UNIT-2

COMPLEX POWER SERIES

Power series

A series of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$.

$= a_0 + a_1(z-a) + a_2(z-a)^2 + \dots$ is called power series in terms of $z-a$.

Definition

If $\sum a_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$. thus R is called radius of converges of power series and $|z| = R$ is called circle of converges of power series.

Theorems.

Taylor's Theorem

Let $f(z)$ be analytic at all points with in a circle C_0 with centre 'a' and radius 'R'. then at each pnt 'z' with in C_0.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \dots + \frac{f^n(a)}{n!}(z-a)^n + \dots,$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n.$$

Note

If $a=0$ in Taylor series expansion, we get

$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots\cdots$ It is called Maclaurin's series expansion of $f(z)$.

→ Expand e^z as a Taylor's series about $z=1$.

Sol: Let $f(z) = e^z$. ⇒

$z=1 \Rightarrow a=1$

$\therefore f(a) = e^1 = e.$

$f'(z) = e^z \Rightarrow f'(a) = e.$

$f''(z) = e^z \Rightarrow f''(a) = e.$

From Taylor's series expansion,

$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots\cdots$

$\Rightarrow f(z) = e + (z-1)e + \frac{(z-1)^2}{2!} e + \cdots\cdots$

$= e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\cdots\right].$

$= e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}.$

$\therefore f(z) = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}.$

II nd Method,

Let $f(z) = e^z$

$z-1 = \omega \Rightarrow z = 1+\omega$

$\therefore f(z) = e^{1+\omega} = e^1 \cdot e^{\omega} = e\left[1 + \omega + \frac{\omega^2}{2!} + \frac{\omega^3}{3!} + \cdots\right]$

$= e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$

$= e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$

$\therefore f(z) = e \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}.$

→ Expand $\log z$ by Taylor series about $z=1$.

Sol: Let $f(z) = \log z$.

$z=1.$

$\Rightarrow f(a) = \log 1 \Rightarrow f(a) = 0.$

$\Rightarrow f'(z) = \frac{1}{z} \Rightarrow f'(a) = 1$

$\Rightarrow f''(z) = -1/z^2 \Rightarrow f''(a) = -1$

$\Rightarrow f'''(z) = -\left(\frac{-2}{z^3}\right) \Rightarrow f'''(a) = 2.$

$f^{(4)}(z) = \frac{-6}{z^3}$

From Taylor series expansion.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots$$

$$\Rightarrow f(z) = 0 + 1(z-1) + \frac{(-1)}{2!}(z-1)^2 + \frac{2}{3!}(z-1)^3 + \cdots$$

$$= (z-1) - \frac{(z-1)^2}{2!} + 2\frac{(z-1)^3}{3!} + \frac{(6)}{(3)}\frac{(z-1)^4}{4!} \cdots$$

$$= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\alpha} (-1)^{n+1} \cdot \frac{(z-1)^n}{n}.$$

$$\therefore f(z) = \sum_{n=1}^{\alpha} \frac{(-1)^{n+1}}{n}(z-1)^n$$

II.

Let $f(z) = \log z$.

Let $z-1 = \omega \Rightarrow z = \omega+1$,

$$f(z) = \log(\omega+1)$$

$\left[\because \log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots\right]$

$$= \omega - \frac{\omega^2}{2} + \frac{\omega^3}{3} - \cdots$$

$$= (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \cdots$$

$$= \sum_{n=1}^{\alpha} (-1)^{n+1} \cdot \frac{(z-1)^n}{n}.$$

$$\therefore f(z) = \sum_{n=1}^{\alpha} \frac{(-1)^{n+1}}{n}(z-1)^n.$$

Date: 19/8/19
Day: thursday

→ Expand $f(z) = \frac{1}{(z+1)^2}$ in Taylor series about $z=-i$.

Sol Let $f(z) = \frac{1}{(z+1)^2}$.

Let $z+i = \omega \Rightarrow z = \omega - i$.

$$\therefore f(z) = \frac{1}{(\omega - i + 1)^2}.$$

$$= \frac{1}{[\omega - (i-1)]^2} = \frac{1}{\left((i-1)\left[\frac{\omega}{i-1} - 1\right]\right)^2} = \frac{1}{(i-1)^2} \cdot \frac{1}{\left(1 - \frac{\omega}{i-1}\right)^2}$$

$$= \frac{-1}{2i}\left[1 - \frac{\omega}{i-1}\right]^{-2}.$$

$[\because (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots]$

$$= -\frac{1}{2i}\left[1 + 2\left(\frac{\omega}{i-1}\right) + 3\left(\frac{\omega}{i-1}\right)^2 + 4\left(\frac{\omega}{i-1}\right)^3 + \cdots\right]$$

$$= -\frac{1}{2i}\left[1 + 2\left(\frac{z+i}{i-1}\right) + 3\left(\frac{z+i}{i-1}\right)^2 + 4\left(\frac{z+i}{i-1}\right)^3 + \cdots\right]$$

$$\therefore f(z) = -\frac{1}{2i}\left[1 + 2\left(\frac{z+i}{i-1}\right) + 3\left(\frac{z+i}{i-1}\right)^2 + 4\left(\frac{z+i}{i-1}\right)^3 + \cdots\right]$$

→ Expand $f(z) = \frac{z-1}{z+1}$ in Taylor series about the point $z=0$ & $z=1$

Sol: Let $f(z) = \frac{z-1}{z+1}$. $\Rightarrow f(z) = \frac{z+1-2}{z+1} = 1 - \frac{2}{z+1}$

Let $z+1 = \omega \Rightarrow z = \omega$

$(1+x)^{-1}$

Sol.

About $z=0$.

$\therefore f(z) = 1 - 2(1+z)^{-1}$

$= 1 - 2[1 - z + z^2 - z^3 + \ldots]$ if $|z|<1$.

$f(z) = -1 + 2z - 2z^2 + 2z^3 - \ldots$ if $|z|<1$

About $z=1$.

Let $z-1=\omega$, $\Rightarrow z = \omega+1$

$\therefore f(z) = 1 - \frac{2}{z+1} = 1 - \frac{2}{\omega+2} = 1 - \frac{2}{2(1+\frac{\omega}{2})} = 1 - \frac{1}{1+\omega/2} = 1 - (1+\omega/2)^{-1}$

$f(z) = 1 - [1 - \frac{\omega}{2} + (\frac{\omega}{2})^2 - (\frac{\omega}{2})^3 + \ldots]$ if $|\omega/2|<1$.

$= \frac{\omega}{2} - (\frac{\omega}{2})^2 + (\frac{\omega}{2})^3 + \ldots$ if $|\omega|<2$

$\therefore f(z) = \frac{z-1}{2} - (\frac{z-1}{2})^2 + (\frac{z-1}{2})^3 + \ldots$ if $|z-1|<2$.

→ Expand $f(z) = \frac{1}{z^2+z-6}$ about $z=-1$.

$z^2+z-6=0$

$z^2-2z+3z-6=0$

$z(z-2)+3(z-2)=0$

$(z-2)(z+3)=0$

Sol: Let $f(z) = \frac{1}{z^2+z-6} = \frac{1}{5(z-2)} - \frac{1}{5(z+3)}$

$= \frac{1}{5}\left[\frac{1}{z-2} - \frac{1}{z+3}\right]$.

Let $z+1=\omega \Rightarrow z = \omega-1$.

$= \frac{1}{5}\left[\frac{1}{(z+1)-3} - \frac{1}{(z+1)+2}\right]$

$1 = 5 \cdot$

$B = -\frac{1}{5}$

$= \frac{1}{5}\left[\frac{1}{\omega-3} - \frac{1}{\omega+2}\right]$.

$= \frac{1}{5}\left[\frac{1}{-3[1-\frac{\omega}{3}]} - \frac{1}{2[1+\frac{\omega}{2}]}\right]$.

$$= \frac{1}{5}\left[-\frac{1}{3}\left(1-\frac{w}{3}\right)^{-1} - \frac{1}{2}\left(1+\frac{w}{2}\right)^{-1}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\cdots\right) - \frac{1}{2}\left(1-\frac{z+1}{2}+\left(\frac{z+1}{2}\right)^2-\cdots\right)\right]$$

$=$. If $\left|\frac{z+1}{3}\right| < 1$ & $\left|\frac{z+1}{2}\right| < 1$.

$$\therefore f(z) = \frac{1}{5}\left[-\frac{1}{3}\left(1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\cdots\right) - \frac{1}{2}\left(1-\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2-\cdots\right)\right]$$

If $|z+1| < 2$.

→ Expand $f(z) = \frac{1}{z^2-z-6}$ about $z=-1$ & $z=1$

Sol) Let $f(z) = \frac{1}{z^2-z-6}$.

$z^2-z-6 = z^2-3z+2z-6$
$= z(z-3)+2(z-3)$
$= (z-3)(z+2)$

$A = 1/5$, $B = -1/5$

$$= \frac{1}{5}\left(\frac{1}{z-3}\right) - \frac{1}{5}\left(\frac{1}{z+2}\right)$$

$$= \frac{1}{5}\left[\frac{1}{z-3} - \frac{1}{z+2}\right]$$

above $f(z)$ express in powers of $z+1$.

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{(z+1)-4} - \frac{1}{(z+1)+1}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-4\left(1-\frac{z+1}{4}\right)} - \frac{1}{1+(z+1)}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{4}\left(1-\frac{z+1}{4}\right)^{-1} - \left[1+(z+1)\right]^{-1}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{4}\left[1+\left(\frac{z+1}{4}\right)+\left(\frac{z+1}{4}\right)^2+\cdots\right] - \left[1-(z+1)+(z+1)^2-\cdots\right]\right]$$

If $\left|\frac{z+1}{4}\right| < 1$ and $|z+1| < 1$.

above $f(z)$. express in powers of $z-1$.

$$\therefore f(z) = \frac{1}{5}\left[\frac{1}{(z-1)-2} - \frac{1}{(z-1)+3}\right]$$

$$= \frac{1}{5}\left[\frac{1}{-2\left[1-\left(\frac{z-1}{2}\right)\right]} - \frac{1}{3\left[1+\frac{z-1}{3}\right]}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{2}\left(1-\frac{z-1}{2}\right)^{-1} - \frac{1}{3}\left(1+\frac{z-1}{3}\right)^{-1}\right]$$

$$= \frac{1}{5}\left[-\frac{1}{2}\left(1+\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2+\cdots\right) - \frac{1}{3}\left(1-\frac{z-1}{3}+\left(\frac{z-1}{3}\right)^2-\cdots\right)\right]$$

If $\left|\frac{z-1}{2}\right| < 1$ and $\left|\frac{z-1}{3}\right| < 1$.

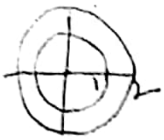
$$\therefore f(z) = \frac{1}{5}\left[-\frac{1}{2}\left(1+\left(\frac{z-1}{2}\right)+\left(\frac{z-1}{2}\right)^2+\cdots\right) - \frac{1}{3}(\text{"})\right]_{//}$$

Expand Taylor series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ about $z=0$.

$f(z) = \frac{1}{(z-1)(z-2)}$

$f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2(1-\frac{z}{2})} + \frac{1}{1-z} = (1-z)^{-1} - \frac{1}{2}(1-z/2)^{-1}$.

$f(z) = (1+z+z^2+z^3+ \cdots) - \frac{1}{2}(1+\frac{z}{2}+(z/2)^2+(z/2)^3+\cdots)$.



$|z|<1$, $|z|<2$

if $|z/2|<1$ and $|z|<1$, $|z|<1$

$\therefore f(z) = (1+z+z^2+z^3+\cdots) - \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+(\frac{z}{2})^3+\cdots)$.

$= -\frac{1}{2}\left[\sum_{n=0}^{\infty}\left(\frac{z^n}{2^n}\right)\right] + \sum_{n=0}^{\infty} z^n$, if $|z|<1$

→ Expand Taylor series expansion $f(z) = \frac{z+1}{(z-3)(z-4)}$ about the $z=2$.

determine the region of convergence.

$f(z) = \frac{z+1}{(z-3)(z-4)}$

$\frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$

$z+1 = A(z-4)+B(z-3)$

$z=3$: $4 = A(3-4) \Rightarrow A=-4$

$z=4$: $5 = B(1) \Rightarrow B=5$

Sol: $f(z) = \frac{z+1}{(z-3)(z-4)} \Rightarrow f(z) = \frac{-4}{z-3} + \frac{5}{z-4}$.

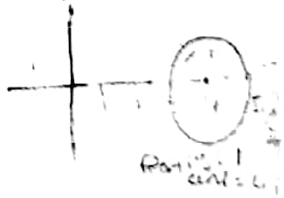
$= \frac{-4}{(z-2)-1} + \frac{5}{(z-2)-2} = \frac{4}{1-(z-2)} + \frac{5}{-2[1-(\frac{z-2}{2})]}$

$= 4(1-(z-2))^{-1} - \frac{5}{2}(1-(\frac{z-2}{2}))^{-1}$.

$= 4[1+(z-2)+(z-2)^2+\cdots] - \frac{5}{2}[1+(\frac{z-2}{2})+(\frac{z-2}{2})^2+\cdots]$ if $|z-2|<1$ and $|\frac{z-2}{2}|<1$.

$= 4(1+(z-2)+(z-2)^2+\cdots] - 5/2(1+(\frac{z-2}{2})+(\frac{z-2}{2})^2+\cdots]$ if $|z-2|<1$.

→ Explain the expansion $\frac{1}{(z-1)(z-3)}$ in a Taylor series in powers of $z-4$ and determine the region of convergence.



Sol: Let $f(z) = \frac{1}{(z-1)(z-3)}$

$\therefore f(z) = \frac{-1}{2(z-1)} + \frac{1}{2(z-3)} = -\frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z-3}\right]$

$= -\frac{1}{2}\left[\frac{1}{(z-4)+3} - \frac{1}{(z-4)+1}\right] = -\frac{1}{2}\left[\frac{1}{3(1+(\frac{z-4}{3}))} - \frac{1}{1+(z-4)}\right]$

$= -\frac{1}{2}\left[\frac{1}{3}(1+(\frac{z-4}{3}))^{-1} - (1+(z-4))^{-1}\right]$.

$= -\frac{1}{2}\left[\frac{1}{3}(1-(\frac{z-4}{3})+(\frac{z-4}{3})^2-(\frac{z-4}{3})^3+\cdots) - (1-(z-4)+(z-4)^2-(z-4)^3+\cdots)\right]$

if $|\frac{z-4}{3}|<1$ & $|z-4|<1$

The region of convergence is $|z-4|<1$

$|z-4|<1$

→ Find the Taylor series expansion of $f(z) = \frac{z-1}{z^2}$ about $z=1$.

find its region of convergence.

Sol: Let $f(z) = \frac{z-1}{z^2}$.

$\frac{z}{z^2} - \frac{1}{z^2} = \frac{1}{z} - \frac{1}{z^2}$

Laurent's Series.

Let C_1 & C_2 be two circles given by $|z'-a|=r_1$ & $|z'-a|=r_2$ resp. where $r_2 < r_1$ and z' is any point on C_1 and C_2. Let $f(z)$ be analytic on C_1 and C_2 through the region between the two circles. Let z be any point in the ring shaped region between two circles C_1 & C_2

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \underbrace{\sum_{n=1}^{\infty} b_n(z-a)^{-n}}_{\text{principle part of } f(z)} \qquad \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-a)^{n+1}} dz'$; $b_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{(z'-a)^{-n+1}} dz'$.

If $f(z) = \frac{1}{(1-z)(z-2)}$ then find Laurent's series expansion

in the annulas region $1<|z|<2$.

in $|z|>2$.

ii, Laurent's series expansion in $0<|z-1|<1$

iii Laurent's series expansion in

Sol Let $f(z) = \frac{1}{(1-z)(z-2)}$.

$$\frac{1}{(1-z)(z-2)} = \frac{A}{1-z} + \frac{B}{z-2} \Rightarrow 1 = A(z-2) + B(1-z).$$

If $z=2 \Rightarrow 1 = 0 + B(1-2) \Rightarrow B=-1$

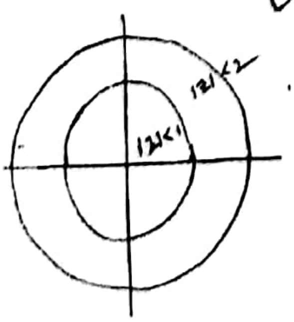
If $z=1 \Rightarrow 1 = A(1-2)+0 \Rightarrow A=-1$.

$$\therefore \frac{1}{(1-z)(z-2)} = -\frac{1}{1-z} - \frac{1}{z-2}.$$

$$\Rightarrow \frac{1}{(1-z)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

i) if $1<|z|<2$.

$f(z)$ is analytic in the region $1<|z|<2$.



i.e $1<|z| \Rightarrow \frac{1}{|z|}<1$.

$|z|<2 \Rightarrow \frac{|z|}{2}<1$.

$$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-z/2)}$$

$$= \frac{1}{z}(1-\tfrac{1}{z})^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$\therefore f(z) = \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right].$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}(z/2)^n.$$

$|z| > 1$

$\frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$

$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-2/z)} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-2/z\right)^{-1}$

$= \frac{1}{z}\left[\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\dots\right) - \left(1+2/z+(2/z)^2+\dots\right)\right]$

$= \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(2/z\right)^n.$

iii) Here given region is $0 < |z-1| < 1$

$|z-1| < 1$

$\frac{1}{|z-1|} > 1$ × (not possible)

$\therefore f(z) = \frac{1}{z-1} - \frac{1}{z-2}.$

$= \frac{1}{z-1} - \frac{1}{(z-1)-1}.$

$= \frac{1}{z-1} + \frac{1}{1-(z-1)}.$

$(1-x)^{-1} = 1+x+x^2+x^3+\dots$

$(1+x)^{-1} = 1-x+x^2-x^3+\dots$

$= \frac{1}{z-1} + \left[1+(z-1)+(z-1)^2+\dots\right]$ if $0 < |z-1| < 1.$

$= \frac{1}{z-1} + \sum_{n=0}^{\infty}(z-1)^n.$ if $0 < |z-1| < 1.$

Date: 23-8-19
Day: Monday.

→ Find the Laurent's series of $f(z) = \frac{z}{(z-1)(z-2)}$ about $z=0$ in 3 different ways.

Sol Given that $f(z) = \frac{z}{(z-1)(z-2)}$

$\therefore f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$

$z = A(z-2) + B(z-1)$

If $z=1$

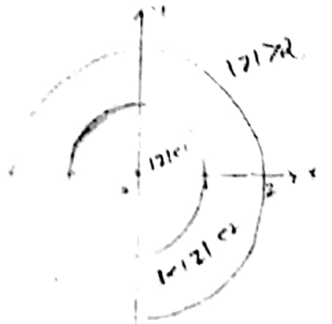
$1 = A(1-2) + 0.$

$A = -1$

If $z = 2.$

$2 = B(1)$

$B = 2$



i, If $|z| < 1$

$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}.$

$= \frac{1}{1-z} - \frac{2}{2(1-z/2)} = \frac{1}{1-z} - \frac{1}{1-z/2}$

$|z| < 1$, $\frac{|z|}{2} < 1$

$= \left[1+(1-z)+(1-z)^2+\dots\right] - \left(1+(z/2)+(z/2)^2+\dots\right)$

since $|z| < 1$, $\frac{|z|}{2} < 1.$

$\therefore f(z) = \sum_{n=0}^{\infty}(1-z)^n - \sum_{n=0}^{\infty}(z/2)^n.$

ii If $1 < |z| < 2.$

$1 < |z|$ & $|z| < 2$

$\frac{1}{|z|} < 1$ & $\frac{|z|}{2} < 1$

$f(z) = \frac{-1}{z-1} + \frac{2}{z-2}$

$= \frac{-1}{z\left[1-\frac{1}{z}\right]} - \frac{2}{2\left[1-z/2\right]}$

$= -\frac{1}{z(1-\frac{1}{z})} - \frac{1}{(1-z/2)}$

$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \left(1-z/2\right)^{-1}$

since $\frac{1}{|z|} < 1$, $\frac{|z|}{2} < 1$

$= -1/z\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\dots\right] - \left[1+(z/2)+(z/2)^2+\dots\right]$

$\therefore f(z) = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$. Since $\frac{1}{|z|}<1$ & $\left|\frac{z}{2}\right|<1$.

If $|z|>2$.

$|z|>2$

$= \frac{|z|}{2}>1$.

$= \frac{2}{|z|}<1$

Also $\frac{1}{|z|}<1$

$\therefore f(z) = -\frac{1}{z-1} + \frac{2}{z-2}$.

$= \frac{-1}{z(1-\frac{1}{z})} + \frac{2}{z(1-\frac{2}{z})}$.

$= \frac{-1}{z}(1-\frac{1}{z})^{-1} + \frac{2}{z}(1-2/z)^{-1}$.

$\Rightarrow f(z) = -\frac{1}{z}\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right) + \frac{2}{z}\left(1+(2/z)+(2/z)^2+\cdots\right)$

$= \frac{-1}{z}.\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{2}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$. Since $\frac{1}{|z|}<1$ & $\frac{2}{|z|}<1$.

→ Expand $f(z) = \frac{1}{z^2-3z+2}$ in the region (i) $0 < |z-1| < 1$

(ii) $1 < |z| < 2$.

Ai:- Let $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$

$$= \frac{1}{-1}\left[\frac{1}{z-1} - \frac{1}{z-2}\right] \quad z-2-z+1$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

(i) $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{(z-1)-1} - \frac{1}{z-1}$$

$$= -\frac{1}{1-(z-1)} - \frac{1}{z-1}$$

$$= -[1-(z-1)]^{-1} - \frac{1}{z-1} \quad [\because (1-x)^{-1} = 1+x+x^2+\cdots]$$

$$= -[1+(z-1)+(z-1)^2+\cdots] - \frac{1}{z-1}$$

(ii) $1 < |z| < 2$.

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1} \qquad 1<|z| \Rightarrow \frac{1}{|z|} < 1, \quad |z|<2 \Rightarrow \frac{|z|}{2} < 1$$

$$= \frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z(1-\frac{1}{z})}$$

$$= \frac{-1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{-1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right]$$

Imp.
→ 1) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$ as a Laurent's series. Also find the region of convergence.

Sol: Let $f(z) = \frac{e^{2z}}{(z-1)^3}$

$= \frac{1}{(z-1)^3} e^{2(z-1)+2}$.

$= \frac{1}{(z-1)^3} e^{2(z-1)} \cdot e^2$

$\left(\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

$= \frac{e^2}{(z-1)^3}\left[1 + [2(z-1)] + \frac{[2(z-1)]^2}{2!} + \frac{[2(z-1)]^3}{3!} + \cdots\right]$

$= \frac{e^2}{(z-1)^3}\left[1 + 2(z-1) + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$

The Laurent's series expansion of $f(z)$ is

$f(z) = e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{z-1} + \frac{4}{3} + \cdots\right]$

The region of convergence is $0 < |z-1| < 1$.

Imp
2) Find the Laurent's Series expansion of the function $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 < |z+2| < 5$.

Sol:- $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)} = \frac{A}{z-1} + \frac{B}{z-3} + \frac{C}{z+2}$

$$z^2-6z-1 = A(z-3)(z+2) + B(z-1)(z+2) + C(z-1)(z-3).$$

put $z=1 \Rightarrow 1-6-1 = A(-2)(3) \Rightarrow A = 1$

put $z=3 \Rightarrow 9-18-1 = A(0)+B(2)(5)+C(0)$

$$\Rightarrow B = \frac{-10}{10} = -1$$

put $z=-2 \Rightarrow 4+12-1 = A(0)+B(0)+C(-3)(-5)$

$$\Rightarrow 15 = C(15) \Rightarrow C = 1.$$

$$\therefore f(z) = \frac{1}{z-1} + \frac{-1}{z-3} + \frac{1}{z+2}$$

$$= \frac{1}{z+2-3} - \frac{1}{z+2-5} + \frac{1}{z+2}$$

$\left(\because 3 < |z+2| \Rightarrow \frac{3}{z+2} < 1 \right.$

$\left. z+2 < 5 \Rightarrow \frac{z+2}{5} < 1 \right)$

$$= \frac{1}{z+2\left(1-\frac{3}{z+2}\right)} - \frac{1}{-5\left(1-\frac{z+2}{5}\right)} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left(1-\frac{3}{z+2}\right)^{-1} + \frac{1}{5}\left(1-\frac{z+2}{5}\right)^{-1} + \frac{1}{z+2}$$

$$= \frac{1}{z+2}\left[1+\frac{3}{z+2}+\left(\frac{3}{z+2}\right)^2+\cdots\right] + \frac{1}{5}\left[1+\frac{z+2}{5}\right.$$

$$\left.+\left(\frac{z+2}{5}\right)^2+\cdots\right] + \frac{1}{z+2}$$

→ 3) Expand $f(z) = \dfrac{7z-2}{(z+1)z(z-2)}$ about the point $z=-1$ in the region $1<|z+1|<3$ as a Laurent's Series.

A:- Let $f(z) = \dfrac{7z-2}{(z+1)(z)(z-2)} = \dfrac{A}{z+1} + \dfrac{B}{z} + \dfrac{C}{z-2}$

$$7z-2 = A(z)(z-2) + B(z+1)(z-2) + Cz(z+1)$$

put $z=-1 \Rightarrow -9 = A(-1)(-3) \Rightarrow A = -3$

$z=0 \Rightarrow -2 = A(0) + B(1)(-2) \Rightarrow B = 1$

$z=2 \Rightarrow 12 = A(0) + B(0) + C(2)(3)$

$\Rightarrow C = \frac{12}{6} = 2.$

$$\therefore \frac{7z-2}{(z+1)(z)(z-2)} = \frac{-3}{z+1} + \frac{1}{z} + \frac{2}{z-2}$$

$$= \frac{-3}{z+1} + \frac{1}{z+1-1} + \frac{2}{z+1-3}$$

$$\left[\because \text{The region is } 1 < |z+1| < 3 \quad 1 < |z+1|, \quad |z+1| < 3 \quad \frac{1}{z+1} < 1 \quad \frac{z+1}{3} < 1. \right]$$

$$= \frac{-3}{z+1} + \frac{1}{z+1\left(1-\frac{1}{z+1}\right)} + \frac{2}{-3\left(1-\frac{z+1}{3}\right)}$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$

$$= \frac{-3}{z+1} + \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{1}{(z+1)^n} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n.$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+x^3+\cdots = \sum x^n\right].$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{-3}{z+1} + \sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} - \frac{2}{3}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

4) Find the Laurent expansion of $\frac{1}{z^2-4z+3}$ for (i) $1<|z|<3$

(ii) $|z|<1$

(iii) $|z|>3$.

Sol: Let $f(z) = \frac{1}{z^2-4z+3}$

$$= \frac{1}{(z-1)(z-3)}$$

$$= \frac{1}{-2}\left[\frac{1}{z-1} - \frac{1}{z-3}\right] \quad z-3-z+1$$

$$= \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right].$$

(i) The region is $1<|z|<3$.

$1<|z| \Rightarrow \frac{1}{|z|}<1$

$|z|<3 \Rightarrow \frac{|z|}{3}<1$

$$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3\left(1-\frac{z}{3}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right].$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right].$$

$$\left(\because (1-x)^{-1} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n\right)$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{2}\left[\frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right]$$

(ii) The region is $|z|<1$

$$|z|<1 \Rightarrow |z|<3 \Rightarrow \frac{|z|}{3}<1$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-3(1-\frac{z}{3})} - \frac{1}{-(1-z)}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\left(1-\frac{z}{3}\right)^{-1} + (1-z)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^{n} + \sum_{n=0}^{\infty} z^{n}\right].$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n\right]$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{2}\left[-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} + \sum_{n=0}^{\infty} z^n\right]$$

(iii) The region is $|z|>3$.

$$|z|>3 \Rightarrow \frac{|z|}{3}>1 \Rightarrow \frac{3}{|z|}<1$$

$$\frac{3}{|z|}<1 \Rightarrow \frac{1}{|z|}<1.$$

$$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z(1-\frac{3}{z})} - \frac{1}{z(1-\frac{1}{z})}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right].$$

$$= \frac{1}{2}\left[\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right].$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n\right]$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right]$$

→ 5) Expand the function $f(z) = \dfrac{4z+4}{z(z-3)(z+2)}$ in powers of z when

(i) $|z| \leq 1$

(ii) $1 \leq |z| \leq 2$.

(iii) $|z| > 2$

Ans:- Let $f(z) = \dfrac{4z+4}{z(z-3)(z+2)} = \dfrac{A}{z} + \dfrac{B}{z-3} + \dfrac{C}{z+2}$

$$4z+4 = A(z-3)(z+2) + B(z)(z+2) + C(z)(z+3)$$

Put $z = 0 \Rightarrow 4 = A(-6) \Rightarrow A = -2/3$

$z = 3 \Rightarrow 16 = B(3)(5) \Rightarrow B = 16/15$

$z = -2 \Rightarrow -4 = C(-2)(-5) \Rightarrow C = \dfrac{-2}{5}$.

$$\therefore f(z) = \frac{-2}{3z} + \frac{16}{15(z-3)} + \frac{-2}{5(z+2)}$$

(i) The region is $|z| \leq 1$

$$|z| \leq 1 \Rightarrow |z| < 3 \Rightarrow \frac{|z|}{3} < 1$$

$$|z| \leq 1 \Rightarrow |z| < 2 \Rightarrow \frac{|z|}{2} < 1.$$

$$\therefore f(z) = \frac{-2}{3z} + \frac{16}{15(z-3)} - \frac{2}{5(z+2)}$$

$$= \frac{-2}{3z} + \frac{16}{15\cdot(-3)\left(1-\frac{z}{3}\right)} - \frac{2}{5\cdot 2\left(1+\frac{z}{2}\right)}$$

$$= \frac{-2}{3z} - \frac{16}{45}\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{5}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{-2}{3z} - \frac{16}{45}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n.$$

$$\left[\because (1-x)^{-1} = 1+x+x^2+\cdots = \sum_{n=0}^{\infty} x^n\right.$$

$$\left.(1+x)^{-1} = 1-x+x^2-\cdots = \sum_{n=0}^{\infty}(-1)^n x^n.\right).$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{-2}{3z} - \frac{16}{45}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

(ii) Given region is $1 \le |z| \le 2$.

$$1 < |z| \Rightarrow \frac{1}{|z|} < 1 \Rightarrow \frac{3}{|z|} < 1$$

$$|z| < 2 \Rightarrow \frac{|z|}{2} < 1.$$

$$f(z) = \frac{-2}{3z} + \frac{16}{15(z-3)} - \frac{2}{5(z+2)}$$

$$= \frac{-2}{3z} + \frac{16}{15(-z)\left(1-\frac{3}{z}\right)} - \frac{2}{5\cdot 2\left(1+\frac{z}{2}\right)}$$

$$= \frac{-2}{3z} - \frac{16}{15z}\left(1-\frac{3}{z}\right)^{-1} - \frac{1}{5}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{-2}{3z} - \frac{16}{15z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

$$= \frac{-2}{3z} - \frac{16}{15}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} - \frac{1}{5}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n.$$

$$\left(\because (1-x)^{-1} = \sum x^n \right.$$
$$\left.(1+x)^{-1} = \sum (-1)^n x^n\right).$$

(iii) Given region is $|z| > 2$.

$$|z| > 2 \Rightarrow \frac{|z|}{2} > 1 \Rightarrow \frac{2}{|z|} < 1$$

$$|z| > 2 \Rightarrow |z| > 3 \Rightarrow \frac{|z|}{3} > 1 \Rightarrow \frac{3}{|z|} < 1.$$

$$\therefore f(z) = \frac{-2}{3z} + \frac{16}{15(z-3)} - \frac{2}{5(z+2)}$$

$$= \frac{-2}{3z} + \frac{16}{15\cdot z\left(1-\frac{3}{z}\right)} - \frac{2}{5z\left(1+\frac{2}{z}\right)}$$

$$= \frac{-2}{3z} + \frac{16}{15z}\left(1-\frac{3}{z}\right)^{-1} - \frac{2}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= \frac{-2}{3z} + \frac{16}{15z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^{n} - \frac{2}{5z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^{n}$$

$$= \frac{-2}{3z} + \frac{16}{15}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} - \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{z^{n+1}}.$$

The Laurent's series expansion of $f(z)$ is

$$f(z) = \frac{-2}{3z} + \frac{16}{15}\sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}} - \frac{2}{5}\sum_{n=0}^{\infty}(-1)^n\frac{2^n}{5^{n+1}}.$$

6) Find the Laurent Series expansion

$\frac{z^2-1}{(z+2)(z+3)}$, if $2<|z|<3$.

Ans: Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

$$= 1 + \frac{-5z-7}{(z+2)(z+3)}$$

$$z^2+5z+6 \,)\, z^2-1 \,(\, 1$$
$$\underline{z^2+5z+6}$$
$$-5z-7$$

$$= 1 - \frac{5z+7}{(z+2)(z+3)} \quad \text{——} \quad (1)$$

consider $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\Rightarrow 5z+7 = A(z+3) + B(z+2)$

$z=-3 \Rightarrow -8 = B(-1) \Rightarrow B = 8$

$z=-2 \Rightarrow -3 = A(1) \Rightarrow A = -3$

$\therefore f(z) = 1 - \left[\frac{-3}{z+2} + \frac{8}{z+3}\right]$

The Laurent series expansion of $f(z)$ is

$\therefore f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$= 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$ $\quad 2<|z| \Rightarrow \frac{2}{|z|}<1$, $\quad |z|<3 \Rightarrow \frac{|z|}{3}<1$

$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n.$

$\left(\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty}(-1)^n x^n\right).$

7) Find the Laurent series expansion of $f(z) = \frac{z^2-4}{(z+1)(z+4)}$ in the region $1<|z|<4$.

Ans: Let $f(z) = \frac{z^2-4}{(z+1)(z+4)}$

$$\begin{array}{r} z^2+5z+4 \,)\, z^2-4 \,(\, 1 \\ -z^2 \pm 5z \pm 4 \\ \hline -5z-8 \end{array}$$

$= 1 + \frac{-5z-8}{(z+1)(z+4)}$

$= 1 - \frac{5z+8}{(z+1)(z+4)}$

consider $\dfrac{5z+8}{(z+1)(z+4)} = \dfrac{A}{z+1} + \dfrac{B}{z+4}$

$$5z+8 = A(z+4) + B(z+1)$$

put $z=-4 \Rightarrow -12 = B(-3) \Rightarrow B=4$

$z=-1 \Rightarrow 3 = A(3) \Rightarrow A=1.$

$$\therefore f(z) = 1 - \left[\frac{1}{z+1} + \frac{4}{z+4}\right]$$

$$= 1 - \frac{1}{z+1} - \frac{4}{z+4} \quad \left(\because 1<|z|<4\right.$$

$1<|z| \Rightarrow \frac{1}{|z|}<1$

$|z|<4 \Rightarrow \frac{|z|}{4}<1$

$$= 1 - \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{4}{4\left(1+\frac{z}{4}\right)}$$

$$= 1 - \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \left(1+\frac{z}{4}\right)^{-1}$$

$$= 1 - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^n - \sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$$

The Laurent series expansion of $f(z)$ is

$$f(z) = 1 - \sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n$$

Practice problems

(1) Expand $f(z) = \frac{z+3}{z(z^2-z-2)}$ in powers of z where (i) $|z|<1$

(ii) $1<|z|<2$

(iii) $|z|>2$.

(2) Expand $f(z) = \frac{1}{z(z^2-3z+2)}$ in the region

(a) $1\le|z|\le 2$ (b) $0\le|z|\le 1$ (iii) $|z|\ge 2$.

(3) Find the Laurent's series expansion of $f(z) = \frac{1}{(z^2+1)(z^2+2)}$ in positive and negative powers of z if $1<|z|<\sqrt{2}$.

(4) Express $f(z) = \frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $z-1$.

5. obtain Laurent's series expansion for $f(z) = \frac{1}{(z+2)(1+z)^2}$ in (i) $|z|<2$

(ii) $|1+z|>1$

(iii) $|z|<1$

(iv) $1<|z|<2$.

Integration using Residues

Def:- The residue of $f(z)$ at $z = z_0$ is defined as the co-efficient of $(z-z_0)^{-1}$ which is a_{-1} in the Laurent's series expansion of $f(z)$. It is denoted by $Res(f : z_0)$ or $[Res\, f(z)]_{z=z_0}$

If f has a pole at z_0, then

$$[Res\, f(z)]_{z=z_0} = a_{-1} = \frac{1}{2\pi i}\int_C f(z)\, dz.$$

Calculation of Residues

(1) If $z = z_0$ is a simple pole (a pole of order one) of $f(z)$ then

$$\underset{z=z_0}{\text{Res}}\, f(z) = \lim_{z\to z_0} (z-z_0) f(z).$$

(2) If $f(z) = \dfrac{\phi(z)}{\psi'(z)}$ where $\psi(z_0)=0$ but $\phi(z_0) \neq 0$.

$$\left[\text{Res}\, f(z)\right]_{z=z_0} = \frac{\phi(z_0)}{\psi'(z_0)}.$$

(3) If $z = z_0$ is a pole of order m of $f(z)$, then

$$\left[\text{Res}\, f(z)\right]_{z=z_0} = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

Problems

1) Find the poles of the function $f(z) = \frac{1}{(z+1)(z+3)}$ and residues at these poles.

Ans: Let $f(z) = \frac{1}{(z+1)(z+3)}$

The poles of $f(z)$ are $z+1=0,\ z+3=0$

$z = -1, -3$ (simple poles)

Res of $f(z)$ at $z=-1$

$$\boxed{\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)}$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1) f(z)$$

$$= \lim_{z\to -1} (z+1) \cdot \frac{1}{\cancel{(z+1)}(z+3)}$$

$$= \frac{1}{-1+3} = \frac{1}{2}$$

Res of $f(z)$ at $z=-3$

$$\operatorname*{Res}_{z=-3} f(z) = \lim_{z\to -3} (z+3) f(z)$$

$$= \lim_{z\to -3} z+3 \cdot \frac{1}{(z+1)\cancel{(z+3)}}$$

$$= \frac{1}{-3+1}$$

$$= \frac{-1}{2}.$$

2) Find the poles of the function $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$ and residue at these poles.

Sol:- Let $f(z) = \dfrac{z^2}{(z-1)(z-2)^2}$

The poles of $f(z)$ are $z-1=0$, $(z-2)^2=0$
$z=1$, $z=2$.

$z=1$ is a simple pole.
$z=2$ is a pole of order 2.

Res of $f(z)$ at $z=1$

$z=1$ is simple pole.

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z).$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) f(z)$$

$$= \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{z^2}{\cancel{(z-1)}(z-2)^2}$$

$$= \frac{1^2}{(1-2)^2} = \frac{1}{(-1)^2} = 1.$$

Res of $f(z)$ at $z=2$

$z=2$ is a pole of order 2.

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{(2-1)!}\lim_{z\to 2}\left[\frac{d}{dz} \cancel{(z-2)^2} \cdot \frac{z^2}{(z-1)\cancel{(z-2)^2}}\right]$$

$$= \lim_{z\to 2} \frac{d}{dz}\left[\frac{z^2}{z-1}\right]$$

$$= \lim_{z\to 2} \frac{(z-1)(2z) - z^2(1)}{(z-1)^2} \qquad \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

$$= \frac{(2-1)(4) - 4}{(2-1)^2} = \cancel{\frac{(\;)4}{\;}} = \cancel{\;}0$$

$$= \frac{0}{1} = 0 .$$

$$\therefore \operatorname*{Res}_{z=2} f(z) = 0$$

Imp
3) Find the residue of $\frac{ze^z}{(z-1)^3}$ as its pole

A:- Let $f(z) = \frac{ze^z}{(z-1)^3}$

$z-1=0 \Rightarrow z=1$ is a pole of order 3.

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-1)^m f(z)\right]$$

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(3-1)!} \lim_{z\to 1} \frac{d^2}{dz^2}\left[\cancel{(z-1)^3} \cdot \frac{ze^z}{\cancel{(z-1)^3}}\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left[\underset{u}{z}\,\underset{v}{e^z} + e^z\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \left[z e^z + e^z(1) + e^z\right] \qquad (uv)' = uv' + vu'$$

$$= \frac{1}{2}\left[e + e + e\right]$$

$$= \frac{3e}{2} .$$

4) Find the residues of $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+1)}$.

Ans: Let $f(z) = \dfrac{z^2-2z}{(z+1)^2(z^2+1)}$

The poles are $(z+1)^2=0$, $z^2+1=0$

$z=-1$ is a pole of order 2, $z^2=-1$

$z=\pm\sqrt{-1}$

$z=\pm i$ are simple poles.

Res of $f(z)$ at $z=i$

$z=i$ a simple pole.

$$\operatorname{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$$

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} \cancel{(z-i)} \cdot \frac{z^2-2z}{(z+1)^2(z+i)\cancel{(z-i)}}$$

$$= \frac{i^2-2i}{(i+1)^2(i+i)}$$

$$= \frac{-1-2i}{(\cancel{i^2}+\cancel{1}+2i)(2i)} \quad (\because i^2=-1)$$

$$= \frac{-1-2i}{-4} = \frac{1+2i}{4}$$

Res of $f(z)$ at $z=-i$

$z=-i$ is a simple pole.

$$\operatorname{Res}_{z=-i} f(z) = \lim_{z\to -i} (z+i) f(z)$$

$$= \lim_{z\to -i} \cancel{(z+i)} \cdot \frac{z^2-2z}{(z+1)^2\cancel{(z+i)}(z-i)}$$

$$=$$

$$= \frac{(-i)^2 - 2(-i)}{(-i+1)^2(-i-i)}$$

$$= \frac{-1+2i}{(\cancel{i^2}+\cancel{1}-2i)(-2i)} = \frac{2i-1}{4i^2} = \frac{2i-1}{-4}.$$

<u>Res of $f(z)$ at $z=-1$</u>

$z=-1$ is a pole of order 2.

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \underset{z\to a}{Lt} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$\underset{z=-1}{\text{Res}}\, f(z) = \frac{1}{(2-1)!} \underset{z\to -1}{Lt} \frac{d}{dz} \cancel{(z+1)^2} \frac{z^2-2z}{\cancel{(z+1)^2}(z^2+1)}$$

$$= \underset{z\to -1}{Lt} \frac{d}{dz}\left[\frac{z^2-2z}{z^2+1}\right] \quad \left[\because \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}\right]$$

$$= \underset{z\to -1}{Lt} \frac{(z^2+1)(2z-2)-(z^2-2z)(2z)}{(z^2+1)^2}$$

$$= \frac{(2)(-4)-(3)(-2)}{4} = \frac{-2}{4} = \frac{-1}{2}$$

(8) Find the poles and residues at each pole of $f(z) = \dfrac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}$

<u>Ans:</u> Let $f(z) = \dfrac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^2}$

$\left(z-\frac{\pi}{6}\right)^2=0 \Rightarrow z=\frac{\pi}{6}$ is a pole of order 2.

$$\cancel{\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{} (z-a)^m}$$

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$\underset{z=\frac{\pi}{6}}{\text{Res}}\, f(z) = \frac{1}{(2-1)!} \lim_{z\to\frac{\pi}{6}} \frac{d}{dz}\cancel{\left(z-\frac{\pi}{6}\right)^2}\cdot\frac{\sin^2 z}{\cancel{\left(z-\frac{\pi}{6}\right)^2}}$$

$$= \lim_{z\to\frac{\pi}{6}} 2\sin z\cos z$$

$$= 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$= 2\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

6) Find the poles and residue at each pole of $f(z)=\frac{1}{(z^2+4)^2}$

Ans:- Let $f(z)=\frac{1}{(z^2+4)^2}$

The poles of $f(z)$ are $z^2+4=0$

$z^2=-4 \Rightarrow z=\pm 2i$

$z=2i,-2i$ are poles of order 2.

Res of $f(z)$ at $z=2i$

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}} (z-1)^m f(z)\right]$$

$$\underset{z=2i}{\text{Res } f(z)} = \frac{1}{(2-1)!} \underset{z\to 2i}{Lt} \frac{d}{dz}\left[\cancel{(z-2i)^2} \frac{1}{(z+2i)^2\cancel{(z-2i)^2}}\right]$$

$$= \underset{z\to 2i}{Lt} \frac{d}{dz}\left[\frac{1}{(z+2i)^2}\right]$$

$$= \underset{z\to 2i}{Lt} \frac{-2}{(z+2i)^3} = \frac{-2}{(2i+2i)^3} = \frac{-2}{(4i)^3}$$

$$= \frac{+2}{+64i} = \frac{1}{32i}$$

Res of $f(z)$ at $z = -2i$

$$\underset{z=-2i}{\text{Res } f(z)} = \frac{1}{(2-1)!} \underset{z\to -2i}{Lt} \frac{d}{dz} \cancel{(z+2i)^2} \frac{1}{\cancel{(z+2i)^2}(z-2i)^2}$$

$$= \underset{z\to -2i}{Lt} \frac{-2}{(z-2i)^3}$$

$$= \frac{-2}{(-2i-2i)^3}$$

$$= \frac{-2}{(-4i)^3} = \frac{-2}{64i} = \frac{-1}{32i}$$

Practices problems

1) Find the poles of $f(z)$ and residue of these poles of

$$f(z) = \frac{z^2+2z}{(z-1)^2(z^2+4)}.$$

2) Determine the poles and residues of

$$f(z) = \frac{z^2}{(z+2)\, z\, (z-1)^2}$$

3) Find the poles of $f(z) = \dfrac{z+1}{z^2(z-2)}$

and residues at these poles.

4) Determine the poles of the function $f(z) = \dfrac{e^z}{z^2+\pi^2}$ and residues

at these poles.

5) Find the residue at $z=0$ of

The function $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}$.

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Singular points

* Isolated singular point
* Removable singular points - (-ve powers = 0)
* Essential singular points - (No. of -ve powers = Indefine)
* Pole of $f(z)$ - negative power = finite)

Singular point: A point z_0 is called a singular point of the complex function $f(z)$. If $f(z)$ is not analytic at z_0.

Isolated singular point: A singular point $z=a$ is said to be isolated singular point if $f(z)$ is analytic every where on the ~~se~~ deleted neighbourhood of "a".

i.e; There exists a neighbourhood of point $z=a$ which contains no other singular point.

Eg:- i, $f(z) = \frac{1}{z}$ is analytic every where except $z=0$. "0" is isolated singular pnt

ii) $f(z) = \frac{z^2+1}{z^3(z+1)}$ is analytic every where except. $z=0$ and $z=-1$.

Therefore $0, -1$ are isolated singular points.

iii) $f(z) = \frac{1}{\sin(\frac{\pi}{z})}$ "0" is a singular point. $\sin\frac{\pi}{z} = 0 = \sin n\pi \Rightarrow \frac{\pi}{z} = n\pi$

is analytic every where except $z = 1, 1/2, 1/3, 1/4, 1/5, \ldots$ and $\Rightarrow z = \frac{1}{n}$

$z = -1, -1/2, -1/3, -1/4, -1/5, -1/6 \ldots$ These points are also isolated singular points.

Pole of an analytic function.

Let $f(z)$ be an analytic function with in a domain D. and $z=a$. be an isolated singular point. Then the Laurent series of expansion of $f(z)$ is. $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$

The second part of R.H.S in above eq'n

i.e; $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is called principle part of Laurent series of $f(z)$.

If the principle part contains a finite no. of terms "m" then the singular point $z=a$ is called a pole of order "m" of $f(z)$.

Example.

i. $f(z) = \frac{\sin(z-a)}{(z-a)^4} := \frac{1}{(z-a)^4}\left[(z-a) - \frac{(z-a)^3}{3!} + \frac{(z-a)^5}{5!} \ldots\ldots\right]$.

$= \frac{1}{(z-a)^3} - \frac{1}{(z-a)3!} + \frac{z-a}{5!} - \frac{(z-a)^3}{7!} + \ldots$

No. of -ve terms $= 2$.

It is a pole of order "2".

ii. $f(z) = \frac{z}{(z+1)^2(z-2)}$ then $z=-1$. is a pole of order "2". $z=2$ is a pole of order "1".

Essential singular points.

If the principle part of $f(z)$ contains an infinite no. of terms. then the point $z=a$. is called essential singular point of $f(z)$.

(or)

. If there is no finite value of "n". such that.

$\lim_{z \to a} (z-a)^n f(z) = c$, $c \neq 0$ [c is constant] then $z=a$ is called essential singular point.

Example.

1. $f(z) = e^{1/z} = 1/z + 1/2! \, z^2 + 1/3! \, z^3 + \ldots$

then $z=0$ is an essential singular point.

(ii) $f(z) = \sin(1/z-a) = \left[\frac{1}{z-a} - \frac{(1/z-a)^3}{3!} + \frac{(1/z-a)^5}{5!} - \frac{(1/z-a)^7}{7!} + \cdots\right]$.

$z=a$ is an essential singular point.

Removable singular point.

If the principle part of $f(z)$ contains no negative term i.e. $b_n = 0 \; \forall n$, then the point $z=a$ is called removable singular point.

(or).

A singular point $z=a$ is called Removable singular point if $\lim_{z \to a} f(z)$ exists.

Example.

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$\therefore \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

No. of negative terms $= 0$.

$z=0$ is removable singular point.

Zero analytic function.

A point "a" is called zero of analytic function $f(z)$ if $f(a)=0$.

Zero of m^{th} order.

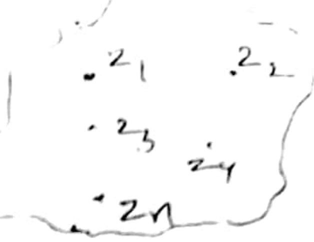
If the analytic function $f(z)$ can be expressed in the form $f(z) = (z-a)^m \phi(z)$.

Where, $\phi(z)$ is analytic function and $\phi(a) \neq 0$. then $z=a$ is called a zero of m^{th} order $f(z)$.

Example

* $f(z) = (z-1)^3$. $[f(z)=0$ at $z=1]$ then $z=1$ is zero of order 3.
* $f(z) = \sin z$, then $z = 0, \pm\pi, \pm 2\pi, \ldots$ are zero of order 1.
* $f(z) = z^2+9$, then $z = +3i, -3i$ are zero of order "1".
* $f(z) = \frac{1}{z-1}$ then $z=\infty$ is a zero of order "1".

Cauchy's Residue Theorem



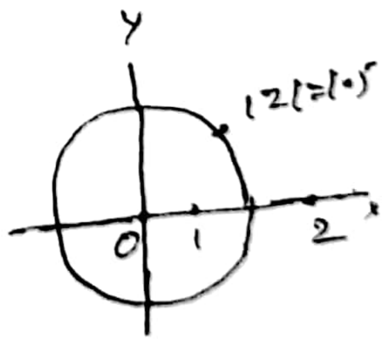
If $f(z)$ is analytic within and on a closed curve c, except at finite no. of poles $z_1, z_2, \dots z_n$ within c and $R_1, R_2, \dots R_n$ be the residues of $f(z)$ at these poles,

then $\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2 \dots + R_n]$

$= 2\pi i\,[\text{sum of residues}]$.

problems

(1) Evaluate $\oint_c \frac{4-3z}{z(z-1)(z-2)}\,dz$, where c is the circle $|z| = \frac{3}{2}$ using Residue theorem.



Ans. Let $f(z) = \frac{4-3z}{z(z-1)(z-2)}$

For the poles, $z(z-1)(z-2)=0$

$\Rightarrow z=0,\ z=1,\ z=2.$

The poles $z=0$, $z=1$ are simple poles and lies inside the circle $|z|=1.5$.

The pole $z=2$ is outside the circle $|z|=1.5$

It is enough to calculate the residues at $z=0$ and $z=1$.

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a} (z-a) f(z).$$

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \lim_{z\to 0} \cancel{(z-0)} \cdot \frac{4-3z}{\cancel{z}(z-1)(z-2)}$$

$$= \frac{4-(0)}{(-1)(-2)} = 2.$$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \lim_{z\to 1} \cancel{(z-1)} \frac{4-3z}{z\cancel{(z-1)}(z-2)}$$

$$= \frac{4-3(1)}{(1)(1-2)} = -1.$$

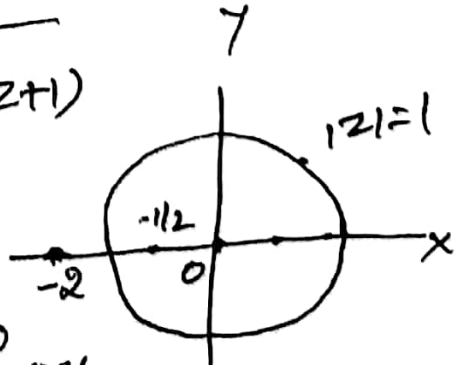
$$\therefore \oint_c \frac{4-3z}{z(z-1)(z-2)}\, dz = 2\pi i \times \text{sum of residues}$$

$$= 2\pi i\,[2+(-1)]$$

$$= 2\pi i$$

(2) Evaluate $\int_c \frac{2z-1}{z(z+2)(2z+1)}\, dz$, where c is the circle $|z|=1$.

Ans:- Let $f(z) = \frac{2z-1}{z(z+2)(2z+1)}$



For the poles,

$z(z+2)(2z+1)=0$

$\Rightarrow z=0,\ z+2=0,\ 2z+1=0$

$\therefore z=0$ is a simple pole & lies inside $|z|=1$.

$z=\frac{-1}{2}$ is a simple pole & lies inside $|z|=1$.

$z=-2$ is a simple pole & it is outside $|z|=1$.

It is enough to calculate the residues at $z=0$, $z=\frac{-1}{2}$.

$$\boxed{\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a} (z-a)\, f(z)}$$

$$\left[\operatorname{Res} f(z)\right]_{z=0} = \lim_{z\to 0} (z-0)\, f(z)$$

$$= \lim_{z\to 0} \cancel{z} \cdot \frac{2z-1}{\cancel{z}(z+2)(2z+1)}$$

$$= \frac{0-1}{(2)(1)} = \frac{-1}{2}.$$

$$\left[\operatorname{Res} f(z)\right]_{z=\frac{-1}{2}} = \lim_{z\to \frac{-1}{2}} \left(z+\frac{1}{2}\right) f(z)$$

$$= \lim_{z\to \frac{-1}{2}} \frac{\cancel{2z+1}}{2} \cdot \frac{2z-1}{z(z+2)(\cancel{2z+1})}$$

$$= \frac{1}{\cancel{2}} \cdot \frac{2\left(\frac{-1}{2}\right)-1}{\frac{-1}{\cancel{2}}\left(\frac{3}{2}\right)}$$

$$= \frac{-2}{\frac{-3}{2}} = \frac{+4}{3}.$$

$$\therefore \int_c \frac{2z-1}{z(z+2)(2z+1)}\, dz = 2\pi i \times \text{sum of residues}$$

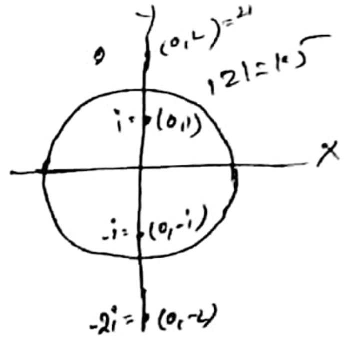
$$= 2\pi i \left[\frac{-1}{2} + \frac{4}{3}\right]$$

$$= 2\pi i \left[\frac{-3+8}{6}\right]$$

$$= 2\pi i \times \frac{5}{6} = \frac{5\pi i}{3}$$

③ Evaluate $\int_C \frac{dz}{(z^2+1)(z^2+4)}$, where c is the circle $|z|=1.5$.

A:- Let $f(z) = \frac{1}{(z^2+1)(z^2+4)}$



For the poles, $(z^2+1)(z^2+4)=0$

$z^2=-1, \; z^2=-4$

$z=\pm i, \; z=\pm 2i$

The poles $z=i,-i$ are simple poles and these are inside the circle $|z|=1.5$.

Res of $f(z)$ at $z=i$

$z=i$ is a simple pole.

$$\left[\text{Res } f(z)\right]_{z=a} = \lim_{z\to a} (z-a) f(z)$$

$$\left[\text{Res } f(z)\right]_{z=i} = \lim_{z\to i} (z-i) \cdot \frac{1}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i} \cancel{(z-i)} \cdot \frac{1}{(z+i)\cancel{(z-i)}(z^2+4)}$$

$$= \frac{1}{(2i)(-1+4)} = \frac{1}{6i}$$

Res of $f(z)$ at $z=-i$

$z=-i$ is a simple pole.

$$\left[\text{Res } f(z)\right]_{z=-i} = \lim_{z\to -i} (z+i) f(z)$$

$$= \lim_{z\to -i} \cancel{(z+i)} \cdot \frac{1}{(z-i)\cancel{(z+i)}(z^2+4)}$$

$$= \frac{1}{(-2i)(3)} = \frac{-1}{6i} \qquad ((-i)^2 = i^2 = -1)$$

$$\therefore \int_C \frac{dz}{(z^2+1)(z^2+4)} = 2\pi i\,[\text{sum of Residues}]$$

$$= 2\pi i\left[\frac{1}{6i} - \frac{1}{6i}\right] = \underline{0}$$

(4) Evaluate the integral $\oint_C \frac{2e^z}{(z+1)^3}\,dz$, where

C is the circle $|z-1|=3$.

Ans:- Let $f(z)=\frac{2e^z}{(z+1)^3}$

For the poles, $(z+1)^3=0$

$z+1=0$

$\Rightarrow z=-1$ is a pole of order 3.

$z=-1$ lies inside the circle $|z-1|=3$. $\left(\because |-1-1| = |-2| = 2<3\right)$

$$\left[\operatorname{Res} f(z)\right]_{z=a}=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=-1}=\frac{1}{2!}\lim_{z\to -1}\frac{d^2}{dz^2}\left[\cancel{(z+1)^3}\frac{2e^z}{\cancel{(z+1)^3}}\right]$$

$$=\frac{1}{2}\lim_{z\to -1}\frac{d}{dz}(2e^z)$$

$$=\frac{1}{2}\lim_{z\to -1}2e^z$$

$$=\frac{1}{2}\cdot 2e^{-1}=\frac{1}{e}.$$

By Residue Theorem,

$$\oint_C \frac{2e^z}{(z+1)^3}\,dz=2\pi i\,[\text{sum of residues}]$$

$$=2\pi i\left(\frac{1}{e}\right)$$

(5) Evaluate $\int_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}\,dz$, where

c is the circle $|z-1|=1$.

A:- Let $f(z) = \dfrac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}$

For the pole, $\left(z-\frac{\pi}{2}\right)^3 = 0 \Rightarrow z = \frac{\pi}{2}$ is a pole of order 3.

$z = \frac{\pi}{2}$ lies inside the circle $|z-1| = 1$.

$\left(\because \left|\frac{\pi}{2}-1\right| = \left|\frac{3.14}{2}-1\right| = |1.57-1| = 0.57 < 1\right)$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=a} = \frac{1}{(m-1)!} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z)\right]$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=\frac{\pi}{2}} = \frac{1}{2!} \lim_{z\to\frac{\pi}{2}} \left[\frac{d^2}{dz^2} \cancel{\left(z-\frac{\pi}{2}\right)^3} \cdot \frac{z\cos z}{\cancel{\left(z-\frac{\pi}{2}\right)^3}}\right]$$

$(uv)' = uv' + vu'$

$$= \frac{1}{2} \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[z(-\sin z) + \cos z(1)\right]$$

$$= \frac{1}{2} \lim_{z\to\frac{\pi}{2}} \frac{d}{dz}\left[(-\underset{u}{z}\ \underset{v}{\sin z}) + \cos z\right]$$

$$= \frac{1}{2} \lim_{z\to\frac{\pi}{2}} \left[-z\cos z + \sin z(-1) + -\sin z\right]$$

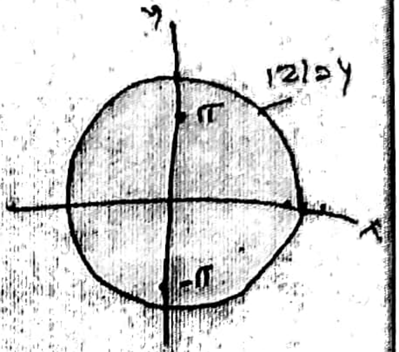
$$= \frac{1}{2}\left[\frac{-\pi}{2}(0) - 1 - 1\right] = -1$$

By Residue theorem,

$$\int_C \frac{z\cos z}{\left(z-\frac{\pi}{2}\right)^3}\, dz = 2\pi i\, [\text{Sum of residues}]$$

$$= 2\pi i\,[-1] = -2\pi i$$

(8) Evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2}dz$ where c is $|z|=4$.



Ans:- Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

The poles,

$z^2+\pi^2=0 \Rightarrow z^2 = -\pi^2$

$\Rightarrow z = \pm\sqrt{-\pi^2} = \pm i\pi$ are poles & order

$z = i\pi, -i\pi$ are inside the circle $|z|=4$.

$\left(\because \begin{array}{l} z=i\pi \Rightarrow |i\pi| = \pi = 3.1 < 4 \\ z=-i\pi \Rightarrow |-i\pi| = \pi = 3.1 < 4 \end{array} \right)$.

<u>Res $f(z)$ at $z=i\pi$</u>

The pole $z=i\pi$ is of order 2.

$$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

$$\therefore [\text{Res } f(z)]_{z=i\pi} = \frac{1}{(2-1)!} \lim_{z\to i\pi} \frac{d}{dz}\left[(z-i\pi)^2 \cdot \frac{e^z}{z^2+\pi^2}\right]$$

$$= \frac{1}{1!} \lim_{z\to i\pi} \frac{d}{dz}\left[\cancel{(z-i\pi)^2} \cdot \frac{e^z}{\cancel{(z-i\pi)^2}(z+i\pi)^2}\right]$$

$\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$ $\boxed{= \frac{e^{i\pi}}{(i\pi+i\pi)^2} = \frac{\cos\pi + i\sin\pi}{(2i\pi)^2}}$ $= \cancel{\frac{-1}{-4\pi^2}}$

$$= \lim_{z\to i\pi} \frac{(z+i\pi)^2(e^z) - e^z \cdot 2(z+i\pi)}{(z+i\pi)^4}$$

$$= \lim_{z\to i\pi} \frac{(z+i\pi)e^z - 2e^z}{(z+i\pi)^3} \quad \left(\because e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1\right)$$

$$= \frac{(2i\pi)e^{i\pi} - 2e^{i\pi}}{(2i\pi)^3} = \frac{2i\pi(-1) - 2(-1)}{-8i\pi^3} = \frac{i\pi+1}{4i\pi^3}$$

Res $f(z)$ at $z=-i\pi$

$z=-i\pi$ sub in Res $f(z)$ at $z=i\pi$.

$$\left[\text{Res } f(z)\right]_{z=i\pi} = \frac{i\pi+1}{4i\pi^3}$$

$$\left[\text{Res } f(z)\right]_{z=-i\pi} = \frac{-i\pi+1}{-4i\pi^3}$$

By Residue Theorem,

$$\int_C \frac{e^z}{(z^2+\pi^2)^2}dz = 2\pi i\,[\text{sum of Residues}]$$

$$= 2\pi i\left[\frac{i\pi+1}{4i\pi^3} + \frac{-i\pi+1}{-4i\pi^3}\right]$$

$$= \frac{2\pi \cancel{i}}{4\cancel{i}\pi^3}\left[i\pi\cancel{+1}+i\pi\cancel{-1}\right]$$

$$= \frac{1}{2\pi^2}\left[2i\pi\right] = \frac{i}{\pi}$$

$$\therefore \int_C \frac{e^z}{(z^2+\pi^2)^2}dz = \frac{i}{\pi}$$

(7) Evaluate $\oint_C \frac{z-3}{z^2+2z+5}dz$, where c is the circle given by

(i) $|z|=1$ (ii) $|z+1-i|=2$ (iii) $|z+1+i|=2$.

Sol: Let $f(z)=\frac{z-3}{z^2+2z+5}$

For the poles ; $z^2+2z+5=0$

$$\Rightarrow z=\frac{-2\pm\sqrt{4-20}}{2}$$

$$=-1\pm 2i$$

The poles $z=-1+2i, -1-2i$ are simple poles.

(i) on the circle $|z|=1$:-

The poles $z=-1\pm 2i$ lies outside the circle $|z|=1$. ($\because$ $|-1+2i| = \sqrt{1+4}=\sqrt{5}>1$
$|-1-2i| = \sqrt{1+4}=\sqrt{5}>1$).

$\therefore$ $f(z)$ is analytic within and on the circle $|z|=1$.

$\therefore$ By cauchy's Theorem $\int_C f(z)\,dz = 0$.

ii) on the circle $|z+1-i|=2$

the pole $z=-1+2i$ lies inside the circle $|z+1-i|=2$.

$(\because |z+1-i| = |-1+2i-i+1| = |-1+i+1| = \sqrt{1+1} = \sqrt{2} < 2)$

The pole $z=-1-2i$ is outside the circle $|z+1-i|=2$

$(\because |z+1-i| = |-1-2i-i| = |-1-3i| = \sqrt{1+9} = \sqrt{10} > 2)$.

$$[\operatorname{Res} f(z)]_{z=a} = \lim_{z\to a} (z-a)\, f(z)$$

$$[\operatorname{Res} f(z)]_{z=-1+2i} = \lim_{z\to -1+2i} (z+1-2i)\cdot \frac{z-3}{z^2+2z+5}$$

$$= \lim_{z\to -1+2i} \cancel{(z+1-2i)}\cdot \frac{z-3}{\cancel{(z+1-2i)}(z+1+2i)}$$

$$= \frac{-1+2i-3}{\cancel{-1}+2i+\cancel{1}+2i}$$

$$= \frac{-4+2i}{4i}$$

By Residue Theorem,

$$\int_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i\,[\text{sum of residues}].$$

$$= \cancel{2}\pi \cancel{i}\left(\frac{-4+2i}{\cancel{4}_2 \cancel{i}}\right)$$

$$= \pi\left(\frac{\cancel{2}(-2+i)}{\cancel{2}}\right)$$

$$= \pi(i-2).$$

(iii) This is left as an exercise to the student.

Problems

1) Evaluate $\int_c \frac{(12z-7)dz}{(2z+3)(z-1)^2}$, where c is $x^2+y^2=4$. $(|z|=2)$.

2) Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$, where c is $|z|=3$.

3) Evaluate $\int_c \frac{2e^z}{z(z-3)} dz$, where c is $|z|=2$ by Residue Theorem.

4) Evaluate $\int_c \frac{\cos \pi z^2}{(z-1)(z-2)} dz$, where c is $|z|=\frac{3}{2}$.

5) Evaluate $\int_c \frac{z}{(z-1)(z-2)^2} dz$, where c is the circle $|z-2|=\frac{1}{2}$ by using Residue Theorem.

6) Evaluate $\int_c \frac{ze^z}{z^2+9} dz$, where c is $|z|=5$ by residue theorem.

7) Evaluate $\int_c \frac{z^2+2z-2}{z(z-4)(z-1)} dz$, where c: $|z|=1.5$.

8) Find the residue of $\frac{z^2}{z^4+1}$ at those poles which lie inside the circle $|z|=2$.

(or)

Evaluate $\int_c \frac{z^2}{z^4+1} dz$, where c is $|z|=2$.

Integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$.

we K.T $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

Let $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2zi}$$

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_C F\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2zi}\right)\frac{dz}{iz},$$

where C is the unit circle $|z|=1$.

$z = e^{i\theta}$
$dz = ie^{i\theta} d\theta$
$dz = iz\, d\theta$
$\frac{dz}{iz} = d\theta$.

problems

1. Show That $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$; $a>b>0$

(or) S.T $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$.

A:- Let $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$

$$dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}.$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)}$$

$$= \int_0^{2\pi} \frac{dz/iz}{\frac{2az+b(z^2+1)}{2z}}$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{dz}{bz^2+2az+b}, \quad \text{Let } f(z) = \frac{1}{bz^2+2az+b}$$

For the poles, $bz^2+2az+b=0$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$, $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$.

$\because a > b > 0$, Then $|\beta| > 1$. It is

outside the circle $|z| = 1$.

$|\alpha\beta| = 1 \Rightarrow |\alpha| < 1 \quad (\because |\beta| > 1)$

$\therefore z = \alpha$ is the only simple pole lies

inside the circle $|z| = 1$.

$$\underset{z=\alpha}{\text{Res}}\, f(z) = \lim_{z\to\alpha} (z-\alpha) f(z)$$

$$= \lim_{z\to\alpha} \cancel{(z-\alpha)} \cdot \frac{1}{b\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{1}{b} \cdot \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\cancel{b}} \cdot \frac{1}{\frac{2\sqrt{a^2-b^2}}{\cancel{b}}} = \frac{1}{2\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\int_C f(z)\,dz$$

$$= \frac{2}{i}\left[2\pi i \times \text{Sum of Residues}\right]$$

$$= \frac{\cancel{2}}{\cancel{i}}\left[2\pi \cancel{i} \times \frac{1}{\cancel{2}\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

$$\therefore \cancel{2}\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\cancel{2}\pi}{\sqrt{a^2-b^2}} \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right.$$

$$\therefore \int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Imp

2) S.T $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$.

A:- Let $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$, $dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_C \frac{1}{2+\frac{z^2+1}{2z}}\cdot\frac{dz}{iz} \quad \text{, where } C \text{ is } |z|=1.$$

$$= \int_C \frac{1}{\frac{4z+z^2+1}{2\cancel{z}}}\cdot\frac{dz}{i\cancel{z}}$$

$$= \frac{2}{i} \int_C \frac{1}{z^2+4z+1} dz$$

Let $f(z) = \dfrac{1}{z^2+4z+1}$

For the poles, $z^2+4z+1=0$

$$z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$= \frac{-4 \pm \sqrt{12}}{2}$$

$$= -2 \pm \sqrt{3}$$

Let $|z| = |-2+\sqrt{3}| = |-2+1.732| = |-0.26| = 0.26 < 1$

$|z| = |-2-\sqrt{3}| = |-2-1.732| = |-3.732| = 3.7 > 1$

$\therefore z = -2+\sqrt{3}$ lies inside the circle $|z|<1$

and $z = -2-\sqrt{3}$ lies outside the circle $|z|<1$.

$$[\text{Res } f(z)]_{z=a} = \underset{z \to a}{Lt} (z-a)\, f(z)$$

$$[\text{Res } f(z)]_{z=-2+\sqrt{3}} = \underset{z \to -2+\sqrt{3}}{Lt} (z+2-\sqrt{3}) \cdot \frac{1}{z^2+4z+1}$$

$$= \underset{z \to -2+\sqrt{3}}{Lt} \cancel{(z+2-\sqrt{3})} \cdot \frac{1}{\cancel{(z+2-\sqrt{3})}(z+2+\sqrt{3})}$$

$$= \frac{1}{\cancel{-2}+\sqrt{3}+\cancel{2}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \int_C f(z)\, dz$$

$$= \frac{2}{i} \cdot (2\pi i \times [\text{Sum of residues}])$$

$$= \frac{2}{\cancel{i}} \left[2\pi \cancel{i} \times \frac{1}{2\sqrt{3}}\right] = \frac{2\pi}{\sqrt{3}}$$

(3) S.T $\int_0^{\pi} \frac{d\theta}{3+2\sin\theta} = \frac{\pi}{\sqrt{5}}$.

Ans: let $z = e^{i\theta}$, $\frac{1}{z} = e^{-i\theta}$

$dz = ie^{i\theta}d\theta = iz\,d\theta \Rightarrow \frac{dz}{iz} = d\theta$.

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$

$\therefore \int_0^{\pi} \frac{d\theta}{3+2\sin\theta} = \int_c \frac{1}{3 + \cancel{2}\cdot\frac{z^2-1}{\cancel{2}iz}} \cdot \frac{dz}{iz}$, where c is $|z|=1$.

$= \int_c \frac{\cancel{iz}}{3iz+z^2-1} \cdot \frac{dz}{\cancel{iz}}$

$= \int_c \frac{dz}{z^2+3iz-1}$, let $f(z) = \frac{1}{z^2+3iz-1}$

For the poles $z^2+3iz-1=0$

$z = \frac{-3i \pm \sqrt{-9+4}}{2}$

$= \frac{-3i \pm \sqrt{-5}}{2}$

$= \frac{-3i+i\sqrt{5}}{2}, \frac{-3i-i\sqrt{5}}{2}$.

$= \alpha, \beta$

For $z = \frac{-3i+i\sqrt{5}}{2}$, $|z| = \left|\frac{i(-3+\sqrt{5})}{2}\right| = |i(-0.38)| < 1$

$z = \frac{-3i-i\sqrt{5}}{2}$, $|z| = \left|-i\frac{(3+\sqrt{5})}{2}\right| = |-i(2.61)| > 1$

$[\text{Res } f(z)]_{z=a} = \underset{z\to a}{Lt}\,(z-a)f(z)$

$[\text{Res } f(z)]_{z=\frac{-3i+i\sqrt{5}}{2}} = \underset{z\to\frac{-3i+i\sqrt{5}}{2}}{Lt}\left(z+\frac{3i-i\sqrt{5}}{2}\right)\cdot\frac{1}{z^2+3iz-1}$

$$= \lim_{z \to \frac{-3i+i\sqrt{5}}{2}} \left(z + \frac{3i-i\sqrt{5}}{2}\right) \cdot \frac{1}{\left(z+\frac{3i-i\sqrt{5}}{2}\right)\left(z+\frac{3i+i\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\frac{-3i+i\sqrt{5}}{2} + \frac{3i+i\sqrt{5}}{2}} = \frac{1}{\frac{\cancel{2}i\sqrt{5}}{\cancel{2}}} = \frac{1}{i\sqrt{5}}$$

$$\therefore \int_0^{\pi} \frac{d\theta}{3+2\sin\theta} = \int_C f(z)\,dz$$

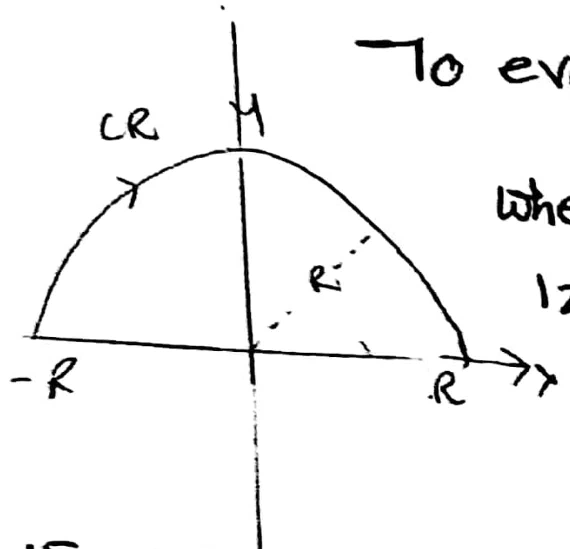
$= 2\pi i \times$ sum of residues

$$= 2\pi \cancel{i} \times \frac{1}{\cancel{i}\sqrt{5}}$$

$$= \frac{2\pi}{\sqrt{5}}$$

4) S.T $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, $a > b > 0$ using Residue Theorem.

Integration of the form $\int_{-\infty}^{\infty} f(x)\,dx$,



To evaluate $\int_{-\infty}^{\infty} f(x)\,dx$, we consider $\int_C f(z)\,dz$ where "c" is the contour of the semicircle C_R; $|z| = R$, together with the diameter $-R\,R$ that closes it. Suppose $f(z)$ has no singularity on the real axis we have that by residue theorem,

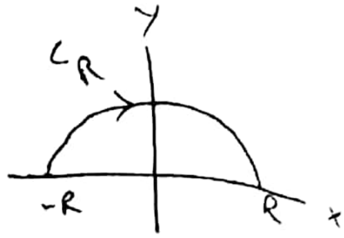
$$\int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{Res}\, f(a).$$

Finally $R \to \infty$, we find the value $\int_{-\infty}^{\infty} f(x)\,dx$, Provides

$$\int_{C_R} |f(z)|\,dz \to 0 \quad \text{as } R \to \infty$$

① Evaluate by contour integration $\int_0^{\infty} \frac{dx}{1+x^2}$.

Ans:- consider $\int_C \frac{dz}{z^2+1} = \int_C f(z)dz$,



where c is the semicircle C_R together with the part of real line $-R$ to R.

For the poles $z^2+1=0$

$\Rightarrow z = \pm i$

$\therefore z = i$ is inside the semicircle C_R.

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a}(z-a) f(z).$$

$$\left[\operatorname{Res} f(z)\right]_{z=i} = \lim_{z\to i}(z-i) \cdot \frac{1}{z^2+1}$$

$$= \lim_{z\to i} \cancel{(z-i)} \frac{1}{\cancel{(z-i)}(z+i)} = \frac{1}{2i}.$$

By Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of Residues}$$

$$= 2\pi i \times \frac{1}{2i} = \pi.$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \pi.$$

$R \to \infty$ then $\int_{C_R} f(z)\,dz = 0$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + 0 = \pi$$ ($\because$ Integrand function is even)

$$\therefore 2\int_0^{\infty} \frac{dx}{1+x^2} = \pi \qquad \therefore \int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$$

(5) Evaluate $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$ by using Residue Theorem.

solu: To evaluate the given integral, we consider $\int_C \frac{z^2\,dz}{(z^2+1)(z^2+4)} = \int_C f(z)\,dz$, where

C is the semicircle C_R of radius R together with the part of the real axis from -R to R.

For the poles, $(z^2+1)(z^2+4)=0$

$$z^2=-1,\ z^2=-4$$

$$z=\pm i,\ z\pm 2i$$

The poles $z=i, 2i$ are only above the upper half of the plane.

$$[\text{Res } f(z)]_{z=a} = \lim_{z\to a}(z-a)\,f(z)$$

$$[\text{Res } f(z)]_{z=i} = \lim_{z\to i}(z-i)\cdot\frac{z^2}{(z^2+1)(z^2+4)}$$

$$= \lim_{z\to i}\cancel{(z-i)}\,\frac{z^2}{\cancel{(z-i)}(z+i)(z^2+4)}$$

$$= \frac{-1}{(2i)(3)} = \frac{-1}{6i} \quad (i^2=-1)$$

$$[\text{Res } f(z)]_{z=2i} = \lim_{z\to 2i}(z-2i)\,f(z).$$

$$= \lim_{z\to 2i}\cancel{(z-2i)}\cdot\frac{z^2}{(z^2+1)(z+2i)\cancel{(z-2i)}}$$

$$= \frac{+4}{(-3)(4i)} = \frac{1}{3i}$$

By Residue Theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{sum of Residue}$$
$$= 2\pi i\left[\frac{-1}{6i}+\frac{1}{3i}\right] = 2\pi i \times \frac{1}{6i} = \frac{\pi}{3}.$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = \frac{\pi}{3}$$

$R \to \infty$ then $\int_{C_R} f(z)\,dz = 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{3}.$$

(4) Prove that $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$.

where $a>0$, $b>0$, $a \neq b$.

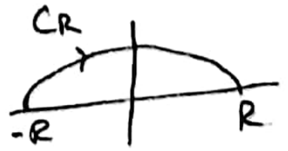
(5) using the method of contour integration prove that $\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$.

A:- Since the integral function is even,

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{1}{2}\int_0^{\infty} \frac{dx}{x^6+1}.$$

consider $\int_C \frac{dz}{z^6+1} = \int_C f(z)$, where c is the semicircle C_R ~~above~~ together with the real axis from $-R$ to R.

Let $f(z) = \frac{1}{z^6+1}$



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For the poles, $z^6+1=0$

$$z^6 = -1 = \cos\pi + i\sin\pi$$

$$\Rightarrow \quad z = (\cos\pi + i\sin\pi)^{1/6}$$

$$= \left[\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\right]^{1/6}$$

$$= \cos\left(\frac{2n\pi+\pi}{6}\right) + i\sin\left(\frac{2n\pi+\pi}{6}\right)$$

$$= e^{i\left(\frac{2n\pi+\pi}{6}\right)}, \quad n=0,1,2,3,4,5.$$

$$\therefore \quad z = e^{\frac{i\pi}{6}},\ e^{\frac{3\pi i}{6}},\ e^{\frac{5\pi i}{6}},\ e^{\frac{7\pi i}{6}},\ e^{\frac{9\pi i}{6}},\ e^{\frac{11\pi i}{6}}$$

Of these poles, $z = e^{\frac{i\pi}{6}}, e^{\frac{3\pi i}{6}}, e^{\frac{5\pi i}{6}}$. lies inside the semicircle above the real axis.

$$\left[\operatorname{Res} f(z)\right]_{z=a} = \lim_{z\to a} (z-a)\, f(z).$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=e^{\frac{i\pi}{6}}} = \lim_{z\to e^{\frac{i\pi}{6}}} \left(z - e^{\frac{i\pi}{6}}\right)\cdot \frac{1}{z^6+1}$$

$$= \lim_{z\to e^{\frac{i\pi}{6}}} \frac{1}{6z^5} \quad \text{(by L-Hospital rule)}$$

$$= \frac{1}{6\left(e^{\frac{i\pi}{6}}\right)^5} = \frac{1}{6\, e^{\frac{5\pi i}{6}}} = \frac{1}{6} e^{\frac{-5\pi i}{6}}$$

$$\therefore \left[\operatorname{Res} f(z)\right]_{z=e^{\frac{3i\pi}{6}}} = \lim_{z\to e^{\frac{3i\pi}{6}}} \left(z - e^{\frac{3i\pi}{6}}\right)\cdot \frac{1}{z^6+1}$$

$$= \lim_{z\to e^{\frac{3i\pi}{6}}} \frac{1}{6z^5}$$

$$= \frac{1}{6\, e^{\frac{15i\pi}{6}}} = \frac{1}{6} e^{\frac{-15i\pi}{6}}$$

$$\left[\operatorname{Res} f(z)\right]_{z=e^{\frac{5\pi i}{6}}} = \lim_{z\to e^{\frac{5\pi i}{6}}} \left(z-e^{\frac{5\pi i}{6}}\right)\cdot\frac{1}{z^6+1}$$

$$= \lim_{z\to e^{\frac{5\pi i}{6}}} \frac{1}{6z^5} = \frac{1}{6e^{\frac{25\pi i}{6}}} = \frac{1}{6}e^{\frac{-25\pi i}{6}}$$

By Residue Theorem,

$$\int_C f(z)\,dz = 2\pi i \times \text{Sum of Residues}$$

$$= 2\pi i\cdot\frac{1}{6}\left[e^{\frac{-5\pi i}{6}}+e^{\frac{-15\pi i}{6}}+e^{\frac{-25\pi i}{6}}\right]$$

$$= \frac{\pi i}{3}\left[(\cos 150^\circ - i\sin 150^\circ) + \cos(450^\circ - i\sin 450^\circ) + \cos 750^\circ - i\sin 750^\circ)\right]$$

$$= \frac{\pi i}{3}\left[\left(\cancel{\frac{-\sqrt{3}}{2}}-i\frac{1}{2}\right)+(0-i(1))+\left(\cancel{\frac{\sqrt{3}}{2}}-i\frac{1}{2}\right)\right]$$

$$= \frac{\pi i}{3}(-2i) = \frac{2\pi}{3}.$$

$$\therefore \int_C f(z)\,dz = \frac{2\pi}{3}$$

$$\therefore \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi/3$$

$R \to \infty$ then $\int_{C_R} f(z)\,dz = 0.$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3}$$

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

$$2\int_0^{\infty}\frac{dx}{x^6+1} = \frac{2\pi}{3} \Rightarrow \int_0^{\infty}\frac{dx}{x^6+1} = \frac{\pi}{3}$$