Sequence and Series and Mean Value Theorem.

Mean Value Theorem:

(1) Rolle's theorem

(2) Lagrange's theorem

(3) cauchy's theorem

(4) Taylor's theorem

(5) Muclaurin's theorem

(1) Rolle's Theorem:

* verify the Rolle's theorem for the following functions.

(1) $f(x) = \frac{\sin x}{e^x}$ in $[0,\pi]$

(2) $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$

(3) $f(x) = x(x+3)\, e^{-x/2}$ in $[-3,0]$

(4) $f(x) = |x|$ in $[-1,1]$

(5) $f(x) = \frac{1}{x^2}$ in $[-1,1]$

(6) $f(x) = \sin x$ in $[-\pi,\pi]$

(7) $f(x) = \tan x$ in $[0,\pi]$

(8) $f(x) = \sec x$ in $[0,2\pi]$

(9) $f(x) = e^x . \sin x$ $[0,\pi]$

(10) $f(x) = (x-a)^m . (x-b)^n$ in $[a,b]$

Rolle's Theorem:

Let $f(x)$ be a function of x defined in (a,b)

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is derivable in (a,b)

(iii) $f(a) = f(b)$

then $\exists$ a $c \in (a,b) \ni f'(c) = 0$.

① $f(x) = \frac{\sin x}{e^x}$ $[0,\pi]$

(i) $f(x) = \frac{\sin x}{e^x}$ is continuous for all 'x'.

$f(x)$ is continuous in $[0,\pi]$

$\Rightarrow f'(x) = \frac{e^x \cdot \cos x - \sin x \cdot e^x}{(e^x)^2}$

$= \frac{e^x(\cos x - \sin x)}{(e^x)^{\cancel{2}}}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$ is exist $\forall x$.

$\Rightarrow f'(x)$ is exist in the interval $[0,\pi]$

$\therefore f(x)$ is derivable in $(0,\pi)$.

$\Rightarrow$ We have to show that $f(0) = f(\pi)$

$f(0) = \frac{\cancel{\cos 0 +}\sin 0}{e^0} = \frac{\cancel{1+}0}{1} = 0.$

$f(\pi) = \frac{\sin\pi}{e^\pi} = \frac{0}{e^\pi} = 0.$

Then $\exists$ exist $c \in (a,b) \ni f'(c) = 0.$

$f(x) = \frac{\sin x}{e^x}$

$f'(x) = \frac{\cos x - \sin x}{e^x}$

$f'(c) = \frac{\cos c - \sin c}{e^c} = 0$

$\cos c - \sin c = 0$

$\sin c = \cos c$

$\tan c = 1$

$c = \tan^{-1}(1)$

$\boxed{c = \pi/4} \in [0,\pi]$

② $f(x) = \log\left(\frac{x^2+ab}{x(a+b)}\right)$ in $[a,b]$ $a>0, b>0.$

$f(x) = \log(x^2+ab) - \log x(a+b)$ is continuous $\forall$ 'x'.

except at $x=0 \notin [a,b]$.

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x) = \log(x^2+ab) - \log x + \log(a+b)$

$= \frac{1}{x^2+ab}(2x) - \frac{1}{x} - 0.$

$f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$ is exist ($\forall$ 'x') in (a,b)

$f(x)$ is derivable in (a,b)

$f(a) = \log(a^2+ab) - \log a(a+b)$

$= \log(a^2+ab) - \log(a^2+ab)$

$= 0.$

$f(b) = \log(b^2+ab) - \log b(a+b)$

$= \log(b^2+ab) - \log(ab+b^2)$

$= 0$

$f(a) = f(b).$

$\exists$ a $c \in (a,b) \ni f'(c) = 0$

We have $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$2c^2 - c^2 - ab = 0$$

$$c^2 = ab$$

$$c = \pm\sqrt{ab}$$

$$c = \sqrt{ab} \text{ (or) } -\sqrt{ab}.$$

$$\boxed{c = \sqrt{ab}} \in (a,b).$$

③ $f(x) = x \cdot (x+3) e^{-x/2}$ in $[-3,0]$

sol: $f(x)$ is continuous $\forall x$.

(i) $f(x)$ is continuous in $[-3,0]$

(ii) $f(x) = (x^2+3x) e^{-x/2}$

$$f'(x) = (x^2+3x) e^{-x/2} \cdot \frac{-1}{2} + e^{-x/2}(2x+3)$$

$$= -\frac{(x^2+3x)}{2} e^{-x/2} + (2x+3) e^{-x/2}$$

$$= e^{-x/2}\left[(2x+3) - \frac{(x^2+3x)}{2}\right]$$

$$= e^{-x/2}\left[\frac{4x+6-x^2-3x}{2}\right]$$

$$= e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$$

$$= \frac{e^{-x/2}}{2}(-x^2+x+6)$$

$f'(x)$ is exist in $[-3,0]$.

$\Rightarrow f(x)$ is derivable in $(-3,0)$.

We have to show that $f(-3) = f(0)$

$$f(-3) = -3(-3+3)e^{-3/2}$$

$$= -3(0)\,e^{-3/2}$$

$$= 0$$

$$f(0) = 0\,(0+3)\,e^{-0/2}$$

$$= 0.$$

$$\boxed{\therefore f(-3) = f(0)}$$

Then $\exists$ a $c \in (a,b) \ni f'(c)=0$.

$$f'(x) = \frac{e^{-x/2}}{2}(-x^2+x+6)$$

$$f'(c) = \frac{e^{-c/2}}{2}(-c^2+c+6) = 0.$$

$$(-c^2+c+6)\,e^{-c/2} = 0.$$

$e^{-c/2} = 0$ and $-c^2+c+6 = 0.$ ✓

$$c^2-c-6=0$$

$$c^2-3c+2c-6=0$$

$$c(c-3)+2(c-3)=0$$

$$(c-3)(c+2)=0$$

$$c=3,\ \boxed{c=-2} \in (-3,0)$$

(4) $f(x) = |x|$ in $[-1,1]$

Sol: We know that $|x| = \begin{cases} -x & \text{if } x<0 \\ x & \text{if } x>0 \end{cases}$

(i) $f(x)=|x|$ is continuous $\forall x$

$\Rightarrow |x|$ is continuous in $[-1,1]$

(ii) The derivative of $|x|$ does not exist.

Because,

L.H.D $\lim\limits_{x\to 0^-} \dfrac{f(x)-f(0)}{x-0}$

$$= \lim_{x\to 0^-}\left(\frac{-x-0}{x}\right)$$

$$= \lim_{x \to 0^-} \frac{-x}{x}$$

$$= \lim_{x \to 0^-} (-1) = \underline{-1}$$

R.H.D $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0^+} \frac{x - 0}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{x}{x}$$

$$= \lim_{x \to 0^+} (1) = 1$$

$$\boxed{\therefore \text{L.H.D} \neq \text{R.H.D}}$$

Hence Rolle's theorem is not verified.

(10). $f(x) = (x-a)^m \cdot (x-b)^n$ in $[a,b]$.

Sol: $f(x)$ is exist $\forall x$.

$\rightarrow f(x)$ is continuous in $[a,b]$

$\rightarrow f(x) = (x-a)^m \cdot (x-b)^n$

$$f'(x) = (x-a)^m \cdot n(x-b)^{n-1}(1-0) + (x-b)^n m(x-a)^{m-1}(1-0)$$

$$= n \cdot (x-b)^{n-1} \cdot (x-a)^m + m \cdot (x-a)^{m-1} \cdot (x-b)^n$$

$$= n \cdot (x-b)^n \cdot (x-b)^{-1} (x-a)^m + m \cdot (x-a)^m \cdot (x-a)^{-1} (x-b)^n$$

$$= (x-a)^m (x-b)^n [n \cdot (x-b)^{-1} + m \cdot (x-a)^{-1}]$$

$$= (x-a)^m (x-b)^n \left(\frac{n}{x-b} + \frac{m}{x-a}\right)$$

$$= (x-a)^n \cdot (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)}\right)$$

$f'(x)$ is exist $\forall x$. except at $x=a$ and $x=b$. $\notin (a,b)$

$\therefore f'(x)$ is exist in (a,b).

$\therefore f(x)$ is derivable in (a,b)

$$f(a) = (a-a)^m \cdot (a-b)^n$$

$$= (0)^m \cdot (a-b)^n$$

$$= 0.$$

$f(b) = (b-a)^m (b-b)^n$

$= (b-a)^m (0)$

$= 0.$

$f(a) = f(b)$

Then $\exists$ a $c \in (a,b) \ni f'(c)=0$

$f(x) = (x-a)^m . (x-b)^n$

$$f'(x) = (x-a)^m . (x-b)^n \left(\frac{n(x-a) + m(x-b)}{(x-a)(x-b)} \right)$$

$$f'(c) = (c-a)^m . (c-b)^n . \left(\frac{n(c-a)+m(c-b)}{(c-a)(c-b)} \right) = 0.$$

$$= (c-a)^m (c-b)^n \left[\frac{nc-na+mc-mb}{(c-a)(c-b)} \right] = 0$$

$$= (c-a)^m . (c-b)^n \left[\frac{(m+n)c-(na+mb)}{(c-a)(c-b)} \right] = 0$$

$(c-a)^m = 0$, $(c-b)^n = 0$ and $(m+n)c - na - mb = 0.$

$\Rightarrow (m+n)c = na + mb.$

$$c = \frac{na+mb}{m+n} \in (a,b)$$

⑤ $f(x) = \frac{1}{x^2}$ in $[-1, 1]$.

Sol: $f(x) = \frac{1}{x^2}$

~~f(x) is does not exist at x.~~

$\Rightarrow f(x)$ is does not continuous in $[-1, 1]$ except at $x=0 \in [-1,1]$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

$\Rightarrow f(x)$ is does not derivable in $(-1,1)$ except at $t=0$.

But $t=0 \in (-1,1)$

$\therefore$ Rolles theorem can not applied.

(6) $f(x) = \sin x$ in $[-\pi, \pi]$

sol: $f(x) = \sin x$

$\therefore f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[-\pi, \pi]$.

$f'(x) = \cos x$.

$\Rightarrow f'(x)$ is derivable in $(-\pi, \pi)$

We have to show that $f(-\pi) = f(\pi)$

$f(-\pi) = \sin(-\pi) = -\sin\pi = 0$

$f(\pi) = \sin\pi = 0$

$f(-\pi) = f(\pi)$

then $\exists$ a $c \in (-\pi, \pi) \ni f'(c) = 0$.

$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f'(c) = 0$

$\cos c = 0$

$c = \cos^{-1}(0)$

$c = \cos^{-1}(\cos \pi/_2)$

$\boxed{c = \pi/_2} \in (-\pi/_2, \pi/_2)$

(7) $f(x) = \tan x$ in $[0, \pi]$.

$f(x) = \tan x$

$f(x)$ is exist $\forall x$. except at $x = \pi/_2 \in (0, \pi)$.

$\therefore f(x)$ is does not continuous in $[0, \pi]$

$f'(x) = \sec^2 x$

$f'(x)$ ~~is~~ is ~~exist~~ does not exist $\forall x$. Except at $x = 0. \in (0, \pi)$

Rolle's theorem can not be verified.

(8) $f(x) = \sec x$ in $[0, 2\pi]$

$f(x) = \sec x$.

$f(x)$ is exist $\forall x$. except at $x = \pi/2 \in (0, 2\pi)$

$\Rightarrow f(x)$ is continuous in $[0, 2\pi]$ except at $x = \pi/2 \in (0, 2\pi)$

$f'(x) = \sec x \cdot \tan x$.

$f'(x)$ is exist $\forall x$. Except at $x = \pi/2 \in (0, 2\pi)$.

$\Rightarrow f'(x)$ is derivable ~~at~~ in $(0, 2\pi)$ Except at $x = \pi/2$.

$\Rightarrow f(0) = f(2\pi)$ (we have to S.T)

$f(0) = \sec 0° = 1$

$f(2\pi) = \sec 2\pi = 1$

$\boxed{f(0) = f(2\pi)}$

Then $\exists$ a $c \in (0, 2\pi) \ni f'(c) = 0$

$\sec c \cdot \tan c = 0$

$\tan c = 0$ and $\sec c = 0$

$c = \tan^{-1}(0)$ $\quad c = \sec^{-1}(0)$

$c = \tan^{-1}(\tan 0)$ $\quad$ ~~sec⁻¹ sec~~

$\boxed{c = 0}$

(9) $f(x) = e^x \cdot \sin x$ in $[0, \pi]$.

$f(x) = e^x \sin x$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, \pi]$

$f'(x) = e^x \cos x + \sin x \, e^x$

$= e^x (\cos x + \sin x)$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, \pi)$.

$\Rightarrow$ we have to show that $f(0) = f(\pi)$.

$f(0) = e^0 \cdot \sin 0 = 0$

$f(\pi) = e^{\pi} \sin \pi = 0$

$\boxed{f(0) = f(\pi)}$

Then $\exists$ a $c \in (0,\pi) \ni f'(c) = 0$

$$e^c(\cos c + \sin c) = 0$$
$$\cos c + \sin c = 0$$

$$\sin c = -\cos c$$
$$\frac{\sin c}{\cos c} = -1$$
$$\tan c = -1$$
$$c = \tan^{-1}(-1)$$
$$c = \tan^{-1}(\tan 3\pi/4)$$
$$\boxed{c = 3\pi/4} \in (0,\pi).$$

Saturday 19/10/19

Lagrange's Mean Value Theorem:

Let $f(x)$ be a function of x. if

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is derivable in (a,b)

(iii) Then $\exists$ a $c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

① $f(x) = x(x-1)(x-2)$ in $[0,1/2]$

② $f(x) = \log x$ $[1,e]$

③ $f(x) = e^x$ $[0,1]$

④ $f(x) = \frac{1}{x}$ $[1,4]$

⑤ $f(x) = x - x^3$ $[-2,1]$

⑥ if $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

By using LMVT, show that $\frac{b-a}{1+b^2} < \tan^{-1}(b) - \tan^{-1}(a) < \frac{b-a}{1+a^2}$ and deduce that

⑦ $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}(4/3) < \frac{\pi}{4} + \frac{1}{6}$

By using L.M.V.T,

⑧ $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}(3/5) > \pi/3 - 1/8.$

⑨ $x \leq \sin^{-1}x \leq \frac{x}{1-x^2}$

① $f(x) = x(x-1)(x-2)$ in $[0, 1/2]$

$f(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is continuous in $[0, 1/2]$

$f(x) = (x^2-x)(x-2)$

$= x^3 - 2x^2 - x^2 + 2x$

$f(x) = x^3 - 3x^2 + 2x$

$f'(x) = 3x^2 - 6x + 2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0, 1/2)$.

$\Rightarrow$ Then $\exists$ a $c \in (0, 1/2) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$$

$f(1/2) = 1/2(1/2 - 1)(1/2 - 2)$

$= \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)$

$= 3/8$

$$3c^2 - 6c + 2 = \frac{3}{\cancel{8}_4} \times \frac{\cancel{2}}{1}$$

$$3c^2 - 6c + 2 - 3/4 = 0$$

$$3c^2 - 6c + 5/4 = 0$$

$$c = \frac{6 \pm \sqrt{36 - 15}}{2(3)}$$

$$= \frac{6 \pm \sqrt{21}}{6}$$

$$= \frac{6}{6} \pm \frac{\sqrt{21}}{6}, \quad \frac{6}{6} - \frac{\sqrt{21}}{6}$$

$$= 1 + \frac{\sqrt{21}}{6}, \quad 1 - \frac{\sqrt{21}}{6}$$

$$\boxed{c = 1 - \frac{\sqrt{21}}{6} \in (0, 1/2)}$$

② $f(x) = \log x$ in $[1, e]$

$f(x)$ is continuous $\forall x$, except at $x = 0 \notin (1, e)$

$\Rightarrow f(x)$ is continuous in $[1, e]$

$f'(x) = \frac{1}{x}$.

$f'(x)$ is exist $\forall x$. except at $x = 0 \notin (1, e)$

$\Rightarrow f'(x)$ is derivable in $(1, e)$.

Then $\exists$ a $c \in (1,e) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{c} = \frac{\log e - \log 1}{e-1}$$

$$\frac{1}{c} = \frac{1-0}{e-1}$$

$$\frac{1}{c} = \frac{1}{e-1}$$

$e = 2.7 \ldots$

$\Rightarrow \boxed{c = e-1} \in (1,e)$. $\quad e-1 = 2.7-1 = 1.7$.

③ $f(x) = e^x$ in $[0,1]$

$f(x)$ is contincious $\forall x$.

$\Rightarrow f(x)$ is contincious in $[0,1]$

$f'(x) = e^x$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(0,1)$.

Then $\exists$ a $c \in (0,1) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$e^c = \frac{e^1 - e^0}{1-0}$$

$$e^c = \frac{e-1}{1}$$

$$e^c = e-1$$

$$\boxed{c = \log(e-1) \in (0,1)}$$

④ $f(x) = \frac{1}{x}$ in $[1,4]$

$f(x)$ is continuous $\forall x$. except at $x=0 \notin (1,4)$

$\Rightarrow f(x)$ is contincious in $[1,4]$

$f'(x) = \frac{-1}{x^2}$ is exist $\forall x$. except at $x=0 \notin (1,4)$

$\Rightarrow f(x)$ is derivable in $(1,4)$

Then $\exists$ a $c \in (1,4) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{-1}{c^2} = \frac{1/4 - 1}{4-1}$$

$$\frac{-1}{c^2} = \frac{\frac{1-4}{4}}{3}$$

$$\frac{-1}{c^2} = \frac{-\cancel{3}/4}{\cancel{3}}$$

$$-\frac{1}{c^2} = -\frac{1}{4}$$

$$c^2 = 4$$

$$c = \sqrt{4} \Rightarrow c = \pm 2$$

$$\boxed{c = 2 \in (1,4)}$$

⑤ $f(x) = x - x^3$ in $[-2,1]$

$f(x) = x - x^3$ is continuous $\forall x$.

$\rightarrow f(x)$ is continuous in $[-2,1]$

$f'(x) = 1 - 3x^2$

$f'(x)$ is exist $\forall x$.

$\Rightarrow f(x)$ is derivable in $(-2,1)$

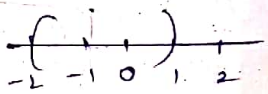
$\exists$ a $c \in (-2,1) \ni f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$1 - 3c^2 = \frac{[1-(1)^3] - [-2-(-2)^3]}{1-(-2)}$$

$$1 - 3c^2 = \frac{(1-1) - [-2-(-8)]}{1+2}$$

$$1 - 3c^2 = \frac{0 - (-2+8)}{3}$$

$$3 - 9c^2 = -6$$



$$\cancel{9}c^2 = \cancel{9}$$

$$c^2 = 1$$

$$c = \pm 1$$

$$\boxed{c = -1 \in (-2,1)}$$

⑥ If $x > 0$ show that $x > \log(1+x) > x - \frac{x^2}{2}$

Sol:- Let us take $f(x) = \log(1+x)$

Since $f(x) = \log(1+x)$ is continuous $\forall x > 0$.

and $f(x)$ is derivable $\forall x > 0$.

By using L.M.V.T,

$\exists$ a $c \in (0,x) \ni f'(c) = \dfrac{f(x) - f(0)}{x-0}$

We have $f(x) = \log(1+x)$

$$f'(x) = \frac{1}{1+x}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - \log(1+0)}{x-0}$$

$$\frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\frac{1}{1+c} = \frac{\log(1+x)}{x} \rightarrow (1)$$

Given that $0 < c < x$

$$1 < c+1 < x+1$$

$$1 < \frac{1}{c+1} < \frac{1}{x+1}$$

$$1 < \frac{\log(1+x)}{x} < \frac{1}{x+1}$$

$$x < \log(1+x) < \frac{x}{1+x}$$

(7) Let $f(x) = \tan^{-1}x$ in $[a,b]$

Given, ~~$f(x)$ is continuous $\forall x$. except~~

$f(x)$ is continuous in $[a,b]$

and $f(x)$ is derivable in (a,b).

By using L.M.V.T,

then $\exists$ a $c \in (a,b) \ni f'(c) = \frac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}$$

Given that, $a < c < b$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+a^2} > \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} > \frac{1}{1+b^2}$$

$$\frac{b-a}{1+a^2} > \tan^{-1}(b) - \tan^{-1}(a) > \frac{b-a}{1+b^2}$$

Given that $a=1$, $b=4/3$

$$\frac{4/3-1}{1+(1)^2} > \tan^{-1}(4/3)-\tan^{-1}(1) > \frac{4/3-1}{1+(4/3)^2}$$

$$\frac{1/3}{2} > \tan^{-1}(4/3)-\pi/4 > \frac{1/3}{\frac{25}{9/3}}$$

$$\frac{1}{6} > \tan^{-1}(4/3)-\pi/4 > \frac{3}{25}$$

$$\frac{1}{6}+\frac{\pi}{4} > \tan^{-1}(4/3) > \frac{3}{25}+\frac{\pi}{4}$$

$$\frac{\pi}{4}+\frac{3}{25} > \tan^{-1}(4/3) > \frac{\pi}{4}+\frac{1}{6}.$$

(8) Let $f(x)=\cos^{-1}x$. in $[a,b]$

Given that, $f(x)$ is continuous in $[a,b]$.

and $f(x)$ is derivable in (a,b)

By using L.M.V.T,

$$\exists \text{ a } c\in(a,b) \ni f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f(x)=\cos^{-1}x \Rightarrow f'(x)=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{-1}{\sqrt{1-c^2}}=\frac{\cos^{-1}(b)-\cos^{-1}(a)}{b-a}$$

We know that, $a<c<b$

$$a^2<c^2<b^2$$

$$-a^2>-c^2>-b^2$$

$$1-a^2>1-c^2>1-b^2$$

$$\sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$$

wrong

$$\frac{1}{\sqrt{1-a^2}}<\frac{1}{\sqrt{1-c^2}}<\frac{1}{\sqrt{1-b^2}}$$

$$\frac{1}{\sqrt{1-a^2}}<\frac{\cos^{-1}(a)-\cos^{-1}(b)}{b-a}<\frac{1}{\sqrt{1-b^2}}$$

$$\frac{-(b-a)}{\sqrt{1-a^2}} > -[\cos^{-1}(a)-\cos^{-1}(b)] > \frac{-(b-a)}{\sqrt{1-b^2}}$$

Given that $a=3/5$, $b=$

$$\frac{a-b}{\sqrt{1-a^2}} > \cos^{-1}(b)-\cos^{-1}(a) > \frac{a-b}{\sqrt{1-b^2}}$$

Given that

Monday
21/10 Cauchy's Mean Value Theorem:

Let $f(x), g(x)$ are a functions of 'x'.

(i) $f(x), g(x)$ are continuous in $[a,b]$

(ii) $f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

verify the cauchy's Mean Value Theorem for the following functions.

① $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$ in $[a,b]$ $0<a<b$.

② $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \pi/2]$

③ $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

④ $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ if $0<a<b$

⑤ $f(x) = x^2+2$, $g(x) = x^3-1$ in $[1,2]$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1,e]$

⑦ $f(x) = x^3$, $g(x) = 2-x$ in $[0,9]$

① $f(x)$ is always continuous $\forall x$.

$g(x)$ is continuous $\forall x$. except at $x=0 \notin (a,b)$ $[0<a<b]$

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$.

$f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2} x^{-3/2}$

$f'(x)$ is exist $\forall x$. except at $x=0 \notin (a,b)$

$f(x)$ is derivable in (a,b).

$g'(x)$ is exist $\forall x$. Except at $x=0 \in (a,b)$

$g(x)$ is derivable in (a,b)

$\Rightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\frac{1}{\cancel{2\sqrt{c}}}}{\frac{-1}{\cancel{2\sqrt{c}}\,c}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$-c = \frac{\sqrt{b}-\sqrt{a}}{\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}\sqrt{b}}}$$

$$\cancel{-}.c = \frac{\sqrt{b}-\cancel{\sqrt{a}}}{\cancel{-}(\cancel{\sqrt{b}-\sqrt{a}})}\sqrt{ab}$$

$$\boxed{c = \sqrt{ab} \in (a,b)}$$

(2) $f(x) = \sin x$, $g(x) = \cos x$. $[0, \pi/2]$

$\Rightarrow f(x), g(x)$ are always continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0, \pi/2]$.

$f'(x) = \cos x$, $g'(x) = -\sin x$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in $(0, \pi/2)$

Then $\exists$ a $c \in (0, \pi/2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\cos c}{-\sin c} = \frac{\sin \pi/2 - \sin 0}{\cos \pi/2 - \cos 0}$$

$$\frac{\cos c}{-\sin c} = \frac{1-0}{0-1}$$

$$\frac{\cos c}{\cancel{-}\sin c} = \frac{1}{\cancel{-}1}$$

$$\cos c = \sin c$$

$$\frac{\sin c}{\cos c} = 1$$

$$\tan c = 1$$

$$c = \tan^{-1}(1)$$

$$\boxed{c = \pi/4 \in (0, \pi/2).}$$

(3) $f(x) = e^x$, $g(x) = e^{-x}$ in $[a,b]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$

$f(x) = e^x \qquad g(x) = e^{-x}$

$f'(x) = e^x \qquad g'(x) = -e^{-x}$

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-x}}$$

$$-e^c \cdot e^c = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$-(e^c)^2 = \frac{e^b - e^a}{\frac{e^a - e^b}{e^a \cdot e^b}}$$

$$\cancel{+} e^{2c} = \frac{\cancel{e^b - e^a}}{\cancel{+}\cancel{(e^b - e^a)}} \quad e^a \cdot e^b$$

$$e^{2c} = e^a e^b$$

$$e^{2c} = e^{a+b}$$

$$2c = a+b$$

$$\boxed{c = \frac{a+b}{2} \in (a,b)}$$

(4) $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ in $[a,b]$ $(0 < a < b.)$

$f(x)$ is continuous $\forall x$. Except at $x=0 \notin (a,b)$

$g(x)$ is continuous $\forall x$, Except at $x=0 \notin (a,b)$

$\Rightarrow f(x), g(x)$ are continuous in $[a,b]$

$f'(x) = -2x^{-3} \qquad g'(x) = \cancel{\log x}\ \frac{-1}{x^2}$

$f'(x)$ is exist $\forall x$. Except at $x=0$.

$g'(x)$ is exist $\forall x$, Except at $x=0$

$\Rightarrow f(x), g(x)$ are derivable in (a,b)

Then $\exists$ a $c \in (a,b) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\cancel{-}2/c^{\cancel{3}}}{\cancel{-}1/\cancel{c^2}} = \frac{1/b^2 - 1/a^2}{1/b - 1/a}$$

$$\frac{2}{x} = \frac{\frac{a^2-b^2}{a^2b^2}}{\frac{a-b}{ab}}$$

$$\frac{2}{x} = \frac{(a+b)\cancel{(a-b)}}{(ab)^{\cancel{2}}} \times \frac{\cancel{ab}}{\cancel{a-b}}$$

$$\boxed{x = \frac{2ab}{a+b} \in (a,b)}$$

⑤ $f(x) = x^2+2$, $g(x) = x^3-1$, in $[1,2]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow$ $f(x), g(x)$ are continuous in $[1,2]$

$f'(x) = 2x$, $g'(x) = 3x^2$.

$f'(x)$ is exist $\forall x$.

$g'(x)$ is exist $\forall x$.

$\Rightarrow$ $f(x), g(x)$ are derivable in $(1,2)$

Then $\exists$ a $c \in (1,2) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{2\cancel{c}}{3c^{\cancel{2}}} = \frac{[(2)^2+2]-[(1)^2+2]}{[(2)^3-1]-[(1)^3-1]}$$

$$\frac{2}{3c} = \frac{(4+2)-(1+2)}{(8-1)-(1-1)}$$

$$\frac{2}{3c} = \frac{6-3}{7-0}$$

$$\frac{2}{3c} = \frac{3}{7}$$

$$9c = 14$$

$$\boxed{c = \frac{14}{9} \in (1,2)}$$

⑥ $f(x) = \log x$, $g(x) = \frac{1}{x}$ in $[1,e]$

$f(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$g(x)$ is continuous $\forall x$. Except at $x=0 \notin (1,e)$

$\Rightarrow$ $f(x), g(x)$ are continuous in $[1,e]$

$f'(x) = \frac{1}{x} \quad , g'(x) = \frac{-1}{x^2}$

$f'(x)$ is exist $\forall x$. Except at $x=0 \notin (1,e)$

$g'(x)$ is exist $\forall x$. Except at $x=0 \notin (1,e)$

$\Rightarrow f(x), g(x)$ is derivable in $(1,e)$.

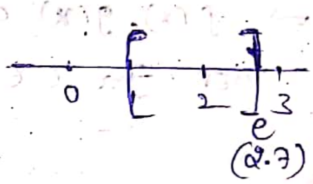
Then $\exists$ a $c \in (1,e) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{1/\cancel{c}}{-1/c^{\cancel{2}}} = \frac{\log e - \log 1}{1/e - 1/1}$$

$$-c = \frac{\log_e e - 0}{\frac{1-e}{e}}$$

$$-c = \frac{1-0}{\frac{1-e}{e}}$$

$$-c = \frac{e}{1-e}$$



$$\boxed{c = \frac{e}{e-1} \in (1,e)}$$

$$\boxed{c = 1.58}$$

⑦ $f(x) = x^3, \; g(x) = 2-x$ in $[0,9]$

$f(x)$ is continuous $\forall x$.

$g(x)$ is continuous $\forall x$.

$\Rightarrow f(x), g(x)$ are continuous in $[0,9]$

$f'(x) = 3x^2$

$f'(x)$ is exist $\forall x$.

$f(x)$ is derivable in $(0,9)$

$g'(x) = 0-1 = -1$

$g'(x)$ is exist $\forall x$.

$g(x)$ is derivable in $(0,9)$.

$\Rightarrow f(x), g(x)$ are derivable in $(0,9)$

Then $\exists$ a $c \in (0,9) \ni \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{3c^2}{-1} = \frac{(9)^3 - (2)^3(2-0)^3}{(2-9)-(2-0)}$$

$$\cancel{+}\cancel{3}\cancel{c^2} \cancel{=} \frac{\cancel{81 \times 9/3} - 8}{\cancel{+7}\ \cancel{+/2}}$$

$$\cancel{c^2} \cancel{=} \frac{\cancel{81 \times 8}}{\cancel{7}}$$

$$-3c^2 = \frac{721}{-7-2}$$

$$\cancel{-}3c^2 = \frac{721}{\cancel{-}9}$$

$$c^2 = \frac{721}{27}$$

$$c = \sqrt{\frac{721}{27}}$$

$$\boxed{c = 5.1675 \in (0,9)}$$

Wednesday
23/10 Taylor's Expansion And Maclurin's:

Taylor's expansion at $x = a$, $x = 1$, $x = \pi/2$.

Taylor's expansion about / at } $x = a$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

This is also called as taylor's expansion in powers of $(x-1)$.

Maclurin's:

at $x = 0$, $f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$

① $f(x) = \sin x$

② $f(x) = \log(1+x)$

③ $f(x) = \tan^{-1} x$

④ $f(x) = e^x$ at $x = 1$.

⑤ $f(x) = (1-x)^{5/2}$

⑥ $f(x) = \log x$ in powers of $x-1$ and hence evaluate $\log 1.1$ correct to four decimal process.

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x = 2$.

⑥ $f(x) = \log x$

By Taylor's expansion at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \dots$$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \Rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -6x^{-4} \Rightarrow f^{iv}(1) = -6$

wrong

$\therefore f(x) = \log a + (x-a)$

---

⑥ $f(x) = \log(x)$

By Taylor's expansion at $x=a$

is $f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \frac{(x-a)^4}{4!}f^{iv}(a) + \dots$

at $a=1$

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \frac{(x-1)^4}{4!}f^{iv}(1) + \dots \rightarrow ①$$

$f(x) = \log x \Rightarrow f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

$f''(x) = \frac{-1}{x^2} \rightarrow f''(1) = -1$

$f'''(x) = 2x^{-3} \Rightarrow f'''(1) = 2$

$f^{iv}(x) = -6x^{-4} \Rightarrow f^{iv}(1) = -6$

from ①, $\log x = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}2 + \frac{(x-1)^4}{4!}(-6) + \dots$

$$\log x = (x-1) - \frac{(x-1)^2}{2!} + 2\cdot\frac{(x-1)^3}{3!} - 6\frac{(x-1)^4}{4!} + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{\cancel{6}\,3}(\cancel{2}) - \cancel{6}\cdot\frac{(x-1)^4}{\cancel{24}\,4} + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots$$

$$\log(1.1) = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} + \frac{(1.1-1)^4}{4} + \cdots$$

$$= 0.1 - \frac{0.01}{2} + \frac{0.001}{3} + \frac{0.00001}{4} + \cdots$$

$$= 0.1 - 0.005 + 0.0003 + 0.000002$$

NO use

$$= \underline{0.105302}$$

$$\log(1.1) = 0.0953101A9$$

$$\boxed{\log(1.1) \simeq 0.094}$$

⑦ $f(x) = 2x^3 - 7x^2 + x + 6$ at $x = 2$.

By Taylor's expansion at $x = 2$

is $f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2)$

$$+ \frac{(x-2)^4}{4!}f^{IV}(2) + \cdots \rightarrow ①$$

$f(x) = 2x^3 - 7x^2 + x + 6 \Rightarrow f(2) = -4$

$f'(x) = 6x^2 - 14x + 1 \Rightarrow f'(2) = 24 - 28 + 1 = -3$

$f''(x) = 12x - 14 \rightarrow f''(2) = 24 - 14 = 10$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$\cancel{f^{IV}(x) = 0} \Rightarrow f^{IV}(2) \neq 0$

from ①,

$$2x^3 - 7x^2 + x + 6 = -4 + (x-2)\overset{(-3)}{\cancel{10}} + \frac{(x-2)^2}{2!}10 + \frac{(x-2)^3}{3!}12$$

$$= -4 + (x-2)\overset{(-3)}{\cancel{10}} + \frac{(x-2)^2}{2!}10 + \frac{(x-2)^3}{3!}$$

$$= -4 - 3(x-2) + 10\frac{(x-2)^2}{2!} + 12\cdot\frac{(x-2)^3}{3!}$$

② $f(x) = \log(1+x)$

Now, the Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \rightarrow ①$$

$f(x) = \log(1+x) \Rightarrow f(0) = \log(1+0) = 0$

$f'(x) = \cancel{\log} \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = \frac{-1}{1}$

$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = -6$

from ①,

$$\log(1+x) = 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \cdots$$

$$= x - \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} - 6 \frac{x^4}{4!} + \cdots$$

⑤ $f(x) = (1-x)^{5/2}$

By Maclurin's expansion is

$$f(x) = f(0) + x.f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{IV}(0) + \cdots \rightarrow ①$$

$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$

$f'(x) = \frac{5}{2}(1-x)^{3/2}(-1) \Rightarrow f'(0) = \frac{-5}{2}$

$f''(x) = \frac{-5}{2} \frac{3}{2}(1-x)^{1/2}(-1) \Rightarrow f''(0) = \frac{15}{4}$

$f'''(x) = \frac{15}{4} \frac{1}{2}(1-x)^{-1/2}(-1) \Rightarrow f'''(0) = -\frac{15}{8}$

⋮

from ①,

$$(1-x)^{5/2} = 1 + x.\left(\frac{-5}{2}\right) + \frac{x^2}{2!}\left(\frac{15}{4}\right) + \frac{x^3}{3!}\left(\frac{-15}{8}\right) + \cdots$$

$$= 1 - \frac{5}{2}x + \frac{15}{4}\frac{x^2}{2!} + -\frac{15}{8} \cdot \frac{x^3}{3!} + \cdots$$

$$= 1 - \frac{5}{2}x + \frac{15}{8}x^2 - \frac{5}{16}x^3 + \cdots$$

(1) $f(x) = \sin x.$

Now, the Maclurin's expansion is

$$f(x) = f(0) + x \cdot f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots \rightarrow (1)$$

$f(x) = \sin x \Rightarrow f(0) = 0$

$f'(x) = \cos x \Rightarrow f'(0) = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$

$f^{iv}(x) = -(-\sin x) \Rightarrow f^{iv}(0) = 0.$

from (1),

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \cdots$$

(2) $f(x) = \tan^{-1} x.$

Now, Maclurin's expansion is

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \cdots$$

$f(x) = \tan^{-1} x. \Rightarrow f(0) = \tan^{-1}(0) = 0$

$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x^2)^2}(2x) \Rightarrow f''(0) = 0.$

$$f'''(x) = \frac{(1+x^2)^2(-2) - (-2x)2(1+x^2)(2x)}{[(1+x^2)^2]^2}$$

$$= \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} \Rightarrow f'''(0) = \frac{-2(1+(0)^2)^2 + 0}{(1+0)^4} = -2$$

from (1),

$$\tan^{-1} x = 0 + x.(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-2) + \cdots$$

$$\tan^{-1} x = x - 2\frac{x^3}{3!} + \cdots$$

④ $f(x) = e^x$ at $x=1$

Now, Taylor's expansion is

$$f(x) = f(a) + \cancel{(x-a)} f'(x-a) + \frac{x^2}{2!} f''(x-a) + \frac{x^3}{3!} f''' x-a$$

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$\Rightarrow$ at $a=1$

$$f(x) = f(1) + (x-1) f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots \longrightarrow ①$$

$f(x) = e^x \rightarrow f(1) = e$

$f'(x) = e^x \rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

$f'''(x) = e^x \Rightarrow f'''(1) = e$

$f^{IV}(x) = e^x \Rightarrow f^{IV}(1) = e$

$\vdots$

from ①,

$$e^x = e + (x-1) e + \frac{(x-1)^2}{2!} e + \frac{(x-1)^3}{3!} e + \cdots$$

$$e^x = e\left[1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \cdots\right]$$