \vec{F}\cdot\hat{n}\,ds = a^2 b^2/4$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \cdots + \iint_{S_6}$$

$$= a^2bc - \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{b^2c^2}{4}} + ab^2c - \cancel{\frac{a^2c^2}{4}} + \cancel{\frac{a^2c^2}{4}} + abc^2 - \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{a^2b^2}{4}}$$

$$\iint \vec{F}\cdot\hat{n}\,ds = abc(a+b+c) \quad \text{—(2)}$$

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From (1) & (2)

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)$$

$$\therefore \iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Hence Gauss divergence theorem is verified.

Ex

Use Gauss divergence theorem to evaluate $\iint_S (yz^2\,\vec{i} + zx^2\,\vec{j} + 2z^2\,\vec{k})\,ds$ where S is the closed surface bounded by the xy-plane and upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol we have by Gauss divergence theorem

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

$$\vec{F} = yz^2\,\vec{i} + zx^2\,\vec{j} + 2z^2\,\vec{k}$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= 0 + 0 + 4z$$

$$\operatorname{div}\vec{F} = 4z$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv = \iiint_V 4z\,dz\,dy\,dx$$

$$= 4\int_{r=0}^{a}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} (r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$(\because$ ~~$x = r\cos\theta, y$~~ $dx\,dy\,dz = r^2\,dr\,d\theta\,d\phi)$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\sin\theta\cos\theta\,(\phi)_0^{2\pi}\,dr\,d\theta$$

$$= 8\pi\int_{r=0}^{a} r^3 \int_{\theta=0}^{2\pi}\frac{1}{2}\,2\sin\theta\cos\theta\,d\theta\,dr$$

$$= \frac{8\pi}{2}\int_{r=0}^{a} r^3\left(\frac{-\cos2\theta}{2}\right)_0^{2\pi} dr$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = 4\pi\int_{r=0}^{a} r^3(-1+1)\,dr = 0$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S (yz^2\,\vec{i} + 2x^2\,\vec{j} + 2z^2\,\vec{k})\,ds = 0.$$

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Ex

verify divergence theorem for $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface bounded by the region $x^2+y^2=4, z=0, z=3$

Sol we have to prove that

$$\iiint_V \operatorname{div}\vec{F}\,dv = \iint_S \vec{F}\cdot\vec{n}\,ds$$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\operatorname{div}\vec{F} = 4 - 4y + 2z$$

$$\iiint_V \operatorname{div}\vec{F}\,dv = \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(4z - 4yz + z^2\right)_0^3\,dy\,dx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12 - 12y + 9)\,dy\,dx$$

$$= \int_{x=-2}^{2}\left(21y - 6y^2\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\,dx$$

$$= \int_{x=-2}^{2}\left[21\sqrt{4-x^2} - 6(4-x^2)\right] - \left[-21\sqrt{4-x^2} - 6\cdot(4-x^2)\right]dx$$

$$= \int_{x=-2}^{2} 42\sqrt{4-x^2}\,dx$$

$$= 84\int_{x=0}^{2}\sqrt{4-x^2}\,dx = 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left(0 + 2\sin^{-1}1 - (0+0)\right) = 2\cdot\frac{\pi}{2}\,84 = 84\pi$$

$$\therefore \iiint_V \operatorname{div}\vec{F}\,dv = 84\pi$$

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To find Surface integral

(1) For the surface S_1 ($z=0$) (xy plane)

Here $\hat{n} = -\vec{k}$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|-\vec{k}\cdot\vec{k}|} = \frac{dx\,dy}{|-1|} = dx\,dy$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot -\vec{k}\,dx\,dy$$

$$= \iint_{S_1} -z^2\,dx\,dy = 0 \quad (\because z=0)$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = 0$$

(2) For the surface S_2 ($z=3$) (plane parallel to xy plane)

Here $\hat{n} = \vec{k}$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|} = \frac{dx\,dy}{|\vec{k}\cdot\vec{k}|} = dx\,dy$$

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \iint_{S_2} (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \iint_{S_2} z^2\,dx\,dy$$

$$= \iint_{S_2} 3^2\,dx\,dy$$

$$= 9\iint_{S_2} dx\,dy = 9A \quad (A = \text{Area of circle } x^2+y^2=4)$$

$$= 9(\pi r^2) = 9(\pi\cdot 2^2) = 36\pi$$

$$\therefore \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = 36\pi$$

(3) For the surface S_3 : $x^2+y^2=4$

Let $\phi = x^2+y^2-4$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k} = 2x\vec{i} + 2y\vec{j}$$

$$\nabla\phi = 2(x\vec{i}+y\vec{j})$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2} = \sqrt{4(x^2+y^2)} = \sqrt{4\cdot 4} = \sqrt{16} = 4$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(x\vec{i}+y\vec{j})}{4} = \frac{x\vec{i}+y\vec{j}}{2}$$

$$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\frac{(x\vec{i}+y\vec{j})}{2}$$

$$\vec{F}\cdot\hat{n} = (4x^2-2y^3)/2 = 2x^2-y^3$$

Now $$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \iint_{S} (2x^2-y^3)\,ds$$

Hence Green's ...

Let $x = 2\cos\theta$, $y = 2\sin\theta$

$ds = r\,d\theta\,dz = 2\,d\theta\,dz \quad \begin{pmatrix} \theta & 0 & \text{to} & 2\pi \\ z & 0 & \text{to} & 3 \end{pmatrix}$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{3} \left[2(2\cos\theta)^2 - (2\sin\theta)^3\right] 2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)\,d\theta\,[z]_0^3$$

$$= 48\left[\int_{\theta=0}^{2\pi}\cos^2\theta\,d\theta - \int_{\theta=0}^{2\pi}\sin^3\theta\,d\theta\right]$$

$$= 48 \times 4 \times \frac{1}{2} \times \pi/2 - 0 = 48\pi$$

$$\int_{s_3} \bar{F}\cdot\hat{n}\,ds = 48\pi$$

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,ds = \iint_{s_1}\bar{F}\cdot\hat{n}\,ds + \iint_{s_2}\bar{F}\cdot\hat{n}\,ds + \iint_{s_3}\bar{F}\cdot\hat{n}\,ds$$

$$= 0 + 36\pi + 48\pi$$

$$\iint_S \bar{F}\cdot\hat{n}\,ds = 84\pi$$

$\therefore$ LHS = RHS = 84π

Hence $\iiint_V \text{divergence of } \bar{F}\,dv = \iint_S \bar{F}\cdot\hat{n}\,ds$

Hence Gauss divergence theorem is verified.

Green's Theorem

(Relation between line integral and double integral)

Statement: If R is a closed region in xy plane bounded by a simple closed curve 'c' and If M and N are continuous functions of x & y having continuous derivatives in R

then $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

where 'c' is traversed in the positive (anti clockwise direction)

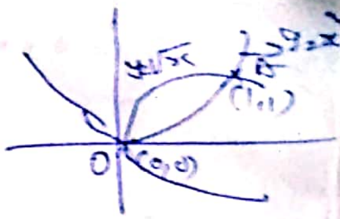
Ex: verify Green's theorem in plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where 'c' is the region bounded by $y = \sqrt{x}$ and $y = x^2$

Sol: $M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$

we have to prove that-

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



To find

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (16y - 6y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} 10y\,dy\,dx$$

$$= \int_{x=0}^{1} \left(10\,\frac{y^2}{2}\right)_{y=x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} (x - x^4)\,dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 5\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = 3/2 \quad \text{(1)}$$

To find Line integral.

(1) Along OA

Equation of OA is $y = x^2$

$\Rightarrow dy = 2x\,dx$

x varies from 0 to 1

$$\int_{\text{along OA}} M\,dx + N\,dy = \int_{\text{along } y=x^2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \left(3x^3/3 - 8x^5/5 + 8\frac{x^4}{4} - \frac{12x^5}{5}\right)_0^1$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5} = 3 - 4 = -1$$

$$\therefore \int_{\text{along OA}} M\,dx + N\,dy = -1$$

(2) Along AB

Equation of AB is $y = \sqrt{x}$

$\Rightarrow dy = \frac{1}{2\sqrt{x}}\,dx$

x varies from 1 to 0

$$\int_{\text{along AB}} M\,dx + N\,dy = \int_{\text{along } y=\sqrt{x}} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{x=1}^{0} (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x^{3/2})\frac{1}{2\sqrt{x}}\,dx$$

$$= \left(3x^3/3 - \frac{8x^2}{2} + 2x - \frac{6x^2}{2\cdot 2}\right)_1^0$$

$$\int_{\text{along AB}} M\,dx + N\,dy = 0 - (1 - 4 + 2 - 3/2) = 5/2$$

$\frac{1}{\sqrt{2}}$

$\partial/\partial y$

verified

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$$\therefore \oint_C M\,dx + N\,dy = \int_{\text{along } OA} M\,dx + N\,dy + \int_{\text{along } AO} M\,dx + N\,dy$$

$$= -1 + 5/2$$

$$\therefore \oint_C M\,dx + N\,dy = 3/2 \quad \text{--- (2)}$$

$\therefore$ From (1) & (2)

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 3/2$$

$$\therefore \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Ex Evaluate by Green's theorem

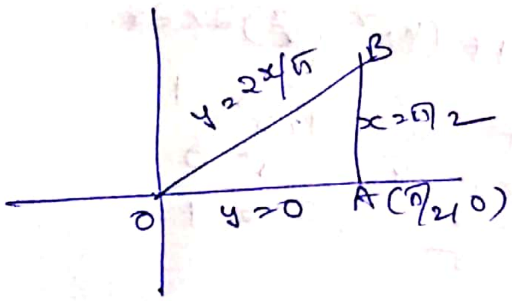
$\oint_C (y - \sin x)\,dx + \cos x\,dy$ where 'C' is the triangle enclosed by the lines $y = 0$, $x = \pi/2$, $\pi y = 2x$

Sol Let $M = y - \sin x$, $N = \cos x$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -\sin x$$

$$\oint_C (y - \sin x)\,dx + \cos x\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{2x/\pi} (-\sin x - 1)\,dx\,dy$$



$$= -\int_{x=0}^{\pi/2} (1 + \sin x)(y)_0^{2x/\pi}\,dx$$

$$= -\int_{x=0}^{\pi/2} (1 + \sin x)\left(\frac{2x}{\pi}\right) dx$$

$$= -\frac{2}{\pi} \int_{x=0}^{\pi/2} (x + x \sin x)\,dx$$

$x \quad 1$

$-\cos x \quad -\sin x$

$$= -\frac{2}{\pi} \left[-x\cos x + \sin x + \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left[0 + 1 + \frac{\pi^2}{8}\right]$$

$$\therefore \oint_C (y - \sin x)\,dx + \cos x\,dy = -\left(\frac{2}{\pi} + \pi/4\right)$$

Ex verify Green's theorem for

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

where 'c' is the region bounded by $x=0, y=0$ and $x+y=1$

Ex verify Green's theorem.

Ex $\int (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where 'c' is the square with vertices $(0,0)\,(2,0)\,(2,2)\,(0,2)$.

Ex Using Green's theorem Evaluate $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where 'c' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Ex verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2\,dy$ where 'c' is bounded by $y = x$, $y = x^2$.

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Ex verify Green's theorem for

$$\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

where 'c' is the region bounded by $x=0$, $y=0$ and $x+y=1$

Ex verify Green's theorem.

Ex $\int (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ where 'c' is the a square with vertices $(0,0)$ $(2,0)$ $(2,2)$ $(0,2)$.

Ex Using Green's theorem Evaluate $\oint_C (2xy - x^2)\,dx + (x^2 + y^2)\,dy$ where 'c' is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Ex verify Green's theorem for $\int_C (xy + y^2)\,dx + x^2\,dy$ where 'c' is bounded by $y = x$, $y = x^2$.

Stoke's theorem (Transformation between line integral and surface integral)

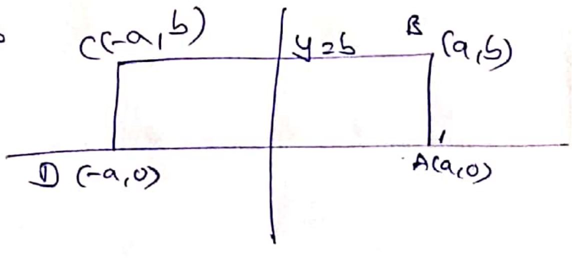
Statement :- let 's' be a open surface bounded by a closed, non-intersecting curve 'c'.

If $\vec{F}$ is any differentiable vector point function

then $\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{Curl}\,\vec{F}\cdot\vec{n}\,ds$

where 'c' is traversed in the positive direction and $\vec{n}$ is unit outward drawn normal at any point of the surface.

Ex. verify stoke's theorem for $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken round the rectangle bounded by the lines $x = \pm a$, $y=0$, $y=b$



Sol let ABCD be the rectangle whose vertices are

$A = (a,0)$, $B = (a,b)$, $C = (-a,b)$

$D = (-a,0)$

we have to prove that

$$\oint_C \vec{F}\cdot d\vec{r} = \int_S \text{Curl}\,\vec{F}\cdot\vec{n}\,ds$$

$$\text{Curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-2xy)\right] - \vec{j}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+y^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2+y^2)\right]$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-2y-2y)$$

$$\therefore \text{Curl}(\vec{F}) = -4y\vec{k}$$

Since the rectangle lies in the xy-plane.

$\hat{n} = \vec{k}$ and $ds = dx\,dy$

$$\int_S \text{Curl}(\vec{F})\cdot\hat{n}\,ds = \iint_S -4y\,\vec{k}\cdot\vec{k}\,ds$$

$$= -4\int_{x=-a}^{a}\int_{y=0}^{b} y\,dx\,dy = -4\int_{x=-a}^{a}\left(\frac{y^2}{2}\right)_0^b dx$$

$$= -2b^2\,[x]_{-a}^{a} \Rightarrow -2b^2(a+a) = -4ab^2$$

$$\therefore \int_S \text{Curl}(\vec{F})\cdot\hat{n}\,ds = -4ab^2 \quad \text{--- (1)}$$

<u>To find line integral</u>

Given $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$

$\vec{F}\cdot d\vec{r} = (x^2+y^2)dx - 2xy\,dy$

(1) <u>Along AB</u>

Equation of AB is $x = a$
$\Rightarrow dx = 0$

$$\therefore \int_{\text{along AB}} \vec{F}\cdot d\vec{r} = \int_{\text{along } x=a} (x^2+y^2)dx - 2xy\,dy = \int_{y=0}^{b} 0 - 2ay\,dy$$

$$= -\cancel{2}a\left(\frac{y^2}{\cancel{2}}\right)_0^b$$

$$\therefore \int_{\text{along AB}} \vec{F}\cdot d\vec{r} = -ab^2$$

(2) <u>Along BC</u>

Equation of BC is $y = b \Rightarrow dy = 0$

$$\int_{\text{along BC}} \vec{F}\cdot d\vec{r} = \int_{\text{along } y=b} (x^2+y^2)dx - 2xy\,dy = \int_{x=a}^{-a}(x^2+b^2)\,dx = \left(\frac{x^3}{3}\right)_a^{-a} + (b^2x)_a^{-a}$$

$$= -2a^3/3 - 2ab^2$$

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$$\therefore \int_{\text{along } BC} \vec{F}\cdot d\vec{r} = -\frac{2a^3}{3} - 2ab^2$$

along CD

Equation of CD is $x = -a$

$\Rightarrow dx = 0$

$$\int_{\text{along } CD} \vec{F}\cdot d\vec{r} = \int_{y=b}^{0} 2ay\,dy = 2a\left(\frac{y^2}{2}\right)_b^0 = -ab^2$$

Along DA

Equation of DA is $y = 0$

$\Rightarrow dy = 0$

$$\int_{\text{along } DA} \vec{F}\cdot d\vec{r} = \int_{x=-a}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

$$\therefore \int_{\text{along } DA} \vec{F}\cdot d\vec{r} = \frac{2a^3}{3}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$\oint_C \vec{F}\cdot d\vec{r} = -4ab^2 \quad \text{—— (2)}$$

$\therefore$ From (1) & (2)

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \hat{n}\,ds = -4ab^2$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \hat{n}\,ds$$

Hence Stoke's theorem is verified.

$\therefore \int_{\text{along } BC} \bar{F}\cdot d\bar{r} = -\frac{\quad}{3}$

along CD

Equation of CD is $x = -a$

$\Rightarrow dx = 0$

$$\int_{\text{along } CD} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 2ay\, dy = 2a\left(\frac{y^2}{2}\right)_b^0 = -ab^2$$

Along DA

Equation of DA is $y = 0$

$\Rightarrow dy = 0$

$$\int_{\text{along } DA} \bar{F}\cdot d\bar{r} = \int_{x=-a}^{a} x^2\, dx = \left(\frac{x^3}{3}\right)_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3} = \frac{2a^3}{3}$$

$$\therefore \int_{\text{along } DA} \bar{F}\cdot d\bar{r} = \frac{2a^3}{3}$$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_{AB} \bar{F}\cdot d\bar{r} + \int_{BC} \bar{F}\cdot d\bar{r} + \int_{CD} \bar{F}\cdot d\bar{r} + \int_{DA} \bar{F}\cdot d\bar{r}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$\oint_C \bar{F}\cdot d\bar{r} = -4ab^2 \quad \text{—— (2)}$$

$\therefore$ From (1) & (2)

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F}\cdot \hat{n}\, ds = -4ab^2$$

$$\Rightarrow \oint_C \bar{F}\cdot d\bar{r} = \iint_S \operatorname{curl} \bar{F}\cdot \hat{n}\, ds$$

Hence Stoke's theorem is verified.

Ex: Verify Stoke's theorem for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol: The boundary 'c' of s is a ⊙ in xy plane ie $x^2+y^2=1, z=0$

The parametric equations are $x = \cos\theta$, $y = \sin\theta$, $\theta = 0 \to 2\pi$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$$\oint_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$$

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Since $z=0 \Rightarrow dz=0$

$$= -\int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)\sin\theta \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \sin^2\theta \, d\theta - \int_{\theta=0}^{2\pi} \sin 2\theta \, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2} d\theta + \left[\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4} + \frac{1}{2}\cos 2\theta\right]_0^{2\pi}$$

$$= \left(\frac{2\pi}{2} - 0 + \cancel{\frac{1}{2}}\right) - \left(0 - 0 + \cancel{\frac{1}{2}}\right)$$

$$= \pi$$

$$\oint_C \vec{F} \cdot d\vec{r} = \pi \quad \text{———— } (1)$$

Again

$$\nabla \times \vec{F} = \text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \vec{i}\,(-2yz + 2yz) - \vec{j}\,[0-0) + \vec{k}\,[0+1]$$

$$\therefore \text{curl}\,\vec{F} = \vec{k}$$

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_S \vec{k} \cdot \vec{k}\, ds = \iint_S ds = A = \pi r^2 = \pi \cdot 1^2 = \pi$$

$$\therefore \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = A \text{ — } (2)$$

From (1) & (2)

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \pi$$

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$

Hence Stoke's theorem is verified.

Ex verify Stoke's theorem for $\vec{F} = (y-z+2)\vec{i} + (yz+4)\vec{j} - xz\vec{k}$ where 'S' is the surface of the cube $x=0, y=0, z=0, x=2, y=2, z=2$ above the ~~xy~~ plane.

Ans = -4.

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Ex

Apply stoke's theorem to evaluate

$\oint_C y\,dx + z\,dy + x\,dz$ where 'C' is the curve of intersection

of the sphere $x^2+y^2+z^2 = a^2$ and $x+z = a$.

Sol The intersection of the sphere $x^2+y^2+z^2=a^2$ and the plane $x+z=a$ is a circle in the plane $x+z=a$

with AB is diametre.

Equation of the plane $x+z=a \Rightarrow \frac{x}{a}+\frac{z}{a}=1$

$OA = OB = a$ ie $A=(a,0,0)$, $B=(0,0,a)$

length of the diametre $AB = \sqrt{a^2+0^2+a^2} = \sqrt{2}\cdot a$

Radius of $\odot = r = a/\sqrt{2}$

$\vec{F}\cdot d\vec{r} = y\,dx + z\,dy + x\,dz$

$\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$

$\therefore$ $\text{Curl}(\vec{F}) = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$

$= -(\vec{i}+\vec{j}+\vec{k})$.

Let $\hat{n}$ be the unit normal $= \frac{\nabla s}{|\nabla s|}$.

$s: x+z-a$

$\nabla s = \vec{i}+\vec{k}$

$|\nabla s| = \sqrt{1^2+1^2} = \sqrt{2}$

$\hat{n} = \frac{\nabla s}{|\nabla s|} = \frac{\vec{i}+\vec{k}}{\sqrt{2}}$

$\iint_S \text{Curl}\vec{F}\cdot\hat{n}\,ds = \iint_S -(\vec{i}+\vec{j}+\vec{k})\cdot\frac{\vec{i}+\vec{k}}{\sqrt{2}}$

$= -\iint_S \frac{+1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\,ds = -\sqrt{2}\iint_S ds = -\sqrt{2}\,A$

$\iint_S \text{Curl}\vec{F}\cdot\hat{n}\,ds = -\sqrt{2}\,\pi r^2 = -\sqrt{2}\,\pi\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{\pi a^2}{\sqrt{2}}$

$\therefore \iint_S \text{Curl}\vec{F}\cdot\hat{n}\,ds = \frac{\pi a^2}{\sqrt{2}}$.

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