UNIT-I

VECTOR CALCULUS

Introduction :

Vector calculus is concerned with differentiation and integration of vector fields primarily in 3-dimensional Euclidean space R^3. The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow. Vector analysis is very important in many fields of engineering such as mechanical, civil, computer, structural and electrical engineering. Scalar values, such as mass and temperature convey only a magnitude, but vectors such as velocity employ both a magnitude and a direction. In physics, the term work is used to describe the energy that is added to or removed from an object or system when a force is applied to it. The work done by a force can be described by the dot product of the force vector and the displacement vector.

Vector finds many applications in Electrical Engineering: The generator that generates Electrical Energy or the Motor that Generates mechanical power work on the principles of physics which are based on vector manipulation. Since vectors and matrices are used in linear algebra, anything that requires the use of arrays that are linear dependent requires vectors. A few well-known examples in Computer engineering are Internet search, Graph analysis, Machine learning, Graphics, Bioinformatics, Data mining, Computer vision, Speech recognition, Compilers, Parallel computing and Scientific computing. Robotics also have Vector Calculus applications. Vectors can be used by air-traffic controllers when tracking planes, by meteorologists when describing wind conditions, and by computer programmers when they are designing virtual worlds.

Definitions :

Scalar : A quantity which is completely specify by its magnitude only.

Ex: Time, Temperature.

Vector : A quantity which is completely specify by its magnitude and direction.

Ex: Force ,Velocity.

Position Vector: Let A and B are two vectors then the position vector of AB is $\overline{AB} = \overline{OB} - \overline{OA}$.

If $\bar{a} = a_1 i + a_2 j + a_3 k$ then $|\bar{a}| = \sqrt{a_1{}^2 + a_2{}^2 + a_3^2}$

If $\bar{a}$ is any vector then its unit vector is given by $\frac{\bar{a}}{|\bar{a}|}$

Dot Product

$\bar{a}.\bar{b} = |\bar{a}||\bar{b}| \cos\theta$ where θ is angle between two vectors

We know $i.i = j.j = k.k = 1$ and $i.j = j.k = k.i = 0$

if $\bar{a} = a_1 i + a_2 j + a_3 k$, $\bar{b} = b_1 i + b_2 j + b\, k$ then $\bar{a}.\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

Cross Product

$\bar{a} \times \bar{b} = |\bar{a}||\bar{b}| \hat{n} \sin\theta$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{since } i \times i = j \times j = k \times k = 0$$

i x j =k; j x k = i; k x i = j; j x i = - k; i x k =- j; k x j = i

Scalar and Vector Point Functions

Consider a region in three dimensional space. To each point $P(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $P(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$, $\bar{f}$ is called vector point functions.

Examples:

For example take a heated solid. At each point P(x,y,z)of the solid, there will be temperature $T(x,y,z)$. This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position $P(x,y,z)$ in space, it will be having some speed, say, v. This **speed** v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

Tangent vector to a curve in space

Consider an interval [a,b].

Let $x = x(t), y = y(t), z = z(t)$ be continuous and derivable for $a \le t \le b$.

Then the set of all points $(x(t), y(t), z(t))$ is called a curve in a space.

Let A = $(x(a), y(a), z(a))$ and B = $(x(b), y(b), z(b))$. These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t + \delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector PQ. As $\overline{Q}$→P, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\lim_{\delta t \to 0} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. If we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

Vector Differential Operator

Def. The vector differential operator ∇(read as del) is defined as

$$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}.$$

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator.

We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably

Gradient of a Scalar Point Function

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

Directional Derivative

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let $\phi+\Delta\phi$ be the value of the function at neighbouring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$. $\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q.

The limiting value of $\frac{\Delta\phi}{\Delta r}$ $as\ \Delta r \to 0$ is called the derivative of ϕ in the direction of $\overline{PQ}$ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

The physical interpretation of $\nabla\phi$

The gradient of a scalar function $\phi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\phi(x,y,z) = c$ at P and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ .

Greatest value of directional derivative of $\bar{\Phi}$ **at a point P = |grad ϕ|** at that point.

NOTE:

1.Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}=dx\bar{i}+dy\bar{j}+dz\bar{k}$ if ϕ is any scalar point function, then

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=\left(\bar{i}\frac{\partial\Phi}{\partial x}+\bar{j}\frac{\partial\Phi}{\partial y}+\bar{k}\frac{\partial\Phi}{\partial z}\right).\left(\bar{i}dx+\bar{j}dy+\bar{k}dz\right)=\nabla\Phi.d\bar{r}$$

2. $grad\Phi$ at any point is a vector normal to the surface $\Phi(x,y,z)=c$ through that point w $P(x,y,z)$ where c is a constant.

3. The directional derivative of a scalar point function ϕ at a point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}$. grad ϕ= $\bar{e}$. $\nabla\phi$.

4.If θ is angle between two surfaces $\emptyset_1,\emptyset_2$ then

$\text{Cos }\theta = \frac{\nabla\phi_1 \,.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_1|}$

5.Unit Normal vector of a surface ϕ is $\frac{\nabla\phi.}{|\nabla\phi|}$

Solved Problems

1.Show that $\nabla[f(r)] = \frac{f'(r)}{r}\bar{r}$ **where** $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ **.**

Sol: Since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r) = \sum \bar{i} f^1(r)\frac{\partial r}{\partial x} = \sum \bar{i} f^1(r)\frac{x}{r}$$

$$= \frac{f^1(r)}{r}\sum \bar{i}x = \frac{f^1(r)}{r}.\bar{r}$$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$, $\nabla(r^n) = nr^{n-2}\,\bar{r}$.

2.Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol: Given $f = xy + yz + zx$.

$$\text{Grad f} = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k}).[(y+z)\bar{i}+(z+x)\bar{j}+(x+y\bar{k})] at (1,2,0)$$

$$= \frac{1}{3}[(y+z)+2(z+x)+2(x+y)](1,2,0) = \frac{10}{3}$$

3. Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1,1,1).

Sol: Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = (y^2+2xz)\bar{i} + (z^2+2xy)\bar{j} + (x^2+2yz)\bar{k}$$

At (1,1,1), $\nabla f = 3\bar{i}+3\bar{j}+3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve $x = t, y = t^2, z = t^3$. then

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i}+2t\bar{j}+3t^2\bar{k} = (\bar{i}+2\bar{j}+3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent $= \nabla f.\bar{e} = \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k}).3(\bar{i}+\bar{j}+\bar{k})$

$$\frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

4. Find the directional derivative of the function f = $x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol: The position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i}+2\bar{j}+3\bar{k}$ and

$\overline{OQ} = 5\bar{i} + 4\bar{k}$; $\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \dfrac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of $\overline{PQ} = \bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\ \frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

5. Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).

Sol: we have

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

6.Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy2 + y = z$ at (0,1,1).

Sol: Let f(x, y, z) ≡ $3xy^2+y - z = 0$

Let us find the unit normal e to this surface at (0,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2,\ \frac{\partial f}{\partial y} = 6xy + 1,\ \frac{\partial f}{\partial z} = -1.$$

$\nabla f = 3y^2$i+($6xy$+1)j-k

$(\nabla f)_{(0,1,1)}$ = 3i+j-k = $\bar{n}$

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i + j - k}{\sqrt{9+1+1}} = \frac{3i + j - k}{\sqrt{11}}$$

Let $g(x, y, z) = xyz^2 + xz$, then

$$\frac{\partial g}{\partial x} = yz^2 + z, \frac{\partial g}{\partial y} = xz^2, \frac{\partial g}{\partial z} = 2xy + x$$

$$\nabla g = (yz^2 + z)i + xz2j + (2xyz + x)k$$

And $[\nabla g]_{(1,1,1)} = 2i+j+3k$

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i + j - k}{\sqrt{11}}\right) = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

7.Evaluate the angle between the normal to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3).

Sol: Given surface is $f(x, y, z) = xy - z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normal to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$ = (grad f) at (4,1,2) $= \bar{i} + 4\bar{j} - 4\bar{k}$

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}.\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

8. Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol: Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x} = 2xi + 2yj + 4zk$

Normal vector at (2,2,3) = $[\nabla f]_{(2,2,3)} = 4\,\bar{i} + 4\,\bar{j} + 12\,\bar{k}$

Unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

9. Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface $ax^2 - byz = (a+2)x$ will orthogonal to $4x^2y + z^3 = 4$ at the point (1,-1,2).

Sol: Let the given surfaces be $f(x,y,z) = ax^2 - byz = (a+2)x$ -------------(1)

And $g(x,y,z) = 4x^2y + z^3 = 4$ ------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

$$a + 2b - (a+2) = 0 \Rightarrow b = 1$$

Now $\frac{\partial f}{\partial x} = 2ax - (a+2),\ \frac{\partial f}{\partial y} = -bz \text{ and } \frac{\partial f}{\partial z} = -by.$

$$\nabla f = \sum \bar{i}\frac{\partial f}{\partial x} = [(2ax - (a+2)]i - bz + bk = (a-2)i - 2bj + bk$$

$= (a-2)i - 2j + k = \bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy,\ \frac{\partial g}{\partial y} = 4x^2,\ \frac{\partial g}{\partial z} = 3z^2.$

$\nabla g = \sum \bar{i}\frac{\partial g}{\partial x} = 8xyi + 4x^2j + 3z^2k$

$(\nabla g)_{(1,-1,2)} = -8i + 4j + 12k = \bar{n}_2$, normal vector to surface 2.

Given the surfaces $f(x,y,z), g(x,y,z)$ are orthogonal at the point (1,-1,2).

$$[\nabla f].[\nabla g] = 0 \Rightarrow ((a-2)i - 2j + k).(-8i + 4j + 12k) = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 \Rightarrow a = 5/2$$

$$Hence\ a = 5/2\ and\ b = 1.$$

Divergence of a vector

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla.\ \bar{f}$

This is a scalar point function.

NOTE: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ **, then *div*** $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Solenoidal Vector

A vector point function $\bar{f}$ is said to be solenoidal if $div\ \bar{f} = 0$.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ ($\nabla.\ \bar{f}$).

Suppose $\bar{F}(x, y, z, t)$ is the velocity of a fluid at a point (x, y, z) and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\overline{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Solved Problems

1. Find $div\ \bar{f}$ when $grad(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$

Then $\frac{\partial\phi}{\partial x} = 3x^2 - 3yz, \frac{\partial\phi}{\partial y} = 3y^2 - 3zx, \frac{\partial\phi}{\partial z} = 3z^2 - 3xy$

$$\text{grad } \phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$$

$= 3(2x)+3(2y)+3(2z) = 6(x+y+z)$

2. If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is Solenoidal, find *P*.

Sol: Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

$$\text{div } \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1 + 1 + p = 2 + p$$

since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0 \Rightarrow 2 + p = 0 \Rightarrow p = -2$

3. Find div $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n\bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially with respect to x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f}$ =r^n ($x\bar{i} + y\bar{j} + z\bar{k}$)

div $\bar{f}$ = $\frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$$

$$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$$

Let $\bar{f} = r^n\bar{r}$ be solenoidal. Then $div\ \bar{f} = 0$

$(n+3)\, r^n = 0 \Rightarrow n = -3$

4. Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$ **.**

Sol: We have $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \frac{\bar{r}}{r^3} = \bar{r}.$$

$$r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\therefore \frac{\partial f_1}{\partial x} = r^{-3} - 3xr^{-4}\frac{x}{y} = r^{-3} - 3x^2r^{-5}$$

$$\nabla.\left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2 = 3r^{-3} - 3r^{-5}r^2 = 0$$

Curl of a Vector

Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by = $\bar{f}$ $f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\text{curl } \bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Note : **curl** $\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ **=∇x** $\bar{f}$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Physical Interpretation of curl

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point $P(x, y, z)$ on the body, then $\bar{w}$ = ½ $curl\ \bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

Any motion in which curl of the velocity vector is a null vector i.e $curl\ \bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if $curl\ \bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function $\varphi(x, y, z)$ such that $\bar{f} = grad\ \phi$. This ϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f} = grad\ \phi$, then curl $\bar{f} = 0$.

Hence $\nabla x\ \bar{f} = 0 \Leftrightarrow$ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force later.

Solved Problems

1. Find $curl\ \bar{f}$ where $\bar{f} = grad(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$ Then

$$grad\ \phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = 3(x^2-yz)\bar{i} + 3(y^2-zx)\bar{j} + 3(z^2-xy)\bar{k}$$

$$curl\ grad\ \phi = \nabla x\ grad\ \phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= 3[\bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)] = \bar{0}$$

$$\therefore curl\ \bar{f} = \bar{0}.$$

Note: We can prove in general that $curl\ (grad\ \phi) = \bar{0}$. $(i.e)$ $grad\ \phi$ is always irrotational.

2.Show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum \bar{i}(-x+x) = \bar{0}$$

$\therefore$ $\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz)dx = \frac{x^3}{3} - xyz + f_1(y,z) \text{......(1)}$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z,x) \text{......(2)}$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y) \text{......(3)}$$

From (1), (2),(3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \quad \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$$

Which is the required scalar potential.

3. Find constants *a*, *b* and *c* if the vector

$\bar{f} =$ $(2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$ **is Irrotational.**

Sol: Given $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$

Curl $\bar{f}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$ =

$(c-3)\bar{i}-(2-a)\,\bar{j}+(b-3)\bar{k}$

If the vector is Irrotational then curl $\bar{f} = \bar{0}$

$\therefore 2-a=0 \Rightarrow a=2, b-3=0 \Rightarrow b=3, c-3=0 \Rightarrow c=3$

4.If $f(r)$ is differentiable, show that $curl\{\ \bar{r}\, f(r)\} = \bar{0}$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$.

Sol: r = $\bar{r} = \sqrt{x^2+y^2+z^2}$

$$\therefore r^2 = x^2+y^2+z^2$$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and\, \frac{\partial r}{\partial z} = \frac{z}{r}$

$$curl\{\bar{r}\, f(r)\} = curl\{f(r)(\ x\bar{i}+y\bar{j}+z\bar{k}\ )\} = curl\ \ (x.f(r)\bar{i}+y.f(r)\,\bar{j}+z.f(r)\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right] = \bar{0}\,.$$

5.Find constants a,b,c so that the vector $\bar{A}$ = $(x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\bar{k}$ **is Irrotational. Also find ϕ such that** $\bar{A}$ = **∇ϕ.**

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\bar{j} + (4x+cy+2z)\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow curl\ \bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\bar{j} + (b-2)\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\bar{j} + (b-2)\bar{k} = 0\bar{i} + 0\bar{j} + 0\bar{k}$

Comparing both sides,

$c + 1 = 0, a - 4 = 0, b - 2 = 0$

$c = -1, a = 4, b = 2$

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k}$, on substituting the values of a, b, c

we have $\bar{A} = \nabla\phi$.

$\Rightarrow \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\bar{j} + (4x-y+2z)\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - 3y^2/2 - yz + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(y,x)$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

Laplacian Operator

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi) = div(grad\ \phi)$

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

Solved Problems

1.Prove that $div.(grad\ r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m1)r^{m-2}$ (or)

$\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $grad(r^m) = \sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i}mr^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i}mr^{m-1}\frac{x}{r} = \sum \bar{i}mr^{m-2}x$

$\therefore div\ (grad\ r^m) = \sum\frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2) + 3r^{m-2}]$

$= m[(m-2)\ r^{m-2} + 3r^{m-2}]$

$= m[(m-2+3)r^{m}-2] = m(m+1)r^{m}-2.$

$Hence\ \nabla^2(r^m) = m(m+1)r^{m}-2$

2. Show that $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r} f^1(r)$ where $r = |\bar{r}|$.

Sol: $grad\,[f(r)] = \nabla f(r) = \sum i \frac{\partial}{\partial x}[f(r)] = \sum i\, f^1(r)\frac{\partial r}{\partial x} = \sum i\, f^1(r)\frac{x}{r}$

$\therefore div\,[grad\, f(r)] = \nabla^2[f(r)] = \nabla.\nabla f(r) = \sum \frac{\partial}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum rf^{11}(r)\frac{x}{r}x + rf^1(r) - x^2}{r^2}.\frac{f^1(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^1(r) - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^1(r) - \frac{1}{r^3}f^1(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^1(r)$$

3. If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and irrotational.

Sol: Given $\nabla^2\phi = 0 \Rightarrow div(grad\ \phi) = 0 \Rightarrow grad\ \phi$ is solenoidal

We know that $curl\ (grad\ \phi) = \bar{0} \Rightarrow grad\ \phi$ is always irrotational

4. Prove that curl grad ϕ = 0.

Sol: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z}$$

$$curl(grad\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \bar{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) - \bar{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right) = \bar{0}$$

Note : Since $Curl(grad\phi) = \bar{0}$, we have $grad\ \phi$ is always irrotational.

5. Prove that $\boldsymbol{div}\ curl\,\bar{f} = 0$

$Proof : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\quad div\ \ curl\ \bar{f} = \nabla.(\nabla \times \bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial y \partial x} + \frac{\partial^2 f_1}{\partial y \partial z} + \frac{\partial^2 f_2}{\partial z \partial x} - \frac{\partial^2 f_1}{\partial z \partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0,$ we have $curl\,\bar{f}$ is always solenoidal.

VECTOR INTEGRATION

Line Integral

Any integral which is to be evaluated over a Curve C is called Line integral of $\bar{F}$.

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

Solved Problems

1. If $\bar{F}$ = $(x^2-27)\,\bar{i}$ $-6yz\,\bar{j}$ $+8xz^2\,\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Sol : Given $\bar{F} = (x^2-27)\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here $y = 0 = z$ and $dy = dz = 0$. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here $x = 1, z = 0 \Rightarrow dx = 0, dz = 0$. y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{1} (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

$x = 1 = y \quad dx = dy = 0$ *and* z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F}\cdot d\bar{r} = \int_{z=0}^{1} 8xz^2 dz = \int_{z=0}^{1} 8xz^2 dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}\cdot d\bar{r} = \frac{88}{3}$$

2. If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}\cdot d\bar{r}$ along the curve C in xy-plane $y = x^3$ from (1,1) to (2,8).

Sol: Given $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$,--------(1)

Along the curve $y = x^3, dy = 3x^2\,dx$

$$\therefore \quad \bar{F} = (5x^4 - 6x^2)\,\bar{i} + (2x^3 - 4x)\,\bar{j},\ [Putting\ y = x^3\ in\ (1)]$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} = dx\bar{i} + 3x^2dx\,\bar{j}$$

$$\therefore \quad \bar{F}\cdot d\bar{r} = [(5x^4 - 6x^2)\,\bar{i} + (2x^3 - 4x)\,\bar{j}].\left[dx\,\bar{i} + 3x^2dx\,\bar{j}\right]$$

$$= (5x^4 - 6x^2)\,dx + (2x^3 - 4x)3x^3dx$$

$$= (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

Hence $\int_{y=x^3} \bar{F}\cdot d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$$

= 16(4+2-3-1) – (1+1-3-2) = 32+3 = 35

3. Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\,\bar{k}$ from $t = 0$ to $t = 2\pi$

Sol : Given force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ and the arc is $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$

$$i.e., x = \cos t, y = \sin t, z = -t$$

$$\therefore d\bar{r} = (-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt$$

$\therefore \bar{F}.d\bar{r} = (-t\,\bar{i} + \cos t\,\bar{j} + \sin t\,\bar{k}).(-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt = (t\sin t + \cos^2 t - \sin t)dt$

Hence work done $= \int_0^{2\pi} \bar{F}.d\bar{r} = \int_0^{2\pi} (t\sin t + \cos^2 t - \sin t)\,dt$

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi}(-\sin t)dt + \int_0^{2\pi}\frac{1+\cos 2t}{2}dt - \int_0^{2\pi}\sin t\,dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

Surface Integral

Any integral which is to be evaluated over a surface S is called surface integral and it is denoted by $\int_S \bar{F}.\bar{n}\,ds$

Let $\bar{F} = F_1\,i + F_2\,j + F_3\,k$, where *F1 ,F2 ,F3* are continuous and differentiable functions of *x,y,z.*

Then $\int_S \bar{F}.ndS = \iint_s F_1dydz + F_2dxdz + F_3dxdy$

Note: 1.Let R be the projection of S on xy plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{k}|}dxdy$

2. Let R be the projection of S on yz plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{i}|}dydz$

3. Let R be the projection of S on zx plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{j}|}dxdz$

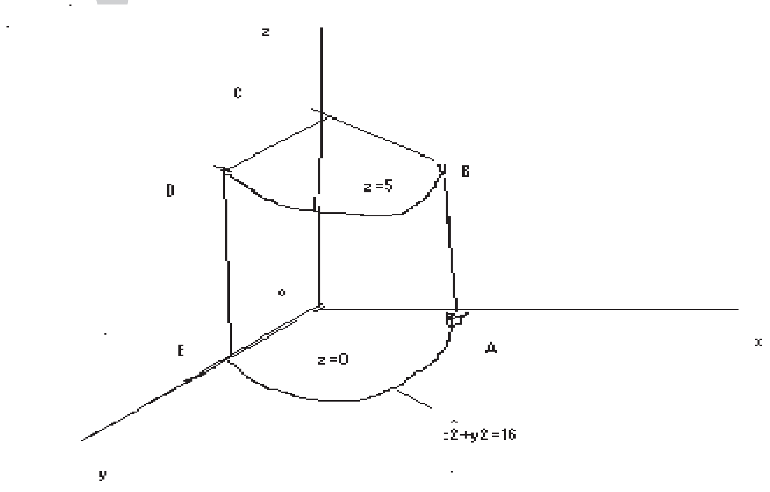
Solved Problems

1.Evaluate $\int \bar{F}.\bar{n}dS$ where $\bar{F} = zi + xj - 3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol: The surface S is $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Let $\varphi = x^2 + y^2 = 16$

Then $$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$$



unit normal

$$\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4} \quad (\because\ x^2 + y^2 = 16)$$

Let R be the projection of S on yz-planeThen

$$\int_S \bar{F}.\bar{n}dS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\ .\ \bar{i}|} \ \ldots\ldots\ldots\ldots\ldots\ *$$

Given $\bar{F} = zi + xj - 3y^2zk$

$$\bar{F}\ .\ \bar{n} = \frac{1}{4}(xz + xy)$$

and $\bar{n}\ .\ \bar{i} = \frac{x}{4}$

In *yz*-plane, $x = 0, y = 4$

In first octant, *y* varies from 0 to 4 and *z* varies from 0 to 5.

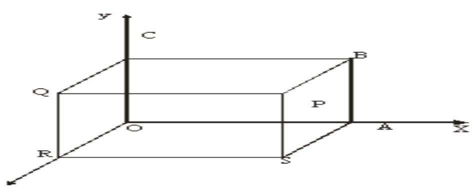
$$\int_S \bar{F}.\bar{n}dS$$

$$= \int_{y=0}^{4}\int_{z=0}^{5}(y+z)dz\,dy = 90.$$

2 : If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\bar{n}dS$ where S is the surface of the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$

Sol: Given that S is the surface of the $x = 0, x = a, y = 0, y = a, z = 0, z = a$, and

$$\bar{F} = zi + xj - 3y2zk$$



we need to evaluate $\int_S \bar{F}.\bar{n}dS$.

(I)For *OABC*

Equation is z = 0 and *dS* = *dxdy*

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.\bar{n}dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\, dxdy = 0$$

(II)For *PQRS*

Equation is *z* = *a* and *dS* = *dxdy*

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)dy \right) dx = \frac{a^4}{2}$$

(III)For *OCQR*

Equation is *x* = *0*, and $\bar{n} = -\bar{i}$, *dS* = *dydz*

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(IV)For *ABPS*

Equation is *x* = *a*, and $\bar{n} = -\bar{i}$, *dS* = *dydz*

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4azdz \right) dy = 2a^4$$

(V)For *OASR* Equation is *y* = *0*, and $\bar{n} = -\bar{j}$, *dS* = *dxdz*

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

(VI)For *PBCQ* Equation is *y* = *a*, and $\bar{n} = -\bar{j}$, *dS* = *dxdz*

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

Adding (i) to (vi)

we get $\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$

Volume Integrals

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots .\delta V_p \ldots .\delta V_m$

Let $P_i(\bar{r}_i)$ be a point in δV_i .Then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(r_i)\delta V_i$. Let $m \to \infty$ in such a way that δV_i shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})dv$ or $\int_V \bar{F}dv$.

Cartesian Form : Let $\bar{F}(\bar{r}) = F_1\bar{i} + F_2\bar{i} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that $dv = dx\,dy\,dz$. The volume integral given by $\int_v \bar{F}dv = \iiint (F_1\bar{i} + F_2\bar{i} + F_3\bar{k})dx\,dydz$

$= \bar{i}\iiint F_1\,dxdydz + \bar{j}\iiint F_2\,dxdydz + \bar{k}\iiint F_3\,dxdydz$

Solved Problems

1.If $\bar{F} = 2xz\,i - x\,j + y^2k$ evaluate $\int \bar{F}\,dv\ over\ V$ where V is the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Given $\bar{F} = 2xz\,i - x\,j + y^2k$.

The volume integral is given by

$$\int \bar{F}\,dv = \int_0^2 \int_{y=0}^{6} \int_{z=x^2}^{4} (2xz\,i - x\,j + y^2k)dxdydz$$

$= i\int_0^2\int_{y=0}^6\int_{z=x^2}^4(2xz\,)dxdydz - j\int_0^2\int_{y=0}^6\int_{z=x^2}^4(x\,)dxdydz + k\int_0^2\int_{y=0}^6\int_{z=x^2}^4(y^2)dxdydz$

$= i\int_0^2\int_{y=0}^6 x(16-x^4)\,dxdy - j\int_0^2\int_{y=0}^6 x(4-x^2)\,dxdy + k\int_0^2\int_{y=0}^6 y^2(x^2-4)\,dxdy$

$= i\int_0^2\int_{y=0}^6(16x-x^5)\,dxdy - j\int_0^2\int_{y=0}^6(4x-x^3)\,dxdy + k\int_0^2\int_{y=0}^6 y^2(x^2-4)\,dxdy$

$= i\int_0^2 6(16x-x^5)\,dx - j\int_0^2 6(4x-x^3)\,dx + k\int_0^2 72(x^2-4)\,dx$

$= i\int_0^2(96x-6x^5)\,dx - j\int_0^2(24x-6x^3)\,dx + k\int_0^2(72x^2-218)\,dx$

$= 128i - 24j - 384k$

Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i)

$\int_S \bar{F}.\bar{n}\,ds$ into a volume integral where S is a closed surface.

(ii)

$\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii)

$\int_S (\nabla \times \bar{A}).\bar{n}\,ds$ into a line integral around the boundary of an open two sided surface.

Gauss Divergence Theorem

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F dv = \int_s \bar{F}.\bar{n}\; dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

Solved Problems

1. Verify Gauss Divergence theorem for $\overline{F} = (x^3 - yz)\bar{\imath} - 2x^2 y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \overline{F}.\bar{n}dS = \int_V div\overline{F}dv$$

Now $div\ \bar{f} = \sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= 3x^2 - 2x^2 + 1$$

Here the cube bounded by the planes $x = y = z = a$ and coordinate planes.

Hence

$x \to 0\ to\ a$

$y \to 0\ to\ a$

$z \to 0\ to\ a$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2 + 1)\,dx\,dy\,dz = \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\,dz$$

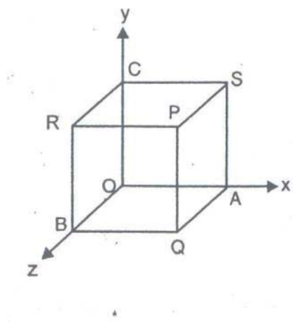
$$\int_0^a\int_0^a \left[\frac{a^3}{3} + a\right]dydz = \int_0^a \left[\frac{a^3}{3} + a\right](y)_0^a dz = \left(\frac{a^3}{3} + a\right)a\int_0^a dz = \left(\frac{a^3}{3} + a\right)(a^2) = \frac{a^5}{3} + a^3 \ldots\ldots(1)$$

Verification: We will calculate the value of $\int_S \overline{F}.\bar{n}dS$ over the six faces of the cube.

(i)

For $S_1 = PQAS$; unit outward drawn normal $\bar{n} = \bar{\imath}$

$x = a;\ ds = dy\,dz;\ 0 \le y \le a, 0 \le z \le a$



$\therefore \overline{F}.\overline{n} = x^3 - yz = a^3 - yz \sin ce x = a$

$$\therefore \iint_{S_1} \overline{F}.\overline{n} dS = \int_{z=0}^{a} \int_{y=0}^{a} (a^3 - yz) dydz$$

$$= \int_{z=0}^{a} \left[a^3 y - \frac{y^2}{2} z \right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z \right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots (2)$$

(ii)

For $S_2 = OCRB$; unit outward drawn normal $\bar{n} = -\bar{i}\ x = 0;\ ds = dy\,dz;\ 0 \le y \le a, y \le z \le a$

$\bar{F}.\bar{n} = -(x^3 - yz) = yz \text{ since } x = 0$

$$\int_{S_2}\int \bar{F}.\bar{n} dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2} \int_{z=0}^{a} zdz = \frac{a^4}{4} \ldots (3)$$

(iii)

For $S_3 = RBQP$; z = a; $ds = dxdy;\ \bar{n} = \bar{k}$

$$0 \le x \le a, 0 \le y \le a$$

$\bar{F}.\bar{n} = z = a \ \ \text{since } z = a$

$$\therefore \iint_{S_3} \overline{F}\overline{n} dS = \int_{y=0}^{a} \int_{x=0}^{a} adxdy = a^3 \ldots..(4)$$

(iv)

For $S_4 = OASC$; z = 0; $\bar{n} = -\bar{k}$, $ds = dxdy;$

$$0 \le x \le a, 0 \le y \le a$$

$\bar{F}.\bar{n} = -z = 0 \;\; since\, z = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}dS = 0 \ldots(5)$$

(v)

For $S_5 = PSCR$; y = a; $\bar{n} = \bar{j},\, ds \;=\; dzdx;$

$$0 \le x \le a, 0 \le z \le a$$

$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \;\; since\, y = a$

$$\int_{S_5}\int \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a}(-2ax^2)dzdx$$

$$\int_{x=0}^{a}(-2ax^2z)_{z=0}^{a}dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \ldots(6)$$

(vi)

For $S_6 = OBQA$; y = 0; $\bar{n} = -\bar{j},\, ds \;=\; dzdx;$

$$0 \le x \le a, 0 \le y \le a$$

$\bar{F}.\bar{n} = 2x^2y = 0 \; since\, y = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0$$

$$\int_{S}\int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv\, using\ (1)$$

Hence Gauss Divergence theorem is verified

2. Use divergence theorem to evaluate $\int\int_S \bar{F}.d\bar{S}$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

Sol: We have

$$div\overline{F} = \nabla.\overline{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

By divergence theorm,

$$\int\int_S \bar{F}.dS = \int\int_V\int \bar{\nabla}.\bar{F}dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4 - 4y + 2z)dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[(4-4y)z + z^2]_0^3\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12(1-y) + 9]\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21 - 12y)dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} ydy\right]dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}} dy - 12(0)\right]dx$$

[**Since the integrans in first integral is even and in 2nd integral it is on add function**]

$$= 42\int_{-2}^{2}(y)_0^{\sqrt{4-x^2}}\,dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}dx$$

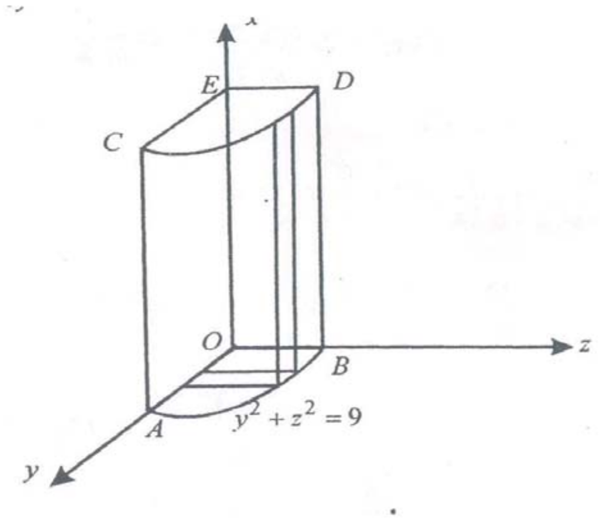
$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2.\frac{\pi}{2} - 0\right] = 84\pi$$

3. Verify divergence theorem for $2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\iint_S \overline{F}.\overline{n}dS$, **where** $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ **and S is the closed surface of the region in the first octant bounded by the cylinder** $y^2+z^2 = 9$ **and the planes** $x = 0, x = 2, y = 0, z = 0$

Sol: Let $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ $\quad \therefore \nabla.\overline{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$



$$\iiint_V \nabla.\overline{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy - 2y)z + 8x\,\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy - 2y)\sqrt{9 - y^2} + 4x(9 - y^2)\right] dy\,dx$$

$$= \int_0^2\int_0^3 [(1 - 2x)(-2y)\sqrt{9 - y^2} + 4x(9 - y^2)]\,dy\,dx$$

$$= \int_0^2 \left\{\left[(1 - 2x)\frac{(9 - y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 + 4x\left(9y - \frac{y^3}{3}\right)_0^3\right\} dx$$

↓

{ Since $\int f^{/}(x[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1}$ }

$$= \int_0^2 \{\frac{2}{3}(1-2x)[0-27] + 4x[27-9]\}\, dx = \int_0^2 [-18(1-2x) + 72x] dx$$

$$\left[-18(x-x^2)+72\frac{x^2}{2}\right]_0^2 = -18(2-4)+36(4) = 36+144 = 180 \ldots(1)$$

Now we sall calculate $\int_S \bar{F}.\bar{n}\, ds$ *for all the five faces.*

$$\int_S \bar{F}.\bar{n}dS = \int_{S_1} \bar{F}.\bar{n}dS + \int_{S_2} \bar{F}.\bar{n}dS + \ldots\ldots + \int_{S_5} \bar{F}.\bar{n}dS$$

Where S_1 is the face *OAB*, S_2 is the face *CED*, S_3 is the face *OBDE*, S_4 is the face *OACE* and S_5 is the curved surface *ABDC*.

(i)

On $S_1 : x = 0, \bar{n} = -i$ $\quad \therefore \bar{F}.\bar{n} = 0$ *Hence* $\int_{S_1} \bar{F}.\bar{n}dS$

(ii) On $S_2 : x = 2, \bar{n} = i$ $\quad \therefore \bar{F}.\bar{n} = 8y$

$$\therefore \int_{S_2} \bar{F}.\bar{n}dS = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dy\,dz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9-z^2)dz = 4\left(9z - \frac{z^3}{3}\right)_0^3 = 4(27-9) = 72$$

(iii) On $S_3 : y = 0, \bar{n} = -j. \therefore \bar{F}.\bar{n} = 0$ *Hence* $\int_{S_3} \bar{F}.\bar{n}dS$

(iv) On $S_4 : z = 0, \bar{n} = -k.$ $\quad \bar{F}.\bar{n} = 0.$ *Hence* $\int_{S_4} \bar{F}.\bar{n}ds = 0$

(v) On $S_5 : y^2 + z^2 = 9, \bar{n} = \frac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|} = \frac{2y\bar{j} + 2z\bar{k}}{\sqrt{4y^2+4z^2}} = \frac{y\bar{j}+z\bar{k}}{\sqrt{4\times 9}} = \frac{y\bar{j}+z\bar{k}}{3}$

$\bar{F}.\bar{n} = \frac{-y^3 + 4xz^3}{3}$ and $\bar{n}.\bar{k} = \frac{z}{3} = \frac{1}{3}\sqrt{9-y^2}$

Hence $\int_{S_5} \bar{F}.\bar{n}ds = \int\int_R \bar{F}.\bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$ Where R is the projection of S_5 on $xy-plane.$

$$= \int_R\int \frac{4xz^3 - y^3}{\sqrt{9-y^2}}dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3}\left[4x(9-y^2) - y^3\left(9-y^2\right)^{-\frac{1}{2}}\right]dy\,dx$$

$Tofind \int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy}$

sub

$\mathrm{y} = 3\sin\theta$

$\mathrm{dy} = 3\cos\theta$

$\int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy} = \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta$

sub

$\sin^3\theta = 3\sin\theta - \sin 3\theta$

We get

$\int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy} = \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = -18$

Hence

$\int_{S3} \bar{F}.\bar{n}ds$

$$= \int_0^2 72x\,dx - 18\int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144 - 36 = 108$$

$Thus \int_S \bar{F}.\bar{n}ds = 0 + 72 + 0 + 0 + 108 = 180 \ldots\ldots (2)$

Hence the Divergence theorem is verified from the equality of (1) and (2).

4. Verify Gauss divergence theorem for $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$ taken over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$.

Sol: We have $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$

$$\nabla.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\iiint_V \bar{\nabla}.\bar{F}\,dv = \iiint_V (3x^2 + 3y^2 + 3z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2 + y^2 + z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^3}{3} + xy^2 + z^2x\right)_0^a dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{a} + ay^2 + az^2\right)dy\,dz$$

$$= 3\int_{z=0}^{a}\left(\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right)_0^a dz$$

$$= 3\int_{0}^{a}\left(\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\right)dz = 3\int_{0}^{a}\left(\frac{2}{3}a^4 + a^2z^2\right)dz$$

$$= 3\left(\frac{2}{3}a^4z + a^2.\frac{z^3}{3}\right)_0^a = 3\left(\frac{2}{3}a^5 + \frac{1}{3}a^5\right)$$

$$= 3a^5$$

To evaluate the surface integral divide the closed surface *S* of the cube into 6 parts.

i.e.,

S_1 : The face *DEFA* ; S_4 : The face *OBDC*

S_2 : The face *AGCO* ; S_5 : The face *GCDE*

S_3 : The face *AGEF* ; S_6: The face *AFBO*

$$\int_S\int \bar{F}.\bar{n}ds = \int_{S_1}\int \bar{F}.\bar{n}ds + \int_{S_2}\int \bar{F}.\bar{n}ds + \cdots + \int_{S_6}\int \bar{F}.\bar{n}ds$$

On S_1, we have $\bar{n} = \bar{i}, x = a$

$$\therefore \int_{S_1}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\bar{i} + y^3\bar{j} + z^3\bar{k}\right).\bar{i}dydz$$

$$\int_{S_1}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\bar{i} + y^3\bar{j} + z^3\bar{k}\right).\bar{i}\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3 dy\,dz = a^3\int_0^a (y)_0^a\,dz$$

$$= a^4(z)_0^a = a^5$$

On S_2, we have $\bar{n} = -\bar{i}, x = 0$

$$\int_{S_2}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(y^3\bar{j} + z^3\bar{k}\right).\left(-\bar{i}\right)dydz = 0$$

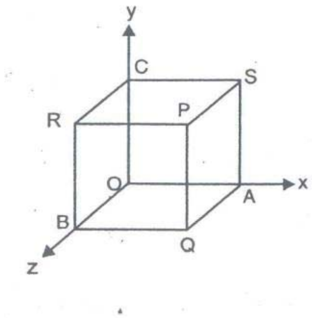
On S_3, we have $\bar{n} = \bar{j}, y = a$

$$\int_{S_3}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + a^3\bar{j} + z^3\bar{k}\right).\bar{j}dxdz = a^3\int_{z=0}^{a}\int_{x=0}^{a}dxdz = a^3\int_0^a adz = a^4\left(z\right)_0^a$$

$$= a^5$$

On S_4, we have $\bar{n} = -\bar{j}, y = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + z^3\bar{k}\right).(-\bar{j})dx\,dz = 0$$



On S_5, we have $\bar{n} = \bar{k}, z = a$

$$\int_{S_5}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + y^3\bar{j} + a^3\bar{k}\right).\bar{k}\,dx\,dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\,dy = a^3\int_0^a (x)_0^a dy = a^4(y)_0^a = a^5$$

On S_6, we have $\bar{n} = -\bar{k}, z = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{\imath}+y^3\bar{\jmath}).(-\bar{k})dx\,dy = 0$$

$$Thus \int_S\int \bar{F}.\bar{n}ds = a^5+0+a^5+0+a^5+0 = 3a^5$$

$$Hence \int_S\int \bar{F}.\bar{n}ds = \int_V\int \bar{V}.\bar{F}\,dv$$

$\therefore The$ Gauss divergence theorem is verified.

5. Compute $\int(ax^2+by^2+cz^2)dS$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int_S \bar{F}.\bar{n}dS = \int_V \bar{V}.\bar{F}\,dv$

$Given\ \bar{F}.\bar{n} = ax^2+by^2+cz^2. Let\ \phi = x^2+y^2+z^2-1$

$\therefore Normal\ vector\ \bar{n}\ to\ the\ surface\ \phi\ is$

$$\bar{V}\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial y}\right)(x^2+y^2+z^2-1) = 2(x\bar{i}+y\bar{j}+z\bar{k})$$

$\therefore$ Unit normal vector $=\bar{n} = \dfrac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2\sqrt{x^2+y^2+z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$ Since $x^2+y^2+z^2=1$

$$\therefore\ \bar{F}.\bar{n} = \bar{F}.(x\bar{i}+y\bar{j}+z\bar{k}) = (ax^2+by^2+cz^2) = (a\,x\bar{i}+by\bar{j}+cz\bar{k}).(x\bar{i}+y\bar{j}+z\bar{k})$$

i.e., $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ $\quad \nabla.\bar{F} = a+b+c$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2+by^2+cz^2)dS = \int_V (a+b+c)dv = (a+b+c)V = \frac{4\pi}{3}(a+b+c)$$

$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

6. Use divergence theorem to evaluate $\int_s\int \bar{F}.d\bar{S}$ where $\bar{F} = x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have $\overline{V}.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2 + y^2 + z^2)$

∴ By divergence theorem,

$\bar{\nabla}.\bar{F}dV = \int\int_V \int \bar{\nabla}.\bar{F}dV$ $= \iiint_v 3(x^2 + y^2 + z^2)dxdydz$

$= 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2(r^2 \sin\theta\, dr\, d\theta\, d\phi)$

Applying spherical coordinates,

$\int_S\int \bar{F}.dS = 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^4 \sin\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\, d\theta$

$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4 \sin\theta(2\pi - 0)drd\theta = 6\pi \int_{r=0}^{a} r^4 \left[\int_0^{\pi} \sin\theta d\theta\right] dr$

$= 6\pi \int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\, dr = -6\pi \int_0^a r^4(\cos\pi - \cos 0)\, dr$

$= 12\pi \int_0^a r^4 dr = 12\pi \left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$

7. **Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.**

Sol: By Gauss theorem, $\int_s \overline{F}.\overline{n}dS = \int_v div\overline{F}dv$

$\frac{\partial\phi}{\partial x} = 1, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 1$

$\therefore grad\phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$

$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$

Let R be the projection of S on *xy*-plane

Then the equation of the given plane will be $x+y=a \Rightarrow y=a-x$

Also when $y=0, x=a$

$$\therefore \int_S \overline{F}.\overline{n}dS = \iint_R \frac{\overline{F}.\overline{n}dxdy}{|\overline{n}.\overline{k}|}$$

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2] dx\, dy$$

$$= \int_{x=0}^{a} \left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)dx$$

$$\therefore \int_S \overline{F}.\overline{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right)dx = \frac{a^4}{4}, \text{ on simplification}\ldots(1)$$

Given $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$

$$\therefore div\, \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\, dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\, dy$$

$$= \int_0^a\int_0^{a-x}[a^2-(x+y)^2]\; dy\, dx = \int_0^a\int_0^{a-x}(a^2-x^2-y^2-2xy)dx\, dy$$

$$= \int_0^a [a^2 y - x^2 y - \frac{y^3}{3} - xy^2]_0^{a-x}\ dx$$

$$= \int_0^a (a-x)(2a^2 - x^2 - ax)dx = \frac{a^4}{4} \dots\dots (2)$$

Hence from (1) and (2), the Gauss Divergence theorem is verified.

8. Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{\imath} + zx^2\bar{\jmath} + 2z^2\bar{k}).ds, where\ S$ is the closed surface bounded by the *xy*-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol: Divergence theorem states that

$$\int\int_S \bar{F}.ds = \int\int\int_V \bar{\nabla}.\bar{F}\ dv$$

Here $\nabla.\overline{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$

$$\therefore \int\int_s \overline{F}.ds = \int\int\int_V 4z dxdydz$$

Introducing spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi,$

$z = r\cos\theta$ then $dxdydz = r^2 dr d\theta d\phi$

$$\therefore \int\int_s \overline{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (r\cos\theta)(r^2\sin\theta dr d\theta d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\ d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta\ (2\pi - 0)dr\ d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3 \left[\int_0^{\pi} \sin 2\theta\ d\theta\right] dr = 4\pi\int_{r=0}^{a} r^3 \left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3 (1-1)dr = 0$$

9. Use Divergence theorem to evaluate $\iint\left(x\bar{i}+y\bar{j}+z^2\bar{k}\right).\bar{n}ds$. Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z = 4$.

Sol: Given $\int\int(x\bar{\imath}+y\bar{\jmath}+z^2\bar{k}).\bar{n}.ds$ *Where* S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z = 4$. Let $\bar{F} = x\bar{\imath}+y\bar{\jmath}+z^2\bar{k}$

By Gauss Divergence theorem, we have

$$\int\int(x\bar{\imath}+y\bar{\jmath}+z^2\bar{k}).\bar{n}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\,dv$$

$$\nabla.\bar{F}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z^2)=1+1+2z=2(1+z)$$

On the cone, $x^2+y^2=z^2$ and $z=4$ $\Rightarrow x^2+y^2=16$

The limits are $z=0$ *to* $4, y=o$ *to* $\sqrt{16-x^2}, x=0$ *to* $4.$

$$\int\int_V\int \bar{\nabla}.\bar{F}\,dv = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^4 2(1+z)dx\,dy\,dz$$

$$=2\int_0^4\int_0^{\sqrt{16-x^2}}\left\{[z]_0^4+\left[\frac{z^2}{2}\right]_0^4\right\}dx\,dy$$

$$=2\int_0^4\int_0^{\sqrt{16-x^2}}[4+8]dxdy=2\times12\int_0^4[y]_0^{\sqrt{16-x^2}}dx$$

$$=24\int_0^4\sqrt{16-x^2}\,dx=24\int_0^{\frac{\pi}{2}}\sqrt{16-16\sin^2\theta}\,.4\cos\theta d\theta$$

$$[put\,x=4\sin\theta\Rightarrow dx=4\cos\theta d\theta.\ Also\ x=0\Rightarrow\theta=0\ and\ x=4\Rightarrow\theta=\frac{\pi}{2}]$$

$$\therefore\int\int_V\int\nabla.\bar{F}dv=96\times4\int_0^{\frac{\pi}{2}}4\sqrt{1-\sin^2\theta}\cos\theta d\theta=96\times4\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$\int\int_V\int\bar{\nabla}.\bar{F}\,dv=96\,X\,4\int_0^{\frac{\pi}{2}}4\sqrt{1-\sin^2\theta}\,\cos\theta d\theta=96\,X\,4\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

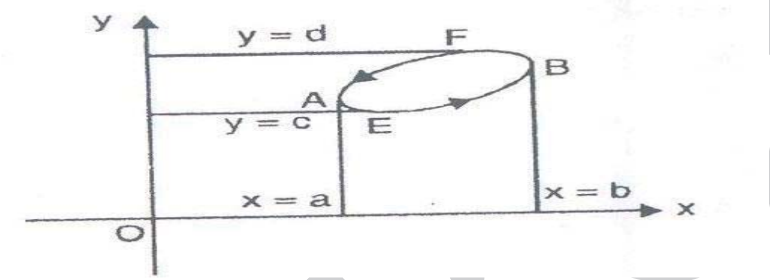
$$= 96\,X\,4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = 96X4\int_0^{\frac{\pi}{2}} \left[\frac{1}{2}+\frac{\cos 2\theta}{2}\right] d\theta$$

$$= 384\left[\frac{1}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 96\pi$$

Green's Theorem in a Plane(Transformation b/w Line Integral and Surface Integral)

If S is Closed region in *xy* plane bounded by a simple closed curve C and if M and N are continuous functions of *x and y* having continuous derivatives in R, then

$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$. Where C is traversed in the anti clock-wise direction

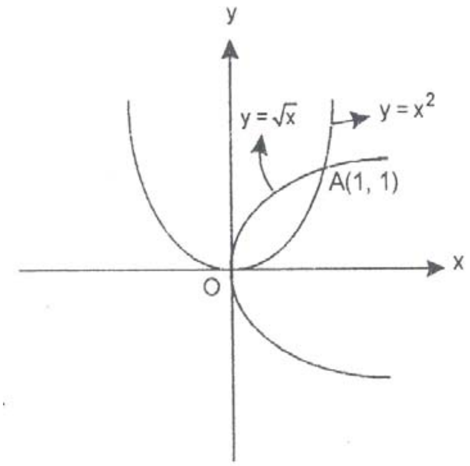


Solved Problems

1.Verify Green's theorem in plane for $\oint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ **where** ***C*** **is the region bounded by y=**$\sqrt{x}$ **and y=**x^2 .

Sol: Let $M=3x^2-8y^2$ and $N=4y-6xy$. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (16y - 6y) dxdy$

$$=10\iint_R ydxdy = 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} ydydx = 10\int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$=5\int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

….(1)

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1}\{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\} \left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$$=\int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$$

And $I_2 = \int_1^0 \left[\left(3x^2 - 8x\right)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}dx\right] = \int_1^0 \left(3x^2 - 11x + 2\right)dx = \frac{5}{2}$

$\therefore\ I_1 + I_2 = -1 + 5/2 = 3/2.$

From(1) and (2), we have $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

Hence the verification of the Green's theorem.

2.Evaluate $\oint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ over triangle enclosed by the $lines$ $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$ using Green's theorem.

Sol : Let M=y-$\sin x$ and $N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

$\therefore$ By Green's theorem $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

$$\Rightarrow \int_c (y - \sin x)dx + \cos x dy = \iint_R (-1 - \sin x) dxdy$$

$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\, dxdy$$

$$= -\int_{x=0}^{\pi/2} (\sin x + 1)\, [y]_0^{2x/\pi}\, dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1) dx$$

$$= \frac{-2}{\pi} \left[x(-\cos x + x) \right]_0^{\pi} - \int_0^{\pi/2} 1(-\cos x + x) dx$$

$$= \frac{-2}{\pi} \left[x(-\cos x + x) + \sin x - \frac{x^2}{2} \right]_0^{\pi/2}$$

$$= \frac{-2}{\pi} \left[-x\cos x + \frac{x^2}{2} + \sin x \right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

3.A Vector field is given by $\overline{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j}$

Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, $z=0$

(i) Directly (ii) By using Green's theorem

Sol: (i) Using the line integral

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1 dx + F_2 dy = \oint_c \sin y dx + x(1+\cos y)dy$$

$$= \oint_c \sin y dx + x\cos y dy + x dy = \oint_c d(x\sin y) + x dy$$

Given Circle is $x^2+y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\, d\theta$ and

dy=a $\cos\theta\, d\theta$ and $\theta = 0 \to 2\pi$

$$\therefore \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\cos\theta\sin(a\sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\cos\theta\, d\theta$$

$$=[a\cos\theta\sin(a\sin\theta)]_0^{2x} + 4a^2\int_0^{\pi/2} cos^2\theta\, d\theta$$

$$=0+4a^2.\frac{1}{2}.\frac{\pi}{2} = \pi a^2$$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=$(1+\cos y)$

By Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\therefore \oint_c \sin y dx + x(1+\cos y)dy = \iint_R (-\cos y + 1 + \cos y)dxdy == \iint dxdy$$

$$= \iint_R dA = A = \pi a^2 (\because area\ of\ circle = \pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

4.Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x= $a\cos\theta, y = b\sin\theta \quad (i.e)\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii)The Circle x=$a\cos\theta, y = a\sin\theta \ (i.e) x^2 + y^2 = a^2$

Sol: We have by Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\int_R dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \to 2\pi$

$\therefore Area, A = \frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a\cos\theta)(b\cos\theta) - (b\sin\theta(-a\sin\theta))]d\theta$

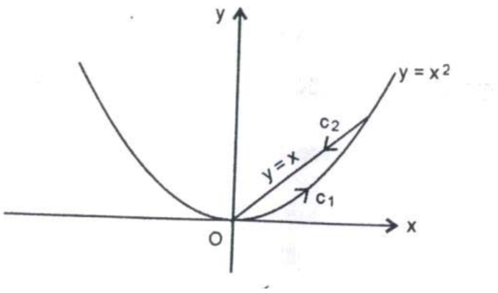
$$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + \sin^2\theta)\,d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$$

(ii)Put a=b to get area of the circle A=πa^2

5. Verify Green's theorem for $\int_c[(xy + y^2)dx + x^2dy]$, where *C* is bounded by *y*=*x* and *y*=x^2

Sol: By Green's theorem, we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\oint_c Mdx + Ndy = \int_{c_1} Mdx + Ndy + \int_{c_2} Mdx + Ndy$......(1) …..(1)

Along C_1 $(i.e.\, y = x^2)$, the line integral is

$$\int_{c_1} Mdx + Ndy = \int_{c_1} [x(x^2) + x^4]dx + x^2 d(x^2) \int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)dx$$

$$= \left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$ …….(2)

Along C_2 $(i.e.\, y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx + Ndy = \int_{c_2} (x.x + x^2)dx + x^2 dx \quad [\because dy = dx]$$

$$= \int_{c_2} 3x^2 dx = 3\int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0 - 1 = -1$$ ….(3)

From (1), (2) and (3), we have

$\int_c Mdx + Ndy = \frac{19}{20} - 1 = \frac{-1}{20}$

…(4)

Now

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (2x - x - 2y)dxdy$$

$$= \int_0^1 [(x^2 - x^2) - (x^3 - x^4)]dx = \int_0^1 (x^4 - x^3)dx$$

$$=\left(\frac{x^5}{5}+\frac{x^4}{4}\right)_0^1=\frac{1}{5}-\frac{1}{4}=\frac{-1}{20} \quad \ldots.(5)$$

From(4)and(5), We have $\oint_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

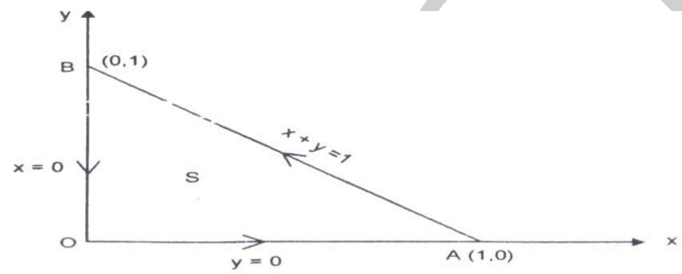
Hence the Green's Theorem is verified.

6. Verify Green's theorem for $\int_c [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by *x=0, y=0 and x+y=1*.

Sol : By Green's theorem, we have

$$\int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here M=$3x^2 - 8y^2$ and N=*4y-6xy*



$$\therefore \frac{\partial M}{\partial y} = -16y \text{ and } \frac{\partial N}{\partial x} = -6y$$

Now $\int_c M dx + N dy = \int_{OA} M dx + N dy + \int_{AB} M dx + N dy + \int_{BC} M dx + N dy \ldots(1)$

Along OA, y=0 $\therefore dy = 0$

$$\int_{OA} M dx + N dy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and y varies from 0 to 1.

$$\int_{AB} M dx + N dy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-1)]dy$$

$$=\int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$$

$$=\int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$$

$$=\frac{11}{3} + 2 - 3 = \frac{8}{3}$$

Along BO, x=0 $\therefore\ dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_c Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_{x=0}^{1}\int_{y=0}^{1-x} (-6y + 16y)dxdy$

$$=10\int_{x=0}^1 \left[\int_{y=0}^{1-x} ydy\right]dx = 10\int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$=5\int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$=-\frac{5}{3}[(1-1)^3 - (1-0)^3] =$$

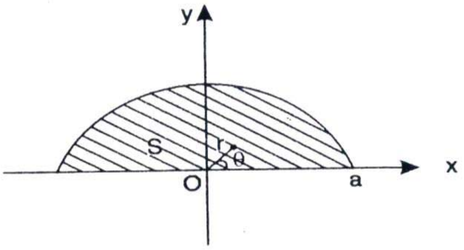
From (2) and (3), we have $\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Hence the Green's Theorem is verified.

7. Apply Green's theorem to evaluate $\oint_c (2x^2 - y^2)dx + (x^2 + y^2)\,dy, where\ c\ is$

the boundary of the area enclosed by the *x*-axis and upper half of the circle $x^2 + y^2 = a^2$

Sol: Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then



Figure

$$\frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = 2x$$

$\therefore$ By Green's Theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_R (2x + 2y)dxdy$$

$$= 2\iint_R (x + y)dy$$

$$= 2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta).rd\theta dr$$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$$\therefore \oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)d\theta$$

$$= 2.\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$$

8. Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$

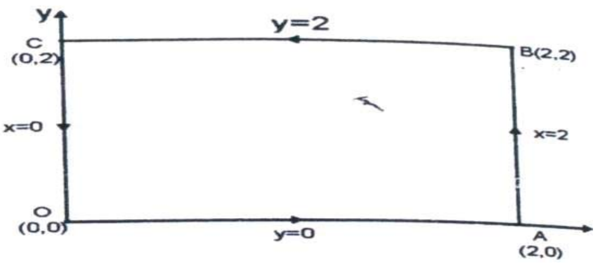
Where *C* is square with vertices (0,0), (2,0), (2,2), (0,).

Sol: The Cartesian form of Green's theorem in the plane is

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore \quad \frac{\partial M}{\partial y} = -3xy^2$ and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (Mdx + Ndy)$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along *OA(y=0)* (ii) along *AB(x=2)* (iii) along *BC(y=2)* (iv) along *CO(x=0).*

(i)Along *OA(y=0)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$$

…..(1)

(ii)Along *AB(x=2)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$$

….(2)

(iii)Along *BC(y=2)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_0^2 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3} \text{......(3)}$$

(iv)Along *CO(x=0)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 y^2\,dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

…..(4)

Adding(1),(2),(3) and (4), we get

$$\int_c \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \qquad \ldots(5)$$

Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here x ranges from 0 to 2 and y ranges from 0 to 2 .

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_0^2\int_0^2 (-2y + 3xy^2)dxdy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2}y^2\right)_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)dy = \left(-2y^2 + 2y^3\right)_0^2$$

$$= -8+16=8 \qquad \ldots(6)$$

From (5) and (6), we have

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Hence the Green's theorem is verified.

Stoke's Theorem (Transformation between Line Integral and Surface Integral)

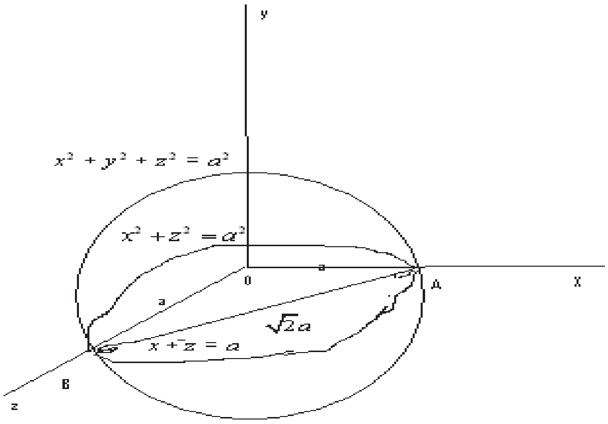
Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function then

$\oint_c \bar{F}.d\,\bar{r} = \int_s curl\ \bar{F}.\bar{n}\ ds$ where c is traversed in the positive direction and $\bar{n}$ is unit outward drawn normal at any point of the surface.

Solved Problems

1. Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ **where *c* is the curve of intersection of the sphere** $x^2 + y^2 + z^2 = a^2$ **and *x+z=a*.**

Sol: The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane *x+z=a* is a circle in the plane *x+z=a* with *AB* as diameter.



Equation of the plane is $x+z=a \Longrightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e., $A = (a, 0, 0)$ and B=(0,0,a)

$\therefore$ Length of the diameter AB $= \sqrt{a^2 + a^2 + 0} = a\sqrt{2}$

Radius of the circle, r=$\frac{a}{\sqrt{2}}$

Let $\bar{F}.d\bar{r} = ydx + zdy + xdz \Longrightarrow \bar{F}.d\bar{r} = \bar{F}.\left(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz\right) = ydx + zdy + xdz$

$\Longrightarrow \bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$

$$\therefore curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\left(\bar{\imath} + \bar{\jmath} + \bar{k}\right)$$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla_S}{|\nabla_S|}$

Then $s=x+z-a$, $\nabla S = \bar{i} + \bar{k}$ $\therefore$ $\bar{n} = \frac{\nabla_S}{|\nabla_S|} = \frac{\bar{\imath}+\bar{k}}{\sqrt{2}}$

Hence $\oint_c \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}\,ds$ (by Stokes Theorem)

$$=-\int(\bar{\imath}+\bar{\jmath}+\bar{k}).\left(\frac{\bar{\imath}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds$$

$$=-\sqrt{2}\int_s ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

2.Prove by Stokes theorem, $Curl\ grad\ \phi = \bar{0}$

Sol: Let S be the surface enclosed by a simple closed curve C.

$\therefore$ By Stokes theorem

$$\int_s(curl\ grand\phi).\bar{n}\,ds = \int_s(\nabla x\nabla\phi).\bar{n}\,dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$$

$$=\oint_c\left(\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right).\left(\bar{i}dx+\bar{j}dy+\bar{k}dz\right)$$

$$=\oint_c\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)=\int d\phi=[\phi]_p$$ where P is any point on C.

$$\therefore \int curl\ grad\phi.\bar{n}\,ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$$

3. Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc $x^2 + y^2 \le 1, z = 0.$

Sol: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \le 1, z = 0.$ We use the parametric co-ordinates x=cos $\theta, y = sin\theta, z = 0, 0 \le \theta \le 2\pi;$

dx=-sin$\theta\ d\theta$ and dy =cos$\theta\ d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1dx + F_2dy + F_3dz = \int_c -y^3dx + x^3dy$$

$$=\int_0^{2\pi}[-sin^3\theta(-sin\theta) + cos^3\theta cos\theta]d\theta = \int_0^{2\pi}(cos^4\theta + sin^4\theta)d\theta$$

$$=\int_0^{2\pi}(1 - 2sin^2\theta\ cos^2\theta)d\theta = \int_0^{2\pi}d\theta - \frac{1}{2}\int_0^{2\pi}(2sin\theta\ cos\theta)^2\,d\theta$$

$$=\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} \sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi}(1-\cos 4\theta)d\theta$$

$$=2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}\sin 4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$$

Put x=r cos$\emptyset$, $y = r\sin\emptyset \quad \therefore dxdy = rdr\, d\emptyset$

r is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$$\therefore \int (\nabla \times \bar{F}).\bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2.rdr\, d\emptyset = \frac{3\pi}{2}$$

L.H.S=R.H.S. Hence the theorem is verified.

4.Verify Stokes theorem for $\bar{F} = (2x-y)\bar{\imath} - yz^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection of the *xy*-plane.

Sol: The boundary C of S is a circle *in xy* plane i.e $x^2 + y^2$=1, *z=0*

The parametric equations are x=$\cos\theta, y = \sin\theta, \theta = 0 \rightarrow 2\pi$

$$\therefore \quad dx = -\sin\theta\, d\theta, dy = \cos\theta\, d\theta$$

$$\int_c \bar{F}.d\bar{r} = \int_c \bar{F_1}dx + \bar{F_2}dy + \bar{F_3}dz = \int_c (2x-y)dx - yz^2dy - y^2zdz$$

$$=\int_c (2x-y)dx(since\ z = 0\ and\ dz = 0)$$

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta - \int_0^{2\pi} \sin 2\theta d\theta$$

$$=\int_{\theta=0}^{2\pi} \frac{1-\cos 2\theta}{2}d\theta - \int_0^{2\pi} \sin 2\theta\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}.\cos 2\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}(2\pi-0)+0+\frac{1}{2}.(\cos 4\pi-\cos 0)=\pi$$

Again $\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}=\bar{\imath}(-2yz+2yz)-\bar{\jmath}(0-0)+\bar{k}(0+1)=\bar{k}$

$\therefore \int_S(\nabla\times\bar{F}).\bar{n}ds=\int_S\bar{k}.\bar{n}ds=\int_R\int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds=dxdy$

Now $\int\int_R dxdy=4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}dydx=4\int_{x=0}^{1}\sqrt{1-x^2}\,dx=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x\right]_0^1$

$$=4\left[\frac{1}{2}\sin^{-1}1\right]=2\frac{\pi}{2}=\pi$$

$\therefore$ *The* Stokes theorem is verified.

5.Evaluate by Stokes theorem $\oint_c(x+y)dx+(2x-z)dy+(y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Sol: Let $\bar{F}.d\bar{r}=\bar{F}.\left(\bar{\imath}dx+\bar{\jmath}dy+\bar{k}dz\right)=(x+y)dx+(2x-z)dy+(y+z)dz$

Then $\bar{F}=(x+y)\bar{\imath}+(2x-z)\bar{\jmath}+(y+z)\bar{k}$

By Stokes theorem, $\oint_c\bar{F}.d\bar{r}=\int\int_S curl\,\bar{F}.\bar{n}\,ds$

Where S is the surface of the triangle *OAB* which lies in the *xy* plane. Since the z Co-ordinates of $O, A\ and\ B$

Are zero. Therefore $\bar{n}=\bar{k}$. Equation of *OA* is *y=0* and

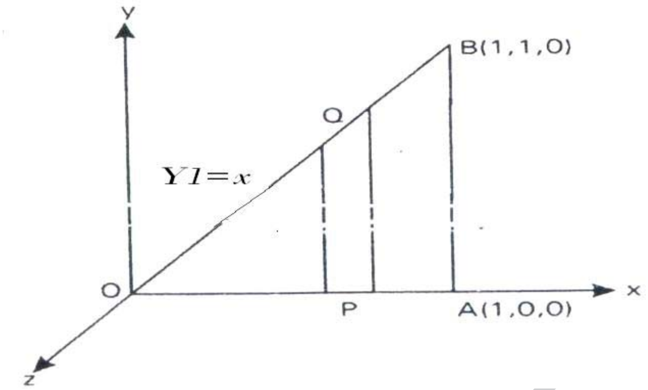
that of *OB*, *y=x* in the *xy* plane.

$\therefore\ curl\,\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x+y & 2x-z & y+z\end{vmatrix}=2\bar{\imath}+\bar{k}$

$\therefore\ curl\,\bar{F}.\bar{n}$ds=curl $\bar{F}.\bar{K}\ dx\,dy=dx\,dy$

$\therefore\oint_c\bar{F}.d\bar{r}=\int\int_s dx\,dy=\int\int_s dA=A=area\ of$ the $\Delta\ OAB$

$=\frac{1}{2}OA \times AB = \frac{1}{2}\times 1\times 1=\frac{1}{2}$



6: Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines x=$\pm a, y = 0, y = b$.

Sol: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0).

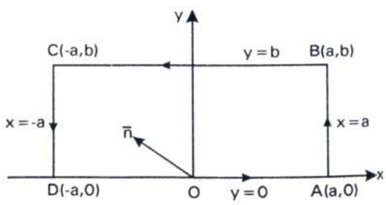
Equations of AB, BC, CD and DA are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r} = \int_S curl\ \bar{F}.\bar{n}ds$

$\oint_c \bar{F}.d\bar{r} = \oint_c \{(x^2 + y^2)\bar{i} - 2xy\bar{j}\}.\{\bar{i}dx + \bar{j}dy\}$

$=\oint_C (x^2 + y^2)\, dx - 2xydy$

$=\int_{AB} + \int_{Bc} + \int_{CD} + \int_{DA}$ …..(1)



(i) Along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\, dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

$$\text{from (1)}, \int_{BC} = \int_{x=a}^{x=-a} (x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$$

(iii) Along CD, x=-a, dx=0

$$\text{from (1)}, \int_{CD} = \int_{y=b}^{0} 2aydy = 2a\left[\frac{y^2}{2}\right]_{y=b}^{0} = -ab^2$$

(iv)Along DA, y=0, dy=0

$$\text{from (1)}, \int_{DA} = \int_{x=-a}^{x=a} x^2dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$$

(i)+(ii)+(iii)+(iv) gives

$$\therefore \oint_c \bar{F}.d\bar{r} = -ab^2 - \frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \qquad \text{….(2)}$$

Consider $\int_S curl\ \bar{F}.\bar{n}\, dS$

Vector Perpendicular to *the xy*-plane is $\bar{n} = k$

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$$

Since the rectangle lies in the *xy* plane,

$\bar{n} = \bar{k}$ and *ds* =*dx dy*

$$\int_S curl\ \bar{F}.\bar{n}dS = \int_S -4y\bar{k}.\bar{k}dx\ dy = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\ dx\ dy$$

$$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y\ dx\ dy = 4\int_{y=0}^{b} y[x]_{-a}^{a}\ dy = -4\int_{y=0}^{b} 2aydy$$

$$= -4a[y^2]_{y=0}^{b} = -4ab^2$$ ...(3).Hence from (2) and (3),

Stoke's theorem verified.

7.Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the *cube x =0, y=0, z=0, x=2, y=2,z=2* above the *xy* plane.

Sol: Given $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube.

x=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\ \bar{F}.\bar{n}ds = \int \bar{F}.d\bar{r}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & y + 4 & -xz \end{vmatrix} = \bar{\imath}(0 + y) - \bar{\jmath}(-z + 1) + \bar{k}(0 - 1) = y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}$$

$\therefore\ \nabla \times \bar{F}.\bar{n} = \nabla \times \bar{F}.k = \left(y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}\right).k = -1$

$\therefore \int \nabla \times \bar{F}.\bar{n}.ds = \int_0^2 \int_0^2 -1\ dx\ dy\ \ (\because z = 0, dz = 0) = -4$

…..(1)

To find $\int \bar{F}.d\bar{r}$

$\int \bar{F}.d\bar{r} = \int \left((y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}\right).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$

$= \int[(y - z + 2)dx + (yz + 4)dy - (xz)dz]$

Sis the surface of the cube above the *xy*-plane

$\therefore z = 0 \Rightarrow dz = 0$

$\therefore \int \bar{F}.d\bar{r} = \int (y + 2)dx + \int 4dy$

Along $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx = 2[x]_0^2 = 4$ ……..(2)

Along $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx = 4[x]_2^0 = -8$ ……. .(3)

Along $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y\ change\ from\ 0\ to\ 2.$

$\int \bar{F}.d\bar{r} = \int_0^2 4dy = [4y]_0^2 = 8$ ……(4)

Along $\overline{CO}, x = 0, z = 0, dx = 0, dz = 0, y\ change\ from\ 2\ to\ 0.$

$.\int_2^0 4dy = -8$ …..(5)

Above the surface When *z=2*

Along $0'A', \int_0^2 \bar{F}\,\overline{dr} = 0$ ….(6)

Along $A'B', x = 2, z = 2, dx = 0, dz = 0,$ y changes from 0 to 2

$$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y+4)dy = 2\left[\frac{y^2}{2}\right]_0^2 + 4[y]_0^2 = 4+8=12$$ ….(7)

Along $B'C', y = 2, z = 2, dy = 0, dz = 0,$ x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$ ….(8)

Along $C'D', x = 0, z = 2, dx = 0, dz = 0,$ y changes from 2 to 0.

$$\int_2^0 (2y+4) = 2\left[\frac{y^2}{2}\right]_2^0 + 4[y]_2^0 = -12$$ …..(9)

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$\int_C \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4$ …..(10)

By Stokes theorem, We have

$\int \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}ds = -4$

Hence Stoke's theorem is verified.