Probability and Distribution

Def:- A random experiment means a repeatable process that yields a result or an observation.

ex:- Tossing a coin, Rolling a die, extract a ball from a box are random experiments.

Def:- (Probability):- The probability of an event is defined as the ratio of no. of favourable chances of the event to total no. of chances (outcomes).

i.e $P(E) = \frac{m}{n} = \frac{\text{no. of favourable outcomes}}{\text{total no. of outcomes}}$.

Note:- 1. The probability of an event lies between 0 & 1.

i.e $0 \le P(E) \le 1$.

2. If $\bar{E}$ denote the event of non-favourable outcomes, then $P(\bar{E}) = 1 - P(E)$.

3. $0 \le P(\bar{E}) \le 1$.

Def:- (Sample space):- The set of all possible outcomes in a random experiment is called a sample space. It is denoted by S.

The elements in a sample space are called Events. It is denoted by E.

Problems

1. In a single throw with two dice, find the probability of a sum (i) 10
(ii) which is a perfect square.

A:- The sample space when two dices are thrown

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$$
$$(2,1)\ (2,2), (2,3), (2,4), (2,5), (2,6)$$
$$(3,1)\ (3,2), (3,3), (3,4), (3,5), (3,6)$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}.$$

Total no. of events $= 6^2 = 36 = n$

(i) Let E be the event of getting a sum 10 on the two dices.

$E = \{ (5,5), (4,6), (6,4) \}$. ie no. of favourable events $= 3 = m$

$$P(E) = \frac{m}{n} = \frac{3}{6} = \frac{3}{36} = \frac{1}{12}.$$

(ii) Let E be the event of getting which is a perfect square.

ie either sum is 4 or 9.

$E = \{ (1,3), (2,2), (3,1), (3,6), (4,5), (5,4), (6,3) \}$.

ie no. of favourable events $= 7 = m$

$$\therefore P(E) = \frac{m}{n} = \frac{7}{36}.$$

(2) Find the probability that at least one head when a toss two coins at a time.

Ans:- Total no. of even when a toss two coins is HH, HT, TH, TT. i.e. $n=4$

Sample space $S = \{HH, HT, TH, TT\}$. $n=4$

Let E be event at least one head is

$E = \{HH, HT, TH\}$ i.e. $m = 3$.

$$P(E) = \frac{m}{n} = \frac{\text{no. of favourable events}}{\text{Total no. of favourable events}} = \frac{3}{4}$$

(3) Find the probability that exactly one head when a toss two coins at a time.

Ans:- Sample space $S = \{HH, HT, TH, TT\}$. $n=4$

Let E be the event for exactly one head.

$E = \{HT, TH\}$, $m=2$.

$$P(E) = \frac{\text{no. of favourable outcomes}}{\text{Total no. of outcomes}} = \frac{m}{n} = \frac{2}{4} = \frac{1}{2}.$$

(4) In a class there are 10 boys and 5 girls. A committee of 4 students is to be selected from the class. Find the probability for the committee to contain at least 3 girls.

Ans:- Total no. of students $= 10+5 = 15$.

A committee of 4 students from 15 formed in $^{15}C_4$ ways. $= n$

Boys	Girls	
10	5	
1	3	= 4
0	4	= 4

The no. of ways, committee containing at least 3 girls $= {}^{10}C_1 \times {}^{5}C_3 + {}^{10}C_0 \cdot {}^{5}C_4 := m$

$\therefore$ The required probability is

$$P(E) = \frac{m}{n} = \frac{{}^{10}C_1 \times {}^5C_3 + {}^{10}C_0 \times {}^5C_4}{{}^{15}C_4}$$

$${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$${}^{10}C_1 = \frac{10!}{1!\,9!} = 10$$

$${}^{10}C_0 = \frac{10!}{0!\,10!} = 1$$

$${}^5C_3 = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times 3!}{3! \times 2} = 10$$

$${}^5C_4 = \frac{5!}{4!\,1!} = 5$$

$$= \frac{10 \times 10 + 1 \times 5}{1365}$$

$$= \frac{105}{1365}$$

$$= 0.0769$$

$${}^{15}C_4 = \frac{15!}{4!\,11!} = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2} = 105 \times 13 = 1365$$

(5) A class consists of 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that

(i) 3 boys are selected

(ii) exactly 2 girls are selected.

A:- Total no. of students $= 6 + 10 = 16$

The total no. of ways of 3 selected from 16 students

$$n = {}^{16}C_3$$

(i) The no. of ways 3 boys are selected

Boys	Girls	
10	6	
3	0	= 3

$$m = {}^{10}C_3 \times {}^6C_0$$

The required probability $P(E) = \frac{m}{n} = \frac{{}^{10}C_3 \times {}^6C_0}{{}^{16}C_3} = \frac{3}{14}$

$$= 0.2143$$

(ii) The no. of ways exactly 2 girls are selected

B	G	
10	6	
1	2	= 3

$$m = {}^{10}C_1 \times {}^6C_2$$

The required probability is

$$P(E) = \frac{m}{n} = \frac{{}^{10}C_1 \times {}^6C_2}{{}^{16}C_3} = \frac{15}{56} = 0.2678$$

Def:- Two events E_1, E_2 are said to be disjoint events, if $E_1 \cap E_2 = \phi$. These are also called mutually exclusive events.

Note:- 1. If E_1, E_2 are any events in a Sample space, then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

2. If E_1, E_2 are any two disjoint events, then $p(E_1 \cup E_2) = p(E_1) + p(E_2)$

3. If E_1, E_2 are any two events of a sample space, then $p(E_2 - E_1) = p(E_2) - p(E_1)$.

Def:- (conditional Event):- If E_1, E_2 are events of a sample space 'S' and if E_2 occurs after the occurrence of E_1, then the event of occurance of E_2 after E_1 is called conditional Event of E_2.

It is denoted by $\frac{E_2}{E_1}$.

llly we define $\frac{E_1}{E_2}$ —

ex:- 1. Two coins are tossed. The event of getting two tails, given that there is at least one tail is conditional Event.

2. Two unbiased dice are thrown. If the sum of the numbers thrown on them is 7, the event of getting 1 on any one of dice is a conditional event.

Def:- (conditional probability):-

If E_1 and E_2 are two events in a sample space S and $P(E_1) \neq 0$, then the probability of E_2 after the event E_1 has occured, is called conditional probability of E_2. It is denoted by $P\left(\frac{E_2}{E_1}\right)$.

we define $P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$

$||^{ly}$ we define $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$.

~~BAYE'S Theorem~~

Multiplication Theorem of probability

In a random experiment if E_1, E_2 are two events such that $P(E_1) \neq 0$, $P(E_2) \neq 0$

Then $P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right)$

$$P(E_2 \cap E_1) = P(E_2) \cdot P\left(\frac{E_1}{E_2}\right).$$

Independent Events, Dependent events

If the occurence of the event E_2 is not effected by the occurance or non-occurance of the event E_1, then the event E_2 is said to be independent of E_1.

In this case, $P\left(\frac{E_2}{E_1}\right) = P(E_2)$.

otherwise, they are said to be dependent events

In this case, $P\left(\frac{E_2}{E_1}\right) \neq P(E_2)$.

BAYE'S Theorem

$E_1, E_2, \cdots E_n$ are n mutually exclusive and exhaustive events such that $P(E_i) > 0$, $i = 1, 2, \ldots n$ in a Sample space S and A is any other event in 'S' intersecting with every E_i such that $P(A) > 0$.

If E_i is any of the events of $E_1, E_2, \cdots E_n$ where $P(E_1), P(E_2), \cdots P(E_n)$ and $P(A|E_1), P(A|E_2), \cdots P(A|E_n)$ are known, then

$$P(E_k|A) = \frac{P(E_k) \cdot P(A|E_k)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \cdots + P(E_n)P(A|E_n)}$$

Problems

(1.) In a certain college, 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the student body.

(a) What is the probability that mathematics is being studied?

(b) If a student is selected at random is found to be studying mathematics, find the probability that the student a girl?

(c) a boy?

A: Boys — 25%

Girls — 10%

Girls constitute 60% of body, $P(G) = \frac{60}{100} = \frac{3}{5}$

Boys constitute 40% of body; $P(B) = \frac{40}{100} = \frac{2}{5}$.

The probability that mathematics studied by a boy is

$$P(M/B) = \frac{25}{100} = \frac{1}{4}.$$

The probability that mathematics studied by a girl is

$$P(M/G) = \frac{10}{100} = \frac{1}{10}.$$

(a) The required probability that the student studied Mathematics is

$$P(M) = P(B).P(M/B) + P(G).P(M/G)$$

$$= \frac{2}{5}\cdot\frac{1}{4} + \frac{3}{5}\cdot\frac{1}{10} = \frac{4}{25}$$

(b) By Baye's Theorem, the probability of mathematics student is a girl is

$$P\left(\frac{G}{M}\right) = \frac{P(G)\cdot P(M/G)}{P(B)\cdot P(\frac{M}{B}) + P(G)\cdot P(\frac{M}{G})}$$

$$= \frac{\frac{3}{5}\cdot\frac{1}{10}}{\frac{4}{25}} = \frac{3}{\cancel{50}_{2}} \times \frac{\cancel{25}}{4} = \frac{3}{8}.$$

(c) By Baye's Theorem, the probability of mathematics student is a Boy is

$$P(B/M) = \frac{P(B)\cdot P(M/B)}{P(B)\,P(\frac{M}{B}) + P(G)\,P(\frac{M}{G})}$$

$$= \frac{\frac{2}{5}\cdot\frac{1}{4}}{\frac{4}{25}}$$

$$= \frac{2}{\cancel{20}_{4}} \times \frac{\cancel{25}^{5}}{\cancel{4}_{2}} = \frac{5}{8}$$

2) In a Bolt factory Machine A, B, C manufacture 20%, 30% and 50% of the total of their out-put and 6%, 3% and 2% are defective. A bolt is drawn at random and found to be defective. Find the probability that it is manufactured from (i) Machine A (ii) Machine B (iii) Machine C.

Ans:- Let $P(A)$, $P(B)$, $P(C)$ be the probabilities of the events that the bolts are manufactured by the Machine A, B, C resp.

Then $P(A) = \frac{20}{100} = \frac{1}{5}$

$P(B) = \frac{30}{100} = \frac{3}{10}$

$P(C) = \frac{50}{100} = \frac{1}{2}$.

Let D denote that the bolt is defective.

$P(D/A) = \frac{6}{100}$

$P(D/B) = \frac{3}{100}$

$P(D/C) = \frac{2}{100}$

(i) The probability of that defective Bolt from Machin A is

$$P(A/D) = \frac{P(D/A) \cdot P(A)}{P(D/A)\,P(A) + P(D/B)\,P(B) + P(D/C)\,P(C)}$$

$$= \frac{\frac{6}{100} \cdot \frac{1}{5}}{\frac{6}{100} \cdot \frac{1}{5} + \frac{3}{100} \cdot \frac{3}{10} + \frac{2}{100} \cdot \frac{1}{2}}$$

$$= \frac{6/500}{31/1000} = \frac{6}{\cancel{500}} \times \frac{\cancel{1000}^{2}}{31} = \frac{12}{31}.$$

(ii) The probability that the defective Bolt from Machine B is

$$P(B/D) = \frac{P(D/B) \cdot P(B)}{P(\frac{D}{A}) P(A) + P(D/B) P(B) + P(D/C) \cdot P(C)}$$

$$= \frac{3/100 \cdot \frac{3}{10}}{31/1000}$$

$$= \frac{9}{31}.$$

(iii) The probability that the defective Bolt from Machine C is

$$P(C/D) = \frac{P(D/C) P(C)}{P(D/A) P(A) + P(D/B) P(B) + P(D/C) \cdot P(C)}$$

$$= \frac{\frac{2}{100} \cdot \frac{1}{2}}{31/1000}$$

$$= \frac{1}{100} \times \frac{1000}{31} = \frac{10}{31}$$

(3) A Businessman goes to hotels X, Y, Z, 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of the rooms in X, Y, Z hotels having faulty plumbings. What is the probability that business man's room having faulty plumbing is assigned to hotel Z.?

Ans:- Let the probability of business man going to hotels X, Y, Z be resp.

$$P(X) = \frac{20}{100}, \quad P(Y) = \frac{50}{100}, \quad P(Z) = \frac{30}{100}$$

$$= \frac{2}{10} \qquad = \frac{5}{10} \qquad = \frac{3}{10}$$

Let E be the event that fault plumbing rooms. Then the probability that hotels x, y, z having faulty plumbing are

$$P(E|x) = \frac{5}{100} = \frac{1}{20}, \quad P(E|y) = \frac{4}{100} = \frac{1}{25}, \quad P(E/z) = \frac{8}{100} = \frac{2}{25}.$$

The probability that the business man's room having faulty plumbbing & assigned to hotel z is

$$P\left(\frac{z}{E}\right) = \frac{P\left(\frac{E}{z}\right) P(z)}{P(x)\,P\left(\frac{E}{x}\right) + P(y)\cdot P\left(\frac{E}{y}\right) + P(z)\,P\left(\frac{E}{z}\right)}$$

$$= \frac{\frac{2}{25}\cdot\frac{3}{10}}{\frac{2}{10}\cdot\frac{1}{20} + \frac{5}{10}\cdot\frac{1}{25} + \frac{3}{10}\cdot\frac{2}{25}}$$

$\frac{10+20+24}{1000} = \frac{54}{1000}$

$$= \frac{6}{250} \Big/ \frac{54}{1000} = \frac{6}{250}\times\frac{1000}{54} = \frac{4}{9}$$

(4) A Bag 'A' contains 2 white and 3 red balls and a bag 'B' contains 4 white and 5 red balls. One ball is drawn at random from one of the bag and it is found to be red. Find the probability that the red ball is drawn from B.

Ans. Let total no. of Bags = 2

$P(A) = \frac{1}{2}$ i.e. the probability of select the bag A.

$P(B) = \frac{1}{2}$ i.e. " " " " " " B.

Let R denote the event of drawing a red ball

$\therefore P\left(\frac{R}{A}\right) = \frac{3}{5}$ i.e The prob. of draw a red ball from A.

$P\left(\frac{R}{B}\right) = \frac{5}{9}$ i.e The prob. of draw a red ball from B.

$\therefore$ The proba of the red ball draw from The Bag.

is $P(B/R) = \dfrac{P(B)\cdot P(R/B)}{P(A)\cdot P\left(\frac{R}{A}\right) + P(B)\, P\left(\frac{R}{B}\right)}$

$= \dfrac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\cdot\frac{5}{9}}$

$=$

$=$

(5) Suppose 5 men out of 100 and 25 women out of 10,000 are colour blind. A colour blind person is chosen at random. What is the probability of the person being a male

Ans:- Given that persons are men & women

$\therefore P(M) = \frac{1}{2}$ i.e the prob. of selecting men.

$P(W) = \frac{1}{2}$ i.e " " " " women.

Let B represented the blind person.

$\therefore P(B/M) = \frac{5}{100}$ i.e The prob. of selecting a blind from men.

$\therefore P(B/W) = \frac{25}{10000}$ is The prob. of selecting a blind from women.

The prob. That The select a Male from Blind is

$$P(M|B) = \frac{P(B|M)\ P(M)}{P(M)\ P(\frac{B}{M}) + P(W)\ P(\frac{B}{W})}$$

$$= \frac{\frac{5}{100} \cdot \frac{1}{2}}{\frac{5}{100} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{25}{10000}}$$

$=$

$=$

(6) In a factory, Machine A produces 40% of the output and Machine B produces 60%. on the average 9 items in 1000 produced by A are defective and '1' item in 250 Produced by B is defective. An item drawn at random from a day's out-put is defective. What is the probability that it was produced by A or B?

Ans:- output from Machine A = 40%.

$$P(A) = \frac{40}{100} = 0.4$$

output from Machine B = 60%.

$$P(B) = \frac{60}{100} = 0.6$$

Let D denote the defective item.

$P(\frac{D}{A}) = \frac{9}{1000} = 0.009$ i.e The prob. That defective item from A.

$P(\frac{D}{B}) = \frac{1}{250} = 0.004$ i.e The prob. That defective item from B.

The prob. that the bolt produced by A and it is defective is

$$P(A|D) = \frac{P(A)\cdot P(D|A)}{P(A)\,P(D|A) + P(B)\,P(D|B)}$$

$$= \frac{0.4 \times 0.009}{(0.4 \times 0.009) + (0.6 \times 0.004)}$$

$$= 0.6.$$

The prob. that the bolt produced by B and it is defective is

$$P(B|D) = \frac{P(B)\,P(D|B)}{P(A)\,P(D|A) + P(B)\,P(D|B)}$$

$$= \frac{0.6 \times 0.004}{0.4 \times 0.009 + (0.6 \times 0.04)}$$

$$= 0.4.$$

Practice problems

1. companies B_1, B_2, B_3 produce 30%, 45% and 25% of the cars resp. It is known that 2%, 3% and 2% of the cars produced from B_1, B_2 and B_3 are defective.
   (i) what is the prob. that a car purchased is defective?
   (ii) If a car purchased is found to be defective what is prob. that this car is produce by company B_3?

(2) First box contains 2 black, 3 red, 1 white balls; second box contains 1 black, 1 red, 2 white balls and third box contains 5 black 3 red, 4 white balls. Of these a box is

selected at random. From it red ball is randomly drawn. If the ball is red, find the prob. That it is from second Box.

3) Of the three men, the chances that a politician, a business man or an academician will be appointed as a vice-chancellor (V.C) of a university are 0.5, 0.3, 0.2 resp. probability that research is promoted by these persons if they are appointed as V.C are 0.3, 0.7, 0.8 resp.

(i) Determine the probability that research is promoted.

(ii) If research is promoted, what is the probability that V.C is an academician.

Random Variables and distribution functions

Def:- (Random variable):- A real variable X whose value is determined by the outcome of a random experiment is called a random variable.

ex:- 1. Tossing of a coin twice.

Sample space $S = \{s_1, s_2, s_3, s_4\}$ where

$s_1 = HH$, $s_2 = HT$, $s_3 = TH$, $s_4 = TT$.

Define a function $X : S \to R$ by $X(s)$ = no. of heads

Then $X(s_1) = 2$, $X(s_2) = 1$, $X(s_3) = 1$, $X(s_4) = 0$.

Range $= \{2, 1, 0\} = \{0, 1, 2\}$

$\therefore$ Random variable $X(s) = \{0, 1, 2\}$.

2. Throw of pair of dice.

Sample space
$$S = \left\{\begin{array}{lll} (1,1), (1,2) & \cdots & (1,6) \\ (2,1), (2,2) & \cdots & (2,6) \\ (3,1) & \cdots & (3,6) \\ (4,1) & \cdots & (4,6) \\ (5,1) & \cdots & (5,6) \\ (6,1) & \cdots & (6,6) \end{array}\right\}.$$

Define a function $X : S \to R$ by $X(s)$ = Sum on the dice.

$X(s) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is a Random variable.

Types of Random variable

There are Two types:
(1) Discrete Random variable
(2) continuous Random variable.

Discrete random variable:- A random variable X which can take only a finite no. of discrete values in an interval of domain is called a discrete random variable.

ex:- → ex 1, ex 2 are discrete r.v.

→ the ran. var denoting the no. of students in a class.

$X(x) = \{x \mid x \text{ is a positive integer}\}$.

Continuous Random variable :- A r.v X which can take values continuously i.e which takes all possible values in a given interval is called a continuous R.V.

ex:- 1. The height, age and weight of individuals

2. Temprature and time etc.

Probability distribution function

The PDF associated with a random variable X is the probability that the outcome of an experiment will be one of the outcomes for which $X(s) \le x$, $x \in R$. It is denoted by $F_X(x)$.

i.e $F_X(x) = P(X \le x) = P\{S : X(s) \le x\} \; -\infty < x < \infty$

is called the distribution function.

Discrete probability distribution

Let X be a random variable assume values $x_1, x_2, \cdots x_n$ with probabilities $p_1, p_2, \cdots \cdots p_n$ resp, where $P(X = x_i) = p_i \ge 0$ for each x_i and $p_1 + p_2 + \cdots + p_n = \sum_{i=1}^{n} p_i = 1$. Thus

x	x_1	x_2	x_3	$\cdots$	x_n
$P(x)$	p_1	p_2	p_3	$\cdots$	p_n

is called the discreate probability distribution of random variable X.

Note:- 1. $P(X < x_i) = P(x_1) + P(x_2) + \cdots + P(x_{i-1})$

2. $P(X \le x_i) = P(x_1) + P(x_2) + \cdots + P(x_i)$

3. $P(X > x_i) = 1 - P(X \le x_i)$.

Expectation of Discrete prob. Distribution

Let X be a random variable. The expection of X is defined as $E(X) = \sum_{i=1}^{n} p_i x_i$.

Mean:- The mean of Discrete distribution function is denote by μ and defined as

$$\mu = \frac{\sum p_i x_i}{\sum p_i} = \sum p_i x_i = E(X) \quad (\because \sum p_i = 1)$$

variance:- The variance of the discrete distribution function is denoted by σ^2 and defined as

$$\begin{aligned}
\sigma^2 &= E(x-\mu)^2 \\
&= \sum p_i (x_i - \mu)^2 \\
&= \sum p_i (x_i^2 + \mu^2 - 2x_i\mu) \\
&= \sum p_i x_i^2 + \sum p_i \mu^2 - 2\sum x_i \mu p_i \\
&= \sum p_i x_i^2 + \mu^2 \sum p_i - 2\mu \sum x_i p_i \\
&= E(x^2) + \mu^2(1) - 2\mu \cdot \mu \\
&= E(x^2) + \mu^2 - 2\mu^2 \quad (\because \sum p_i = 1, \ \sum x_i p_i = \mu = E) \\
&= E(x^2) - \mu^2
\end{aligned}$$

$\therefore$ The variance $\boxed{\sigma^2 = E(x^2) - \mu^2 = \sum p_i x_i^2 - \mu^2}$

Standard deviation :- The standard deviation of discrete distribution function is defined as positive square root of variance.

$\therefore$ Standard deviation $\sigma = \sqrt{\sigma^2} = \sqrt{\text{variance}}$

Cummulative distribution function

The cummulative distribution function is $F(x) = P(X \le x)$.

problems

1. Construct a prob. distribution for a random variable defining $X : S \to R$ has $X(i, j) = i+j$, where 'S' is the sample space obtained when two dice are thrown. Find (i) $P(2 \le x \le 7)$ (ii) $P(x \ge 7)$ (iii) $P(x \le 5)$ (iv) $P(x < 12)$ (v) Mean (vi) variance (vii) Standard deviation (viii) $F(x=4)$ (ix) $F(x=12)$ (x) $P(x \le 10)$.

Ans:- Let 'S' be the sample space obtained by two dice are thrown.

The sample space $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}.$

Define $X : S \to R$ by $X(i, j) = i + j$

$X(1,1) = 1+1 = 2,$
$X(1,2) = 1+2 = 3$
$\vdots$

For sum $2 \to (1,1)$
$3 \to (1,2), (2,1)$
$4 - (1,3), (3,1), (2,2)$
$5 - (1,4), (4,1), (2,3), (3,2)$
$6 - (1,5), (5,1), (2,4), (4,2), (3,3)$
$7 - (1,6), (6,1), (2,5), (5,2), (3,4), (4,3)$
$8 - (2,6), (6,2), (3,5), (5,3), (4,4)$
$9 - (3,6), (6,3), (4,5), (5,4)$
$10 - (4,6), (6,4), (5,5)$
$11 - (5,6), (6,5)$
$12 - (6,6)$.

$\therefore X(S) = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

The probability distribution table is

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(i) $P(2 \le x \le 7)$

$= P(x=2) + P(x=3) + P(x=4) + P(x=5) + P(x=6) + P(x=7)$

$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36}$

$= \frac{21}{36}$.

(ii) $P(x \ge 7)$

$= P(x=7) + P(x=8) + P(x=9) + P(x=10) + P(x=11) + P(x=12)$

$= \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$

$= \frac{21}{36}$

(iii) $P(x \le 5)$

$= P(x=2) + P(x=3) + P(x=4) + P(x=5)$

$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} = \frac{10}{36}$.

(iv) $P(x < 12)$

$= P(x=2) + P(x=3) + \cdots + P(x=11)$

$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36}$

$= \frac{35}{36}$.

(iv)

$$P(x < 12) = 1 - P(x = 12)$$
$$= 1 - \frac{1}{36}$$
$$= \frac{35}{36}$$

(v) Mean:- The mean $\mu = \sum x_i p_i$

$$= x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$$

$$= 2\left(\frac{1}{36}\right) + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + \cancel{\frac{5}{36}} + 6 \times \frac{5}{36}$$
$$+ 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$

$$= 2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12 / 36$$

$$= \frac{252}{36} = 7 \quad \therefore \boxed{\text{Mean } \mu = 7}$$

(vi) variance:- The variance is

$$\sigma^2 = E(x^2) - \mu^2$$
$$= \sum x_i^2 p_i - \mu^2$$

$$= \left[4\left(\frac{1}{36}\right) + 9 \times \frac{2}{36} + 16 \times \frac{3}{36} + 25 \times \frac{4}{36} + 36 \times \frac{5}{36} + 49 \times \frac{6}{36}\right.$$
$$\left. + 64 \times \frac{5}{36} + 81 \times \frac{4}{36} + 100 \times \frac{3}{36} + 121 \times \frac{2}{36} + 144 \times \frac{1}{36}\right] - 49$$

$$= \frac{1974}{36} - 49$$
$$= 49.83 - 49 = 0.83$$
$$\therefore \boxed{\text{variance } \sigma^2 = 0.83}$$

(vii) Standard deviation $\sigma = \sqrt{\text{variance}} = \sqrt{0.83}$

$$= 0.911$$

$$\therefore \boxed{S.D \ \sigma = 0.911}$$

(viii) $F(x=4) = P(x \le 4)$

$= P(x=2) + P(x=3) + P(x=4)$

$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = \frac{1}{6}$

(ix) $F(x=12) = P(X \le 12) = P(X=2) + P(X=3) + \cdots + P(X=12)$

$= \frac{1}{36} + \frac{2}{36} + \cdots\cdots + \frac{1}{36} = \frac{36}{36} = 1$

(x) $P(x \le 10) = P(x=2) + P(x=3) + \cdots\cdots + P(x=10)$

$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36} + \frac{3}{36}$

$= \frac{33}{36}$.

(or) $P(x \le 10) = 1 - [P(x=11) + P(x=12)]$

$= 1 - \left[\frac{2}{36} + \frac{1}{36}\right]$

$= 1 - \frac{3}{36} = \frac{33}{36}$

(2) construct a probability distribution for a random variable, Defineing $X : S \to R$ by

$X(i,j) = \text{minimum}\{i,j\}$, Find

(i) Mean (ii) variance (iii) Standard deviation

(iv) $P(2 \le x \le 5)$ (v) $P(x \ge 2)$ (vi) $P(x \le 5)$

(vii) $F(x=3)$.

Ai:- Let S be the sample space, when two dice are thrown.

The sample space $S=$

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}.$

Define $X : S \to R$ by $X(s) = \min\{i, j\}$.

$\therefore X(S) = \{1, 2, 3, 4, 5, 6\}$.

The prob. distribution function is

X	1	2	3	4	5	6
P(x)	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

$\therefore 0 \le P(x) \le 1$

$\therefore \Sigma P(x_i) = \frac{11}{36} + \frac{9}{36} + \frac{7}{36} + \frac{5}{36} + \frac{3}{36} + \frac{1}{36} = \frac{36}{36} = 1.$

(i) Mean:- The mean of prob. dist. function is

$$\mu = \Sigma P_i x_i = P_1 x_1 + P_2 x_2 + \cdots + P_6 x_6$$

$$= 1 \times \frac{11}{36} + 2 \times \frac{9}{36} + 3 \times \frac{7}{36} + 4 \times \frac{5}{36} + 5 \times \frac{3}{36} + 6 \times \frac{1}{36}$$

$$= \frac{91}{36} = 2.527 \quad \text{i.e.} \quad \boxed{\mu = 2.527}$$

(ii) variance:- the variance is

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \Sigma P_i x_i^2 - \mu^2$$

$$= P_1 x_1^2 + P_2 x_2^2 + \cdots + P_6 x_6^2 - \mu^2$$

$$= 1 \times \frac{11}{36} + \frac{9}{36} \times 4 + \frac{7}{36} \times 9 + \frac{5}{36} \times 16 + \frac{3}{36} \times 25 + \frac{1}{36} \times 36 - (2.527)^2$$

$$\cancel{\frac{301}{36} = 8.361}$$

$$= \frac{301}{36} - (2.527)^2 = 8.361 - 6.385 = 1.976.$$

$$\therefore \boxed{\sigma^2 = 1.976}$$

(iii) Standard deviation

$$\sigma^2 = 1.976$$

$$S.D = \sqrt{\text{variance}} = \sqrt{1.976} = 1.405$$

$$\therefore \sigma = 1.405.$$

(iv) $P(2 \le x \le 5)$

$$= P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

$$= \frac{9}{36} + \frac{7}{36} + \frac{5}{36} + \frac{3}{36} = \frac{24}{36} = 0.66$$

(v) $P(x \ge 2)$

$$= 1 - P(x \le 1)$$

$$= 1 - [P(x=1)] \cancel{+ P(x=2)}$$

$$= 1 - \left[\frac{11}{36}\right] \cancel{+ \frac{9}{36}} \cancel{- 1 - \frac{20}{36} = \frac{16}{36}} =$$

$$= \frac{25}{36} = 0.69.$$

(vi) $P(x \le 5)$

$$= P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

$$= \frac{11}{36} + \frac{9}{36} + \frac{7}{36} + \frac{5}{36} + \frac{3}{36} = \frac{35}{36} = 0.972$$

(or)

$$P(x \le 5) = 1 - P(x > 5)$$

$$= 1 - P(x=6) = 1 - \frac{1}{36} = \frac{35}{36} = 0.972.$$

(viii) $F(x=3) = P(x \le 3)$

$$= P(x=1) + P(x=2) + P(x=3)$$

$$= \frac{7}{36} + \frac{5}{36} + \frac{3}{36} = \frac{15}{36} = 0.41$$

(3) A random variable x has the following prob. distribution function:

x	0	1	2	3	4	5	6	7
$P(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Determine (i) K (ii) Evaluate $P(X<6)$, $P(X\geq 6)$, $P(0<X<5)$ and $P(0\leq X\leq 4)$ (iii) Mean (iv) variance of distribution function.

(i) we know that $\sum P(x_i)=1$

$\therefore P(x_1)+P(x_2)+\cdots+P(x_n)=1$

$0+K+2K+2K+3K+K^2+2K^2+7K^2+K=1$

$\Rightarrow 10K^2+9K-1=0$

$\Rightarrow 10K^2+10K-K-1=0$

$\Rightarrow 10K(K+1)-1(K+1)=0$

$\Rightarrow (10K-1)(K+1)=0$

$\Rightarrow K=\frac{1}{10}\ ,\ K=-1.$

$\therefore K=\frac{1}{10}=0.1$ ($\because P(x)\geq 0$; so $K\neq -1$).

(ii) $P(X<6) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)$

$=0+K+2K+2K+3K+K^2$

$=K^2+8K=(0.1)^2+8(0.1)=0.01+0.8$
$=0.81.$

$P(X\geq 6) = P(X=6)+P(X=7)$

$=2K^2+7K^2+K$

$=9K^2+K=9(0.1)^2+0.1=9(0.01)+0.1=0.19.$

$P(0<X<5) = P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$=K+2K+2K+3K=8K=8(0.1)=0.8.$

$P(0\leq X\leq 4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)$

$=0+K+2K+2K+3K=8K=8(0.1)=0.8$

(iii) Mean $\mu = \sum p_i x_i$

$= p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$

$= 0 + 1.k + 2.2k + 3.2k + 4.3k +$

$5.k^2 + 6.2k^2 + 7(7k^2 + k)$

$= 66k^2 + 30k$

$= 66(0.1)^2 + 30(0.1) = 3.66$

(iv) Standard deviation $= E(x^2) - [E(x)]^2$

$= \sum p_i x_i^2 - \mu^2$

$= 0 + k(1) + 2k(4) + 2k(9) + 3k(16) + k^2(25)$

$+ 2k^2(36) + (7k^2 + k)(49) - (3.66)^2$

$= 440k^2 + 124k - (3.66)^2$

$= 440(0.1)^2 + 124(0.1) - (3.66)^2$

$= 3.4044.$

(4) The probability density function of a variate x is

x	0	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k	13k

(i) Find $P(x<4)$, $P(x \geq 5)$, $P(3 < x \leq 6)$.

(ii) what will be the minimum value of k so that $P(x \leq 2) > 0.3$?

Ans:- We K.T $\sum P(x_i) = 1$

$\therefore k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

$49k = 1 \Rightarrow k = \frac{1}{49}$

(i) $P(x<4) = P(x=0) + P(x=1) + P(x=2) + P(x=3)$

$= k + 3k + 5k + 7k$

$= 16k = 16\left(\frac{1}{49}\right) = \frac{16}{49}.$

$P(x \geq 5) = P(x=5) + P(x=6)$

$= 11k + 13k = 24k = 24\left(\frac{1}{49}\right) = \frac{24}{49}$

$P(3 < x \leq 6) = P(x=4) + P(x=5) + P(x=6)$

$= 9K + 11K + 13K$

$= 33K = \frac{33}{49}$.

(ii) $P(x \leq 2) > 0.3$

$\Rightarrow P(x=0) + P(x=1) + P(x=2) > 0.3$

$\Rightarrow K + 3K + 5K > 0.3$

$\Rightarrow 9K > 0.3$

$\Rightarrow K > \frac{0.3}{9} = \frac{3}{10} \times \frac{1}{9_3} = \frac{1}{30}$

The minimum value of K is $\frac{1}{30}$.

(5) Let x denote the no. of heads in a single toss of 4 fair coins. Determine

(i) $P(x < 2)$

(ii) $P(1 < x \leq 3)$

Sol:- Total no. of coins $= 4$

Total no. of events $= 2^4 = 16$

For $x=0$ heads $= {}^4C_0 = \frac{4!}{0!\,4!} = 1$ i.e TTTT.

For $x=1$ head $= {}^4C_1 = \frac{4!}{1!\,3!} = 4$ i.e HTTT, THTT, TTHT, TTTH.

For $x=2$ heads $= {}^4C_2 = \frac{4!}{2!\,2!} = \frac{4\times3}{2} = 6$ i.e HHTT, HTTH, THHT, HTHT, THTH, TTHH.

For $x=3$ heads $= {}^4C_3 = \frac{4!}{3!\,1!} = 4$ i.e HHHT, THHH, HTHH, HHTH.

For $x=4$ heads $= {}^4C_4 = \frac{4!}{0!\,4!} = 1$ i.e HHHH.

The prob. distribution function is

x	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(i) $P(X<2) = P(X=0) + P(X=1)$

$$= \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$$

(ii) $P(1 < X \leq 3) = P(X=2) + P(X=3)$

$$= \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}.$$

(6) A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected number of defective items?

Ans:- Total no. of items = 12

Good items = 7

defective items = 5

Let x denote the no. of defective items among 4 items drawn from items.

$\therefore$ x can be taken 0, 1, 2, 3 or 4 defective items.

For $x=0$ i.e. no defective item $= {}^7C_4 = \frac{7!}{4!\,3!} = \frac{7\times6\times5}{3\times2} = 35$

For $x=1$ (one defective, 3 good) $= {}^5C_1 \times {}^7C_3 = \frac{5!}{1!\,4!} \times \frac{7!}{3!\,4!}$

$$= 5 \times \frac{7\times6\times5}{3\times2}$$

$$= 175$$

For $x=2$ (2 defective, 2 good) $= {}^5C_2 \times {}^7C_2$

$$= \frac{5!}{2!\,3!} \times \frac{7!}{2!\,5!} = \frac{5\times4}{2} \times \frac{7\times6}{2}$$

$$= 210.$$

For $x=3$ (3 defective, 1 good) $= {}^5C_3 \times {}^7C_1$

$$= \frac{5!}{3!\,2!} \times \frac{7!}{1!\,6!}$$

$$= 10 \times 7 = 70$$

for $x = 4$ (4 defective) $= {}^5C_4 = \frac{5!}{4!\,1!} = 5$

The total no. of ways 4 items selected from 12 items $= {}^{12}C_4 = \frac{12!}{4!\,8!} = \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2} = 495$.

The probability distribution function is

x	0	1	2	3	4
$p(x)$	$\frac{35}{495}$	$\frac{175}{495}$	$\frac{210}{495}$	$\frac{70}{495}$	$\frac{5}{495}$

$\therefore$ The Expected (Mean) no. of defective items $E(x) = \sum p_i x_i$

$= p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4 + p_5 x_5$

$= 0\left(\frac{35}{495}\right) + 1\left(\frac{175}{495}\right) + 2\left(\frac{210}{495}\right) + 3\left(\frac{70}{495}\right) + 4\left(\frac{5}{495}\right)$

$= \frac{825}{495} = \frac{165}{99}$

Practice problems

(1) Given That $f(x) = \frac{k}{2x}$, is a prob. distribution for a random variable x That can take on the values $x = 0, 1, 2, 3, 4$.

(i) Find k

(ii) Find Mean and variance of x.

(2) Calculate Expectation and variance of x, if the prob. distribution of the random variable x is given by

x	-1	0	1	2	3
$p(x)$	0.3	0.1	0.1	0.3	0.2

③ A random variable x has the following Prob. distribution.

X	1	2	3	4	5	6	7	8
P(x)	K	2K	3K	4K	5K	6K	7K	8K

Find the value of

(i) k (ii) $P(X \leq 2)$ (iii) $P(2 \leq X \leq 5)$.

(iv) Mean (v) Variance (vi) Standard deviation.

④ A Box contains 8 items of which 2 are defective. A man draws 3 items from the Box. Find the expected no. of defective items he has drawn.

⑤ A fair coin is tossed until a head or five tails occurs. Find the expected number r of tosses of the coin.

Binomial Distribution

Let x be a discrete random variable. A probability density function defined by $p(x=r) = {}^nC_r\, p^r q^{n-r}$ is called a binomial distribution.

where n = total no. of times the experiment is repeated

r = no. of success

p = probability of success

q = probability of failure.

And $p+q=1 \Rightarrow q = 1-p$.

The binomial prob. distribution table is

x	0	1	2	...	n
$p(x)$	${}^nC_0\, p^0 q^n$	${}^nC_1\, p^1 q^{n-1}$	${}^nC_2\, p^2 q^{n-2}$	...	${}^nC_n\, p^n q^0$

Constants of Binomial distribution

(1) The mean of the Binomial distribution is

$$\boxed{\mu = np}$$

(2) The variance of the Binomial distribution is

$$\boxed{\sigma^2 = npq}$$

(3) The standard deviation of the Binomial distribution is $\sqrt{npq}$.

(4) The Mode of the binomial distribution is The value of x at which $P(x)$ has the maximum value.

Mode = Integral part of $(n+1)p$, if $(n+1)p$ is not an integer

= $(n+1)p$ and $(n+1)p-1$, if $(n+1)p$ is an integer.

Recurrence relation of Binomial distribution

we know that $P(r) = {}^nC_r \, p^r q^{n-r}$

$$P(r+1) = {}^nC_{r+1} \, p^{r+1} q^{n-(r+1)}$$

$$\therefore \frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r} \cdot \frac{p^{r+1}}{p^r} \cdot \frac{q^{n-r}}{q^{n-(r+1)}}$$

$$= \frac{n-r}{r+1} \cdot p \cdot \frac{1}{q}$$

$$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{\frac{n!}{(r+1)!\,(n-r-1)!}}{\frac{n!}{r!\,(n-r)!}} = \frac{r!\,(n-r)!}{(r+1)!\,(n-r-1)!} = \frac{n-r}{r+1}$$

$$\therefore \boxed{P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q} \cdot P(r)}$$

Problems

(1) A fair coin is tossed six times. Find the probability of getting (i) 4 heads. (ii) at least 4 heads (iii) at most 4 heads.

Solu:- No. of trails of a coin = $n = 6$

p = prob. of getting head = $\frac{1}{2}$

q = " " " non head = $1 - \frac{1}{2} = \frac{1}{2}$

(i) Here required heads $r = 4$.

By Binomial distribution,

$$\boxed{P(x=r) = {}^nC_r \, p^r q^{n-r}}$$

$\therefore P(x=4) = {}^6C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{6-4}$ ${}^6C_4 = 15$

$= 15 \cdot \frac{1}{2^6} = \frac{15}{64}$

$\therefore$ The prob. of getting 4 heads $= \frac{15}{64}$.

(ii) Here the required prob. is at least 4 heads.

$$P(x \geq 4) = P(x=4) + P(x=5) + P(x=6).$$

$$= {}^6C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^{6-4} + {}^6C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^{6-5} + {}^6C_6 \left(\tfrac{1}{2}\right)^6.$$

$$= 15 \cdot \frac{1}{2^6} + 6 \cdot \frac{1}{2^6} + 1 \cdot \frac{1}{2^6}$$

$$= \frac{22}{64}.$$

$\therefore$ The prob. of getting at least 4 heads $= \frac{22}{64} = \frac{11}{32}$.

(iii) Here the required prob. is atmost 4 heads. (0 or 1 or 2 or 3 or 4 heads)
ie maximum 4 heads.

$$P(x \leq 4) = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4).$$

$$= {}^6C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{6-0} + {}^6C_1 \tfrac{1}{2} \cdot \left(\tfrac{1}{2}\right)^5 + {}^6C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^4 + {}^6C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^3 + {}^6C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^6$$

$$= \frac{1}{2^6} \left[1 + 6 + 15 + 20 + 15 \right] = \frac{57}{64}$$

$\therefore$ The prob. of getting is maximum 4 heads $= \frac{57}{64}$.

(2) A die is thrown 6 times. If getting an even number is a success, find the prob. of (i) at least one success
(ii) ≤ 3 success
(iii) 4 success. (iv) at most 4 success

Ans:- Here no. of trials $n = 6$

P = prob. of getting even no. in one throw $= \frac{3}{6} = \frac{1}{2}$
(2 or 4 or 6)

q = prob. of getting non even no. $= 1 - \frac{1}{2} = \frac{1}{2}$.

By Binomial theorem, $\boxed{P(X=r) = {}^{n}C_{r}\, p^{r} \cdot q^{n-r}}$

(i) At least one success

$$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + \cdots + P(X=6)$$

(or)

$$= 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^{6}C_{0} \left(\tfrac{1}{2}\right)^{0} \left(\tfrac{1}{2}\right)^{6-0} = 1 - \frac{1}{2^6} = 1 - \frac{1}{64} = \frac{63}{64}$$

$$= 0.984$$

(ii) ≤ 3 success

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= {}^{6}C_{0} \cdot \left(\tfrac{1}{2}\right)^{0} \left(\tfrac{1}{2}\right)^{6} + {}^{6}C_{1}\, \tfrac{1}{2} \cdot \left(\tfrac{1}{2}\right)^{5} + {}^{6}C_{2} \left(\tfrac{1}{2}\right)^{2} \cdot \left(\tfrac{1}{2}\right)^{4} + {}^{6}C_{3} \left(\tfrac{1}{2}\right)^{3} \left(\tfrac{1}{2}\right)^{3}$$

$$= \frac{1}{2^6}\left[1 + 6 + 15 + 20\right] = \frac{42}{64} = \frac{21}{32} = 0.656$$

(iii) 4 success

$$P(X=4) = {}^{6}C_{4} \cdot \left(\tfrac{1}{2}\right)^{4} \left(\tfrac{1}{2}\right)^{2} = \frac{15}{64} = 0.234$$

(iv) At most 4 success

$$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

(or)

$$= 1 - [P(X=5) + P(X=6)]$$

$$= 1 - \left[{}^6C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right) + {}^6C_6 \left(\tfrac{1}{2}\right)^6 \right].$$

$$= 1 - \left[\frac{6}{64} + \frac{1}{64} \right] = 1 - \frac{7}{64} = \frac{57}{64}$$

(3) If 3 of 20 tyres are defective and 4 of them are randomly chosen for inspection, what is the prob. that

(i) only one defective tyre will be included?

(ii) at least two defective tyres will be included?

Ans:- Given that no. of tyres selected $n=4$.

p = The prob. of defective tyre $= \frac{3}{20}$

q = " " " non " " $= 1-p = 1-\frac{3}{20} = \frac{17}{20}$

By Binomial distribution,

$$P(X=r) = {}^nC_r\, p^r\, q^{n-r}.$$

(i) only one defective tyre included is

$$P(X=1) = {}^4C_1 \left(\tfrac{3}{20}\right)^1 \left(\tfrac{17}{20}\right)^{4-1} = 4 \cdot \tfrac{3}{20} \cdot \left(\tfrac{17}{20}\right)^3 = 0.36$$

(ii) At least two defective tyres included is

$$P(X \ge 2) = P(X=2) + P(X=3) + P(X=4)$$

$$= {}^4C_2 \left(\tfrac{3}{20}\right)^2 \left(\tfrac{17}{20}\right)^2 + {}^4C_3 \left(\tfrac{3}{20}\right)^3 \left(\tfrac{17}{20}\right)^1 + {}^4C_4 \left(\tfrac{3}{20}\right)^4$$

④ out of 800 families with 5 children each, how many would you expect to have (a) 3 boys (b) either 2 or 3 boys (c) atleast 1 boy (d) 5 girls. Assume equal prob. for boys and girls.

A:- Here no. of children $n=5$

P = The prob. of a boy $= \frac{1}{2}$

q = The prob. of a girl (not a boy) $= 1-P = 1-\frac{1}{2} = \frac{1}{2}$

By Binomial distribution,

$$P(x=r) = {}^nC_r \cdot p^r \cdot q^{n-r}$$

(i) 3 boys

$$P(x=3) = {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{2^5} = \frac{5}{16} \text{ per family}$$

∴ The total no. of families having 3 boys $= 800 \times \frac{5}{16}$

$= 250$ family

(ii) either 2 or 3 boys.

$$P(\text{either } 2 \text{ or } 3 \text{ boys}) = P(x=2) + P(x=3)$$
$$2 \le x \le 3$$

$$= {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$$

$$= \frac{10}{32} + \frac{10}{32} = \frac{20}{32} = \frac{5}{8} \text{ per family}$$

The total no. of families having either 2 or 3 boys

$= 800 \times \frac{5}{8} = 500$ families

(iii) At least 1 boy

$$P(x \ge 1) = P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5)$$

(or)

$$= 1 - P(x=0)$$

$$= 1 - {}^5c_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$$

$\therefore$ The no. of families having at least 1 boy $= 800 \times \frac{31}{32}$
$= 775$ families.

(d) 5 girls i.e. no boy

$$p(x=0) = {}^5c_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = \frac{1}{32} \text{ per family}$$

$\therefore$ The total no. of families having 5 girls
$= 800 \times \frac{1}{32} = 25$ families.

(5) Six dice are thrown 729 times. How many times do you expect at least three dice to show a 5 or 6?

Ans- No. of dice $n=6$

p = The prob. of getting 5 or 6 $= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ The prob. of getting at least three dice to show 5 or 6 $= P(X \geq 3)$

$$= P(x=3) + P(x=4) + P(x=5) + P(x=6)$$

$$= {}^6c_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 + {}^6c_4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 + {}^6c_5 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right) + {}^6c_6 \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0$$

$$= \frac{1}{3^6} \left[20(8) + 15(4) + 6(2) + 1\right]$$

$$= \frac{233}{729}$$

The expected no. of such cases in 729 times $= \frac{233}{729} \times 729$
$= 233.$

(6) The mean and variance of a binomial distribution are 4 and $\frac{4}{3}$ resp. Find $P(x \geq 1)$.

Ans:-

Mean $np = 4$ — (1)

variance $npq = \frac{4}{3}$.

$\therefore \frac{npq}{np} = \frac{4/3}{4} \Rightarrow \boxed{q = \frac{1}{3}}$

we K.T $p+q = 1 \Rightarrow p = 1-q = 1-\frac{1}{3} = \frac{2}{3} \Rightarrow \boxed{p = \frac{2}{3}}$

from (1) $np = 4 \Rightarrow n \cdot \frac{2}{3} = 4 \Rightarrow \boxed{n = 6}$

$P(X \geq 1) = P(x=1) + P(x=2) + \cdots + P(x=6)$

(or)

$= 1 - P(x=0)$

$= 1 - {}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6$ $\boxed{P(x=r) = {}^nC_r\, p^r q^{n-r}}$

$= 1 - \frac{1}{3^6} = 0.9986$.

(7) A discrete random variable x has the mean 6 and variance 2. If it is assumed that the distribution is binomial, find the prob. that $5 \leq x \leq 7$.

Ans:-

Mean $np = 6$ — (1)

variance $npq = 2$.

$\therefore \frac{npq}{np} = \frac{2}{6} \Rightarrow q = \frac{1}{3}$

we K.T $p+q = 1 \Rightarrow p = 1-q = 1-\frac{1}{3} = \frac{2}{3}$.

From (1) $np = 6 \Rightarrow n = \frac{6}{p} = \frac{6}{\frac{2}{3}} = 9$

$\therefore \boxed{n = 9}, \boxed{p = \frac{2}{3}}, \boxed{q = \frac{1}{3}}$

From binomial distribution,

$$P(X=r) = n_{C_r} p^r q^{n-r}.$$

$$\therefore P(5 \le X \le 7) = P(X=5) + P(X=6) + P(X=7)$$

$$= 9_{C_5}\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^4 + 9_{C_6}\left(\frac{2}{3}\right)^6\left(\frac{1}{3}\right)^3 + 9_{C_7}\left(\frac{2}{3}\right)^7\left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3^9}\left[126(32) + 84(64) + 36(128)\right]$$

$$= 0.7121$$

⑧ 20% of items produced from a factory are defective. Find the prob. that in a sample of 5 chosen at random

(i) none is defective

(ii) one is defective

(iii) $P(1 < x < 4)$.

A:- The prob. of defective items $p = 20\% = \frac{20}{100} = \frac{1}{5}$

The prob. of non " " $q = 1-p$

$= 1 - \frac{1}{5} = \frac{4}{5}$.

no. of samples $n = 5$.

By Binomial distribution,

$$P(X=r) = n_{C_r} p^r q^{n-r}$$

(i) None is defective

$$P(X=0) = 5_{C_0}\left(\frac{1}{5}\right)^0\left(\frac{4}{5}\right)^5 = \frac{4^5}{5^5} = 0.327$$

(ii) one is defective

$$P(X=1) = 5_{C_1}\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^4 = 5 \cdot \frac{4^4}{5^5} = 0.4096$$

(iii) $P(1 < x < 4) = \cancel{P(2) + P(3)}\ P(x=2) + P(x=3)$

$$= {}^5C_2 \left(\frac{1}{5}\right)^2 \left(\frac{4}{5}\right)^3 + {}^5C_3 \left(\frac{1}{5}\right)^3 \left(\frac{4}{5}\right)^2$$

$$= \frac{1}{5^5}\left[10(64) + 10(16)\right] = 0.256$$

(9) In 256 sets of 12 tosses of a coin, In how many cases one can expect 8 heads and 4 tails?

Sol:- no. of trials = 12

p = The prob. of getting head = $\frac{1}{2}$

q = The prob. " " tail = $\frac{1}{2}$.

By Binomial distribution is

$$P(X=r) = {}^nC_r\, p^r q^{n-r}.$$

The prob. of getting 8 heads & 4 tails is

$$P(X=8) = {}^{12}C_8 \left(\frac{1}{2}\right)^{8\text{-heads}} \left(\frac{1}{2}\right)^{4\text{-tails}}$$

$$= 495 \cdot \frac{1}{2^{12}} = 0.1208.$$

$\therefore$ The Expected no. of such cases in 256 sets

$$= 256 \times 0.1208 = 30.9375 \cong 31.$$

(10) Four coins are tossed 160 times. The no. of times x heads occur is given below.

x	0	1	2	3	4
No. of times	8	34	69	43	6

Fit a binomial distribution to this data on the hypothesis that coins are unbiased.

Ans:- The coin is unbiased:

Here $n = 4$ = no. of coins

p = The prob. of getting head $= \frac{1}{2}$

q = " " " " tail $= 1-p = 1-\frac{1}{2} = \frac{1}{2}$.

$N = \sum f_i = 8 + 34 + 69 + 43 + 6 = 160$.

By Binomial distribution

$$P(x=r) = {}^{n}C_{r}\, p^{r} q^{n-r}$$

<u>Binomial</u> <u>distribution</u> <u>table</u> is

No. of heads x	frequency f	Probability $P(x=r)$	Expected no. of times (or) The oretical frequency. $f(x) = N \cdot P(x=r)$
0	8	${}^{4}C_{0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{4} = \frac{1}{16}$	$f(0) = 160 \times \frac{1}{16} = 10$
1	34	${}^{4}C_{1}\,\frac{1}{2}\left(\frac{1}{2}\right)^{3} = \frac{1}{4}$	$f(1) = 160 \times \frac{1}{4} = 40$
2	69	${}^{4}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{2} = \frac{6}{16}$	$f(2) = 160 \times \frac{6}{16} = 60$
3	43	${}^{4}C_{3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{1} = \frac{4}{16}$	$f(3) = 160 \times \frac{4}{16} = 40$
4	6	${}^{4}C_{4}\left(\frac{1}{2}\right)^{4}\left(\frac{1}{2}\right)^{0} = \frac{1}{16}$	$f(4) = 160 \times \frac{1}{16} = 10$

(11) 7 coins are tossed at a time, this experiment repeated 128 times. The no. of heads observed at each throw is recorded and the result are given below:

No. of heads	0	1	2	3	4	5	6	7
Frequency	7	6	19	35	30	23	7	1

Fit a binomial distribution to the data assuming that the coins are (i) unbiased (ii) biased (or) the nature of the coin is not known.

~~(i) unbiased~~ ~~(ii) biased~~

Ans:- Here no. of coins = 7 = n

(i) The coins are unbiased :-

p = the prob. of getting head = $\frac{1}{2}$

q = " " " tail = $\frac{1}{2}$.

$N = \Sigma f_i = 7 + 6 + \dots + 1 = 128$

By Binomial distribution

$$P(x = r) = {}^nC_r\, p^r q^{n-r}.$$

The binomial distribution table for unbiased

No. of heads x	frequency f	Probability $P(x=r)$	Expected or Theoretical frequency $f(x) = N\,P(x=r)$
0	7	${}^7C_0 (\frac{1}{2})^0 (\frac{1}{2})^7 = \frac{1}{128}$	$f(0) = 128 \times \frac{1}{128} = 1$
1	6	${}^7C_1 (\frac{1}{2})^1 (\frac{1}{2})^6 = \frac{7}{128}$	$f(1) = 128 \times \frac{7}{128} = 7$
2	19	${}^7C_2 (\frac{1}{2})^2 (\frac{1}{2})^5 = \frac{21}{128}$	$f(2) = 128 \times \frac{21}{128} = 21$
3	35	${}^7C_3 (\frac{1}{2})^3 (\frac{1}{2})^4 = \frac{35}{128}$	$f(3) = 128 \times \frac{35}{128} = 35$
4	30	${}^7C_4 (\frac{1}{2})^4 (\frac{1}{2})^3 = \frac{35}{128}$	$f(4) = 128 \times \frac{35}{128} = 35$
5	23	${}^7C_5 (\frac{1}{2})^5 (\frac{1}{2})^2 = \frac{21}{128}$	$f(5) = 21$
6	7	${}^7C_6 (\frac{1}{2})^6 (\frac{1}{2}) = \frac{7}{128}$	$f(6) = 7$
7	1	${}^7C_7 (\frac{1}{2})^7 (\frac{1}{2})^0 = \frac{1}{128}$	$f(7) = 1$

(ii) The coins are biased & the nature of the coin is not known.

no. of coins $n = 7$

$N = \Sigma f_i = 128$.

$$\text{Mean} = \frac{\sum f_i x_i}{\sum f_i} = \frac{0+6+38+105+120+115+42+7}{128}$$

$$= 3.383.$$

ie $np = 3.383$

$$p = \frac{3.383}{n} = \frac{3.383}{7} = \boxed{0.4833}$$

And $p+q=1 \Rightarrow q = 1-p = 1-0.4833 = \boxed{0.5167}$.

By Binomial distribution

$$P(x=r) = {}^nC_r\, p^r q^{n-r}.$$

$\therefore$ The binomial distribution table to biased is.

$$P(x=0) = {}^7C_0 (0.4833)^0 (0.5167)^7 = 0.0098$$

$$P(x=1) = {}^7C_1 (0.4833)^1 (0.5167)^6 = \cancel{0.1808}\ 0.0644$$

$$P(x=2) = {}^7C_2 (0.4833)^2 (0.5167)^5 = 0.1808$$

$$P(x=3) = {}^7C_3 (0.4833)^3 (0.5167)^4 = 0.2817$$

$$P(x=4) = {}^7C_4 (0.4833)^4 (0.5167)^3 = 0.2634$$

$$P(x=5) = {}^7C_5 (0.4833)^5 (0.5167)^2 = 0.1477$$

$$P(x=6) = {}^7C_6 (0.4833)^6 (0.5167)^1 = 0.0460$$

$$P(x=7) = {}^7C_7 (0.4833)^7 = 0.0062.$$

x	0	1	2	3	4	5	6	7
$f(x) = NP(x=r) = 128\,P(x=r)$	1	8	23	36	34	19	6	1

Practice problems

(1) The mean of a Binomial distribution is 3 and the variance is $\frac{9}{4}$. Find

(i) The value of n.

(ii) $P(X \geq 7)$

(iii) $P(1 \leq X < 6)$.

(2) The prob. that life of a bulb is 100 days is 0.05. Find the prob. that out of 6 bulbs (i) At least one (ii) ≥ 4 (iii) None, will be having a life of 100 days.

(3) Assume that 50% of all engineering students are good in mathematics. Determine the prob. that among 18 engineering students (i) exactly 10 (ii) at least 2 and at most 9.

(4) In a eight throws of a die 5 or 6 is considered a success. Find the mean of number of success and the standard deviation.

(5) It has been claimed that 60% of all solar heat installations the ~~...~~ utility bill is reduced at least one-third. Accordingly, what are the prob. that the utility bill will be reduced at least one-third in

(i) ~~for~~ four of five installations

(ii) at least four of five installations.

(6) Determine the prob. of getting a sum of 9 exactly twice in 3 throws with a pair of fair dice.

(7) Fit a binomial distribution to the following frequency data.

x	0	1	2	3	4
f	28	62	46	10	4

Poisson distribution

Let x be a random variable. The prob. density function is given by

$$P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}, \quad r=0,1,2,\ldots n,$$

where λ is a parameter, is called a poisson distribution.

Ex:- 1. The no. of defective electric bulbs manufactured by a factory.

2. The no. of printing mistakes per page in a large text.

Condition's of poisson distribution

The poisson distribution is used under the following conditions

(1) The no. of trials (n) is large.

(2) The prob. of success (p) is very small.

(3) $np = \lambda$ is finite.

Constants of The poisson Distribution

(1) The mean of the poisson distribution is

$$E(x) = \lambda \; (= np)$$

The parameter λ is the Arithematic Mean of the ~~pd~~ poisson distribution.

(2) The variance of the poisson distribution is λ.

$\therefore$ Mean = variance in poisson distribution.

(3) The standard deviation of ~~pois~~ poisson distr. is $\sqrt{\lambda}$.

(4) The Mode of the poisson distribution

$= \lambda-1, \lambda$ if λ is an integer

= Integral part of λ, if λ is not an integer.

(5) The Recurrance relation for the poisson distr. is

$$\boxed{P(x) = \frac{\lambda}{x} P(x-1)}$$

problems

1) A Hospital Switch board receives an average of 4 emergency calls in a 10 min. interval. what is the prob. that

(i) There are at-most 2 emergency calls in a 10 min. interval.

(ii) There are exactly 3 emergency calls in a 10 min. interval.

A:- Given that $\lambda = 4$ calls in 10 min.

[∴ possion distribution is used only when n is not given (or) n is a large no. If n is given it is binomial distribution.]

$$\boxed{P(x=r) = \frac{e^{-\lambda} \lambda^{r}}{r!}}$$

(i) At-most 2 emergency calls

$$P(x \leq 2) = P(x=0) + P(x=1) + P(x=2).$$

$$= \frac{e^{-\lambda} \cdot \lambda^{0}}{0!} + \frac{e^{-\lambda} \cdot \lambda^{1}}{1!} + \frac{e^{-\lambda} \lambda^{2}}{2!}$$

$$= e^{-\lambda}\left[1 + \lambda + \frac{\lambda^{2}}{2!}\right]$$

$$= e^{-4}\left[1 + 4 + \frac{4^2}{2!}\right] \quad (\because \lambda = 4)$$

$$= 0.2381$$

(ii) Exactly 3 calls

$$P(x=3) = \frac{e^{-\lambda}.\lambda^3}{3!} = \frac{e^{-4}\,4^3}{3!} = 0.1953.$$

(2) Average no. of accidents on any day on a national highway is 1.8. Determine the prob. that the no. of accidents are

(i) at least one (ii) at most one.

Ans:- Given that, Mean $\lambda = 1.8$

By poisson distribution,

$$P(x=r) = \frac{e^{-\lambda}\,\lambda^r}{r!} = \frac{e^{-1.8}.(1.8)^r}{r!} \quad (1)$$

(i) at least one.

$$P(x \geq 1) = P(x=1) + P(x=2) + \ldots\ldots$$

$$= 1 - P(x=0)$$

$$= 1 - \frac{e^{-1.8}.(1.8)^0}{0!}$$

$$= 1 - e^{-1.8} = 1 - 0.1653 = 0.831$$

(ii) at most one

$$P(x \leq 1) = P(x=0) + P(x=1)$$

$$= e^{-1.8} + \frac{e^{-1.8}.(1.8)^1}{1!}$$

$$= 0.4628.$$

(3) 2 % of The items of a factory are defective. The items are packed in boxes. what is The prob. That There will be

(i) 2 defective items

(ii) at least 3 defective items in a box of 100 items?

Ans:- p = The prob. of defective item = 2%

$= \frac{2}{100} = 0.02.$

n = no. of items = 100.

Mean $\lambda = np = 100(0.02) = 2.$

From The ~~pdf~~ poisson distribution is

$$P(x=r) = \frac{e^{-\lambda}\lambda^{r}}{r!} = \frac{e^{-2}.2^{r}}{r!} \quad —(1)$$

(i) 2 defective items

$$P(x=2) = \frac{e^{-2}.2^{2}}{2!} = \frac{2}{e^{2}} = 0.2706$$

(ii) at least 3 defective items

$$P(x \geq 3) = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[\frac{e^{-2}.2^{0}}{0!} + \frac{e^{-2}.2^{1}}{1!} + \frac{e^{-2}.2^{2}}{2!}\right]$$

$$= 1 - \frac{1}{e^{2}}[1 + 2 + 2] = 1 - \frac{5}{e^{2}} = 0.3233$$

(4) If the prob. that an individual suffers a bad reaction from a certain injection is 0.001, determine the prob. that out of 2000 individuals

(i) exactly 3

(ii) more than 2 individuals

(iii) none

(iv) more than one individual suffer a bad reaction

A:- Given p = Prob. of a bad reaction = 0.001.

n = Total no. of persons = 2000.

Mean $\lambda = np = 2000(0.001) = 2$.

From, poisson distribution,

$$P(X=r) = \frac{e^{-\lambda}.\lambda^r}{r!} = \frac{e^{-2}.2^r}{r!} \quad \text{—(1)}$$

(i) $P(X=3) = \frac{e^{-2}.2^3}{3!} = \frac{8}{6e^2} = 0.1804$

(ii) $P(X\geq 2) = 1-[P(X=0)+P(X=1)+P(X=2)]$

$$=1-\left[\frac{e^{-2}.2^0}{0!}+\frac{e^{-2}.2^1}{1!}+\frac{e^{-2}.2^2}{2!}\right]$$

$$=1-\frac{1}{e^2}[1+2+2] = 1-0.67667$$

$$= 0.3233$$

(iii) None

$$P(X=0) = \frac{e^{-2}.2^0}{0!} = e^{-2} = 0.1353$$

(iv) more Than one

$$P(x > 1) = 1 - [P(x=0) + P(x=1)]$$

$$= 1 - \left[\frac{e^{-2} \cdot 2^0}{0!} + \frac{e^{-2} \cdot 2^1}{1!}\right]$$

$$= 1 - \frac{1}{e^2}[1+2] = 1 - 0.406 = \underline{0.594}$$

(5) If a random variable has a poisson distribution such That $P(1) = P(2)$, find

(i) Mean of The distribution.

(ii) $P(x \geq 1)$ (iii) $P(1 < x < 4)$ (iv) $P(x=4)$.

As:- Given That $P(2) = P(1)$

From recurrence relation,

$$P(x) = \frac{\lambda}{x} P(x-1)$$

$x = 2$, $P(2) = \frac{\lambda}{2} P(1) \Rightarrow \cancel{P(1)} = \frac{\lambda}{2} \cancel{P(1)}$

$\Rightarrow \lambda = 2.$

(or)

$$\frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{e^{-\lambda} \cdot \lambda^1}{1!}$$

$\lambda = 2$

$\therefore$ Mean $\lambda = 2$.

From, poisson distribution,

$$P(X = r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!} = \frac{e^{-2} \cdot 2^r}{r!} \quad \text{--- (1)}$$

(ii) $P(x \geq 1) = 1 - P(x=0)$

$$= 1 - \frac{e^{-2} \cdot 2^0}{0!} = 1 - \frac{1}{e^2} = 0.8647$$

(iii) $P(1 < x < 4) = P(x=2) + P(x=3)$

$=$ ____

(iv) $P(x=4) =$ ____

(6) using poisson's distribution, find the prob. that the ace of spades will be drawn from a pack of well shuffled cards at least once in 104 consecutive trials?

Ans: Total no. of pack of cards = 52

p = The prob. of getting ace = $\frac{1}{52}$

n = no. of trials = 104.

Mean $\lambda = np = 104 \times \frac{1}{52} = 2$.

$P(x \geq 1) = 1 - P(x=0)$ $\quad P(x=r) = \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$

at least one

$= 1 - \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$

$= 1 - \frac{e^{-2} \cdot 2^{0}}{0!} = 1 - \frac{1}{e^{2}} = 1 - 0.1353$

$= 0.8647$.

(7) Fit a ~~[illegible]~~ poisson distribution to the following data and calculate the expected frequencies.

x	0	1	2	3	4
$f(x)$	109	65	22	3	1

Ans: Here N = total frequences $= \Sigma f_i = f_1 + f_2 + \cdots + f_n$

$= 109 + 65 + 22 + 3 + 1$

$= 200$.

Mean $\lambda = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{0+65+44+9+4}{200} = \frac{122}{200} = 0.61$

Mean of the poisson dist. $\lambda = 0.61$

From poisson dist. $p(x=r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$

x	observed frequency f(x)	$P(x=r)$	Expected no. of freq $F(x) = N \cdot p(x=r)$
0	109	$\frac{e^{-0.61} \cdot (0.61)^0}{0!} = 0.5433$	$200 \times 0.5433 = 108.67 = 109$
1	65	$\frac{e^{-0.61} (0.61)^1}{1!} = 0.3314$	$200 \times 0.3314 = 66.28 = 66$
2	22	$\frac{e^{-0.61} (0.61)^2}{2!} = 0.1010$	$200 \times 0.1010 = 20.21 = 22$
3	3	$\frac{e^{-0.61} (0.61)^3}{3!} = 0.1233$	$200 \times 0.1233 = 4.11 = 4$
4	1	$\frac{e^{-0.61} (0.61)^4}{4!} = 0.0031$	$200 \times 0.0031 = 0.63 = 1$
			200

(8) The distribution of typing mistakes committed by a typist is given below. Assuming the distribution to be poisson, find the expected frequencies.

x	0	1	2	3	4	5
f(x)	42	33	14	6	4	1

(9) wireless sets are manufactured with 25 soldered Joints each. on the average 1 Joint in 500 is defective. How many sets can be expected to be free from defective Joints in a consignment of 10,000 sets?

(10) If x is a poisson variate such that $3P(X=4) = \frac{1}{2}P(X=2) + P(X=0)$, find (i) the mean (ii) $P(X \leq 2)$.

(11) If a bank received on the average 6 bad cheques per day, find the prob. that it will receive 4 bad cheques on any given day.

(12) A manufacturer of cotter pins knows that 5 % of his product is defective. pins are sold in box of 100. He guarantees that not more than 10 pins will be defective. what is the approximate prob. that a box will fail to meet the guaranteed quality?

(13) using Recurrence formula, find the prob. when $x = 0, 1, 2, 3, 4$ and 5; if the mean of poisson distribution is 3.

Continuous random variable

A random variable x which takes continuous values is called a continuous random variable.

Probability density function:- The prob. distribution function defined on a continuous random variable is called probability density function.

If it satisfies (i) $f(x) \geq 0 \ \forall x \in R$.

(ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Note:-

(i) $P(x \geq a) = \int_{a}^{\infty} f(x)\,dx$.

$(\text{or } (P x > a))$

(ii) $P(x \leq a) = \int_{-\infty}^{a} f(x)dx$

(iii) $P(a \leq x \leq b) = \int_{a}^{b} f(x)dx$.

Measures of continuous prob. distribution

(1) Mean $\mu = \int_{-\infty}^{\infty} x f(x)dx = E(x)$

In general, Mean of expectation of $\phi(x) = \int_{-\infty}^{\infty} \phi(x) f(x)dx$

(2) Expectation = mean.

(3) variance $\sigma^2 = E(x-\mu)^2$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x^2+\mu^2-2x\mu) f(x)dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx - 2\mu \int_{-\infty}^{\infty} x f(x)dx$$

$$= E(x^2) + \mu^2(1) - 2\mu \cdot \mu$$

$$= E(x^2) - \mu^2$$

$\therefore$ variance $\boxed{\sigma^2 = E(x^2) - \mu^2}$

(4) Median:- A point M is said to be a median, if it divides total prob. into two equal parts.

i.e $\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)dx = \frac{1}{2}$.

(5) Standard deviation :- The S.D is the positive square root of variance.

(6) Mode:- Mode is a point at which the prob. has a maximum value.

Problems:-

(1) If the probability density of a random variable is given by $f(x) = K(1-x^2)$, $0 < x < 1$
$= 0$, otherwise.

(i) Find the value of K

(ii) Find the prob. that it will take on a value between 0.1 and 0.2

(iii) greater than 0.5.

Ans:- Let $f(x) = K(1-x^2),\ 0 < x < 1$
$= 0$, otherwise.

(i) We know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_0^1 f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 K(1-x^2)\,dx = 1 \Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow K\left[1-\frac{1}{3}\right] = 1$$

$$\Rightarrow K = 3/2$$

(ii) The prob. that the ~~value~~ variate will take on a value between 0.1 and 0.2 is

$$P(0.1 < x < 0.2) = \int_{0.1}^{0.2} f(x)dx = \int_{0.1}^{0.2} K(1-x^2)dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2} \qquad \left(\because K = \frac{3}{2}\right)$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{(0.2)^3}{3}\right) - \left(0.1 - \frac{(0.1)^3}{3}\right)\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right] = 0.1465$$

(iii) $$P(x > 0.5) = \int_{0.5}^{\infty} f(x)dx$$

$$= \int_{0.5}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx$$

$$= \int_{0.5}^{1} K(1-x^2)dx + 0$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1-\frac{1}{3}\right) - \left(0.5 - \frac{(0.5)^3}{3}\right)\right]$$

$$= \frac{3}{2}\left(\frac{2}{3} - 0.4583\right) = 0.3125$$

(2) A continuous random variable has the prob. density function

$$f(x) = Kxe^{-\lambda x}, \quad \text{for } x \geq 0, \lambda > 0$$
$$= 0, \quad \text{otherwise}$$

Determine (i) K (ii) Mean (iii) variance.

Ans:- Let $f(x) = Kxe^{-\lambda x}, \quad x \geq 0, \lambda > 0$

$= 0$, otherwise

(i) we know that $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{0}^{\infty} f(x)dx = 1$$

$$\Rightarrow \int_{0}^{\infty} K \underset{u}{x}\, \underset{v}{e^{-\lambda x}} = 1$$

$\because \int uv = uv_1 - u'v_2 + \dots$

$v_1 = \int v$

$v_2 = \int v_1, \dots$

$$\Rightarrow K\left[x\frac{e^{-\lambda x}}{-\lambda} - 1\cdot\frac{e^{-\lambda x}}{\lambda^2}\right]_0^{\infty} = 1$$

$$\Rightarrow K\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$$

$$\Rightarrow \frac{1}{\lambda^2} = \frac{1}{K} \Rightarrow \lambda^2 = K.$$

$$\therefore f(x) = \lambda^2 x\cdot e^{-\lambda x}, \quad x \geq 0$$
$$= 0 \quad \text{otherwise}.$$

(ii) Mean of the distribution $\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{0}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x\, \lambda^2 x\, e^{-\lambda x}\, dx$$

$$= \lambda^2 \int_{0}^{\infty} \underset{u}{x^2}\, \underset{v}{e^{-\lambda x}}\, dx \qquad \int uv = uv_1 - u'v_2 + u''v_3 - \cdots$$

$$= \lambda^2 \left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x \cdot \frac{e^{-\lambda x}}{\lambda^2} + 2 \cdot \frac{e^{-\lambda x}}{-\lambda^3} \right]_0^{\infty}$$

$$= \lambda^2 \left[0 - \left(0 - 0 + \frac{2}{-\lambda^3}\right) \right]$$

$$= \frac{2}{\lambda}.$$

$\therefore$ Mean $\mu = \dfrac{2}{\lambda}$

(iii) variance $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$

$$= \int_{0}^{\infty} x^2 \lambda^2 x\, e^{-\lambda x}\, dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x}\, dx - \frac{4}{\lambda^2}$$

$$= \lambda^2 \left[x^3 \frac{e^{-\lambda x}}{-\lambda} - 3x^2 \frac{e^{-\lambda x}}{\lambda^2} + 6x \frac{e^{-\lambda x}}{-\lambda^3} - 6 \frac{e^{-\lambda x}}{\lambda^4} \right]_0^{\infty}$$

$$= \lambda^2 \left[(0) - \left(0 - 0 + 0 - 6 \cdot \frac{1}{\lambda^4}\right) \right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

$\therefore$ variance $= \dfrac{2}{\lambda^2}$

③ The prob. density $f(x)$ of a continuous random variable is given by $f(x) = c e^{-|x|}$, $-\infty < x < \infty$. Show that $c = \frac{1}{2}$. And find the mean and variance of the distribution. Also find the prob. that the variate lies between 0 and 4.

Ans:- Let $f(x) = c e^{-|x|}$, $-\infty < x < \infty$

To find c :-

We K.T $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{-\infty}^{\infty} c e^{-|x|}\,dx = 1$

$\Rightarrow 2c \int_{0}^{\infty} e^{-x}\,dx = 1$ $\quad$ ($\because e^{-|x|}$ is even function. $e^{-|x|} = e^{-x}$, $0 < x < \infty$) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, f is even

$\Rightarrow 2c \left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$

$\Rightarrow -2c\,[0 - 1] = 1$

$\Rightarrow \boxed{c = \frac{1}{2}}$

To find Mean :-

Mean $= \int_{-\infty}^{\infty} x\, f(x)\,dx$

$= \frac{1}{2}\int_{-\infty}^{\infty} x\, e^{-|x|}\,dx$

$= 0$

$f(x) = x e^{-|x|}$

$f(-x) = -x e^{-|-x|}$

$= -x e^{-|x|}$

$= -f(x)$

f is odd function

$\because \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if f is even

$= 0$, if f is odd

To find variance

$$\text{variance } \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2} \cdot 2 \int_{0}^{\infty} x^2 e^{-x}\,dx \quad \left[\because x^2 e^{-|x|} \text{ is even function}\right]$$

$$= \left[x^2 \cdot \frac{e^{-x}}{-1} - 2x \cdot \frac{e^{-x}}{1} + 2\frac{e^{-x}}{-1} \right]_0^{\infty}$$

$$= (0 - 0 + 0) - (0 - 0 - 2)$$

$$= 2$$

To find probability b/w 0 & 4

$$P(0 \le x \le 4) = \int_0^4 f(x)\,dx$$

$$= \int_0^4 \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2} \int_0^4 e^{-x}\,dx$$

$$= \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_0^4 = \frac{-1}{2}\left(e^{-4} - 1\right)$$

$$= 0.4908$$

(4) Is the function defined by

$$f(x) = 0, \quad x < 2$$
$$= \frac{1}{18}(2x+3), \quad 2 \le x \le 4$$
$$= 0, \quad x > 4.$$

) a prob. density function? Find the prob. that a variate having $f(x)$ as density function will fall in the interval $2 \le x \le 3$.

Ans:- (i) $\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx$

$$= 0 + \int_{2}^{4} \frac{1}{18}(2x+3)\,dx + 0$$

$$= \frac{1}{18}\left[\cancel{2}\cdot\frac{x^2}{\cancel{2}} + 3x\right]_2^4$$

$$= \frac{1}{18}[28 - 10] = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ $f(x)$ is a prob. density function.

(ii) $P(2 \le x \le 3) = \int_{2}^{3} f(x)\,dx$

$$= \int_{2}^{3} \frac{1}{18}(2x+3)\,dx$$

$$= \frac{1}{18}\left[\cancel{2}\cdot\frac{x^2}{\cancel{2}} + 3x\right]_2^3$$

$$= \frac{1}{18}[18 - 10] = \frac{8}{18} = \frac{4}{9}$$

(5) If X is a continuous random variable and $Y = aX + b$, prove that $E(Y) = a E(X) + b$ and $V(Y) = a^2 \cdot V(X)$, where V stands for variance and a, b are constants.

A:- Let X be a continuous random variable.

Let $Y = aX + b$.

(i) We know that $E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx$

Consider $E(Y) = \int_{-\infty}^{\infty} (ax+b) f(x)\, dx$

$= a \int_{-\infty}^{\infty} x f(x)\, dx + b \int_{-\infty}^{\infty} f(x)\, dx$

$= a E(X) + b(1)$ $\left(\because \text{Total prob.} = 1\right)$

$\therefore E(Y) = a E(X) + b.$

(ii) Let $Y = aX + b$ —— (1)

And $E(Y) = a E(X) + b$ —— (2)

(1) - (2) $\Rightarrow$ $Y - E(Y) = aX + \cancel{b} - aE(X) - \cancel{b}$

$= a(X - E(X))$

Squaring on both sides

$[Y - E(Y)]^2 = a^2 (X - E(X))^2$

Taking expectation on both sides

$E(Y - E(X))^2 = a^2 E(X - E(X))^2$

$V(Y) = a^2 V(X)$ $(\because \text{variance} = E[X - E(X)]^2)$

(6) If the prob. density function of X is given by $f(x) = \frac{x}{2}, \; 0 < x \leq 1$

$$= \frac{1}{2}, \quad 1 < x \leq 2$$

$$= \frac{3-x}{2}, \quad 2 < x < 3$$

$$= 0, \quad \text{otherwise.}$$

Find the expected value of $f(x) = x^2 - 5x + 3$.

Ans: The Expectation of $\phi(x)$ is

$$E(\phi(x)) = \int_{-\infty}^{\infty} \phi(x) f(x)\, dx$$

$$\therefore E(x^2 - 5x + 3) = \int_{-\infty}^{\infty} (x^2 - 5x + 3) f(x)\, dx$$

$$= \int_0^1 (x^2 - 5x + 3)\frac{x}{2}\, dx + \int_1^2 (x^2 - 5x + 3)\frac{1}{2}\, dx$$

$$+ \int_2^3 (x^2 - 5x + 3)\frac{3-x}{2}\, dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - 5x^2 + 3x)\, dx + \frac{1}{2}\left[\frac{x^3}{3} - 5\frac{x^2}{2} + 3x\right]_1^2$$

$$+ \frac{1}{2}\int_2^3 (3x^2 - 15x + 9 - x^3 + 5x^2 - 3x)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - 5\frac{x^3}{3} + 3\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[\left(\frac{8}{3} - 10 + 6\right) - \left(\frac{1}{3} - \frac{5}{2} + 3\right)\right]$$

$$+ \frac{1}{2}\int_2^3 (-x^3 + 8x^2 - 18x + 9)\, dx$$

$\frac{8}{3} - 4$

$\frac{2 - 15 + 18}{6}$

$$= \frac{1}{2}\left[\frac{1}{4} - \frac{5}{3} + \frac{3}{2}\right] + \frac{1}{2}\left[\frac{-4}{3} - \frac{5}{6}\right]$$

$$+ \frac{1}{2}\left[\frac{-x^4}{4} + 8\frac{x^3}{3} - 18\frac{x^2}{2} + 9x\right]_2^3 = \frac{-11}{6}$$

(check this answer)

(7) The prob. density function of a random variable x is $f(x) = \frac{1}{2}\sin x$, $0 \le x \le \pi$

$= 0$, otherwise.

Find (i) The Mean (ii) Mode (iii) Median.

(iv) The probability between 0 and $\frac{\pi}{2}$.

(8) A random variable x gives the prob. density function $f(x) = 12x^3 - 21x^2 + 10x$, $0 \le x \le 1$

$= 0$, otherwise

(i) Find $p(x \le \frac{1}{2})$ and $p(x > \frac{1}{2})$

(ii) Find a number K such that $p(x \le K) = \frac{1}{2}$.

(9) If prob. density function

$f(x) = Kx^3$, $0 \le x \le 3$

$= 0$, otherwise. Find

(i) The value of K

(ii) The prob. between $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Normal distribution

Let x be a random variable.

The probability density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2c^2}}$$ is called normal distribution.

Constants of a normal distribution

(1) The Mean of normal distribution $\mu = E(X) = b$.

(2) The variance of normal distribution $= \sigma^2$

(3) The standard deviation of normal distribution $= \sqrt{\sigma^2} = \sigma$.

(4) Mode of normal distribution $= \mu$

(5) Median of normal distribution $= \mu$.

Hence in normal distribution,

Mean = Median = Mode

(6)

Def:- The normal distribution mean $\mu = 0$ and standard deviation $\sigma = 1$ is called standard normal distribution.

Probabilities of normal distribution

The probability that the normal variate x with mean μ and S.D σ lies between two specific values x_1 and x_2 ... $x_1 < x_2$ can be obtained using area under the standard normal curve as follows:

step 1

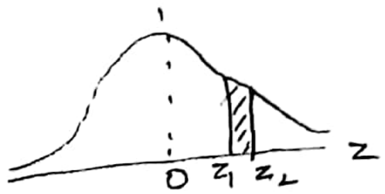
1) perform the change of scale $z = \frac{x-\mu}{\sigma}$ and find z_1 and z_2 corr. to the values of x_1 and x_2 resp.

step 2

2) a) To find $P(x_1 \le X \le x_2) = P(z_1 \le z \le z_2)$

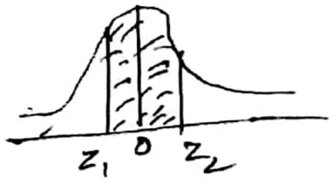
case 1:- If both z_1 and z_2 are ~~to~~ positive (or negative)

$P(x_1 \le x \le x_2) = A(z_2) - A(z_1)$



case 2:- If ~~both~~ z_1 is Negative and z_2 is positive.

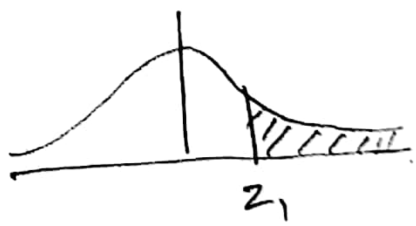
$P(x_1 \le x \le x_2) = A(z_2) + A(z_1)$



2) b) To find $P(z \ge z_1)$

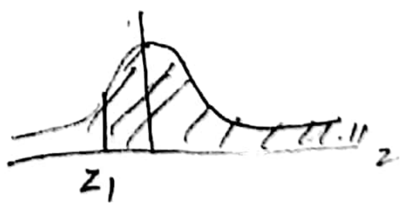
case 1:- If $z_1 > 0$, then

$P(z > z_1) = 0.5 - A(z_1)$



case 2:- If $z_1 < 0$, then

$P(z > z_1) = 0.5 + A(z_1)$



2) c) To find $P(z < z_1)$

case 1:- If $z_1 > 0$, then

$P(z < z_1) = 0.5 + A(z_1)$

case 2:- If $z_1 \le 0$, Then

$P(z < z_1) = 0.5 - A(z_1)$