UNIT-1

MATHEMATICAL LOGIC

Propositional calculus

Statements and Notations:

A number of words making a complete gra-mmatical structure having a sense and meaning and also meant an assertion in logic or mathematics is called a sentence. This assertion may be of two types declarative and non-declarative.

Definition: A proposition (or) statement is a declarative sentence that is either true (or) false, but not both. The truth or falsity of a statement is called its truth value.

Examples:

(1) $8+3=11$

(2) Paris is in England

(3) where are you going ?

(4) $4-x=8$

(5) close the door

(6) what a hot day

The sentences (1) and (2) are statements, the first is true and the second is false.

(3) is a question, not a declarative sentence, hence it is not a statement.

(4) is a declarative sentence, but not a statement, since it is true or false depends on the value of x.

(5) is not a statement, it is a command.

(6) is not a statement, it is exclamation.

Note: Statements are denoted by P, Q, R, ... (or) p, q, r, ... which is also known as proposition variables. propositional variables can assume only two values, true (or) false. There are also two propositional constants, T and F, that represent true or false respectively. If p denotes the proposition " The Capital of U.P is Agra" then instead of saying the proposition " The Capital of U.P is Agra" is false, we can simply say that the value of p is F.

Connectives

A proposition consisting of only a single propositional variable (or) a single propositional constant is called an atomic (primary) proposition. A proposition obtained from the combination of two (or) more propositions by means of Logical operators (or) Connectives of two (or) more propositions (or) by negating a single proposition is called a molecular (or) Compound (or) Composite proposition.

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The words and Symbols used to form Compound proposition are called connectives. The following Symbols are used to represent connectives.

S.No	Symbol used	Connective word	Name of the compound statement formed by the connective	Symbolic form
1	$\neg$	not	Negation	$\sim p$
2	$\wedge$	and	Conjunction	$p \wedge q$
3	$\vee$	or	Disjunction	$p \vee q$
4	$\Rightarrow, \rightarrow$	if .... then	Implication (or) Conditional	$p \Rightarrow q$
5	$\Leftrightarrow, \leftrightarrow$	if and only if	Equivalence (or) Bi-Conditional	$p \Leftrightarrow q$

If p and q are any two statements then $\neg p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$ and $p \rightleftarrows q$ are also statements.

Negation : If p is any proposition, the negation of p, denoted by $\neg p$ (or) $\sim p$ and read as not p, is a proposition which is false when p is true and true when p is false.

For example, consider the statement p: Paris is in France. Then the negation of p is the statement $\sim p$: Paris is not in France.

The negation of the proposition q: No student is intelligent is $\sim q$: Some students are intelligent.

The truth table of $\sim p$ is

p	$\sim p$
T	F
F	T

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Find the negation of the following propositions.

(i) It is cold

(ii) Ravi is rich.

(iii) To day is Thursday.

(iv) There are 12 months in a year.

(v) Vijayawada is the capital of A.P.

Conjunction

If p and q are two statements, then Conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. The truth values of p and q ($p \wedge q$) are given in the following truth table.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Note: $p \wedge q$, $q \wedge p$ have same truth values.

Conjunction is Symmetric.

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Example: (1) The conjunction of

p : It is raining today.

q : There are 20 tables in this room is

$p \wedge q$: It is raining today and there are 20 tables in this room.

(2) If p : Jack went up the hill

q : Jill went up the hill then

$p \wedge q$: Jack and Jill went up the hill.

Disjunction

If p and q are two statements, then the disjunction of p and q is the compound statement denoted by $p \vee q$ and read as "p or q". The compound statement $p \vee q$ is true if at least one of p or q is true. It is false when both p and q are false. The truth values of $p \vee q$ are given the following truth table.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Note: $p \vee q$, $q \vee p$ have same truth values. Disjunction is Symmetric.

Example: 1) The Disjunction of

p : I shall watch the game on T.V

q : go to the stadium is $p \vee q$: I shall watch the game on T.V (or) go to the stadium.

2) There is something wrong with the bulb or wiring.

Conditional Proposition

If p and q are propositions the compound proposition "if p then q" denoted by $p \Rightarrow q$ is called the conditional proposition (or) Implication and the connective is the conditional connective.

The proposition 'p' is called antecedent (or) hypothesis and the proposition q is called the consequence (or) conclusion.

For example, consider if it rains then I will carry an umbrella.

Here p : It rains, is antecedent

q : I will carry an umbrella, is consequence.

The proposition $p \to q$ has a truth value F when p has the truth value T and q has the truth value F. otherwise its is truth value T.

The truth table is

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Example</u>: 1) If I get the book then I begin to read.

Here p : I get the book

q : I begin to read

Symbolic form is $p \rightarrow q$.

2) Express in English the statement $p \rightarrow q$ where

p : The sun rises in the East.

Q : $4+3 = 7$.

<u>Sol</u>: If the sun rises in the East then $4+3 = 7$.

3) Write the following statement in symbolic form.

<u>statement</u>: If either John prefers tea or Jim prefers coffee, then Rita prefers milk.

<u>Sol</u>:- p : John prefers tea

q : Jim prefers coeffee

r : Rita prefers milk.

Symbolic form is $(p \vee q) \rightarrow r$.

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Bi- Conditional statement

The compound statement formed by using the connective 'if and only if' is called Bi-conditional statement. If p and q are any two statements then the statement $p \rightleftarrows q$, which is read as p if and only if q is called a Bi-conditional statement. The truth value of the statement $p \rightleftarrows q$ is T whenever both p, q have identical truth values. The truth table is

p	q	$p \rightleftarrows q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: The statement "p if and only if q" may also be expressed as

p is necessary and sufficient for q

if p then q and conversely.

Exercises

1. Find the truth values of each of the following statements.

(i) Tirupati is in A.P if and only if $1+3=4$

(ii) Hyderabad is in karnataka if and only if $1+3=4$

(iii) Chennai is in Tamilnadu iff $4 \div 2 = 2$

(iv) New Delhi is the capital of SriLanka iff $5-2=3$.

Sol: (i) and (iii) are true.

and (ii) and (iv) are false.

(2) Determine the truth value of each of the following statements:

(i) Paris is in France and $2+2=4$

(ii) Paris is in France and $2+2=5$

(iii) Paris is in England and $2+2=4$

(iv) Paris is in England and $2+2=5$.

Sol: (i) T $(T \wedge T = T)$ (ii) F $(T \wedge F = F)$

(iii) F $(F \wedge T = F)$ (iv) F $(F \wedge F = F)$

(3) Determine the truth value of each of the following statements:

(i) $1+1=5$ (or) $2+2=4$

(ii) $7+1=5$ or $3+7=8$

(iii) $1+1=5$ or $3+3=4$

(iv) $2+5=7$ or $1+7=8$

Sol: (i) T $(F \vee T = T)$

(ii) F $(F \vee F = F)$

(iii) F $(F \vee F = F)$

(iv) T $(T \vee T = T)$

4. Let P : Ravi is rich , Q : Ravi is happy
Write each of the following in Symbolic form
(i) Ravi is poor but happy
(ii) Ravi is neither rich nor happy .
(iii) Ravi is rich and unhappy .

Sol :- (i) $\neg P \wedge Q$ (ii) $\neg P \wedge \neg Q$ (iii) $P \wedge \neg Q$

5. Write the Symbolic statement of
" If Rita and Sita go to I.T Camp and jim and John go to P.C Camp then the College gets the good name".

Sol: Consider p : Rita go to I.T Camp
q : Sita go to I.T camp
r : jim go to P.C Camp
s : John go to P.C Camp
t : college gets good name .

Symbolic form of given statement is
$(p \wedge q) \wedge (r \wedge s) \rightarrow t$.

6. Write the following statements in Symbolic form.
(i) If the Sun is shining today then $2+3>4$.
(ii) The crop will be destroyed if there is a flood.

Sol: (i) Take p: The sun is shining today.

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$q: 2+3 > 4$

Given statement can be written as in Symbolic form as $p \to q$.

(ii) Consider p: The crop will be destroyed

q: There is a flood.

Given statement can be written as in Symbolic form as $q \to p$.

Well formed formulas

Statement formulas: A statement formula is an expression denoted by a string consisting of variables, parentheses and connective symbols.

Well-formed formulas:

A well-formed formula can be generated by the following Rules:

(i) A statement variable standing alone is a well-formed formula.

(ii) If A is a well formed formula, then $\neg A$ is a well formed formula.

(iii) If A and B are well-formed formulas then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \leftrightarrow B)$ are well-formed formulas.

(iv) A string of symbols containing the statement variables, connectives and parentheses is a well-formed formula if and only if it can be obtained by finite applications of rules (1), (2) and (3).

<u>Example</u>: From the formulas given below select those which are well-formed formula according to the definition.

(1) $(p \to (p \vee q))$ ✓

(2) $((p \to (\sim p)) \to (\sim p))$ ✓

(3) $((p \to (q \to r)) \to ((p \to q) \to (p \to r)))$

It is not well-formed formula, because one of the parentheses in the beginning is missing.

(4) $((\sim p \to q) \to (q \to p)))$.

It is not well-formed formula, because one more parentheses in the end.

(5) $((p \wedge q) \rightleftarrows p)$ ✓

Truth tables

The truth value of a proposition is either true (denoted by T) or false (denoted by F). A truth table is a table that shows the truth value of a compound proposition for all possible cases.

1. Construct the truth table for $\sim p \wedge q$.

Sol:-

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

2. Construct the table for $(p \vee q) \vee \sim p$.

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

3. Construct the truth tables for the following formulas.

(i) $\sim(\sim p \vee \sim q)$ (ii) $\sim(\sim p \wedge \sim q)$

4. Construct the truth tables for the following formulas:

(i) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$

(ii) $(p \vee q) \wedge (r \vee s)$.

5. Construct a truth table for each of the following compound statements

(i) $(p \rightarrow q) \wedge (\sim p \rightarrow q)$ (ii) $p \rightarrow (\sim q \vee r)$

Sol: 4 (ii)

p	q	r	s	$p \vee q$	$\sim r \vee s$	$(p \vee q) \wedge (\sim r \vee s)$
T	T	T	T	T	T	T
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	T	F	F	T	F	F
F	F	T	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	T	F
F	F	F	F	F	F	F

5 (ii)

p	q	r	$\sim q$	$\sim q \vee r$	$p \rightarrow (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

6. Construct the truth table for the following :

$[(p \vee q) \wedge (\sim r)] \leftrightarrow (q \rightarrow r)$.

Tautologies

<u>Definition</u> : A compound proposition that is always true for all possible truth values of its variables or, in otherwords contain only T in the last column of its truth table is called a tautology. A compound proposition that is always false for all possible truth values of its variables or in otherwords contain only F in the last column of its truth table is called a contradiction. A proposition that is neither a tautology nor a contradiction is called a contingen-cy.

propositions like

(i) The professor is either a woman or a man.

(ii) People either like watching TVs or they don't.

are always true and are called tautologies.

<u>Example</u>: $p \vee \sim p$ is a tautology since it always has truth value T.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Propositions like

(i) x is prime and x is an even integer greater than 8.

(ii) All men are good and all men are bad are always false and are called contradictions.

<u>Example</u>: $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

<u>Exercise</u>

Indicate which of the following formulas are tautologies or contradictions.

(i) $(p \to (p \vee q))$

(ii) $((p \vee q) \to p)$

(iii) $((p \to \sim p) \to \sim p)$

(iv) $((\sim q \wedge p) \wedge q)$

(v) $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$.

(vi) $(\sim p \to q) \to (q \to p)$

(vii) $((p \wedge q) \leftrightarrow p)$

<u>Sol</u>:- (i) The truth table for $(p \to (p \vee q))$ is

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since all the entries in the last column of the truth table of $(p \rightarrow (p \vee q))$ are true.

Hence $(p \rightarrow (p \vee q))$ is a tautology.

(ii) The truth table for $((p \vee q) \rightarrow p)$

p	q	$p \vee q$	$p \vee q \rightarrow p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since all the entries in the last column of the truth table of $((p \vee q) \rightarrow p)$ are true and false.

Hence $((p \vee q) \rightarrow p)$ is a contingency.

(iii) The truth table for $((\sim q \wedge p) \wedge q)$

p	q	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Since all the entries in the last column of the truth table of $((\sim q \wedge p) \wedge q)$ are false.

Hence $((\sim q \wedge p) \wedge q)$ is a contradiction.

Equivalence formulas

The propositions p and q are said to be logically equivalent if $p \leftrightarrow q$ is a tautology and is denoted by $p \Leftrightarrow q$ (or) $p \equiv q$

Example : $\sim(\sim p)$ is equivalent to p.

$p \wedge p$ is equivalent to p.

p	$\sim p$	$\sim(\sim p)$	$p \leftrightarrow \sim(\sim p)$
T	F	T	T
F	T	F	T

$p \leftrightarrow \sim(\sim p)$ is a tautology.

Hence $\sim(\sim p) \equiv p$.

p	p	$p \wedge p$	$p \leftrightarrow p \wedge p$
T	T	T	T
F	F	F	T

$p \leftrightarrow p \wedge p$ is a tautology.

Hence $p \wedge p \equiv p$.

Note : $\Leftrightarrow$ is only symbol, but not connective symbol.

1. Show that the propositions are $p \to q$ and $\sim p \vee q$ are logically equivalent.

(OR)

prove that $p \to q \equiv \sim p \vee q$.

Sol: The truth table for $(p \to q) \leftrightarrow (\sim p \vee q)$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology.

Hence, $(p \to q) \equiv (\sim p \vee q)$.

2. Prove that $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

3. Show the following equivalences

(i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

(ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$.

(iii) $\sim (p \to q) \equiv p \wedge \sim q$

(iv) $\sim (p \rightleftarrows q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

Sol:- (iv) Let $A = (p \wedge \sim q) \vee (\sim p \wedge q)$, $B = \sim (p \rightleftarrows q)$.

To show A and B are logically equivalent.

i.e $A \leftrightarrow B$ is a tautology.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	A	$p \rightleftarrows q$	B	$A \leftrightarrow B$
T	T	F	F	F	F	F	T	F	T
T	F	F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	F	T	T
F	F	T	T	F	F	F	T	F	T

All the truth values of $(A \leftrightarrow B)$ are true, hence $A \leftrightarrow B$ is a tautology.

$\therefore A \equiv B$.

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4. Show that p is equivalent to the following formulas $\sim(\sim p)$, $p \wedge p$, $p \vee p$, $p \vee (p \wedge q)$, $p \wedge (p \vee q)$, $(p \wedge q) \vee (p \wedge \sim q)$, $(p \vee q) \wedge (p \vee \sim q)$.

Sol:

p	$\sim p$	$\sim(\sim p)$	$p \wedge p$	$p \vee p$
T	F	T	T	T
F	T	F	F	F

$\therefore p \Leftrightarrow \sim(\sim p) \Leftrightarrow p \wedge p \Leftrightarrow p \vee p$.

Consider
A : $(p \vee (p \wedge q))$
B : ~~$p \vee$~~ $p \wedge (p \vee q)$
C : $(p \wedge q) \vee (p \wedge \sim q)$
D : $(p \vee q) \wedge (p \vee \sim q)$.

p	q	$p \wedge q$	$p \vee q$	$\sim q$	$p \wedge \sim q$	$p \vee \sim q$	A	B	C	D
T	T	T	T	F	F	T	T	T	T	T
T	F	F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	F	F	F
F	F	F	F	T	F	T	F	F	F	F

$\therefore p \Leftrightarrow A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$

Hence, all the formulas are equivalent.

Equivalent formulas

Let p, q and r be any three statements. Then all possible formulas may be written as

1. $p \vee p \Leftrightarrow p$, $p \wedge p \Leftrightarrow p$ (Idempotent laws)
2. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
   $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ (Associative laws)
3. $p \vee q \Leftrightarrow q \vee p$
   $p \wedge q \Leftrightarrow q \wedge p$ (Commutative laws).
4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ (Distributive laws)
5. $p \wedge T \Leftrightarrow p$, $p \vee F \Leftrightarrow p$ (Identity laws)
6. $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$ (Domination laws)
7. $p \vee \sim p \Leftrightarrow T$, $p \wedge \sim p \Leftrightarrow F$ (Negation laws)
8. $\sim(\sim p) \Leftrightarrow p$ (Double Negation law)
9. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
   $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ (Demorgan's laws)
10. $p \vee (p \wedge q) \Leftrightarrow p$
    $p \wedge (p \vee q) \Leftrightarrow p$ (Absorption laws).

* Use truth tables to prove the distributive law,

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

Sol:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical.

$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$ is a tautology.

Therefore $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

i.e $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

<u>Replacement process</u>:

consider the formula $A : p \rightarrow (q \rightarrow r)$.

The formula $q \rightarrow r$ is a part of the formula A. If we replace $q \rightarrow r$ by an equivalent formula $\sim q \vee r$ in A, we get another formula $B : p \rightarrow (\sim q \vee r)$. Here, we can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A is known as the Replacement process.

1. Show that $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r)$
$\Leftrightarrow (p \wedge q) \to r$

Sol: Consider $p \to (q \to r)$

$\Leftrightarrow p \to (\sim q \vee r)$ $\quad (\because q \to r \Leftrightarrow \sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$

$\Leftrightarrow (\sim p \vee \sim q) \vee r$ $\quad$ (Associative Law)

$\Leftrightarrow \sim (p \wedge q) \vee r$ $\quad$ (Demorgan's Law)

$\Leftrightarrow p \wedge q \to r$.

Hence, $p \to (q \to r) \Leftrightarrow (p \to (\sim q \vee r)) \Leftrightarrow (p \wedge q) \to r$

2. Show that the following equivalences

(i) $p \to (q \to p) \Leftrightarrow \sim p \to (p \to q)$.

(ii) $p \to (q \vee r) \Leftrightarrow (p \to q) \vee (p \to r)$

Sol:- (i) Consider $p \to (q \to p)$

$\Leftrightarrow p \to (\sim q \vee p)$

$\Leftrightarrow \sim p \vee (\sim q \vee p)$

$\Leftrightarrow (\sim p \vee \sim q) \vee p$ $\quad$ (Associative law)

$\Leftrightarrow p \vee (\sim p \vee \sim q)$ $\quad$ (Commutative law)

$\Leftrightarrow (p \vee \sim p) \vee \sim q$ $\quad$ ($\because$ Associative law)

$\Leftrightarrow T \vee \sim q$ $\quad$ (Negation law)

$\Leftrightarrow T$ $\quad$ (Domination law)

Also, $\sim p \to (p \to q)$

$\Leftrightarrow \sim p \to (\sim p \vee q)$ $\quad (\because p \to q \Leftrightarrow \sim p \vee q)$

$\Leftrightarrow \sim (\sim p) \vee (\sim p \vee q)$

$\Leftrightarrow p \vee (\sim p \vee q)$ $\quad$ (Double Negation law).

$\Leftrightarrow (p \vee \sim p) \vee q$ $\quad$ (Associative law)

$\Leftrightarrow T \vee q \Leftrightarrow T$ ($\because$ Domination law)

Hence, $p \rightarrow (q \rightarrow p) \Leftrightarrow \sim p \rightarrow (p \rightarrow q)$.

(ii) Consider $p \rightarrow (q \vee r)$

$\Leftrightarrow \sim p \vee (q \vee r)$

$\Leftrightarrow (\sim p \vee \sim p) \vee (q \vee r)$

$\Leftrightarrow (\sim p \vee q) \vee (\sim p \vee r)$

$\Leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$

Hence, $p \rightarrow (q \vee r) \Leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$.

(3) Show that $(\sim p \wedge (\sim q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r))$ $\Leftrightarrow r$ by using Replacement process.

<u>Sol:</u>

Consider, $(\sim p \wedge (\sim q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r))$

$\Leftrightarrow ((\sim p \wedge \sim q) \wedge r) \vee ((q \wedge r) \vee (p \wedge r))$ ($\because$ Associative)

$\Leftrightarrow (\sim (p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$ ($\because$ Demorgans Law Distributive)

$\Leftrightarrow (\sim (p \vee q) \wedge r) \vee ((p \vee q) \wedge r)$ ($\because$ Commutative law)

$\Leftrightarrow (\sim (p \vee q) \vee (p \vee q)) \wedge r$ (Distributive law).

$\Leftrightarrow T \wedge r$ (by negation Law)

$\Leftrightarrow r$ (Identity law).

Hence, $(\sim p \wedge (\sim q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r)) \Leftrightarrow r$.

(4) Show that $\sim (p \rightleftarrows q) \Leftrightarrow (p \vee q) \wedge \sim (p \wedge q)$.

Sol: Consider, $\sim(p \rightleftarrows q)$

$\Leftrightarrow \sim((p \to q) \wedge (q \to p))$

$\Leftrightarrow \sim((\sim p \vee q) \wedge (\sim q \vee p))$

$\Leftrightarrow \sim(((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$ ($\because$ Distributive Law)

$\Leftrightarrow \sim((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))$ ($\because$ Distributive law).

$\Leftrightarrow \sim(((\sim p \wedge \sim q) \vee F) \vee (F \vee (q \wedge p))$ ($\because p \wedge \sim p \Leftrightarrow F$)

$\Leftrightarrow \sim(\sim(p \vee q) \vee F \vee (p \wedge q))$ (De morgan Law) Commutative

$\Leftrightarrow \sim(\sim(p \vee q) \vee (p \wedge q))$ ($F \vee P \equiv P$)

$\Leftrightarrow (p \vee q) \wedge \sim(p \wedge q)$.

Hence, $\sim(p \rightleftarrows q) \Leftrightarrow (p \vee q) \wedge \sim(p \wedge q)$.

5\. * Are $(p \to q) \to r$ and $p \to (q \to r)$ logically equivalent? Justify your answer by using the rules of Logic to simply both expressions and also by using truth tables.

Sol: $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent because

Consider $(p \to q) \to r$

$\Leftrightarrow (\sim p \vee q) \to r$ ($p \to q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim(\sim p \vee q) \vee r$.

$\Leftrightarrow (\sim(\sim p) \wedge \sim q) \vee r$.

$\Leftrightarrow (p \wedge \sim q) \vee r$

$\Leftrightarrow (p \wedge r) \vee (\sim q \wedge r)$

Also, $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$.

$\Leftrightarrow \sim p \vee \sim q \vee r$.

Truth table Method :

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Hence the truth values of $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ are not identical.

Duality Law

Two formulas A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other. If the formula A contains the special variables T or F, then

A^* its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Ex: 1. Write the duals of (i) $(p \vee q) \wedge r$
(ii) $(p \wedge q) \wedge T$

Sol: Duals are (i) $(p \wedge q) \vee r$
ii) $(p \vee q) \vee F$.

2. If the formula A is given by $A: \sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim r))$, then find A^*.

Sol: Dual of A is $A^* : \sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim r))$.

Tautological Implications

· In $p \to q$, the statement p is called antecedent and q is called consequent.

For any statement formula $p \to q$, the statement formula $q \to p$ is called its converse.

$\sim p \to \sim q$ is called the inverse.

And $\sim q \to \sim p$ is called the contra-positive.

Ex:- Write the converse, inverse and contra-positive of the following proposition.

* If ΔABC is a right triangle then $|AB|^2 + |BC|^2 = |AC|^2$.

Converse: If $|AB|^2 + |BC|^2 = |AC|^2$ then ΔABC is a right triangle.

Inverse: If ΔABC is not a right triangle, then $|AB|^2 + |BC|^2 \neq |AC|^2$

Contra-positive: If $|AB|^2 + |BC|^2 \neq |AC|^2$ then ΔABC is not a right triangle.

Definition: A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology. We shall denote it by $A \Rightarrow B$, which is read as "A implies B".

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

* Prove that $(p \to q) \Rightarrow (\sim q \to \sim p)$.

Sol:

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \to (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true.

$(p \to q) \to (\sim q \to \sim p)$ is a tautology.

Hence $(p \to q) \Rightarrow (\sim q \to \sim p)$.

Normal Forms

In this section, we will denote the word "Product" in place of "Conjunction" and "sum" in place of "disjunction".

Definition: A product of the variables and their negations in a formula is called an elementary product.

Example: Let p, q be any two atomic variables. Then p, $\sim p$, $\sim p \wedge q$, $\sim q \wedge p$, $\sim q \wedge p \wedge \sim p$, $p \wedge \sim p$, $\sim p \wedge \sim q$ are some examples of elementary products.

Definition: A sum of the variables and their negations in a formula is called an elementary sum.

Example: Let p, q be any two atomic variables. Then p, $\sim p$, $\sim p \vee q$, $\sim q \vee p$, $p \vee \sim p$, $\sim q \vee p \vee \sim p$ are some examples of elementary sums.

Definition: Any part of an elementary sum (product) which is itself an elementary sum (product) is called a factor of the original sum (product).

Example: q, $q \vee \sim p$, $\sim p \vee \sim q$ are some of the factors of $q \vee \sim p \vee \sim q$ and $\sim q$, $p \wedge \sim p$, $\sim q \wedge p$

are some of the factors of $\sim q \wedge p \wedge \sim p$.

Disjunctive Normal Form (d.n.f)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

Procedure to obtain a disjunctive normal form of a given logical expression:

Three steps are required to obtain a disjunctive normal form through algebraic manipulations.

1. Remove all $\Rightarrow$ and $\Leftrightarrow$ by an equivalent expression containing the connectives $\wedge, \vee, \sim$ only.
2. Eliminate $\sim$ before sums and products by using the double negation or by using De Morgan's Laws.
3. Apply the distributive law until a sum of elementary product is obtained.

Note: 1) The d.n.f of a given formula is not unique because different d.n.f's can be obtained for given formula if the distributive laws are applied in different ways.

2) Extended distributive law

$(p \wedge q) \vee (r \wedge s) = (p \wedge r) \vee (p \wedge s) \vee (q \wedge r) \vee (q \wedge s)$ is is very useful to obtain the disjunctive normal form:

*1. Obtain the disjunctive normal forms of the following.

(i) $p \wedge (p \to q)$

(ii) $\sim (p \to (q \wedge r))$

(iii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$.

Sol:- (i) $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q)$

$\Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$ (∵ distributive)

which is the required disjunctive normal form.

(ii) $\sim (p \to (q \wedge r))$

$\Leftrightarrow \sim (\sim p \vee (q \wedge r))$ (∵ $p \to q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim(\sim p) \wedge (\sim q \vee \sim r)$

$\Leftrightarrow p \wedge (\sim q \vee \sim r)$

$\Leftrightarrow (p \wedge \sim q) \vee (p \wedge \sim r)$ (∵ distributive).

Which is the required disjunctive normal form.

(iii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$

$\Leftrightarrow (\sim (p \vee q) \wedge (p \wedge q)) \vee (\sim(\sim(p \vee q)) \wedge \sim (p \wedge q))$

(∵ $r \rightleftarrows s \Leftrightarrow (r \wedge s) \vee (\sim r \wedge \sim s)$).

$\Leftrightarrow ((\sim p \wedge \sim q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge (\sim p \vee \sim q))$

(∵ by De morgans Law)

$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee ((p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q))$

(∵ by distributive Law)

$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$

which is the required d.n.f of the given formula.

2. obtain disjunctive normal form of

$p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

Sol:-

$p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

$\Leftrightarrow \sim p \vee ((p \to q) \wedge \sim(\sim q \vee \sim p))$ $(\because p \to q \Leftrightarrow \sim p \vee q)$

$\Leftrightarrow \sim p \vee ((\sim p \vee q) \wedge (q \wedge p))$ ($\because$ double negation law).

$\Leftrightarrow \sim p \vee ((\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p))$ ($\because$ distributive law).

$\Leftrightarrow \sim p \vee (\sim p \wedge p \wedge q) \vee (q \wedge p)$ $(\because F \wedge q \Leftrightarrow F$, $p \vee F \Leftrightarrow p)$

$\Leftrightarrow \sim p \vee (F \wedge q) \vee (p \wedge q) \Leftrightarrow \sim p \vee (p \wedge q)$.

which the required d.n.f of the given formula.

Conjunctive Normal Form (c.n.f)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

The procedure of obtaining a c.n.f of a given formula is similar to the one given for d.n.f.

Example: $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q)$

which is the required c.n.f

Note: The conjunctive normal form is not unique.

1. obtain c.n.f of the following formulas:

(i) $((p \to q) \wedge \sim q) \to \sim p$ (ii) $((p \to q) \wedge \sim p) \to \sim q$.

Sol: (i) $((p \to q) \wedge \sim q) \to \sim p$

$\Leftrightarrow ((\sim p \vee q) \wedge \sim q) \to \sim p$

$\Leftrightarrow \sim((\sim p \vee q) \wedge \sim q) \vee \sim p$

$\Leftrightarrow (\sim(\sim p \vee q) \vee \sim(\sim q)) \vee \sim p$

$\Leftrightarrow ((p \wedge \sim q) \vee q) \vee \sim p$

$\Leftrightarrow ((p \vee q) \wedge (\sim q \vee q)) \vee \sim p$

$\Leftrightarrow (p \vee q \vee \sim p) \wedge (\sim q \vee q \vee \sim p)$

which is the required c.n.f.

(ii) $((p \to q) \wedge \sim p) \to \sim q$

$\Leftrightarrow ((\sim p \vee q) \wedge \sim p) \to \sim q$

$\Leftrightarrow \sim((\sim p \vee q) \wedge \sim p) \vee \sim q$

$\Leftrightarrow \sim(\sim p \vee q) \vee \sim(\sim p)) \vee \sim q$

$\Leftrightarrow ((p \wedge \sim q) \vee p) \vee \sim q$

$\Leftrightarrow ((p \vee p) \wedge (\sim q \vee p)) \vee \sim q$

$\Leftrightarrow (p \vee p \vee \sim q) \wedge (\sim q \vee p \vee \sim q)$

$\Leftrightarrow (p \vee \sim q) \wedge (\sim q \vee p)$

which is the required c.n.f.

2. obtain c.n.f. of the following

$\sim(p \vee q) \rightleftarrows (p \wedge q)$

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Principal Disjunctive normal form
(p.d.n.f)

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ and $\sim p \wedge \sim q$ are called minterms of p and q. It may be noted that none of the minterms should contain both a variable and its negation. For given two variables, there are 2^2 minterms. The number of minterms in n variables are 2^n. For example, minterms for the three variables p, q and r are

$p \wedge q \wedge r$, $p \wedge q \wedge \sim r$, $p \wedge \sim q \wedge r$, $p \wedge \sim q \wedge \sim r$
$\sim p \wedge q \wedge r$, $\sim p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge r$, $\sim p \wedge \sim q \wedge \sim r$

<u>Definition</u>: Principal disjunctive normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms only. This is also called the <u>Sum of products canonical form</u>.

<u>Methods to obtain P.d.n.f of a given formula</u>:

1. <u>Truth table method</u>: For every truth value T of the given formula, select the minterm which is also has the value T for the same combination of the

truth values of the variables.

* obtain p.d.n.f of each of the following

(i) $p \to q$ (ii) $p \vee q$ (iii) $\sim(p \wedge q)$.

Sol:- The truth table is

p	q	$p \to q$	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	T

Hence $p \to q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$.

$p \vee q \Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$.

$\sim(p \wedge q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

which are p.d.n.forms of given formulas.

* obtain the p.d.n.f of the following formula.

$p \vee (\sim p \to (q \vee (\sim q \to r)))$.

Sol: Consider $A : \sim p \to (q \vee (\sim q \to r))$

$B : p \vee (\sim p \to (q \vee (\sim q \to r)))$.

p	q	r	$\sim p$	$\sim q$	$\sim q \to r$	$q \vee (\sim q \to r)$	A	B	Necessary minterms
T	T	T	F	F	T	T	T	T	$p \wedge q \wedge r$
T	T	F	F	F	T	T	T	T	$p \wedge q \wedge \sim r$
T	F	T	F	T	T	T	T	T	$p \wedge \sim q \wedge r$
T	F	F	F	T	F	F	T	T	$p \wedge \sim q \wedge \sim r$
F	T	T	T	F	T	T	T	T	$\sim p \wedge q \wedge r$
F	T	F	T	F	T	T	T	T	$\sim p \wedge q \wedge \sim r$
F	F	T	T	T	T	T	T	T	$\sim p \wedge \sim q \wedge r$
F	F	F	T	T	F	F	F	F	—

Therefore, p.d.n.f of $p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r)))$ is $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$.

2. Replacement Method (without constructing truth table): To obtain the p.d.n.f of a given formula without using its truthtable, we use the following steps.

1. First replace the Conditionals and BiConditionals by their equivalent formulas Containing only $\wedge, \vee$ and $\sim$.
2. The negations are applied to the variables by using De morgan's Laws followed by the application of distributive laws.
3. Any elementary product which is a Contradiction is droped.
4. Min terms are obtained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

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truth values of the variables.

* obtain p.d.n.f of each of the following

(i) $p \to q$ (ii) $p \vee q$ (iii) $\sim(p \wedge q)$

Sol:- The truth table is

p	q	$p \to q$	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	T

Hence $p \to q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$.

$p \vee q \Leftrightarrow (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$.

$\sim(p \wedge q) \Leftrightarrow (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

which are p.d.n.forms of given formulas.

* obtain the p.d.n.f of the following formula.

$p \vee (\sim p \to (q \vee (\sim q \to r)))$.

Sol: Consider $A : \sim p \to (q \vee (\sim q \to r))$

$B : p \vee (\sim p \to (q \vee (\sim q \to r)))$.

p	q	r	$\sim p$	$\sim q$	$\sim q \to r$	$q \vee (\sim q \to r)$	A	B	Necessary minterms
T	T	T	F	F	T	T	T	T	$p \wedge q \wedge r$
T	T	F	F	F	T	T	T	T	$p \wedge q \wedge \sim r$
T	F	T	F	T	T	T	T	T	$p \wedge \sim q \wedge r$
T	F	F	F	T	F	F	T	T	$p \wedge \sim q \wedge \sim r$
F	T	T	T	F	T	T	T	T	$\sim p \wedge q \wedge r$
F	T	F	T	F	T	T	T	T	$\sim p \wedge q \wedge \sim r$
F	F	T	T	T	T	T	T	T	$\sim p \wedge \sim q \wedge r$
F	F	F	T	T	F	F	F	F	—

1. Obtain the p.d.n.f. for the following formulas, without using truth tables.

(i) $p \rightleftarrows q$ (ii) $\sim p \vee q$ (iii) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$.

Sol: (i) $p \rightleftarrows q$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim q \vee p)$

$\Leftrightarrow ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$

$\Leftrightarrow ((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))$.

$\Leftrightarrow (\sim p \wedge \sim q) \vee F \vee F \vee (p \wedge q)$

$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

which is the required p.d.n.f.

(ii) $\sim p \vee q \Leftrightarrow (\sim p \wedge T) \vee (T \wedge q)$ $(\because \sim p \Leftrightarrow \sim p \wedge T)$

$\Leftrightarrow (\sim p \wedge (q \vee \sim q)) \vee ((p \vee \sim p) \wedge q)$

$\Leftrightarrow ((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee ((p \wedge q) \vee (\sim p \wedge q))$

$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

which is the required p.d.n.f.

(iii) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

$\Leftrightarrow ((p \wedge q) \wedge T) \vee ((\sim p \wedge r) \wedge T) \vee ((q \wedge r) \wedge T)$

$\Leftrightarrow ((p \wedge q) \wedge (r \vee \sim r)) \vee ((\sim p \wedge r) \wedge (q \vee \sim q)) \vee ((q \wedge r) \wedge (p \vee \sim p))$.

$\Leftrightarrow ((p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r)) \vee ((\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)) \vee ((q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p))$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$.

which is the required p.d.n.f.

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Principal Conjunctive Normal Form (P.C.N.F)

The dual of a minterm is called a maxterm. For a given number of variables the maxterm consists of disjunctions in which each variable or its negation, but not, appears only once.

For two variables p and q, there are 2^2 maxterms given by $p \vee q$, $p \vee \neg q$, $\neg p \vee q$, $\neg p \vee \neg q$.

Maxterms for three variables p, q and r are

$p \vee q \vee r$, $p \vee q \vee \neg r$, $p \vee \neg q \vee r$, $p \vee \neg q \vee \neg r$

$\neg p \vee q \vee r$, $\neg p \vee q \vee \neg r$, $\neg p \vee \neg q \vee r$, $\neg p \vee \neg q \vee \neg r$.

Each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Definition: Principal conjunctive normal form of a given formula can be defined as an equivalent formula consists of conjunctive of maxterms only. This is also called the Product of Sums Canonical form. The process for obtaining P.C.N.F is similar to the one followed for P.D.N.F. For obtaining P.C.N.F of α, one can also construct the P.D.N.F of $\neg \alpha$ and apply negation.

1. obtain the principal conjunctive normal forms of the following

(a) $p \wedge q$ using truth table.

(b) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p)$ without using truth table.

Sol:- a) The truth table of $p \wedge q$ is

p	q	$p \wedge q$	Necessary Maxterms
T	T	T	—
T	F	F ✓	$p \vee \sim q$
F	T	F ✓	$\sim p \vee q$
F	F	F ✓	$\sim p \vee \sim q$

$\therefore p \wedge q \Leftrightarrow (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

which is the required p.c.n.f.

(b) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p)$

$\Leftrightarrow (\sim(\sim p) \vee r) \wedge ((q \rightarrow p) \wedge (p \rightarrow q))$

$\Leftrightarrow (p \vee r) \wedge ((\sim q \vee p) \wedge (\sim p \vee q))$

$\Leftrightarrow [(p \vee r) \vee F] \wedge [(\sim q \vee p) \vee F] \wedge [(\sim p \vee q) \vee F]$ $[\because p \vee F \equiv p]$

$\Leftrightarrow [(p \vee r) \vee (q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)]$.

$\Leftrightarrow [(p \vee r \vee q) \wedge (p \vee r \vee \sim q)] \wedge [(\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)] \wedge [(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)]$.

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

which is the required p.c.n.f.

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2. obtain the PCNF of the following formula $(\sim p \to r) \wedge (q \to p)$ by (i) using Truth table (ii) without using Truth table.

Sol: The truth table is

p	q	r	$\sim p$	A $\sim p \to r$	B $q \to p$	$A \wedge B$	Necessary maxterms
T	T	T	F	T	T	T	—
T	T	F	F	T	T	T	—
T	F	T	F	T	T	T	—
T	F	F	F	T	T	T	—
F	T	F	T	F	F	F ✓	$\sim p \vee q \vee \sim r$
F	T	T	T	T	F	F ✓	$\sim p \vee q \vee r$
F	F	T	T	T	T	T	—
F	F	F	T	F	T	F ✓	$\sim p \vee \sim q \vee \sim r$

$$(\sim p \to r) \wedge (q \to p) \Leftrightarrow (\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee \sim r) \wedge (\sim p \vee \sim q \vee \sim r)$$

which the required p.c.n.f.

(ii) $(\sim p \to r) \wedge (q \to p)$

$\Leftrightarrow (\sim(\sim p) \vee r) \wedge (\sim q \vee p)$

$\Leftrightarrow ((p \vee r) \vee T) \wedge ((\sim q \vee p) \vee T)$

$\Leftrightarrow ((p \vee r) \vee (q \wedge \sim q)) \wedge ((\sim q \vee p) \vee (r \wedge \sim r))$

$\Leftrightarrow ((p \vee r \vee q) \wedge (p \vee r \vee \sim q)) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r)$

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Theory of Inference for Statement Calculus

Consider a set of propositions $p_1, p_2, \ldots\ldots p_n$ and a proposition q. Then a compound proposition of the form $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots\cdots p_n) \rightarrow q$ is called an <u>argument</u>. Here $p_1, p_2, \cdots p_n$ are called the <u>Premises</u> of the argument and q is called a <u>Conclusion</u> of the argument.

It is a practice to write the above argument in the following tabular form:

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

Here, the three-dot symbol stands for "Therefore".

An argument with Premises $p_1, p_2, \cdots\cdots, p_n$ and Conclusion q is said to <u>Valid</u> if whenever each of Premises $p_1, p_2, \cdots, p_n$ is true, then the conclusion q is also true. otherwise the argument is invalid.

checking the validity of an argument form:

step1: construct truth table for the premises and the conclusion.

2: Find the rows in which all the premises are true (critical rows)

3: check conclusion of all critical rows

a) If in each critical row the conclusion is true then the argument form is valid.

b) If there is a row in which conclusion is false then the argument form is invalid.

1. using truth table, show that the conclusion $C: \sim p$ follows from the premises $H_1: \sim q$, $H_2: p \to q$.

Sol:-

$$\left.\begin{array}{l} H_1: \sim q \\ H_2: p \to q \end{array}\right\} \text{premises}$$

$\therefore C: \sim p$ } Conclusion

we construct a truth table for the premises and conclusion.

p	q	H_1 $\sim q$	H_2 $p \to q$	C $\sim p$
T	T	F	T	F
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T ✓

We see in the above table that there is only one case in which both premises are true, namely, the last case and in this case the conclusion is true, hence the argument is valid.

2. Determine whether the following statement form is valid (or) invalid

$$\left.\begin{array}{l} p \to q \vee \sim r \\ q \to p \wedge r \end{array}\right\} \text{premises}$$

$$\therefore\ p \to r \quad \} \text{ Conclusion}$$

Sol:- Construct truth table for the premises and the conclusion

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	~~F~~ T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	F	F	T	T	T

The above table, in fourth critical row the conclusion is false.

Hence the given argument is invalid.

3. Show that the conclusion C follows from the premises $H_1, H_2, \ldots\ldots$ in the following cases.

(a) $H_1 : p \rightarrow q$ $\quad$ $C : p \rightarrow p \wedge q$

(b) $H_1 : \sim p$, $H_2 : p \vee q$ $\quad$ $C : q$

(c) $H_1 : \sim p \vee q$, $H_2 : \sim(q \wedge \sim r)$ $\quad$ $H_3 : \sim r$ $\quad$ $C : \sim p$

Sol: a) Given $H_1 : p \rightarrow q$

$C : p \rightarrow p \wedge q$.

We draw a truth table for the premises and conclusion.

		H_1		C
p	q	$p \rightarrow q$	$p \wedge q$	$p \rightarrow p \wedge q$
T	T	T ✓	T	T ✓
T	F	F	F	F
F	T	T ✓	F	T ✓
F	F	T ✓	F	T ✓

Therefore, C follows from the premises H_1 because in first, third, fourth rows H_1 value is T and C value T.

b) Given $H_1 : \sim p$, $H_2 : p \vee q$ } premises

$\therefore$ $C : q \rightarrow$ Conclusion.

We draw a truth table for the premises and conclusion

p	q	$\sim p$	$p \vee q$
T	T	F	T
T	F	F	T
F	T ✓	T	T
F	F	T	F

Therefore, C follows from the premises H_1 and H_2 because in third row H_1, H_2 values are T and C value is T.

c) Given $H_1 : \sim p \vee q$

$H_2 : \sim(q \wedge \sim r)$

$H_3 : \sim r$

} premises

$\therefore C : \sim p$ Conclusion

We draw a truth table for the premises and conclusion

p	q	C $\sim p$	H_1 $\sim p \vee q$	r	H_3 $\sim r$	$q \wedge \sim r$	H_2 $\sim(q \wedge \sim r)$
T	T	F	T	T	F	F	T
T	T	F	T	F	T	T	F
T	F	F	F	T	F	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	T
F	F	T	T	T	F	F	T
F	T	T	T	F	T	T	F
F	F	T ✓	T ✓	F	T ✓	F	T ✓

Therefore, c follows from the premises H_1, H_2, H_3 because in last row H_1, H_2, H_3 have value T and c value T.

Rules of Inference

(i) Method (without using truth table):

The truth table technique becomes tedious when the numbers of atomic variables present in all the formulae representing the premises and the conclusion is large. To overcome this disadvantage, we need to investigate other possible methods, without using the truth table. We now describe the process of derivation by which one demonstrates that a particular formula is a valid consequence of a given set of premises. Before we do this, we give two rules of inference which are called rules P and T.

Rule P: A Premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is a tautologically implied by one or more of the preceding formulas in the derivation.

Before we Proceed with the actual process of derivation, we list some important implications and equivalences that will be referred to frequently.

Implication formulas :

$I_1 : p \wedge q \Rightarrow p$

$I_2 : p \wedge q \Rightarrow q$ } (Simplification)

$I_3 : p \Rightarrow p \vee q$

$I_4 : q \Rightarrow p \vee q$ } (addition)

$I_5 : \sim p \Rightarrow p \to q$

$I_6 : q \Rightarrow p \to q$

$I_7 : \sim(p \to q) \Rightarrow p$

$I_8 : \sim(p \to q) \Rightarrow \sim q$

$I_9 : p, q \Rightarrow p \wedge q$

$I_{10} : \sim p, p \vee q \Rightarrow q$ (disjunctive Syllogism)

$I_{11} : p, p \to q \Rightarrow q$ (modus ponens)

$I_{12} : \sim q, p \to q \Rightarrow \sim p$ (modus tollens)

$I_{13} : p \to q, q \to r \Rightarrow p \to r$ (hypothetical Syllogism)

$I_{14} : p \vee q, p \to q, q \to r \Rightarrow r$ (dilemma)

Equivalent formulas

$E_1 : \sim(\sim p) \Leftrightarrow p$ (double negation law)

$E_2 : p \wedge q \Leftrightarrow q \wedge p$

$E_3 : p \vee q \Leftrightarrow q \vee p$ } (Commutative laws)

$E_4 : (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

$E_5 : (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ } (associative laws)

$E_6 : p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$E_7 : p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ } (distributive laws)

$E_8 : \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
$E_9 : \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ } (De Morgan's laws)

$E_{10} : p \vee p \Leftrightarrow p$
$E_{11} : p \wedge p \Leftrightarrow p$ } (idempotent laws)

$E_{12} : r \vee (p \wedge \sim p) \Leftrightarrow r$

$E_{13} : r \wedge (p \vee \sim p) \Leftrightarrow r$

$E_{14} : r \vee (p \vee \sim p) \Leftrightarrow T$

$E_{15} : r \wedge (p \wedge \sim p) \Leftrightarrow F$

$E_{16} : p \rightarrow q \Leftrightarrow \sim p \vee q$

$E_{17} : \sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$

$E_{18} : p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$

$E_{19} : p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

$E_{20} : \sim(p \leftrightarrow q) \Leftrightarrow (p \leftrightarrow \sim q)$

$E_{21} : (p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$E_{22} : (p \leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

$E_{23} : (p \wedge T) \Leftrightarrow p$
$E_{24} : (p \vee F) \Leftrightarrow p$ } (identity laws)

$E_{25} : (p \vee T) \Leftrightarrow T$
$E_{26} : (p \wedge F) \Leftrightarrow F$ } (domination laws)

$E_{27} : (p \vee \sim p) \Leftrightarrow T$
$E_{28} : (p \wedge \sim p) \Leftrightarrow F$ } (negation laws)

$E_{29} : p \vee (p \wedge q) \Leftrightarrow p$
$E_{30} : p \wedge (p \vee q) \Leftrightarrow p$ } (absorption laws)

1. Demonstrate that r is a valid inference from the premises $p \to q$, $q \to r$ and p.

Sol:

$\{1\}$ (1) $p \to q$ rule P

$\{2\}$ (2) p rule P

$\{1,2\}$ (3) q rule T, (1), (2) and $p, p \to q \Rightarrow q$.

$\{4\}$ (4) $q \to r$ rule P

$\{1,2,4\}$ (5) r rule T (3), (4) and $p, p \to q \Rightarrow q$.

Hence the result.

2. Show that t is a valid conclusion from the premises $p \Rightarrow q$, $q \Rightarrow r$, $r \Rightarrow s$, $\sim s$ and $p \vee t$.

Sol:

$\{1\}$ (1) $p \Rightarrow q$ rule P

$\{2\}$ (2) $q \Rightarrow r$ rule P

$\{1,2\}$ (3) $p \Rightarrow r$ rule T, (1),(2) and I_{13}.

$\{4\}$ (4) $r \Rightarrow s$ rule P

$\{1,2,4\}$ (5) $p \Rightarrow s$ rule T, (4),(5) and I_{13}.

$\{6\}$ (6) $\sim s$ rule P

$\{1,2,4,6\}$ (7) $\sim p$ rule T, (5),(6) and I_{12}.

$\{8\}$ (8) $p \vee t$ rule P

$\{1,2,4,6,8\}$ (9) t rule T, (7), (8) and I_{10}.

Hence the result.

*3. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Sol:

{1}	1.	$C \vee D$	rule P
{2}	2.	$(C \vee D) \rightarrow \neg H$	rule P
{1, 2}	3.	$\neg H$	rule T, (1), (2) and I_{11}
{4}	4.	$\neg H \rightarrow (A \wedge \neg B)$	rule P
{1, 2, 4}	5.	$(A \wedge \neg B)$	rule T, (3), (4) and I_{11}
{6}	6.	$(A \wedge \neg B) \rightarrow (R \vee S)$	rule P
{1, 2, 4, 6}	7.	$(R \vee S)$	rule T, (5), (6) and I_{11}

Hence the result.

(OR)

{1}	1.	$(C \vee D) \rightarrow \neg H$,	rule P.
{2}	2	$\neg H \rightarrow (A \wedge \neg B)$	rule P
{1, 2}	3.	$(C \vee D) \rightarrow (A \wedge \neg B)$	rule T, (1), (2) and I_{13}
{4}	4.	$(A \wedge \neg B) \rightarrow (R \vee S)$	rule P
{1, 2, 4}	5.	$(C \vee D) \rightarrow (R \vee S)$	rule T, (3), (4) and I_{13}
{6}	6.	$(C \vee D)$	rule P
{1, 2, 4, 6}	7.	$(R \vee S)$	rule T, (5), (6) and I_{11}

Hence the result.

(4) * Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.

Sol:

{1}	1.	$P \vee Q$	Rule P
{1}	2.	$\sim P \to Q$	Rule T, E_1 and E_{16}
{3}	3.	$Q \to S$	Rule P
{1, 3}	4.	$\sim P \to S$	Rule T, (2), (3) and I_{13}
{1, 3}	5.	$\sim S \to P$	Rule T, (4) and E_{18}, E_1
{6}	6.	$P \to R$	Rule P
{1, 3, 6}	7.	$\sim S \to R$	Rule T, (5), (6) and I_{13}
{1, 3, 6}	8.	$S \vee R$	Rule T, (7) and E_{16}, E_1

Hence the result.

(5) Show that r logically follows from premises $p \to (q \to r)$, $p \wedge q$

Sol:

{1}	1.	$p \wedge q$	Rule P
{1}	2.	p	Rule T and I_1
{3}	3.	$p \to (q \to r)$	Rule P
{1, 3}	4.	$q \to r$	Rule T, (2), (3) and I_{11}
{1}	5.	q	Rule T and I_1
{1, 3}	6.	r	Rule T and I_{11}

Hence, r logically follows from given premises.

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(6) Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Sol:

$\{1\}$	1. $P \to M$	Rule P	
$\{2\}$	2. $\neg M$	Rule P	
$\{1,2\}$	3. $\neg P$	Rule T, (1), (2) and I_{12}	
$\{4\}$	4. $P \vee Q$	Rule P	
$\{1,2,4\}$	5. Q	Rule T, (3), (4) and I_{10}	
$\{6\}$	6. $Q \to R$	Rule P	
$\{1,2,4,6\}$	7. R	Rule T, (5), (6) and I_{11}	
$\{1,2,4,6\}$	8. $R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9	

Hence the result.

(7) Show that $\neg Q$, $P \to Q \Rightarrow \neg P$.

Sol:

$\{1\}$	1. $P \to Q$	Rule P	
$\{1\}$	2. $\neg Q \to \neg P$	Rule T, (1) and $P \to Q \Leftrightarrow \neg Q \to \neg P$	
$\{3\}$	3. $\neg Q$	Rule P	
$\{1,3\}$	4. $\neg P$	Rule T, (2), (3) and I_{11}	

Hence the result.

8. Test whether the following is a valid argument.
"If Sachin hits a century, then we gets a free car. Sachin hits a century. Therefore, Sachin gets a free car".

Sol: Let us indicate the statements as follows:

p: Sachin hits a century.

q: Sachin gets a free car.

Hence, the given argument is of the form

$p \to q,\ p \Rightarrow q$.

{1}	1.	$p \to q$	Rule P
{2}	2.	p	Rule P
{1, 2}	3.	q	Rule T, (1), (2) and I_{11}.

Hence the result.

9. Test the validity of the following argument:

If a man is a bachelor, he is worried.
If a man is worried, he dies young.
Therefore, Bachelors die young.

Sol: Let us indicate the statements as follows:

p: a man is a bachelor

q: he is worried

r: he dies young

Hence, the given premises are $p \to q$, $q \to r$.

The conclusion is $p \to r$.

Thus, we need to prove that $p \to q, q \to r \Rightarrow p \to r$.

I^{st} **Method:**

$\{1\}$	1.	$p \to q$,	Rule P
$\{2\}$	2.	$q \to r$,	Rule P
$\{1, 2\}$	3.	$p \to r$,	Rule T, (1), (2) and I_{13}

Hence, the given statements constitute a valid argument.

II^{nd} **Method:** $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since all the truth values of $(p \to q) \wedge (q \to r) \to (p \to r)$ are true,

$(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Hence, the given statements constitute a valid argument.

10. "If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game."

Show that these statements constitute a valid argument.

Sol:- Let us indicate the statements as follows:

p: There was a ball game

q: Travelling was difficult

r: They arrived on time.

Hence, the given premises are $p \to q$, $r \to \sim q$ and r.

The conclusion is $\sim p$.

Thus, we need to prove that $p \to q,\ r \to \sim q,\ r \Rightarrow \sim p$.

$\{1\}$	1.	$r \to \sim q$	Rule P	
$\{2\}$	2.	r	Rule P	
$\{1,2\}$	3.	$\sim q$	Rule T,	(1), (2) and I_{11}
$\{4\}$	4.	$p \to q$	Rule P	
$\{4\}$	5.	$\sim q \to \sim p$	Rule T,	(4) and E_{18}
$\{1,2,4\}$	6.	$\sim p$	Rule T,	(3), (5) and I_{11}

Hence, the given statements constitute a valid argument.

11. By using the method of derivation, show that the following statements constitute a valid argument:

" If A works hard, then either B or C will enjoy. If B enjoys, then A will not work hard. If D enjoys, then C will not. Therefore, if A works hard, D will not enjoy".

Sol: Let us indicate the statements as follows:

p: A works hard

q: B will enjoy

r: C will enjoy

s: D will enjoy

Given premises are $p \to (q \vee r)$, $q \to \sim p$, $s \to \sim r$.

The Conclusion is $p \to \sim s$.

Thus, we need to prove that $p \to (q \vee r)$, $q \to \sim p$,

$s \to \sim r \Rightarrow p \to \sim s$.

{1}	1. p	rule P	(additional premise)
{2}	2. $p \to (q \vee r)$	rule P	
{1,2}	3. $q \vee r$	rule T, (1), (2) and I_{11}.	
{1,2}	4. $\sim q \to r$	rule T, (3) and '$\sim p \to q \Leftrightarrow p \vee q$'.	
{1,2}	5. $\sim r \to q$	rule T, (4) and '$p \to q \Leftrightarrow \sim q \to \sim p$'	
{6}	6. $q \to \sim p$,	rule P	
{1,2,6}	7. $\sim r \to \sim p$,	rule T, (5), (6) and I_{13}.	
{1,2,6}	8. $p \to r$,	rule T, (7) and '$p \to q \Leftrightarrow \sim q \to \sim p$'	
{9}	9. $s \to \sim r$,	rule P	
{9}	10. $r \to \sim s$	rule T, (9) and $p \to q \Leftrightarrow \sim q \to \sim p$	
{1,2,6,9}	11. $p \to \sim s$	rule T, (8), (10) and I_{13}	

Hence the statements constitute a valid argument

Rules of Conditional Proof (or Deduction Theorem)

Another important rule used in Logic is the rule of Conditional proof (or Rule CP). This rule can be defined as follows.

Rule CP: If we can derive S from R and a set of Premises, then we can derive $R \to S$ from a set of Premises alone.

Note: Rule CP is generally used if the conclusion is of the form $R \to S$. In such Cases, R is taken as an additional premise and S is derived from the given premises and R.

12. Show that $r \to s$ can be derived from the premises $p \to (q \to s)$, $\sim r \vee p$ and q.

Sol: To prove $r \to s$, we can ~~to~~ include r as an additional premise.

$\{1\}$	1.	r	Rule P (assumed premise)
$\{2\}$	2.	$\sim r \vee p$	Rule P
$\{3\}$	3.	$r \to p$	Rule T, (2) and $p \to q \Leftrightarrow \sim p \vee q$.
$\{1,3\}$	4.	p	Rule T, (1), (3) and $p, p \to q \Rightarrow q$
$\{5\}$	5.	$p \to (q \to s)$	Rule P
$\{1,3,5\}$	6.	$q \to s$	Rule T, (4), (5) and $p, p \to q \Rightarrow q$.
$\{7\}$	7.	q	Rule P
$\{1,3,5,7\}$	8.	s	Rule T, (6), (7) and $p, p \to q \Rightarrow q$
$\{1,3,5,7\}$	9.	$r \to s$	Rule CP, (1) and (8).

Hence the result.

Consistency of Premises

A set of formulas $H_1, H_2, \ldots H_m$ is said to be consistent if their conjunction has truth value T for some assignment of truth values to the atomic variables appearing in $H_1, H_2, \ldots\ldots, H_m$.

If for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \ldots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \ldots H_m$ are called inconsistent if their conjunction implies a contradiction.

1. Show that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$, $a \wedge d$ are inconsistent.

Sol:

$\{1\}$	1.	$a \wedge d$	Rule P
$\{1\}$	2.	a	Rule T, (1) and $p \wedge q \Rightarrow p$
$\{1\}$	3.	d	Rule T, (1) and $p \wedge q \Rightarrow q$
$\{4\}$	4.	$a \to (b \to c)$	Rule P.
$\{1, 4\}$	5.	$b \to c$	Rule T, (1), (4) and $p, p \to q \Rightarrow q$
$\{1, 4\}$	6.	$\neg b \vee c$	Rule T, (5) and $p \to q \Leftrightarrow \neg p \vee q$
$\{7\}$	7.	$d \to (b \wedge \neg c)$	Rule P
$\{7\}$	8.	$\neg(b \wedge \neg c) \to \neg d$	Rule T, (7) and $p \to q \Leftrightarrow \neg q \to \neg p$
$\{7\}$	9.	$\neg b \vee c \to \neg d$	Rule T, (8) and by De Morgan Law
$\{1, 4, 7\}$	10.	$\neg d$	Rule T, (6), (9) and $p, p \to q \Rightarrow q$
$\{1, 4, 7\}$	11.	$d \wedge \neg d$	Rule T, (3), (10) and $p, q \Rightarrow p \wedge q$

But, $d \wedge \sim d \Leftrightarrow F$

Therefore our assumption is wrong.

Hence, the given premises are inconsistent.

<u>Indirect Method of Proof</u>:-

Proof by Contradiction is often referred to as indirect proof; in contrast with the direct proofs which have appeared earlier. A Contradiction is typically in the form $R \wedge \sim R$, where R is any statement.

1. using indirect method of proof, show that if $p \Rightarrow q \wedge r$, $q \vee s \Rightarrow t$ and $p \vee s$, then t.

Sol:-

$\{1\}$	1.	$p \Rightarrow q \wedge r$	Rule P
$\{2\}$	2.	$q \vee s \Rightarrow t$	Rule P
$\{3\}$	3.	$p \vee s$	Rule P
$\{4\}$	4.	$\sim t$	Rule P (additional premise)
$\{2, 4\}$	5.	$\sim (q \vee s)$	Rule T, (2),(4) and "$\sim q, p \rightarrow q \Rightarrow \sim p$"
$\{2, 4\}$	6.	$\sim q \wedge \sim s$	Rule T, (5) and De Morgan's law
$\{2, 4\}$	7.	$\sim q$	Rule T, (6) and "$p \wedge q \Rightarrow p$"
$\{2, 4\}$	8.	$\sim s \wedge \sim q$	Rule T, (6) Commutative
$\{2, 4\}$	9.	$\sim s$	Rule T, (6) and "$p \wedge q \Rightarrow q$"
$\{3\}$	10.	$s \vee p$	Rule T, (3) and Commutative
$\{2, 3, 4\}$	11.	p	Rule T, (9),(10) and $\sim p, p \vee q \Rightarrow q$
$\{1, 2, 3, 4\}$	12.	$q \wedge r$	Rule T, (11), (1) and I_{11}
$\{1, 2, 3, 4\}$	13.	q	Rule T, (12) and $p \wedge q \Rightarrow q$
$\{1, 2, 3, 4\}$	14.	$q \wedge \sim q$	Rule T, (7), (13)

Contradiction.

2. Prove by indirect method

$\sim q$, $p \Rightarrow q$ and $p \vee t$, then t.

Sol:

{1}	1.	$\sim q$	Rule P
{2}	2.	$p \Rightarrow q$	Rule P
{3}	3.	$p \vee t$	Rule P
{4}	4.	$\sim t$	Rule P. Additional premise
{3}	5.	$t \vee p$	Rule T, (3) and Commutative
{4, 3}	6.	p	Rule T (4), (5) and I_{10}
{2, 4, 3}	7.	q	Rule T, (2), (6) and I_{11}
{1, 2, 4, 3}	8.	$q \wedge \sim q$	Rule T, (1), (7)

($\because q \wedge \sim q \Leftrightarrow F$).

Contradiction.

3. Show that the following set of premises ~~is inconsistent~~ inconsistent:

"If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money."

Sol:- Let us indicate the statements as follows:

p : The contract is valid

q : John is liable for penalty

r : John will go bankrupt

s : Bank will loan him money.

Then, the given premises are $p \to q$, $q \to r$, $s \to \sim r$, $p \wedge s$.

{1}	1.	$p \to q$	Rule P
{2}	2.	$q \to r$	Rule P
{1,2}	3.	$p \to r$	Rule T (1),(2) and I_{13}
{4}	4.	$s \to \sim r$	Rule P
{4}	5.	$r \to \sim s$	Rule T, (4) and $p \to q \Leftrightarrow \sim q \to \sim p$
{1,2,4}	6.	$p \to \sim s$	Rule T, (3),(5) and I_{13}
{1,2,4}	7.	$\sim p \vee \sim s$	Rule T, (6) and "$p \to q \Rightarrow \sim p \vee q$"
{1,2,4}	8.	$\sim (p \wedge s)$	Rule T, (7) and De morgan Laws
{9}	9.	$(p \wedge s)$	Rule P
{1,2,4,9}	10.	$(p \wedge s) \wedge \sim (p \wedge s)$	Rule T, (8),(9) and "$p, q \Rightarrow p \wedge q$"

But, $(p \wedge s) \wedge \sim (p \wedge s) \Leftrightarrow F$.

Thus, our assumption is wrong.

Hence, the given premises are inconsistent.

4. Show that the following premises are inconsistent.

1. If Jack misses many classes because of illness, then he fails high school.
2. If Jack fails high school, then he is uneducated
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes because of illness and reads a lot of books.

Sol: Let us indicate the statements as follows.

p : Jack misses many classes because of illness

q : Jack fails high school

r : Jack is uneducated

s : Jack reads a lot of books.

Then, the given premises are

$p \rightarrow q$, $q \rightarrow r$, $s \rightarrow \neg r$, $p \wedge s$.

{1}	1. $p \rightarrow q$	Rule P	
{2}	2. $q \rightarrow r$	Rule P	
{1,2}	3. $p \rightarrow r$	Rule T, (1), (2) and I_{13}	
{4}	4. $s \rightarrow \neg r$	Rule P	
{4}	5. $r \rightarrow \neg s$	Rule T, (4) and "$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$"	
{1,2,4}	6. $p \rightarrow \neg s$	Rule T, (3), (5) and I_{13}	
{1,2,4}	7. $\neg p \vee \neg s$	Rule T, (6) and "$p \rightarrow q \Leftrightarrow \neg p \vee q$"	
{1,2,4}	8. $\neg(p \wedge s)$	Rule T, (7) and De morgan's law	

{9} 9. $p \wedge s$ Rule P

{1,2,4,9} 10. $\sim(p \wedge s) \wedge (p \wedge s)$ Rule T, (8), (9) and $p, q \Rightarrow p \wedge q$

But, $\sim(p \wedge s) \wedge (p \wedge s) \Leftrightarrow F$; which is a contradiction

Thus, our assumption is wrong.

Hence, the given premises are inconsistent.

Predicate Calculus

A part of a declarative sentence describing the properties of an object (or) relation among objects is called a Predicate.

The Logic based upon the analysis of predicate in any statement is called Predicate logic.

Consider the following two statements

1. Radha is a girl.
2. Seeta is a girl.

Here, the first part Radha (or) Seeta is the Subject of the statement. The second part " is a girl", which refers to a property that the subject can have, is called the predicate. We symbolize a predicate by a capital letter and the names of individuals or objects in general by lower case letters. Here, the predicate " is a girl " is denoted by G, Radha is r and Seeta by s.

Hence, the statement "Radha is a girl" is denoted by $G(r)$, while "Seeta is a girl" is denoted by $G(s)$.

In general, any statement of the type p is Q, where Q is a predicate and p is the subject, can be denoted by $Q(p)$.

A predicate requiring m $(m>0)$ names is called an m-place predicate.

Examples: Consider the examples

(i) Indu is a Student.

The Predicate S: is a student is a 1-place predicate because it is related to one object (or) name, say Indu (i).

The statement can be translated as $S(i)$.

(ii) phaneendhar is taller than Mohan.

The predicate " is taller than" is a 2-place predicate. This statement can be translated as $T(p, m)$ where T symbolizes "taller than", p denotes "phaneendhar" and m denotes "Mohan".

Examples of 3-place predicates and 4-place predicates are

(i) Suman sits between Rahul and Nithin.

(ii) Venkat and Karthik played bridge against Rohith and Raj.

* A Statement is called a 0-place predicate when no names are associated with the statement.

Statement Function, Variables and Quantifiers

Consider the following statements:

1. Somu is mortal
2. India is mortal
3. A table is mortal

Let H be the Predicate 'is mortal', s be the name 'Somu', i be the name 'India' and t be the table. Then $H(s)$, $H(i)$ and $H(t)$ denote the above statements. If we write $H(x)$ for 'x is mortal', then $H(s)$, $H(i)$ and $H(t)$ can be obtained from $H(x)$ by replacing x by an appropriate name.

Note that $H(x)$ is not a statement, but when x is replaced by the name of an object, it becomes a statement.

* A simple statement function of one variable

is defined to be an expression consisting of a predicate symbol and an individual variable.

Such a statement function becomes a statement when the variable is replaced by the name of any object. We can form compound statement functions by combining one or more simple statement functions and the logical connectives.

Note:- 1. $M(x) \wedge H(x)$, $M(x) \rightarrow H(x)$, $\neg H(x)$, $M(x) \vee \neg H(x)$, etc, are all examples of compound statement functions, where $M(x)$ represents "x is a man" and $H(x)$ represents "x is mortal".

2. Consider the statement function of two variables

$G(x, y)$: x is taller than y.

If both x and y are replaced by the names of objects, we get a statement.

If m represents Mr. Madhu and f represents Mr. Gopi, then we have $G(m, f)$: Mr. Madhu is taller than Mr. Gopi; and

$G(f, m)$: Mr. Gopi is taller than Mr. Madhu.

* Form a compound statement from the following

(i) $M(x)$: x is a man

(ii) $H(y)$: y is mortal.

Given $M(x)$: x is a man

and $H(y)$: y is a mortal. Then we write

$M(x) \wedge H(y)$: x is a man and y is mortal.

2. Let $P(x)$ denote the statement '$x > 3$'. What are the truth values of $P(4)$ and $P(2)$?

Sol: Given $P(x)$ denote the statement $x > 3$.

$P(4)$ is the statement $4 > 3$, which is true

$P(2)$ is the statement $2 > 3$, which is false.

Hence, the truth values of $P(4)$ and $P(2)$ are T and F, respectively.

3. Let $Q(x, y)$ denote the statement '$x = y + 3$'. What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Sol: $Q(x, y)$: $x = y + 3$.

$Q(1, 2)$: $1 = 2 + 3$, which is False

$Q(3, 0)$: $3 = 0 + 3$, which is True.

Hence, the truth values of $Q(1, 2)$ and $Q(3, 0)$ are F and T respectively.

Quantifiers

There are two types of quantifiers. They are the following:

1. Universal quantifiers
2. Existential quantifiers

Universal quantifiers:

Consider the following statements:

1. All men are mortal
2. Every apple is red.
3. An integer is either positive or negative.

These statements can be written as

for all x, if x is a man, then x is mortal.

for all x, if x is an apple, then x is red.

for all x, if x is an integer, then x is either positive or negative.

These statements can be symbolized as

$(x)\ (M(x) \rightarrow H(x))$

$(x)\ (A(x) \rightarrow R(x))$

$(x)\ (Z(x) \rightarrow (P(x) \vee N(x)))$, where

$M(x)$: x is a man ; $H(x)$: x is mortal

$A(x)$: x is an apple ; $R(x)$: x is red

$Z(x)$: x is an integer ; $P(x)$: x is positive

$N(x)$: x is negative.

We symbolize 'for all x' by the symbol $(\forall x)$ (or) (x) and it is called universal quantifier.

The universal quantification of $P(x)$ is the proposition $P(x)$ that is true for all values of x in the universe of discourse.

The notation '$(\forall x)\ P(x)$' denotes the universal quantification of $P(x)$.

The proposition $\forall x\ P(x)$ is also expressed as 'for all x $P(x)$' (or) 'for every x $P(x)$' (or) 'for each x $P(x)$'

Existential quantifiers

Consider the following statements:

1. There exists a man
2. Some men are clever.
3. Some real numbers are rational.

These statements can be written as

* There exists an x such that x is a man.
* There exists an x such that x is a man and x is clever.
* There exists an x such that x is a real number and x is a rational number.

The above statements can be symbolized as

$(\exists x)\ M(x)$; $(\exists x)\ (M(x) \wedge C(x))$

$(\exists x)\ (R_1(x) \wedge R_2(x))$, respectively.

Where $M(x)$: x is a man

$C(x)$: x is clever

$R_1(x)$: x is a real number

$R_2(x)$: x is a rational number.

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Existential quantifier is denoted by the symbol ($\exists x$) and stands for 'there is at least one x such that' (or) 'there exists an x such that' (or) "for some x".

The existential quantification of $P(x)$ is the proposition "there exists an element 'x' in the universe of discourse such that $P(x)$ is true."

We use the notation $\exists x\, P(x)$ for the existential quantification; here, '$\exists$' is called the existential quantifier.

Ex: What is the truth value of the following quantifications?

(i) $(\forall x)\, Q(x)$, where $Q(x)$ is the statement '$x < 2$' and the universe of discourse consists of all real numbers?

(ii) $(\forall x)\, P(x)$, where $P(x)$ is the statement '$x^2 < 10$' and the universe of discourse consists of the positive integers not exceeding 4?

Sol: (i) Given, $(\forall x)\, Q(x) : x < 2$

$Q(x)$ is not true for every real number x, since $Q(3)$ is false.

Thus, $(\forall x)\, Q(x)$ is false

(ii) Given $(\forall x)\ P(x) : x^2 < 10$ is $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.

Since the universe of discourse is $\{1, 2, 3, 4\}$.

$P(4) : 4^2 < 10$, which is not true i.e false

Hence, $(\forall x)\ P(x)$ is false.

(2) What is the truth value of the following quantifications?

(i) $\exists x\ P(x)$, where $P(x)$ denotes the statement '$x > 3$' and the universe of discourse consists of all real numbers?

(ii) $\exists x\ Q(x)$, where $Q(x)$ denotes the statement '$x = x+1$' and the universe of discourse is the set of all real numbers?

Sol: (i) Given, $P(x) : x > 3$.

when $x = 4$, $4 > 3$ which is true.

Hence $x = 4$, $\exists x\ P(x)$ is true.

(ii) Given $Q(x) : x = x + 1$.

Since $Q(x)$ is false for every real number x,

i.e $\exists x\ Q(x)$ is false.

(3) Write the following statements in Symbolic form:

(i) Something is good

(ii) Everything is good

(iii) Nothing is good

(iv) Something is not good.

Sol: Statement (i) means "There exists at least one x such that, x is good"

Statement (ii) means, "for all x, x is good."

Statement (iii) means, "for all x, x is not good."

Statement (iv) means, "There is at least one x such that, x is not good."

Thus if $G(x)$: x is good, then

Statement (i) Can be denoted by $(\exists x)G(x)$

Statement (ii) Can be denoted by $(\forall x)G(x)$

Statement (iii) can be denoted by $(\forall x) \sim G(x)$.

Statement (iv) Can be denoted by $(\exists x) \sim G(x)$.

(4). Let $K(x)$: x is a man

$L(x)$: x is mortal

$M(x)$: x is an integer

$N(x)$: x is either positive (or) negative.

Express the following using quantifiers

a) All men are mortal

b) Any integers is either positive (or) Negative.

Sol: a) The given statement Can be written as for all x, if x is a man then x is mortal and this can be expressed as $(x)(K(x) \rightarrow L(x))$

(b) The given statement can be written as for all x, if x is an integer then x is either positive or negative and this can be expressed as $(x)(M(x) \rightarrow N(x))$.