UNIT-I

Electrostatic Fields

- Coulomb's Law
- Electric Field Intensity –Fields due to line and surface Charge Distributions
- Work done in moving a point charge in an electrostatic field
- Electrostatic Potential & Properties of potential function – Potential gradient
- Gauss's law – Application of Gauss's Law
- Maxwell's first law
- Divergence
- Laplace's and Poison's equations – Solution of Laplace's equation in one variable
- Electric dipole – Dipole moment – potential and EFI due to an electric dipole
- Torque on an Electric dipole in an electric field
- Behavior of conductors in an electric field – Conductors and Insulators

Introduction:

The electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges.

(Note: Almost all real electric fields vary to some extent with time. However, for many problems, the field variation is slow and the field may be considered as static. For some other cases spatial distribution is nearly same as for the static case even though the actual field may vary with time. Such cases are termed as quasi-static.)

In this chapter we first study two fundamental laws governing the electrostatic fields, viz, (1) Coulomb's Law and (2) Gauss's Law. Both these law have experimental basis. Coulomb's law is applicable in finding electric field due to any charge distribution, Gauss's law is easier to use when the distribution is symmetrical

.

Coulomb's Law :

Statement:

Coulomb's Law states that the force between two point charges Q1and Q2 is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

Mathematically,

$$F = \frac{kQ_1Q_2}{R^2} \qquad k = \frac{1}{4\pi\varepsilon_0}$$

Where k is the proportionality constant. And ε_0 , is called the permittivity of free space

In SI units, Q1 and Q2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newton's (N)

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0\varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ .....(1)

As shown in the Figure 1 let the position vectors of the point charges Q1and Q2 are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\overrightarrow{F_{12}}$ represent the force on Q1 due to charge Q2.

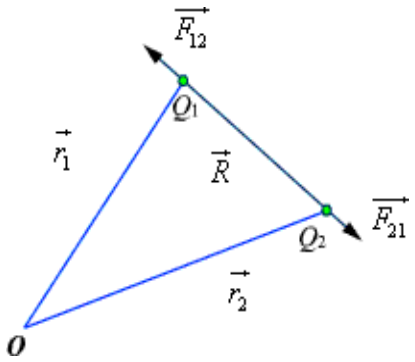


Fig 1: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$\hat{a}_{12} = \frac{\left(\vec{r_2} - \vec{r_1}\right)}{R}$ and $\hat{a}_{21} = \frac{\left(\vec{r_1} - \vec{r_2}\right)}{R}$ .....(2)

$\overrightarrow{F_{12}}$ can be defined as $\overrightarrow{F_{12}} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\hat{a}_{12} = \frac{Q_1Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r_2} - \vec{r_1})}{\left|\vec{r_2} - \vec{r_1}\right|^3}$.

Similarly the force on Q_1 due to charge Q_2 can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

Force Due to 'N 'no.of point charges:

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have *N* number of charges $Q_1, Q_2, \ldots\ldots Q_N$ located respectively at the points represented by the position vectors $\vec{r_1}, \vec{r_2}, \ldots\ldots\vec{r_N}$, the force experienced by a charge *Q* located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i(\vec{r}-\vec{r}_i)}{\left|\vec{r}-\vec{r}_i\right|^3} \quad \text{.....(3)}$$

Electric Field intensity:

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q\to 0}\frac{\vec{F}}{Q} \text{ or, } \vec{E} = \frac{\vec{F}}{Q} \text{ ..... (4)}$$

The electric field intensity E at a point r (observation point) due a point charge Q located at $\vec{r'}$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r}-\vec{r'})}{4\pi\varepsilon_0\left|\vec{r}-\vec{r'}\right|^3} \quad \text{.....(5)}$$

For a collection of N point charges $Q_1, Q_2, \ldots\ldots Q_N$ located at $\vec{r_1}, \vec{r_2}, \ldots\ldots \vec{r_N}$, the electric field intensity at point $\vec{r'}$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_k(\vec{r}-\vec{r_i})}{\left|\vec{r}-\vec{r_i}\right|^3} \quad \text{.....(6)}$$

The expression (6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2 we consider a continuous volume distribution of charge *(t)* in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r'})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r}-\vec{r'})}{4\pi\varepsilon_0\left|\vec{r}-\vec{r'}\right|^3} = \frac{\rho(\vec{r'})dv'(\vec{r}-\vec{r'})}{4\pi\varepsilon_0\left|\vec{r}-\vec{r'}\right|^3} \quad \text{.....(7)}$$

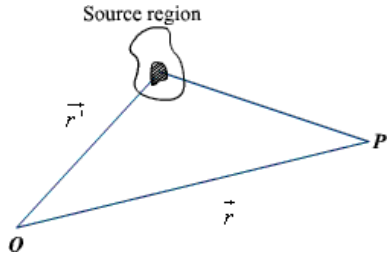


Fig 2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overrightarrow{E(r)} = \int_V \frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dv' \quad \text{.....(8)}$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overrightarrow{E(r)} = \int_L \frac{\rho_L(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} dl' \quad \text{.....(9)}$$

$$\overrightarrow{E(r)} = \int_S \frac{\rho_s(\vec{r}')(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^3} ds' \quad \text{.....(10)}$$

Electric flux density:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux density vector is defined as:

$$\vec{D} = \varepsilon\vec{E} \quad \text{.....(11)}$$

We define the electric flux as

$$\psi = \int_S \vec{D}.d\vec{s} \quad \text{.....(12)}$$

Gauss's Law:

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

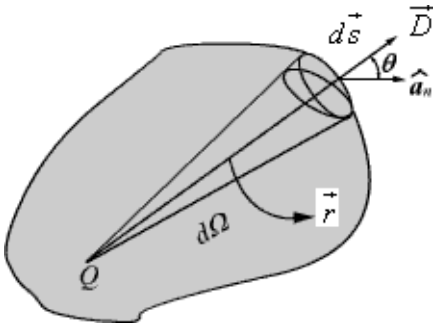


Fig 3: Gauss's Law

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant . The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \quad \text{.....(13)}$$

If we consider an elementary area ***ds***, the amount of flux passing through the elementary area is given by

$$d\psi = \vec{D}.d\vec{s} = \frac{Q}{4\pi r^2} ds\cos\theta \quad \text{.....(14)}$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of Q.

Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_S d\psi = \frac{Q}{4\pi}\oint_S d\Omega = Q$

Which can seen to be same as what we have stated in the definition of Gauss's Law.

Application of Gauss's Law:

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some

symmetry. We shall illustrate the application of Gauss's Law with some examples.

1. An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density $_L$C/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 4(a) (next slide). Since the line charge is assumed to be infinitely long, the electric field will be of the form as shown in Fig. 4(b) (next slide).

If we consider a close cylindrical surface as shown in Fig. 2.4(a), using Gauss's theorm we can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s} \quad \text{.....(15)}$$

Considering the fact that the unit normal vector to areas S_1 and S_3 are perpendicular to the electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hence we can write,

$$\rho_L l = \varepsilon_0 E.2\pi\rho l$$

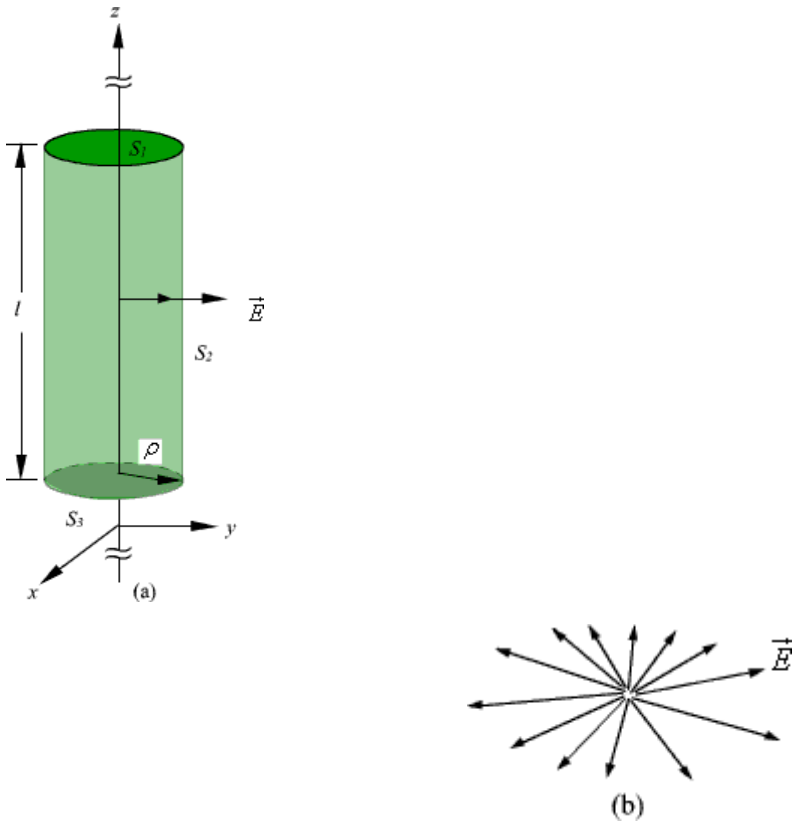


Fig 4: Infinite Line Charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho$$ .....(16)

2. Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinite charged sheet covering the *x*-*z* plane as shown in figure 5. Assuming a surface charge density of ρ_s for the infinite surface charge, if we consider a cylindrical volume having sides Δs placed symmetrically as shown in figure 5, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s \Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y$$ .....(17)

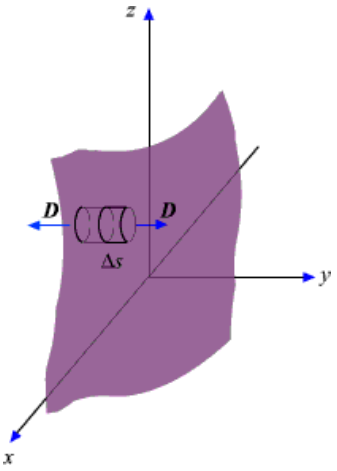


Fig 5: Infinite Sheet of Charge

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

3. Uniformly Charged Sphere

Let us consider a sphere of radius r0 having a uniform volume charge density of rv C/m3. To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius r < r0 and r > r0 as shown in Fig. 6 (a) and Fig. 6(b).

For the region $r \le r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \quad \text{.....(18)}$$

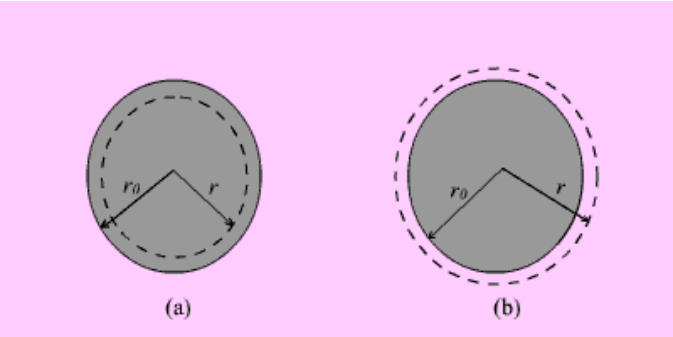


Fig 6: Uniformly Charged Sphere

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en} \quad \text{..... (19)}$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0 \quad \text{..... (20)}$$

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3 \quad \text{..... (21)}$$

By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0 \quad \text{..... (22)}$$

Electrostatic Potential:

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field. Let us suppose that we wish to move a positive test charge Δq from a point P to another point Q as shown in the Fig. 8.The force at any point along its path would cause the particle to accelerate and move it out of the region if unconstrained. Since we are dealing with an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while Δq moves from P to Q. The work done by this external agent in moving the charge by a distance $d\vec{l}$ is given by:

$$dW = -\Delta q \vec{E}.d\vec{l} \quad \text{..... (23)}$$

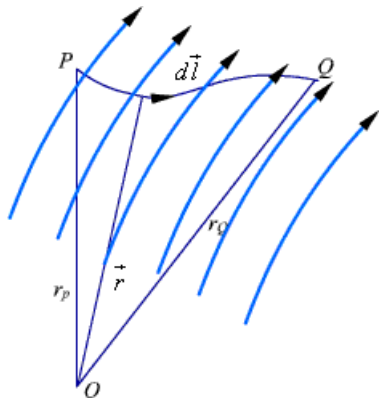


Fig 8: Movement of Test Charge in Electric Field

The negative sign accounts for the fact that work is done on the system by the external agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \quad \text{..... (24)}$$

The potential difference between two points P and Q , VPQ, is defined as the work done per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_P^Q \vec{E} \cdot d\vec{l} \quad \text{..... (25)}$$

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in Joules/Coulomb which is referred to as Volts.

Let us consider a point charge Q as shown in the Fig. 9.

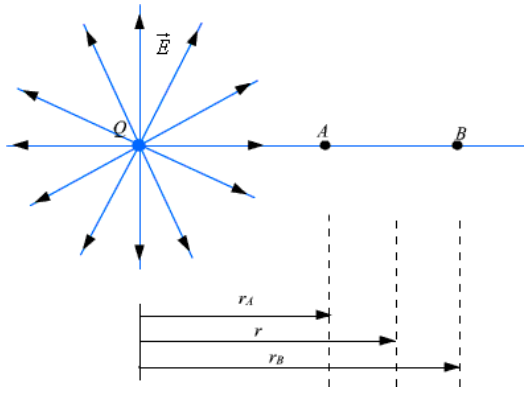


Fig 9: Electrostatic Potential calculation for a point charge

Further consider the two points A and B as shown in the Fig. 9. Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E} \cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} \hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B \quad \text{.....(26)}$$

It is customary to choose the potential to be zero at infinity. Thus potential at any point (rA = r) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. rB = 0).

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} \quad \text{..... (27)}$$

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl$$ .....(28)

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. 10.

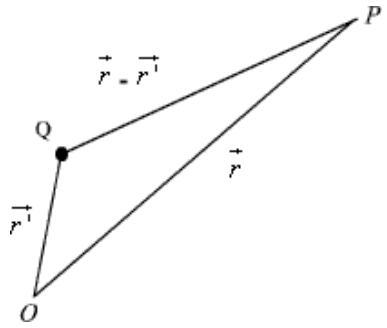


Fig 10: Electrostatic Potential due a Displaced Charge

The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r} - \vec{r'}\right|}$$ ..... (29)

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also. Let us first consider N point charges Q1, Q2 ,. QN located at points with position vectors $\vec{r_1}, \vec{r_2}$, ...$\vec{r_N}$....The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r} - \vec{r_1}\right|} + \frac{Q_2}{\left|\vec{r} - \vec{r_2}\right|} + \ldots\ldots\frac{Q_N}{\left|\vec{r} - \vec{r_N}\right|}\right)$$ ..... (30a)

OR

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r} - \vec{r_n}\right|}$$ .....(30b)

For continuous charge distribution, we replace point charges Qn by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{|\vec{r}-\vec{r}_n|}$$ .....(31)

For surface charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{|\vec{r}-\vec{r}_n|}$$ ..... (32)

For volume charge, $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{|\vec{r}-\vec{r}_n|}$$ .....(33)

It may be noted here that the primed coordinates represent the source coordinates and the unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential at infinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$ .....(34)

where C is a constant. In the same manner when potential is computed from a known electric field we can write:

$$V = -\int \vec{E}\cdot d\vec{l} + C$$ …….. (35)

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l} = \frac{W}{Q}$$ .....(36)

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point P1 to P2 in one path and then from point P2 back to P1 over a different path. If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position P1. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable; the work

must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, VAB = - VBA , and over a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0 \quad \text{.....(37)}$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_S \left(\nabla \times \vec{E} \right) \cdot d\vec{s} = 0 \quad \text{..... (38)}$$

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0 \quad \text{.....(39)}$$

Any vector field that satisfies is called an irrotational field.

From our definition of potential, we can write

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial x} dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x} \hat{a}_x + \frac{\partial V}{\partial y} \hat{a}_y + \frac{\partial V}{\partial z} \hat{a}_z \right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z \right) = -\vec{E} \cdot d\vec{l}$$

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l} \quad \text{.....(40)}$$

from which we obtain,

$$\vec{E} = -\nabla V \quad \text{..... (41)}$$

From the foregoing discussions we observe that the electric field strength at any point is the negative of the potential gradient at any point, negative sign shows that is directed from higher to lower values of $\vec{V}$. This gives us another method of computing the electric field , i. e. if we know the potential function, the electric field may be computed. We may note here that that one scalar function $\vec{V}$ contain all the information that three components of $\vec{E}$ carry, the same is possible because of the fact that three components of $\vec{E}$ are interrelated by the relation $\nabla \times \vec{E}$.

Equipotential Surfaces

An equipotential surface refers to a surface where the potential is constant. The intersection of an equipotential surface with an plane surface results into a path called an equipotential line. No work is done in moving a charge from one point to the other along an equipotential line or surface.

In figure 12, the dashes lines show the equipotential lines for a positive point charge. By symmetry, the equipotential surfaces are spherical surfaces and the equipotential lines are circles. The solid lines show the flux lines or electric lines of force.

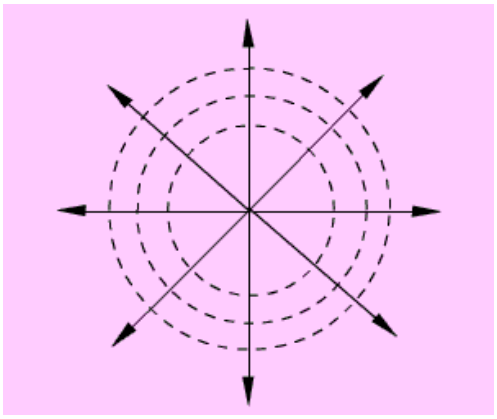


Fig 12: Equipotential Lines for a Positive Point Charge

Michael Faraday as a way of visualizing electric fields introduced flux lines. It may be seen that the electric flux lines and the equipotential lines are normal to each other. In order to plot the equipotential lines for an electric dipole, we observe that for a given Q and d, a constant V requires that $\frac{\cos\theta}{r^2}$ is a constant. From this we can write $r = c_v\sqrt{\cos\theta}$ to be the equation for an equipotential surface and a family of surfaces can be generated for various values of cv.When plotted in 2-D this would give equipotential lines.

To determine the equation for the electric field lines, we note that field lines represent the direction of $\vec{E}$ in space. Therefore,

$d\vec{l} = k\vec{E}$, k is a constant .....(42)

$\hat{a}_r dr + r d\theta \hat{a}_\theta + \hat{a}_\phi r\sin\theta = k(\hat{a}_r E_r + \hat{a}_\theta E_\theta + \hat{a}_\phi E_\phi) = d\vec{l}$ .....(43)

For the dipole under consideration $E_\phi = 0$, and therefore we can write,

$$\frac{dr}{E_r} = \frac{r d\theta}{E_\theta}$$

$$\frac{dr}{r} = \frac{2\cos\theta d\theta}{\sin\theta} = \frac{2d(\sin\theta)}{\sin\theta} \quad \text{..... (44)}$$

Work done in moving a point charge in an electrostatic field:

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q1, Q2,. , QN are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q1 is zero. Q2 is brought in the presence of the field of Q1, the work done W1= Q2V21 where V21 is the potential at the location of Q2 due to Q1. Proceeding in this manner, we can write, the total work done $W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots\ldots\ldots\ldots + (V_{N1}Q_N + \ldots\ldots + V_{N(N-1)}Q_N)$ .....(45)

Had the charges been brought in the reverse order,

$W = (V_{1N}Q_1 + \ldots\ldots + V_{12}Q_1) + \ldots\ldots\ldots\ldots\ldots\ldots + (V_{(N-2)(N-1)}Q_{N-2} + V_{(N-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$

.....(46)

Therefore,

$2W = (V_{1N} + V_{1(N-1)} + \ldots\ldots + V_{12})Q_1 + (V_{2N} + V_{2(N-1)} + \ldots\ldots + V_{23} + V_{21})Q_1 \ldots\ldots$

$\ldots\ldots\ldots\ldots\ldots\ldots + (V_{N1} + \ldots\ldots V_{N2} + V_{N(N-1)})Q_N$ ....

.....(47)

Here VIJ represent voltage at the Ith charge location due to Jth charge. Therefore,

$2W = V_1Q_1 + \ldots\ldots\ldots\ldots + V_NQ_N = \sum_{I=1}^{N} V_IQ_I$ Or, $W = \frac{1}{2}\sum_{I=1}^{N} V_IQ_I$ .....(48)

If instead of discrete charges, we now have a distribution of charges over a volume v then we can write,

$W = \frac{1}{2}\int_v V\rho_v dv$ .....(49)

where ρ_v is the volume charge density and V represents the potential function.

Since, $\rho_v = \nabla \cdot \vec{D}$, we can write

$W = \frac{1}{2}\int_v (\nabla \cdot \vec{D})V dv$ .....(50)

$\nabla.(V\vec{D}) = \vec{D} \cdot \nabla V + V\nabla \cdot \vec{D}$ Using the vector identity,

, we can write

$$W = \frac{1}{2}\int_v \left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right)dv$$

$$= \frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v \left(\vec{D}\cdot\nabla V\right)dv \quad \text{.....(51)}$$

In the expression $\frac{1}{2}\oint_s \left(V\vec{D}\right).d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, the term V $\vec{D}$ varies as $\frac{1}{r^3}$ while the area varies as r2. Hence the integral term varies at least as $\frac{1}{r}$ and the as surface becomes large (i.e. $r \to \infty$) the integral term tends to zero.

Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_v \left(\vec{D}.\nabla V\right)dv = \frac{1}{2}\int_v \left(\vec{D}.\vec{E}\right)dv = \frac{1}{2}\int_v \left(\varepsilon E^2\right)dv = \int_v w_e dv \quad \text{.....(52)}$$

$w_e = \frac{1}{2}\varepsilon E^2$, is called the energy density in the electrostatic field.

Maxwell's first law:

Statement: The following Electrostatic Field equations will be developed in this section:

Integral form

$$\oint_{\text{Surface}} \boldsymbol{D} \bullet d\boldsymbol{a} = \int_{\text{Volume}} \rho\, dv.$$

Differential forms

$$\nabla \bullet \boldsymbol{D} = \rho.$$

$$\text{div } \boldsymbol{D} = \rho$$

Maxwell's first equation is based on Gauss' law of electrostatics published in 1832, wherein Gauss established the relationship between static electric charges and their accompanying static fields.

The above integral equation states that the electric flux through a closed surface area is equal to the total charge enclosed.

The differential form of the equation states that the divergence or outward flow of electric flux from a point is equal to the volume charge density at that point.

Divergence:

The divergence represents the volume density of the outward fluxof a vector field from an infinitesimal volume around a given point.

The following properties can all be derived from the ordinary differentiation rules of calculus. Most importantly, the divergence is a linear operator, i.e.

$$\operatorname{div}(a\mathbf{F} + b\mathbf{G}) = a \operatorname{div} \mathbf{F} + b \operatorname{div} \mathbf{G}$$

for all vector fields **F** and **G** and all real numbers *a* and *b*.

There is a product rule of the following type: if φ is a scalar-valued function and **F** is a vector field, then

$$\operatorname{div}(\varphi\mathbf{F}) = \operatorname{grad} \varphi \cdot \mathbf{F} + \varphi \operatorname{div} \mathbf{F},$$

or in more suggestive notation

$$\nabla \cdot (\varphi\mathbf{F}) = (\nabla\varphi) \cdot \mathbf{F} + \varphi(\nabla \cdot \mathbf{F}).$$

Another product rule for the cross product of two vector fields **F** and **G** in three dimensions involves the curl and reads as follows:

$$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \operatorname{curl} \mathbf{F} \cdot \mathbf{G} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G},$$

or

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G}).$$

The Laplacian of a scalar field is the divergence of the field's gradient:

$$\operatorname{div}(\nabla\varphi) = \Delta\varphi.$$

The divergence of the curl of any vector field (in three dimensions) is equal to zero:

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

.

Poisson's and Laplace's Equations:

For electrostatic field, we have seen that

$\nabla \cdot \vec{D} = \rho_v$

$\vec{E} = -\nabla V$ .....(53)

Form the above two equations we can write

$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v$ .....(54)

Using vector identity we can write, $\varepsilon\nabla\cdot\nabla V+\nabla V\cdot\nabla\varepsilon=-\rho_v$ .....(55)

For a simple homogeneous medium, ε is constant and $\nabla\varepsilon=0$. Therefore,

$$\nabla\cdot\nabla V=\nabla^2V=-\frac{\rho_v}{\varepsilon}$$ .....(56)

This equation is known as Poisson's equation. Here we have introduced a new operator ∇^2, (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2V=\nabla\cdot\nabla V=(\frac{\partial}{\partial x}\hat{a}_x+\frac{\partial}{\partial y}\hat{a}_y+\frac{\partial}{\partial z}\hat{a}_z)\cdot(\frac{\partial V}{\partial x}\hat{a}_x+\frac{\partial V}{\partial y}\hat{a}_y+\frac{\partial V}{\partial z}\hat{a}_z)$$ .....(57)

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial x^2}+\frac{\partial^2V}{\partial x^2}=-\frac{\rho_v}{\varepsilon}$$ .....(58)

In cylindrical coordinates,

$$\nabla^2V=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2V}{\partial\phi^2}+\frac{\partial^2V}{\partial z^2}$$ .....(59)

In spherical polar coordinate system,

$$\nabla^2V=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2V}{\partial\phi^2}$$ .....(60)

At points in simple media, where no free charge is present, Poisson's equation reduces to

$$\nabla^2V=0$$ .....(61)

Which is known as Laplace's equation.

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found hroughout the volume. We shall consider such applications in the section where we deal with boundary value problems.

Solutions to Laplace's Equation in CartesianCoordinates:

Having investigated some general properties of solutions to Poisson's equation, it is now appropriate to study specific methods of solution to Laplace's equation subject to boundary conditions. Exemplified by this and the next section are three standard steps often used in representing EQS fields. First, Laplace's equation is set up in the coordinate system in which the boundary surfaces are coordinate surfaces. Then,

the partial differential equation is reduced to a set of ordinary differential equations by separation of variables. In this way, an infinite set of solutions is generated. Finally, the boundary conditions are satisfied by superimposing the solutions found by separation of variables.

In this section, solutions are derived that are natural if boundary conditions are stated along coordinate surfaces of a Cartesian coordinate system. It is assumed that the fields depend on only two coordinates, *x* and *y*, so that Laplace's equation is (Table I)

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \qquad (1)$$

This is a partial differential equation in two independent variables. One time-honored method of mathematics is to reduce a new problem to a problem previously solved. Here the process of finding solutions to the partial differential equation is reduced to one of finding solutions to ordinary differential equations. This is accomplished by the *method of separation of variables*. It consists of assuming solutions with the special space dependence

$$\Phi(x, y) = X(x)Y(y) \qquad (2)$$

In (2), *X* is assumed to be a function of *x* alone and *Y* is a function of *y* alone. If need be, a general space dependence is then recovered by superposition of these special solutions. Substitution of (2) into (1) and division by Φ then gives

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} \qquad (3)$$

Total derivative symbols are used because the respective functions *X* and *Y* are by definition only functions of *x* and *y*.

In (3) we now have on the left-hand side a function of *x* alone, on the right-hand side a function of *y* alone. The equation can be satisfied independent of *x* and *y* only if each of these expressions is constant. We denote this "separation" constant by k^2, and it follows that

$$\frac{d^2X}{dx^2} = -k^2X \qquad (4)$$

and

$$\frac{d^2Y}{dy^2} = k^2Y \qquad (5)$$

These equations have the solutions

$$X \sim \cos kx \quad \text{or} \quad \sin kx \qquad (6)$$

$$Y \sim \cosh ky \quad \text{or} \quad \sinh ky \qquad (7)$$

If $k = 0$, the solutions degenerate into

$$X \sim \text{constant} \quad \text{or} \quad x \qquad (8)$$
$$Y \sim \text{constant} \quad \text{or} \quad y \qquad (9)$$

The product solutions, (2), are summarized in the first four rows of Table 5.4.1. Those in the right-hand column are simply those of the middle column with the roles of x and y interchanged. Generally, we will leave the prime off the k' in writing these solutions. Exponentials are also solutions to (7). These, sometimes more convenient, solutions are summarized in the last four rows of the table.

Electric dipole:

The name given to two point charges of equal magnitude and opposite sign, separated by a distance which is small compared to the distance to the point P, at which we want to know the electric and potential fields

Dipole moment:

A stronger mathematical definition is to use vector algebra, since a quantity with magnitude and direction, like the dipole moment of two point charges, can be expressed in vector form

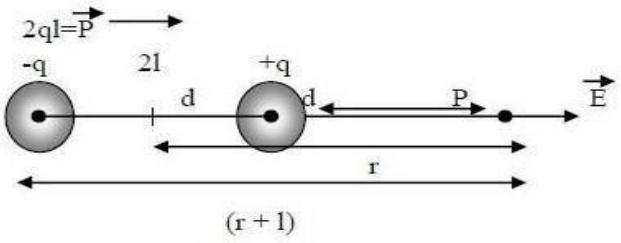
$$\mathbf{p} = q\mathbf{d}$$

Where d is the displacement vectorpointing from the negative charge to the positive charge. The electric dipole moment vector p also points from the negative charge to the positive charge**.**

EFI due to an electric dipole:

To calculate electric field created by a dipole on the axial line (on the same line joining the two charges**),**

- All the measurement ofdistancesareto be takenfromthe centre(O).
- Let the distance between O to +q and O to –q be 'l'. So, total lengthbetween +q and –q will be '2l'.
- Take a point 'p' onthe axial line at the distance 'r' from the centre as shown in figure.



Now, we wish to calculate electric field at point 'P'.

By using the formula for electric field due to point charge,

Electric field due to $+q = \frac{+1}{4\pi\varepsilon_0} \frac{q}{(r - l)^2}$

The distance between (P and +q) = (r-l)

Electric field due to $-q = \frac{-1}{4\pi\varepsilon_0} \frac{q}{(r + l)^2}$

The distance between (P and -q) = (r + l)

(Electric field due to +q will be positive and electric field due to- q will be negative)

Since electric field is a vector quantity so, the net electric field will be the vector addition of the two.

So, the net electric field $E = E_1 + E_2$

$$\mathbf{E = \frac{1}{4\pi\varepsilon_0} \frac{q}{(r - l)^2} - \frac{1}{4\pi\varepsilon_0} \frac{q}{(r + l)^2}}$$

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right]$$

On solving the equation we get −

$$E = \frac{q}{4\pi\varepsilon_0}\left[\frac{(r+l)^2 - (r-l)^2}{(r-l)^2(r+l)^2}\right]$$

$$E = \frac{q}{4\pi\varepsilon_0}\ \frac{4rl}{(r^2 - l^2)^2} \quad .....(1)$$

We know that the dipole moment or effectiveness of dipole (P) is given by −

$$P = 2ql$$

Therefore, putting this value in eq(1), we get

$$E = \frac{1}{4\pi\varepsilon_0}\ \frac{2Pr}{(r^2 - l^2)^2} \quad .....(2)$$

Certain assumptions are made based on this equation −

Since, the dipole is very small so 'l' is also very small as compared to the distance 'r'.

So, on neglecting 'r' with respect to 'l' we get –

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2Pr}{r^4} \text{ (from eq(2))}$$

$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{2P}{r^2}$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{2P}{r^2}$$

Note – Electric field on the axial line of dipole is not 0. Its **magnitude** is resultant as expressed above.

Torque:

An object with an electric dipole moment is subject to a torque τ when placed in an external electric field. The torque tends to align the dipole with the field. A dipole aligned parallel to an electric field has lower potential energy than a dipole making some angle with it. For a spatially uniform electric field **E**, the torque is given by

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}\,,$$

where p is the dipole moment, and the symbol "×" refers to the vector cross product. The field vector and the dipole vector define a plane, and the torque is directed normal to that plane with the direction given by the right-hand rule.

A dipole oriented co- or anti-parallel to the direction in which a non-uniform electric field is increasing (gradient of the field) will experience a torque, as well as a force in the direction of its dipole moment. It can be shown that this force will always be parallel to the dipole moment regardless of co- or anti-parallel orientation of the dipole.

Torque on an Electric dipole in an electric field:

Let us assume an electric dipole is placed in a uniform magnetic field as shown in figure. Each charge of dipole experience a force qE in electric field. Since points of action of these forces are different, these equal and anti paralel forces give rise to a couple that rotate the dipole and make the dipole to align in the direction of field.
The torque τ experienced by the dipole is (qE)×(2dsinθ), where 2d is the length of dipole and θ is the angle between dipole and field direction.

$$\tau = qE \times 2d\sin\theta = (q \times 2d) \times E\sin\theta = p \times E\sin\theta = \boldsymbol{p} \times \boldsymbol{E}$$

we have used the definition of dipole moment p = q×2d in the above equation. **p** and **E** are vectors representing the dipole moment and Electric field respectively.
Last step shown above is the cross product of two vectors