State Variable Analysis And Design Control Systems.

Syllabus :-

1) Introduction — Concept of State, State Variables and State Models.

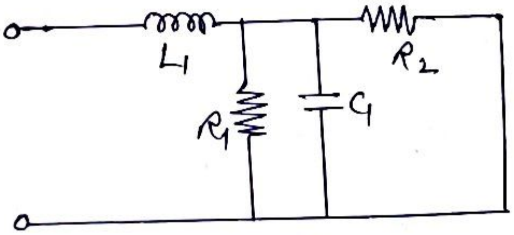
2) State Models for linear Continuous Time Systems
3) State Variables and Linear discrete Time Systems — Solution to State Equations and Concept of Controllability and Observability.

Introduction :-

The Future Behaviour of the system is based on present input and past history of the System. Past history of the System can be described by State Variables.

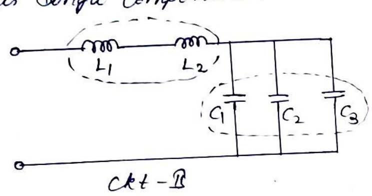
No. of State Variables :-

1) If any electrical network is given, the no. of State Variables is equal to the sum of the inductors and capacitors.



Ckt - I

2) If there are 3 storage elements then there are 3 states. If same kind of elements are Connected in series (or) parallel then it should be treated as Single Component.



Ckt - II

For example, From the above circuit, there are 5 storage elements are present but there are only two States exists.

3) If differential Equation is given, the no. of State Variables is equal to the order of differential Equation.

* Limitations of Transfer Function Analysis :-

1) Transfer Function (T/F) analysis is more suitable for Single Input and Single Output (SISO) systems, whereas State analysis is used for multi-Input and multi-output (MIMO) Systems.

2) T/F analysis can't give any idea about Controllability and Observability.

3) Initial Conditions :- T/F Analysis is valid for only LTI systems whereas State Space Analysis is valid for dynamic systems i.e. system may be linear (or) non-linear, Time Variant (or) Time Invariant.

* Standard Form of State Model :-

$$\dot{X}_{n\times1} = A_{n\times n} X_{n\times1} + B_{n\times m} U_{m\times1} \quad \text{—}(1)$$

$$Y_{p\times1} = C_{p\times n} X_{n\times1} + D_{p\times m} U_{m\times1} \quad \text{—}(2)$$

Eq-(1) is called as "State Equation".

Eq-(2) is called as "output Equation".

Where, A — State Matrix

$\dot{X} = \frac{dx}{dt}$ — Differential State Vector

X — State Vector

U — Input Vector

Y — output Vector

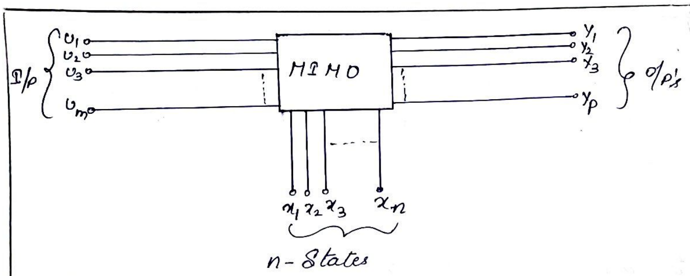
B — Input Matrix

C — Output Matrix

and D — Transition Matrix

Order of the matrix :-

Consider a MIMO System with 'm' inputs, 'n' states and 'p' outputs as shown in figure.



Then,

$$U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_m \end{bmatrix}_{m \times 1} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

And $$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_p \end{bmatrix}_{p \times 1}$$

Note :- If the 'D' matrix is always zero because if the network not Consists of any active element.

* State Space Models :-

1) Differential Equations
2) Transfer Functions
3) Signal Flow Graph
4) Electrical Networks

③

Problems on State Model to Differential Equations :-

1. Write the State model for the following Differential Equation (D.E).

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 10\,u(t)$$

Sol:- Given D.E is,

$$\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 10\,u(t) \quad \text{--- (1)}$$

The order of given D.E is '3'.

$$\Rightarrow \boxed{n = 3}$$

Let, $y = x_1$

$$\Rightarrow \dot{y} = \frac{dx_1}{dt} = x_2$$

$$\ddot{y} = \frac{dx_2}{dt} = x_3$$

$$\dddot{y} = \frac{dx_3}{dt} = \dot{x}_3$$

By Substituting the above values in Eq-(1),

we get,

$$\dot{x}_3 + 2x_3 + 3x_2 + 4x_1 = 10u(t)$$

$$\dot{x}_3 = [-4x_1 - 3x_2 - 2x_3] + 10\,u(t)$$

The above Equation is in the form of State Equation given by,

$$\dot{X}_{n\times 1} = A_{n\times n}\, X_{n\times 1} + B_{n\times m}\, U_{m\times 1}$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}_{3\times1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}_{3\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}_{3\times1} [U]_{1\times1}$$

Now, the output Equation is given by,

$$y = x_1$$

which is in the form of,

$$Y = CX + DU$$

Where, $D = [0]$

$$\Rightarrow Y = CX$$

$$\therefore [Y]_{1\times1} = [1 \; 0 \; 0]_{1\times3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times1}$$

2. Obtain the State Equation for the differential Equation given by,

$$\ddddot{y} + 6\dddot{y} + 5\ddot{y} + 3\dot{y} + 2y = 5u(t)$$

Sol: Given D.E is,

$$\ddddot{y} + 6\dddot{y} - 5\ddot{y} + 3\dot{y} + 2y = 5u(t) \quad \text{--- (1)}$$

Order of D.E is 4.

$$\Rightarrow \boxed{n = 4}$$

Let, $y = x_1$

$$\Rightarrow \dot{y} = \frac{dx_1}{dt} = x_2$$

$$\ddot{y} = \frac{dx_2}{dt} = x_3$$

$$\dddot{y} = \frac{dx_3}{dt} = x_4$$

$$\ddddot{y} = \frac{dx_4}{dt} = \dot{x}_4$$

Substitute the above values in Eq-(1),

$$\Rightarrow \dot{x}_4 + 6x_4 - 5x_3 + 3x_2 + 2x_1 = 5u(t)$$

$$\therefore \dot{x}_4 = [-2x_1 - 3x_2 + 5x_3 - 6x_4] + 5u(t)$$

This is the required State Equation which is in the form of,

$$\dot{X} = AX + BU$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix}_{4\times1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & 5 & -6 \end{bmatrix}_{4\times4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}_{4\times1} [U]_{1\times1}$$

Now, The output Equation is,

$$y = x_1$$

which is in the form of

$$Y = CX + DU$$

where, $D = [0]$

$$\therefore [Y]_{1\times1} = [1 \; 0 \; 0 \; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

* **Problems on State Model to T/F :-**

1. Write the State Model to the given T/F,

$$G(S)H(S) = \frac{2S+3}{S^3+5S^2+6S+7}$$

Sol: Given,

$$T/F = \frac{Y(S)}{U(S)} = \frac{2S+3}{S^3+5S^2+6S+7}$$

$$\Rightarrow Y(S) = 2S + 3S^0 \text{ --- (1)}$$

and $U(S) = S^3 + 5S^2 + 6S + 7S^0$ --- (2)

Let, $S^0 = x_1$

$$\Rightarrow S^1 = \frac{dx_1}{dt} = x_2$$

$$S^2 = \frac{dx_2}{dt} = x_3$$

$$S^3 = \frac{dx_3}{dt} = \dot{x}_3$$

By substituting above values in Eq-(2), we get,

$$U(S) = \dot{x}_3 + 5x_3 + 6x_2 + 7x_1$$

$$\Rightarrow \dot{x}_3 = [-7x_1 - 6x_2 - 5x_3] + U(S)$$

The above Eqn is the required State Equation which is in the form of,

$$\dot{x} = AX + BU$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [U]$$

By Substituting S^0, S^1, S^2 & S^3 values in Eq-(1), The output Equation is obtained as,

$$Y(S) = [2x_2 + 3x_1]$$

Which is in the form of,

$$Y = CX + DU$$

where, $[D] = 0$

$$\Rightarrow [Y] = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}_{1\times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3\times 1} + [0]_{1\times 1}[U]_{1\times 1}$$

2. Write the State Model for the given T/F,

$$T/F = \frac{S^3+6S^2+10}{S^5+10S^4-8S^2+7S+9}$$

Sol: Given,

$$T/F = \frac{Y(S)}{U(S)} = \frac{S^3+6S^2+10}{S^5+10S^4-8S^2+7S+9}$$

$$\Rightarrow Y(S) = 6S^2 + S^3 + 10S^0 \text{ --- (1)}$$

and $U(S) = S^5 + 10S^4 - 8S^2 + 7S + 9S^0$ --- (2)

The order of the given System is 5.

$$\Rightarrow \boxed{n = 5}$$

Let, $S^0 = x_1$

$$\Rightarrow S^1 = \frac{dx_1}{dt} = x_2$$

$$s^2 = \frac{dx_2}{dt} = x_3$$

$$s^3 = \frac{dx_3}{dt} = x_4$$

$$s^4 = \frac{dx_4}{dt} = x_5$$

and $$s^5 = \frac{dx_5}{dt} = \dot{x}_5$$

By substituting the above values in Eq-②, we get,

$$U(s) = \dot{x}_5 + 10x_5 - 8x_3 + 7x_2 + 9x_1$$

$$\Rightarrow \dot{x}_5 = [-9x_1 - 7x_2 + 8x_3 - 10x_5] + U(s)$$

This is the required State Equation which is in the form of,

$$\dot{X} = AX + BU$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix}_{5\times1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -9 & -7 & 8 & 0 & -10 \end{bmatrix}_{5\times5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times1} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{5\times1} [U]_{1\times1}$$

By substituting the s^3, s^2 and s^0 values in Eq-①, the output Equation is obtained as,

$$Y(s) = [x_4 + 6x_3 + 10x_1] + 0\cdot U(s)$$

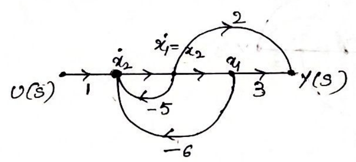
$$\Rightarrow [Y] = [10 \;\; 0 \;\; 6 \;\; 1 \;\; 0]_{1\times5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times1} + [0]_{1\times1}[U]_{1\times1}$$

Procedure for the Conversion of Signal Flow Graph to State Model:-

1) First node should be Input node.
2) Next node is the highest power of 's' Equivalent to differential State Variable.
3) Successive nodes are integrable nodes untill to get 'x_1'.
4) Last node should be the output node.

Problems:-

Q. Obtain the State model for the following Signal Flow Graph



Sol: From the given Signal Flow Graph (SFG),

$$\dot{x}_2 = [-5x_2 - 6x_1] + U(s)$$

$$\Rightarrow \dot{x}_2 = [-6x_1 - 5x_2] + U(s)$$

This is the required State Equation which is in the form of,

$$\dot{X} = AX + BU$$

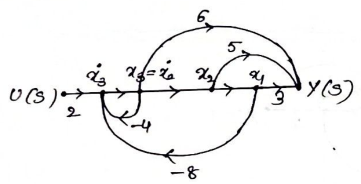
$$\Rightarrow \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times1} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times1} [U]_{1\times1}$$

Now, The o/p Equation is,

$$Y(s) = 3x_1 + 2x_2$$

$$\Rightarrow \underbrace{[Y]}_{1\times1} = \underbrace{[3 \quad 2]}_{1\times2} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times1} + [0]_{1\times1}[U]_{1\times1}$$

2. Obtain the State Model for the following Signal Flow Graph (SFG),



Sol:- From the given SFG, we have,

$$\dot{x}_3 = [-4x_3 - 8x_1] + 2U(s)$$

$$\Rightarrow \dot{x}_3 = [-8x_1 - 4x_3] + 2U(s)$$

$$\Rightarrow \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}}_{3\times1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & 0 & -4 \end{bmatrix}}_{3\times3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times1} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}_{3\times1} [U]_{1\times1}$$

Now, The output Equation is,

$$Y(s) = 3x_1 + 5x_2 + 6x_3$$

$$\Rightarrow \underbrace{[Y]}_{1\times1} = \underbrace{[3 \quad 5 \quad 6]}_{1\times3} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{3\times1} + [0]_{1\times1}[U]_{1\times1}$$

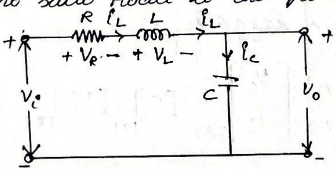
State Model for Electrical Networks :-

Procedure :-

1) Select the State variables as voltage across capacitor, current through Inductor.
2) No. of State Variables equals to sum of inductor and Capacitors.
3) Write Independent KCL and KVL at Capacitor junction. Apply KCL across inductor and KVL through inductor.
4) The resultant Equation should consists of only State variables, differential State variables, input variables and output variables.

Problems :-

Obtain the state Model to the given circuit.



Sol:- Apply KVL to the input loop,

$$\Rightarrow -V_i + i_L R + L\frac{di_L}{dt} + V_C = 0$$

$$\Rightarrow L\frac{di_L}{dt} = -V_C - i_L R + V_i$$

$$\Rightarrow \dot{i}_L = \left[-\frac{1}{L}V_C - \frac{R}{L}i_L\right] + \frac{V_i}{L} \quad \text{--- (1)}$$

Here, The state variables are V_C and i_L

Hence, The Corresponding state vector is $\begin{bmatrix} V_C \\ i_L \end{bmatrix}$

From the given circuit, we have

$$i_L = i_C$$

$$\Rightarrow i_L = C\frac{dV_C}{dt}$$

$$\therefore \dot{V}_C = \frac{1}{C}i_L \quad \text{--- (2)}$$

Hence, From Eq-(1) & -(2), The state Equation is obtained as,

$$\begin{bmatrix} \dot{V}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V_i$$

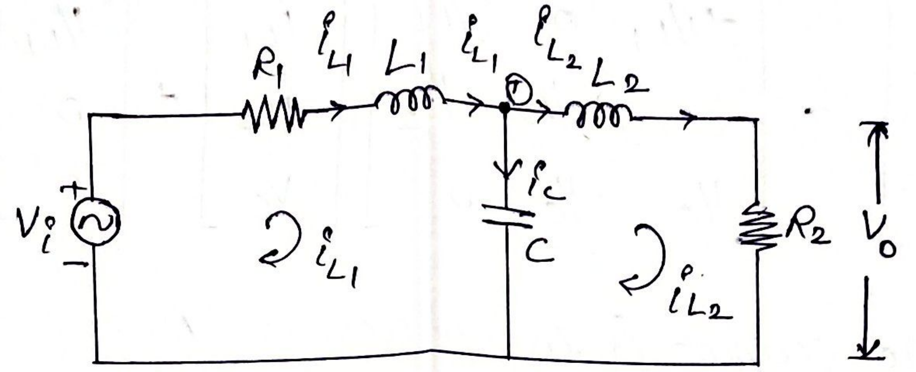
As the output is taken across 'C',

$$\therefore V_0 = V_C.$$

Hence, the output Equation is,

$$[V_0]_{1\times1} = [1 \quad 0]_{1\times2} \begin{bmatrix} V_C \\ i_L \end{bmatrix}_{2\times1} + [0]_{1\times1}[V_i]_{1\times1}$$

2. Obtain the State Model for the given electrical circuit.



Sol:- Apply KVL to the loop - 1,

$$\Rightarrow -V_i + i_{L1} R_1 + L_1 \frac{di_{L1}}{dt} + V_C = 0$$

$$\Rightarrow L_1 \dot{i}_{L1} = -V_C - i_{L1} R_1 + V_i$$

$$\therefore \dot{i}_{L} = \left[-\frac{1}{L_1} V_C - \frac{R_1}{L_1} i_{L1}\right] + \frac{1}{L_1} V_i \quad \text{--- (1)}$$

Apply KVL at loop - 2,

$$\Rightarrow -V_C + L_2 \frac{di_{L2}}{dt} + i_{L2} R_2 = 0$$

$$\Rightarrow L_2 \dot{i}_{L2} = V_C - i_{L2} R_2$$

$$\therefore \dot{i}_{L2} = \frac{1}{L_2} V_C - \frac{R_2}{L_2} i_{L2} \quad \text{--- (2)}$$

Apply KCL at node - 1,

$$\Rightarrow i_{L1} = i_C + i_{L2}$$

$$i_{L1} = C \frac{dV_C}{dt} + i_{L2}$$

$$\Rightarrow C \dot{V}_C = i_{L1} - i_{L2}$$

$$\therefore \dot{V}_C = \frac{1}{C} i_{L1} - \frac{1}{C} i_{L2} \quad \text{--- (3)}$$

From Eq - 1, - 2 & - 3, the State Equation is obtained as,

$$\begin{bmatrix} \dot{i}_{L_1} \\ \dot{i}_{L_2} \\ \dot{V}_c \end{bmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & -1/L_1 \\ 0 & -R_2/L_2 & 1/L_2 \\ 1/C & -1/C & 0 \end{bmatrix} \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + \begin{bmatrix} 1/L_1 \\ 0 \\ 0 \end{bmatrix} [V_i]$$

Since, the output is taken across R_2,

$$\therefore\ V_o = i_{L_2} R_2.$$

Hence, the output Equation is,

$$[V_o] = [0 \quad R_2 \quad 0] \begin{bmatrix} i_{L_1} \\ i_{L_2} \\ V_C \end{bmatrix} + [0][V_i]$$

Transfer function for State Model :-

Consider,

$$\dot{X} = AX + BU \quad \text{—} (1)$$

$$Y = CX + DU \quad \text{—} (2)$$

Apply Laplace Transform for Eq - (1),

$$\Rightarrow SX(S) = AX(S) + BU(S)$$

$$\Rightarrow X(S)[SI - A] = BU(S)$$

$$\therefore\ \boxed{X(S) = B[SI-A]^{-1} U(S)} \quad \text{—} (3)$$

Now, Apply Laplace Transform for Eq - (2),

$$\Rightarrow Y(S) = CX(S) + DU(S) \quad \text{—} (4)$$

Now, Substitute Eq - 3 in Eq - 4,

$$\Rightarrow Y(S) = CB[SI-A]^{-1}U(S) + DU(S)$$

$$\Rightarrow Y(S) = U(S)\left[C[SI-A]^{-1}B + D\right]$$

$$\therefore \boxed{\frac{Y(S)}{U(S)} = C[SI-A]^{-1}B + D}$$

But, as the matrix 'D' is Equals to '0'

$$\therefore \boxed{\frac{Y(S)}{U(S)} = C[SI-A]^{-1}B}$$

Note :-

1) The Equation $|SI-A| = 0$ is known as "Characteristic Equations".

2) The roots of above Equation (C.E) are called as Eigen Values.

Problems :-

Find the T/F to the given State Model,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} [U]$$

$$[Y] = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Sol :- From the given State Model, we have

$$A = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \; ; \; B = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

we have,

$$\frac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B \quad \text{——— (1)}$$

Now, $[sI - A]^{-1} = \dfrac{\text{Adj}\,[sI - A]}{|sI - A|}$

Now, $\text{Adj}\,[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$

$$\Rightarrow [sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix}$$

$$[sI - A] = \begin{bmatrix} s+2 & 3 \\ -4 & s-2 \end{bmatrix}$$

Now, $\text{Adj}\,[sI - A] = \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$

And $|sI - A| = \begin{vmatrix} s+2 & 3 \\ -4 & s-2 \end{vmatrix} = (s+2)(s-2) + 12$

$$\Rightarrow |sI - A| = s^2 - 4 + 12 = s^2 + 8$$

$$\Rightarrow [sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}$$

$$\Rightarrow [sI - A]^{-1} = \frac{1}{s^2 + 8}\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$$

$$\Rightarrow [SI-A]^{-1}\cdot B = \frac{1}{S^2+8}\begin{bmatrix} S-2 & -3 \\ 4 & S+2 \end{bmatrix}_{2\times 2}\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{S^2+8}\begin{bmatrix} 3S-6-15 \\ 12+5S+10 \end{bmatrix}_{2\times 1}$$

$$\therefore [SI-A]^{-1}\cdot B = \frac{1}{S^2+8}\begin{bmatrix} 3S-21 \\ 5S+22 \end{bmatrix}$$

Now, From Eq-1,

$$\frac{Y(S)}{U(S)} = C\,[SI-A]^{-1}\cdot B$$

$$\Rightarrow \frac{Y(S)}{U(S)} = \frac{1}{S^2+8}\begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2}\begin{bmatrix} 3S-21 \\ 5S+22 \end{bmatrix}_{2\times 1}$$

$$= \frac{1}{S^2+8}\begin{bmatrix} 3S-21+5S+22 \end{bmatrix}_{1\times 1}$$

$$\therefore \boxed{\frac{Y(S)}{U(S)} = \frac{8S+1}{S^2+8}}$$

Obtain the T/F to the given State Model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} S & -3 \\ 2 & S+5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}[U]$$

$$[Y] = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And also find the Step response.

Sol:- From the given State Model, we have

$$[SI - A] = \begin{bmatrix} S & -3 \\ 2 & S+5 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

we have,

$$\frac{Y(S)}{U(S)} = C \cdot [SI - A]^{-1} B \quad \text{---} (1)$$

where, $[SI-A]^{-1} = \dfrac{\text{Adj}\,[SI-A]}{|SI-A|}$

Now, $\text{Adj}\,[SI-A] = \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$

and $|SI-A| = \begin{vmatrix} S & -3 \\ 2 & S+5 \end{vmatrix} = S(S+5)+6$

$$|SI-A| = S^2+5S+6$$

$$\Rightarrow [SI-A]^{-1} = \frac{1}{S^2+5S+6} \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$\Rightarrow C[SI-A]^{-1} = \frac{1}{S^2+5S+6} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} S+5 & 3 \\ -2 & S \end{bmatrix}$$

$$= \frac{1}{S^2+5S+6} \begin{bmatrix} 0-2 & 0+S \end{bmatrix}$$

$$\Rightarrow C\,[SI-A]^{-1} = \frac{1}{S^2+5S+6} \begin{bmatrix} -2 & S \end{bmatrix}$$

Now,

$$C \cdot [SI-A]^{-1} B = \frac{1}{S^2+5S+6} \begin{bmatrix} -2 & S \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \frac{Y(S)}{U(S)} = C \cdot [SI-A]^{-1} \cdot B = \frac{1}{S^2+5S+6} \cdot [-2+S]$$

$$\therefore \boxed{\frac{Y(S)}{U(S)} = \frac{S-2}{S^2+5S+6}}$$

To Find Step Response :-

Given that,

Input is step signal.

$$\Rightarrow U(S) = \frac{1}{S}$$

Now, we have,

$$\frac{Y(S)}{U(S)} = \frac{S-2}{S^2+5S+6}$$

$$\Rightarrow Y(S) = \frac{S-2}{S^2+5S+6} \cdot U(S)$$

$$Y(S) = \frac{S-2}{S^2+5S+6} \cdot \frac{1}{S}$$

$$\Rightarrow Y(S) = \frac{S-2}{S(S^2+5S+6)}$$

Consider, $\dfrac{S-2}{S(S^2+5S+6)} = \dfrac{S-2}{S(S+2)(S+3)}$

$$\Rightarrow \frac{S-2}{S(S+2)(S+3)} = \frac{A}{S} + \frac{B}{S+2} + \frac{C}{S+3}$$

$$\Rightarrow S-2 = A(S+2)(S+3) + BS(S+3) + CS(S+2)$$

At $S=0$,

$$\Rightarrow -2 = A(2)(3)$$

$$\therefore \boxed{A = -1/3}$$

At $S=-2$,

$$\Rightarrow +4_2 = B(-2)(1)$$

$$\therefore \boxed{B = 2}$$

At $S=-3$,

$$\Rightarrow +5 = C(-3)(-1)$$

$$\therefore \boxed{C = -5/3}$$

$$\Rightarrow Y(S) = \frac{S-2}{S(S+2)(S+3)} = \frac{-1}{3S} + \frac{2}{S+2} - \frac{5}{3(S+3)}$$

Apply Inverse Laplace Transform on b.s

$$\Rightarrow y(t) = L^{-1}\left\{ \frac{-1}{3S} + \frac{2}{S+2} - \frac{5}{3(S+3)} \right\}$$

$$y(t) = -\frac{1}{3}u(t) + 2e^{-2t}u(t) - \frac{5}{3}e^{-3t}u(t)$$

$$\therefore \boxed{y(t) = u(t)\left[-\frac{1}{3} + 2e^{-2t} - \frac{5}{3}e^{-3t}\right]}$$

(12)

Solution To State Equations :-

1. Laplace Transform Method :-

Consider the state Equation and o/p Equation

$$\Rightarrow \dot{X} = AX + BU$$

$$Y = CX + DU.$$

Apply L.T for both Equations

$$\Rightarrow SX(S) - X(0) = AX(S) + BU(S) \quad \text{--- (1)}$$

$$Y(S) = CX(S) + DU(S)$$

From Eq -(1),

$$\Rightarrow X(S)[SI-A] = BU(S) + X(0)$$

$$X(S) = [SI-A]^{-1} BU(S) + [SI-A]^{-1}X(0)$$

Apply Inverse Laplace Transform,

$$\Rightarrow x(t) = L^{-1}\{[SI-A]^{-1}BU(S)\} + L^{-1}\{[SI-A]^{-1}X(0)\} \quad \text{--- (2)}$$

The Eq -(2) Contains two parts.

1st part is called as "Zero State Response"

2nd Part is called as "Zero Input Response"

2. Classical Method :-

we have,

$$x(t) = \underbrace{\int_0^t e^{A(t-\tau)} BU(\tau)d\tau}_{ZSR} + \underbrace{e^{At}x(0)}_{ZIR} \quad \text{--- (3)}$$

From Eq-(2) and Eq-(3), we have,

$$\mathcal{L}^{-1}\{[sI-A]^{-1}x(0)\} = e^{At}x(0) \quad \text{—— (4)}$$

And $\mathcal{L}^{-1}\{[sI-A]^{-1}BU(s)\} = \int_0^t e^{A(t-\tau)}BU(\tau)d\tau) \quad \text{—— (5)}$

From Eq-(4),

$$\Rightarrow \mathcal{L}^{-1}\{[sI-A]^{-1}\}\cancel{x(0)} = e^{At}\cancel{x(0)}$$

$$\therefore \mathcal{L}^{-1}\{[sI-A]^{-1}\} = e^{At} = \phi(t)$$

$$\Rightarrow [sI-A]^{-1} = \mathcal{L}\{e^{At}\} = \mathcal{L}\{\phi(t)\}$$

$$\therefore [sI-A]^{-1} = \phi(s)$$

where, $\phi(t)$ — State Transition Matrix

$\phi(s)$ — Resolvent Matrix

Now, From Eq-(5),

$$\Rightarrow \mathcal{L}^{-1}\{[sI-A]^{-1}BU(s)\} = \int_0^t e^{A(t-\tau)}BU(\tau)d\tau$$

we have, $\mathcal{L}^{-1}\{[sI-A]^{-1}\} = \phi(t)$

$$\Rightarrow [sI-A]^{-1} = \phi(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{\phi(s)BU(s)\} = \int_0^t e^{A(t-\tau)}BU(\tau)d\tau.$$

$$\therefore \boxed{x(t) = \mathcal{L}^{-1}\{\phi(s)BU(s)\} + \mathcal{L}^{-1}\{\phi(s)x(0)\}}$$

Properties of STM :-

i) we have,

$$\phi(t) = e^{At} = \mathcal{L}^{-1}\left\{[sI - A]^{-1}\right\}$$

1) $\phi(0) = I$

2) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = e^{A(-t)} = \phi(-t)$

$\Rightarrow \phi^{-1}(t) = \phi(-t)$

3) $\phi^{k}(t) = \phi(kt)$

4) $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1 + At_2} = e^{At_1} \cdot e^{At_2}$

$\Rightarrow \phi(t_1 + t_2) = \phi(t_1) \cdot \phi(t_2)$.

5) $\phi(t_2 - t_1) = -\phi(t_2 - t_0) * \phi(t_1 - t_0)$

Problems on STM :-

Obtain the Complete Time response of the S/m given by,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x$$

$$y = [1 \quad -1] x$$

$$\text{and } x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Given that,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x \quad ; \quad y = [1 \quad -1] x \text{ and}$$

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ and $c = [1 \ -1]$

Now, The required General solution is,

$$x(t) = L^{-1}\{\phi(s) B U(s)\} + L^{-1}\{\phi(s) X(0)\}.$$

From the given State Equation, i.e.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x,$$

Since, $B = 0$

$$\Rightarrow L^{-1}\{\phi(s) B U(s)\} = 0$$

Hence, $x(t) = L^{-1}\{\phi(s) X(0)\}$ —— (1)

But, we K.T,

$$\phi(s) = [sI - A]^{-1} = \frac{Adj\,[sI-A]}{|sI-A|}$$

Now, $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

$$\Rightarrow sI - A = \begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}$$

$$\Rightarrow Adj\,[sI-A] = \begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$$

$$\Rightarrow |sI - A| = \begin{vmatrix} s & -1 \\ 2 & s \end{vmatrix} = s^2 + 2$$

Therefore, $\phi(s) = \dfrac{Adj\,[sI-A]}{|sI-A|} = \dfrac{1}{s^2+2}\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}$

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Now, $\phi(s)X(0) = \dfrac{1}{s^2+2}\begin{bmatrix} s & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\Rightarrow \phi(s) X(0) = \frac{1}{s^2+2}\begin{bmatrix} s+1 \\ s-2 \end{bmatrix}$$

Apply ILT on b.s

$$\Rightarrow L^{-1}\{\phi(s) X(0)\} = L^{-1}\left\{\frac{1}{s^2+2}\begin{bmatrix} s+1 \\ s-2 \end{bmatrix}\right\}.$$

From Eq-(1),

$$\Rightarrow x(t) = L^{-1}\{\phi(s) X(0)\} = L^{-1}\left\{\begin{bmatrix} \frac{s+1}{s^2+2} \\ \frac{s-2}{s^2+2} \end{bmatrix}\right\}$$

$$x(t) = \begin{bmatrix} L^{-1}\left\{\frac{s+1}{s^2+2}\right\} \\ L^{-1}\left\{\frac{s-2}{s^2+2}\right\} \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} L^{-1}\left\{\frac{s}{s^2+2}\right\} + L^{-1}\left\{\frac{1}{s^2+2}\right\} \\ L^{-1}\left\{\frac{s}{s^2+2}\right\} - L^{-1}\left\{\frac{2}{s^2+2}\right\} \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix}$$

To find the Complete Time response, we have

$$y(t) = [1 \ -1]\, x.$$

i.e. $y(t) = [1 \ -1]\, x(t)$

$$\Rightarrow y(t) = [1 \;\; -1]\begin{bmatrix} \cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t \\ \cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{bmatrix}$$

$$\Rightarrow y(t) = \cancel{\cos\sqrt{2}t} + \frac{1}{\sqrt{2}}\sin\sqrt{2}t - \cancel{\cos\sqrt{2}t} + \sqrt{2}\sin\sqrt{2}t$$

$$\therefore \boxed{y(t) = \frac{3}{\sqrt{2}}\sin\sqrt{2}t}$$

2. Find out the time response, for unit step input of the system given by;

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and $Y = [-2 \;\; -3]\, X$.

Sol: Given State Equation is,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 5 \end{bmatrix} u$$

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

The given system is Non-Homogeneous S/m.

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Hence, The General Solution is,

$$x(t) = L^{-1}\{\phi(s)BU(s)\} + L^{-1}\{\phi(s)X(0)\} \quad \text{—— (1)}$$

$$\text{But, } \phi(s) = [sI - A]^{-1} = \frac{\text{Adj}[sI - A]}{|sI - A|} \quad \text{—— (2)}$$

$$\text{Now, } sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\Rightarrow sI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\Rightarrow \text{Adj}[sI - A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\Rightarrow |sI - A| = \begin{vmatrix} s+3 & 1 \\ -2 & s \end{vmatrix} = s(s+3) + 2 = s^2 + 3s + 2$$

From Eq-(2),

$$\Rightarrow \phi(s) = \frac{\text{Adj}[sI - A]}{|sI - A|} = \frac{1}{s^2+3s+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\Rightarrow \phi(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\text{Now, } \phi(s)X(0) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \phi(s)X(0) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 \\ -2 \end{bmatrix}$$

Apply ILT on both sides.

$$\Rightarrow L^{-1}\{\phi(s)X(0)\} = L^{-1}\left\{\frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 \\ -2 \end{bmatrix}\right\}$$

$$\Rightarrow L^{-1}\{\phi(s)X(0)\} = L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} \end{bmatrix}$$

$$\Rightarrow L^{-1}\{\phi(s)X(0)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{s+3}{(s+1)(s+2)}\right\} \\ L^{-1}\left\{\dfrac{-2}{(s+1)(s+2)}\right\} \end{bmatrix}$$

$$\Rightarrow L^{-1}\{\phi(s)X(0)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{2}{s+1} - \dfrac{1}{s+2}\right\} \\ L^{-1}\left\{\dfrac{-2}{s+1} + \dfrac{2}{s+2}\right\} \end{bmatrix}$$

$$\therefore L^{-1}\{\phi(s)X(0)\} = \begin{bmatrix} 2e^{-t}u(t) - e^{-2t}u(t) \\ -2e^{-t}u(t) + 2e^{-2t}u(t) \end{bmatrix} \quad \text{--- (3)}$$

Now, $$\phi(s)B = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\Rightarrow \phi(s)\cdot B = \frac{1}{(s+1)(s+2)}\begin{bmatrix} (s+3)0+5 \\ 5s \end{bmatrix}$$

$$\Rightarrow \phi(s)\cdot B\cdot U(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix} U(s)$$

Given that,

The I/p of the S/m is a Step Input.

$$\Rightarrow U(s) = \frac{1}{s}.$$

$$\Rightarrow \phi(s)BU(s) = \frac{1}{(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix}\cdot\frac{1}{s}$$

$$\Rightarrow \phi(s)BU(s) = \frac{1}{s(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix}$$

Apply ILT on both sides.

$$\Rightarrow L^{-1}\{\phi(s)BU(s)\} = L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\begin{bmatrix} 5 \\ 5s \end{bmatrix}\right\}$$

$$\Rightarrow L^{-1}\{\phi(s)BU(s)\} = L^{-1}\begin{bmatrix} \dfrac{5}{s(s+1)(s+2)} \\ \dfrac{5\cancel{s}}{\cancel{s}(s+1)(s+2)} \end{bmatrix}$$

$$\Rightarrow L^{-1}\{\phi(s)BU(s)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{5}{s(s+1)(s+2)}\right\} \\ L^{-1}\left\{\dfrac{5}{(s+1)(s+2)}\right\} \end{bmatrix}$$

$$\Rightarrow L^{-1}\{\phi(s)BU(s)\} = \begin{bmatrix} L^{-1}\left\{\dfrac{5}{2s} - \dfrac{5}{s+1} + \dfrac{5}{2(s+2)}\right\} \\ L^{-1}\left\{\dfrac{5}{s+1} - \dfrac{5}{s+2}\right\} \end{bmatrix}$$

$\therefore \mathcal{L}^{-1}\{\phi(s)BU(s)\} = \begin{bmatrix} \frac{5}{2}u(t) - 5e^{-t}u(t) + \frac{5}{2}e^{-2t}u(t) \\ 5e^{-t}u(t) - 5e^{-2t}u(t) \end{bmatrix}$ — (4)

Substitute Eq-(3) & —(4) in Eq-(1),

$$\Rightarrow x(t) = \begin{bmatrix} \frac{5}{2}u(t) - 5e^{-t}u(t) + \frac{5}{2}e^{-2t}u(t) \\ 5e^{-t}u(t) - 5e^{-2t}u(t) \end{bmatrix} + \begin{bmatrix} 2e^{-t}u(t) - e^{-2t}u(t) \\ -2e^{-t}u(t) + 2e^{-2t}u(t) \end{bmatrix}$$

$$\therefore x(t) = \begin{bmatrix} \frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) \\ 3e^{-t}u(t) - 3e^{-2t}u(t) \end{bmatrix}$$

For Time Response, we have

$$y(t) = [-2 \quad -3]\, x$$

$$\Rightarrow y(t) = [-2 \quad -3] \begin{bmatrix} \frac{5}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t) \\ 3e^{-t}u(t) - 3e^{-2t}u(t) \end{bmatrix}$$

$$= -5u(t) + 6u(t)e^{-t} - 3e^{-2t}u(t) - 9e^{-t}u(t) + 9e^{-2t}u(t)$$

$$= -5u(t) - 3e^{-t}u(t) + 6e^{-2t}u(t)$$

$$\therefore \boxed{y(t) = [6e^{-2t} - 3e^{-t} - 5]\, u(t)}$$

Controllability :-

→ A system is said to be Controllable if it is possible to transfer the initial state to any other state in a finite time by Controlled Vector.

→ Controllability is verified by Kalmn's Test.

$$Q_c = [B \quad AB \quad A^2B \; \ldots\ldots \; A^{n-1}B]$$

where, n – order of the matrix 'A'

Conditions :-

1) Rank of Q_c = Rank of A.
2) $|Q_c| \neq 0$

If the system is satisfying the above two Conditions, then it is said to be Controllable.

Problems of Controllability :-

Check the Controllability of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Sol- Given system is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$\Rightarrow A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}_{2\times 2}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore$ The order of $A = 2$

$\Rightarrow \boxed{n=2}$

Now, $Q_c = [B \quad AB]$ —— (1)

But, $AB = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Substitute B & AB Values in Eq-(1),

$\Rightarrow Q_c = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Rank of Q_c :-

$|Q_c| = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0 + (-1) = -1 \neq 0$.

$\therefore$ Rank of $Q_c = 2$

Rank of A :-

$|A| = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3 \neq 0$

$\therefore$ Rank of $A = 2$.

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Conclusions :-

1) Rank of Q_c = Rank of A.

2) $|Q_c| \neq 0$

Therefore, The Given System is Controllable.

Check the Controllability of

$\dot{x}_1 = -2x_1 + u$

$\dot{x}_2 = -3x_2 + 5x_1$

Sol:- Given,

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$

$\Rightarrow A = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\therefore$ Order of matrix A, $n = 2$

$\therefore Q_c = [B \quad AB]$

Now, $AB = \begin{bmatrix} -2 & 0 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Rank of Qc :-

$$|Q_c| = \begin{vmatrix} 1 & -2 \\ 0 & 5 \end{vmatrix} = 5 + 0 = 5 \neq 0$$

$\therefore$ Rank of $Q_c = 2$

Rank of A :-

$$|A| = \begin{vmatrix} -2 & 0 \\ 5 & -3 \end{vmatrix} = 6 - 0 = 6 \neq 0$$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of Q_c = Rank of A.
2) $|Q_c| \neq 0$

Hence, The Given System is Controllable.

* Observability :-

A System is said to be Observable if it is possible to determine initial states of the S/m by observing the output for a finite time interval.

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \ldots\ldots (A^T)^{n-1} C^T \right]$$

(OR)

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Conditions :-

1) Rank of Q_o = Rank of A.
2) $|Q_o| \neq 0$

Problems on Controllability and observability :-

Check the Controllability and observability for,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -01 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} U$$

and $Y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Given,

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Controllability :-

$Q_c = \begin{bmatrix} B & AB \end{bmatrix}$ $\quad [\because n = 2]$

Now, $AB = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\therefore\ Q_C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

Now, $|Q_C| = \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} = \cancel{2}+\cancel{2} = 0$

$\therefore$ Rank of $Q_C = 1$.

Now, $|A| = \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2+0 = 2 \neq 0$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of $Q_C \neq$ Rank of A.

2) $|Q_C| = 0$

Hence, the Given System is not Controllable.

Observability :-

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Now, $CA = [1 \quad 1]\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow CA = [2-1 \quad 1] = [1 \quad 1]$

$\therefore\ Q_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Now, $|Q_0| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \cancel{1}-\cancel{1} = 0$

$\therefore$ Rank of $Q_0 = 1$.

We have,

Rank of $A = 2$.

Conclusions :-

1) Rank of $Q_0 \neq$ Rank of A.

2) $|Q_0| = 0$

Hence, The Given System is not Observable.

Q. Check the Controllability and Observability for,

$$\dot{x}_1 = -2x_1 + x_2 + u$$
$$\dot{x}_2 = -x_2 + u$$
$$y = x_1 + x_2$$

Sol: Given that,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

and $[Y] = [1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

and $C = [1 \quad 1]$

Controllability :-

Since, $n = 2$

$\Rightarrow Q_c = [B \;\; AB]$

Now, $AB = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\therefore Q_c = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

Now, $|Q_c| = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = \cancel{-1}+\cancel{1} = 0$

$\therefore$ Rank of $Q_c = 1$.

Now, $|A| = \begin{vmatrix} -2 & 1 \\ 0 & -1 \end{vmatrix} = 2-0 = 2$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of $Q_c \neq$ Rank of A.
2) $|Q_c| = 0$

Hence, The given System is not Controllable.

Observability :-

$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$

Now, $CA = [1 \;\; 1]\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} = [-2 \;\; 0]$

$\therefore Q_o = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$

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Now, $|Q_o| = \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 0+2 = 2 \neq 0$

$\therefore$ Rank of $Q_o = 2$

we have,

Rank of $A = 2$

Conclusions :-

1) Rank of Q_o = Rank of A
2) $|Q_o| \neq 0$

Hence, The given System is observable.

Determine the Controllability and Observability of

$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = [1 \;\; 2]$

Sol: Given,

$A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $C = [1 \;\; 2]$

Order of the matrix A, $n = 2$

Controllability :-

$Q_c = [B \;\; AB]$

Now, $AB = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

$\therefore Q_c = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$

Now, $|Q_c| = \begin{vmatrix} 0 & 0 \\ 1 & -3 \end{vmatrix} = 0-0 = 0$

$\therefore$ Rank of $Q_c = 1$

Now, $|A| = \begin{vmatrix} -1 & 0 \\ 0 & -3 \end{vmatrix} = 3-0 = 3 \neq 0$

$\therefore$ Rank of $A = 2$

Conclusions :-

1) Rank of $Q_c \neq$ Rank of A.

2) $|Q_c| = 0$

Hence, The Given System is not Controllable.

Observability :-

$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix}$

Now, $CA = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -6 \end{bmatrix}$

$\therefore Q_o = \begin{bmatrix} 1 & 2 \\ -1 & -6 \end{bmatrix}$

Now, $|Q_o| = \begin{vmatrix} 1 & 2 \\ -1 & -6 \end{vmatrix} = -6+2 = -4 \neq 0$

$\therefore$ Rank of $Q_o = 2$.

And also we have,

Rank of $A = 2$

Conclusions :-

1) Rank of Q_o = Rank of A

2) $|Q_o| \neq 0$.

Hence, The Given System is observable

* Additional Problems :-

1. Determine the State model for

$$\frac{Y(s)}{U(s)} = \frac{2s^2+s+5}{s^3+6s^2+11s+4}$$

Sol:- Given,

$$\frac{Y(s)}{U(s)} = \frac{2s^2+s+5}{s^3+6s^2+11s+4}$$

$$\Rightarrow Y(s) = 2s^2+s+5s^0 \quad \text{--- (1)}$$

$$U(s) = s^3+6s^2+11s+4s^0 \quad \text{--- (2)}$$

Let, $s^0 = x_1$

$\Rightarrow s' = \frac{dx_1}{dt} = x_2$