UNIT - IV

CORRELATION

Autocorrelation & Cross-correlation Applications

Cross-correlation and autocorrelation are commonly used for measuring the similarity of signals especially for "pattern recognition" and for "signal detection".

Example:- Autocorrelation used to extract radar signals to improve sensitivity. Makes use of radar signals being periodic so the signal is a pulse train (parameters: amplitude, pulse width and interval between pulses).

Example:- Cross-correlation used to establish symbol timing by comparing an incoming signal with a known bit sequence to identify a known bit pattern to reference to for system timing.

Example:- Correlation is used for analyzing fractal patterns.

AUTO CORRELATION FUNCTION :-

Consider a time-limited (or band-limited) signal $x(t)$,

$$x(t) = \begin{cases} x(t) & 0 \le t \le T \\ 0 & \text{, otherwise} \end{cases}$$

its autocorrelation function is defined as

$$C_{xx}(t, t+\tau) = E\left[x(t)\, x(t+\tau)\right]$$

$$\approx \frac{1}{T}\int_0^T x(t)\, x(t+\tau)\, dt$$

where the definition equation in the first line is specified for random signals whereas the second line is more general and also applicable for deterministic signals. If the random signal $x(t)$ is drawn from an ergodic stochastic process, then the ensemble average can be approximated by the time average by allowing the duration T to approach infinity.

Some important concepts and properties related to the autocorrelation are summarized here:

Properties :-

* If $x(t)$ is drawn from a wide-sense stationary process, then its autocorrelation function is shift invariant, namely,

$$C_{xx}(t, t+\tau) = C_{xx}(\tau)$$

* The autocorrelation function is symmetric, namely,

$$C_{xx}(\tau) = C_{xx}(-\tau) \text{ and } C_{xx}(\tau) \leq C_{xx}(0) = \sigma_x^2,$$

where $\sigma_x^2 = Var[x(t)]$ denotes the variance of the $x(t)$.

* The normalized autocorrelation function is defined as

$$\bar{C}_{xx}(\tau) = \frac{C_{xx}(\tau)}{C_{xx}(0)}$$

* The decaying rate and the limit of the autocorrelation function can be characterized by

$$|C_{xx}(\tau)| \leq C_{xx}(0) \cos\left(\frac{\pi}{1+T/\tau}\right) \quad (\tau < T)$$

* If $x(t)$ is wide-sense stationary, its autocorrelation function can be written in terms of spectral representations in light of the Wiener-Khinchin theorem.

$$C_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} \, d\omega,$$

where $S_{xx}(\omega)$ denotes the power spectral density of $x(t)$.

* Let $x_1(t)$ denote the Hilbert transform of $x(t)$.

$$x_1(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \, d\tau \, ;$$

then it can be proved that the autocorrelation of $x_1(t)$ is equal to that of $x(t)$, namely,

$$C_{x_1x_1}(\tau) = C_{xx}(\tau),$$

Whereas $x_1(t)$ is orthogonal (or uncorrelated) to $x(t)$, namely, $E[x_1(t)\,x(t)] = 0$.

CROSS - CORRELATION FUNCTION :-

For two time-limited signals $x(t)$ and $y(t)$, the cross-correlation function may be defined as

$$C_{xy}(t, t+\tau) = E[x(t)\, y(t+\tau)]$$

$$\approx \frac{1}{T}\int_0^T x(t)\, y(t+\tau)\, dt,$$

$$C_{xy}(t+\tau, t) = E[y(t)\, x(t+\tau)]$$

$$\approx \frac{1}{T}\int_0^T x(t+\tau)\, y(t)\, dt.$$

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\boxed{C_{xy} \text{ (or) } R_{xy} = \int x(t)\, y(t-\tau)\, dt}$$

It is noted that the cross-correlation function is generally nonsymmetric, namely, $C_{xy}(t, t+\tau) \neq C_{xy}(t+\tau, t)$.

Properties of Cross-correlation :-

* The cross-correlation function is bounded by the cross-correlation inequality

$$|C_{xy}(\tau)|^2 \leq C_{xx}(0)\, C_{yy}(0) = \sigma_x^2 \sigma_y^2, \quad \text{—①}$$

Where $\sigma_x^2 = E[x^2(t)]$ and $\sigma_y^2 = E[y^2(t)]$ denote the power of $x(t)$ and $y(t)$, respectively.

* In terms of spectral representations, the cross-correlation function can be written as the inverse Fourier transform

$$C_{xy}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega,$$

Where $S_{xy}(\omega)$ denotes the cross-spectrum density.

* The correlation coefficient (also called normalized cross-correlation) between two random signals $x(t)$ and $y(t)$ is defined as

$$\rho_{xy} = \frac{C_{xy}(0)}{\sqrt{\operatorname{var}[x(t)]\, \operatorname{var}[y(t)]}}.$$

From eq ①, it follows that the correlation coefficient ρ_{xy} ranges between -1 and 1. Positive / negative ρ_{xy} indicates $x(t)$ and $y(t)$ are positively / negatively correlated; $\rho_{xy}=0$ indicates that they are uncorrelated.

In the frequency domain, let $X(\omega)$ and $Y(\omega)$ denote the Fourier transform of $x(t)$ and $y(t)$, respectively; then the cross-spectrum of $X(\omega)$ and $Y(\omega)$ is defined as

$$S_{XY}(\omega) = E\left[X(\omega)Y^*(\omega)\right],$$

Where the asterisk denotes the complex conjugate. In a similar vein, the normalized cross-spectrum is defined as

$$\bar{S}_{XY}(\omega) = \frac{S_{XY}(\omega)}{\sqrt{Var[X(\omega)]\, Var[Y(\omega)]}}$$

and its magnitude $|\bar{S}_{XY}(\omega)|$ is a real function between 0 and 1 that gives a measure of correlation between $x(t)$ and $y(t)$ at each frequency ω. Observe that $|\bar{S}_{XY}(\omega)|^2$ bears some similarity to ρ^2_{xy}; however, $|\bar{S}_{XY}(\omega)|^2$ takes into account out-of-phase relationships and can examine the variance of two signals in a selected frequency range.

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Auto correlation function and its properties

* The Mean of $x(t)$ is defined as

$$\mu_x(t) = E[x(t)]$$
$$= \int_{-\infty}^{\infty} x\, f_{x(t)}(x)\, dx$$
$$= \mu_x$$

for all t.

* The Autocorrelation Function of $x(t)$ is

$$R_x(t_1, t_2) = E[x(t_1)x(t_2)]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_{x(t_1)x(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(0)X(t_2-t_1)}(x_1, x_2)\, dx_1 dx_2$$

$$= R_X(t_2 - t_1)$$

for all t_1 and t_2

* The Autocovariance Function

$$C_X(t_1, t_2) = E[(X(t_1) - \mu_X)(X(t_2) - \mu_X)]$$
$$= R_X(t_2 - t_1) - \mu_X^2$$

Which is a function of time difference $(t_2 - t_1)$.
We can determine $C_X(t_1, t_2)$ if μ_X and $R_X(t_2 - t_1)$ are known.

Properties of the autocorrelation function :-
For convenience of notation, we redefine

$$R_X(\tau) = E[X(t)\, X(t+\tau)] \quad , \text{ for all } t$$

Property 1:- The mean-square value

$$R_X(0) = E[X^2(t)], \quad \tau = 0$$

Property-2:-

$R_X(\tau)$ is an even function of τ

$$R_X(\tau) = R(-\tau)$$

$$R(-\tau) = E[X(t)\, X(t-\tau)]$$

put $t - \tau = u$

$$\therefore R(-\tau) = E[X(u)\, X(u+\tau)]$$

$$= R_X(\tau)$$

Property-3:-

The maximum value of $R_X(\tau)$ is attained at $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

Consider $E[(X(t+\tau) \pm X(t))^2] \geq 0$

$$\Rightarrow E[X^2(t+\tau)] \pm 2E[X(t+\tau)\, X(t)] + E[X^2(t)] \geq 0$$

$$\Rightarrow 2E[X^2(t)] \pm 2R_X(\tau) \geq 0$$

$$\Rightarrow 2R_X(0) \pm 2R_X(\tau) \geq 0$$

$$\Rightarrow -R_X(0) \leq R_X(\tau) \leq R_X(0)$$

$$\therefore |R_X(\tau)| \leq R_X(0)$$

Property - 4 :-

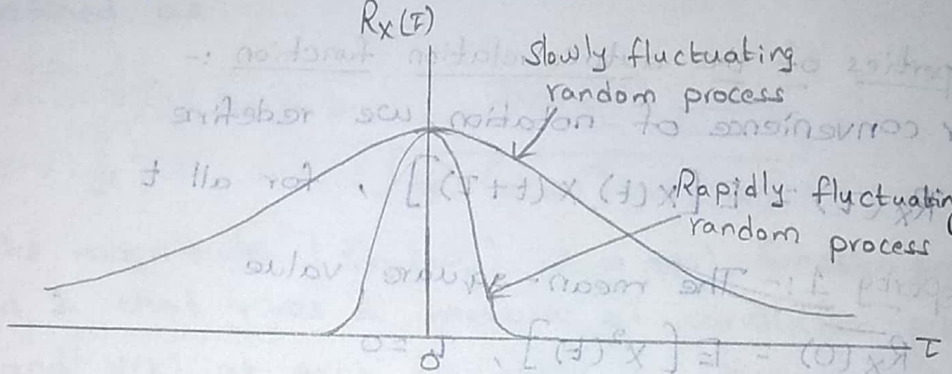
If a random process $\{X(t)\}$ has no periodic components and $E[X(t)] = \bar{X}$ then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = (\bar{X})^2 = \mu_X^2$$

Property - 5 :-

The auto correlation function of a random process cannot have an arbitrary shape.

The $R_X(t)$ provides the interdependence information of two random variables obtained from $X(t)$ at times t seconds apart.



Cross correlation and its properties :-

Let $\{X(t)\}$ and $\{Y(t)\}$ be two random process. Then the cross correlation between them is defined by

$$R_{XY}(t, t+\tau) = E[X(t)\, Y(t+\tau)] = R_{XY}(\tau)$$

$$R_{YX}(t, t+\tau) = E[Y(t)\, X(t+\tau)] = R_{YX}(\tau)$$

Note : Cross correlation function of two R.P is defined as a measure of the similarity between a signal and a time delayed version of a second signal.

Property - 1 :-

Cross correlation function is not an even function

$$R_{XY}(\tau) \neq R_{YX}(\tau)$$

But $R_{XY}(\tau) = R_{YX}(-\tau)$

Proof:- Let $t - \tau = \mu$,

$$\Rightarrow R_{YX}(-\tau) = E[Y(t)\, X(t-\tau)]$$

$R_{YX}(-\tau) = E[Y(\mu+\tau)X(\mu)]$

$= E[X(\mu)Y(\mu+\tau)]$

$= R_{XY}(\tau)$

Property-2:-

If $\{X(t)\}$ and $\{Y(t)\}$ are 2 R.P's then

$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

Proof:- We know that $E[(X(t)+KY(t+\tau))^2] \geq 0$ for any real K.

$\Rightarrow E[X^2(t)] + 2KE[X(t)Y(t+\tau)] + K^2 E[Y^2(t+\tau)] \geq 0$

$\Rightarrow R_{XX}(0) + 2R_{XY}(\tau) + K^2 R_{YY}(0) \geq 0$

~~If $a\lambda^2 + 2b(\lambda)$~~ If $a\lambda^2 + 2b\lambda + c \geq 0$ for all real λ

then $b^2 \leq ac$

$\Rightarrow R_{XY}(\tau)^2 \leq R_{XX}(0)R_{YY}(0)$

$|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$

Property-3:-

If $\{X(t)\}$ and $\{Y(t)\}$ are 2 R.P's. then

$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

Proof:- We know that the G.M of 2 positive numbers does not exceed their A.M. So we've

$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

So from Property 2 we've

$|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

Property-4:- If the process $\{X(t)\}$ and $\{Y(t)\}$ are orthogonal then $R_{XY}(\tau) = 0$.

Property-5:- If the process $\{X(t)\}$ and $\{Y(t)\}$ are independent then $R_{XY}(\tau) = \mu_X \mu_Y$.

Problem :-

1) A stationary process has an autocorrelation function given by $R_{XX}(\tau) = \dfrac{25\tau^2 + 36}{6.25\tau^2 + 4}$. Find the mean and variance of the process.

Sol:- $(\bar{X})^2 = \lim_{|\tau| \to \infty} R_{XX}(\tau) = \lim_{|\tau| \to \infty} \dfrac{\tau^2\left(25 + \frac{36}{\tau^2}\right)}{\tau^2\left(6.25 + \frac{4}{\tau^2}\right)}$

$= 4 \Rightarrow \text{Mean} = 2$

$E[X^2(t)] = R_{XX}(0) = 9$

$\text{Variance} = E[X^2(t)] - E[X(t)]^2 = 5$

2.) If $\{X(t)\}$ is a WSS process with autocorrelation function $R_{XX}(\tau)$ and if $Y(t) = X(t+a) - X(t-a)$, show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$

Sol:- $R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

$= E[(X(t+a) - X(t-a)) \cdot (X(t+\tau+a) - X(t+\tau-a))]$

Apply

$E[X(t+a) \cdot X(t+\tau+a)] = R_{XX}(t+\tau+a-(t+a))$

$\Rightarrow R_{XX}(\tau)$

So $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$

3.) Two R.P's $\{X(t)\}$ and $\{Y(t)\}$ are given by $X(t) = A\cos(\omega t+\theta)$ and $Y(t) = A\sin(\omega t+\theta)$ where A and ω are constants and θ is uniformly distributed over $(0, 2\pi)$. Find the cross correlation.

Sol:- $R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

$= E[A\cos(\omega t+\theta) \cdot A\sin(\omega t + \omega\tau + \theta)]$

$= \dfrac{A^2}{2}[\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega\tau)]$

Since θ is uniform in $(0, 2\pi)$, $f(\theta) = \dfrac{1}{2\pi}$

$R_{XY}(\tau) = \dfrac{A^2}{4\pi}\int_0^{2\pi}[\sin(2\omega t + \omega\tau + 2\theta) + \sin(\omega\tau)]\,d\theta$

$$= \frac{A^2}{4\pi}\left[\frac{\cos(2\omega t + \omega\tau + 2\theta)}{\theta} + \sin(\omega\tau)(\theta)\right]_0^{2\pi}$$

$$= \frac{A^2}{4\pi}\left[2\pi \sin(\omega\tau)\right]$$

$$= \frac{A^2}{2}\sin(\omega\tau)$$

Relation between convolution and correlation

Let $x_1(t)$ & $x_2(t)$

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Now, convolution of $x_1(t)$ & $x_2(-t)$

$$x_1(t) * x_2(-t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(\tau - t)\, d\tau$$

Let $\tau = l$

$$= \int_{-\infty}^{\infty} x_1(l)\, x_2(l-t)\, dl$$

Now change t to τ

$$= \int_{-\infty}^{\infty} x_1(l)\, x_2(l-\tau)\, dl$$

$$x_1(t) * x_2(-t) = R_{x_1 x_2}(\tau)$$

$$\boxed{x_1(t) * x_2(-t) = R_{x_1 x_2}(\tau)}$$

Suppose $x_2(t)$ is an even function

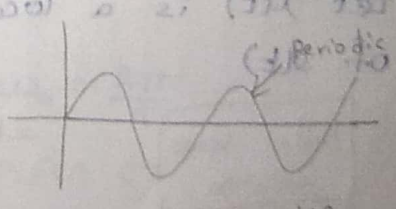
$$x_2(t) = x_2(-t)$$

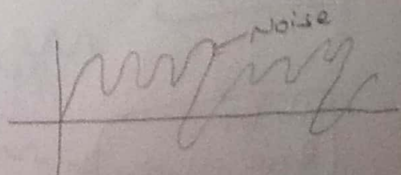
then convolution = correlation.

Detection of periodic signals in the presence of noise by correlation

If signal is $s(t)$

& noise is $n(t)$





Then the noise effected signal is

$$y(t) = s(t) + n(t) \quad \text{—— ①}$$

Case-1:- Detection by Auto correlation

Let $R_{yy}(\tau)$, $R_{ss}(\tau)$ & $R_{nn}(\tau)$ are auto correlations of $y(t)$, $s(t)$ & $n(t)$ respectively.

Now autocorrelation of periodic signal $y(t)$ is

$$R_{yy}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T y(t)\,y(t-\tau)\,dt \quad \text{— ②}$$

substitute ① in ②

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[s(t)+n(t)\right]\left[s(t-\tau)+n(t-\tau)\right]dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[s(t)\cdot s(t-\tau) + s(t)\,n(t-\tau) + n(t)\,s(t-\tau) + n(t)\,n(t-\tau)\right]dt$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\int_0^T s(t)\cdot s(t-\tau)\,dt + \int_0^T s(t)\,n(t-\tau)\,dt + \int_0^T n(t)\,s(t-\tau)\,dt + \int_0^T n(t)\,n(t-\tau)\,dt\right]$$

$$R_{yy}(\tau) = R_{ss}(\tau) + R_{sn}(\tau) + R_{ns}(\tau) + R_{nn}(\tau) \quad \text{— ③}$$

Since the periodic signal & noise are uncorrelated, then $R_{sn}(\tau) = R_{ns}(\tau) = 0$

$$③ \Rightarrow R_{yy}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$$

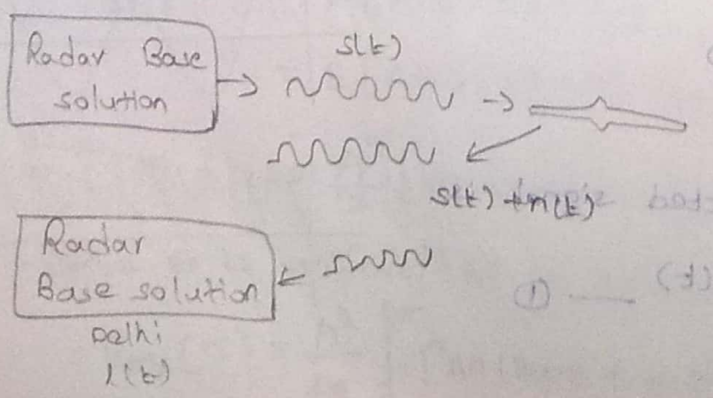
for larger values of τ, the non periodic noise $R_{nn}(\tau) = 0$

then $$R_{yy}(\tau) = R_{ss}(\tau)$$

Case-2:- Detection of periodic signal by Cross correlation

let $y(t) = s(t) + n(t)$ — ①

Let $l(t)$ is a locally generated signal of same frequency of $s(t)$



Now the cross correlation of $y(t)$ & $l(\tau)$

$$R_{yl}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T y(t)\, l(t-\tau)\, dt \quad \text{--- (2)}$$

Substitute (1) in (2)

$$R_{yl}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T [s(t) + n(t)]\, l(t-\tau)\, dt.$$

$$= R_{sl}(\tau) + R_{nl}(\tau)$$

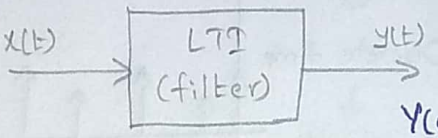
since n & l are uncorrelated

$$R_{nl}(\tau) = 0$$

$$\therefore \quad R_{yl}(\tau) = R_{sl}(\tau)$$

Since s & l are same frequency $R_{sl}(\tau)$.

<u>Extraction of a signal from Noise by filtering</u>



$$Y(\omega) = X(\omega)\, H(\omega)$$

$$R_{xh}(\tau) = X(\omega)\, H(-\omega).$$

Let $s(t)$ = periodic signal

$n(t)$ = Noise signal

The received signal

$$y(t) = s(t) + n(t)$$

$s(t)$ can be detected in the presence of noise by cross correlating $y(t)$ with $l(t)$ which is same as $s(t)$

$$l(t) \xleftrightarrow{F.T} L(\omega)$$

$$l(-t) \xleftrightarrow{F.T} L(-\omega)$$

Since $l(t)$ is a periodic signal, then F.S of $l(t)$ is

$$l(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

The F.T of $l(t)$ is

$$L(\omega) = 2\pi \sum_{k=-\infty}^{\infty} C_k\, \delta(\omega - k\omega_0)$$

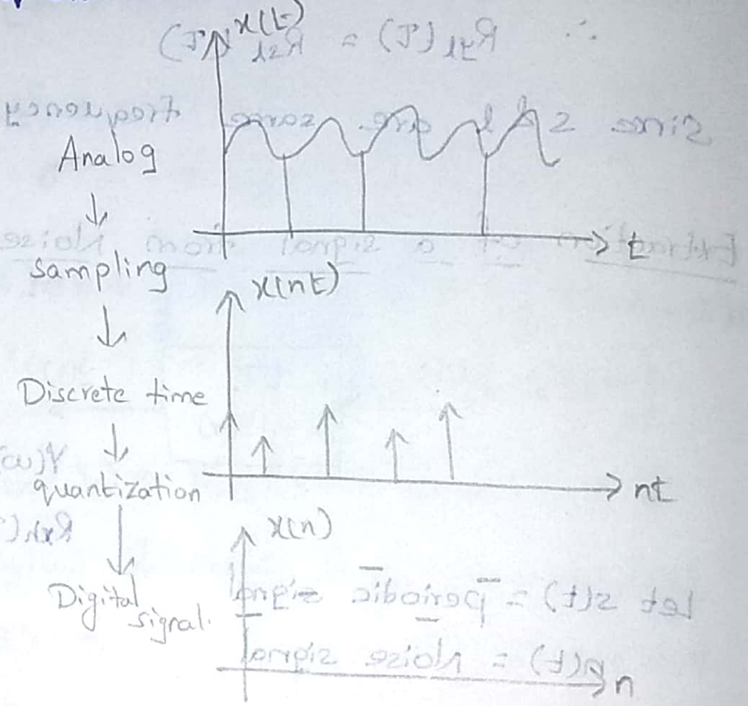
$$L(-\omega) = L^*(\omega)$$

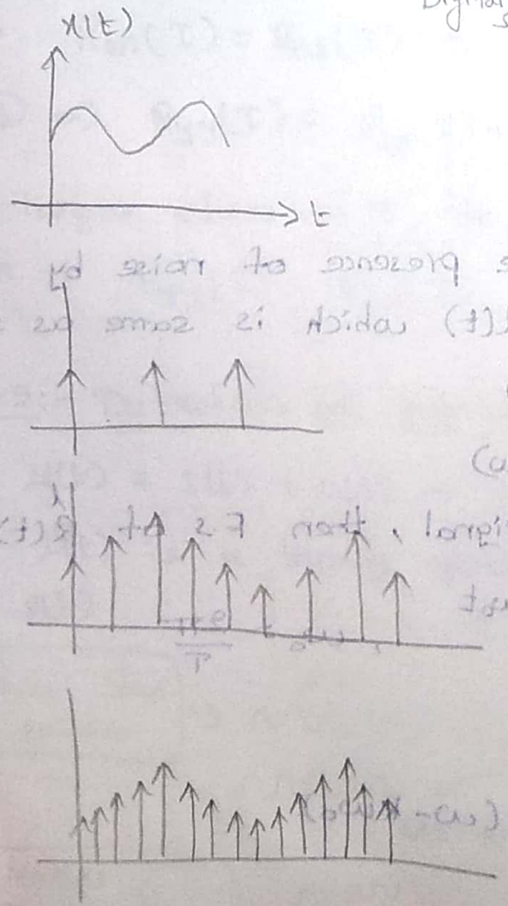
$$= 2\pi \sum_{k=-\infty}^{\infty} C_k^*\, \delta(\omega - k\omega_0)$$

The cross correlation in the time domain is equivalent to filtering in the frequency domain, which allows the frequencies of $s(t)$ and its hormonics.

Sampling:-

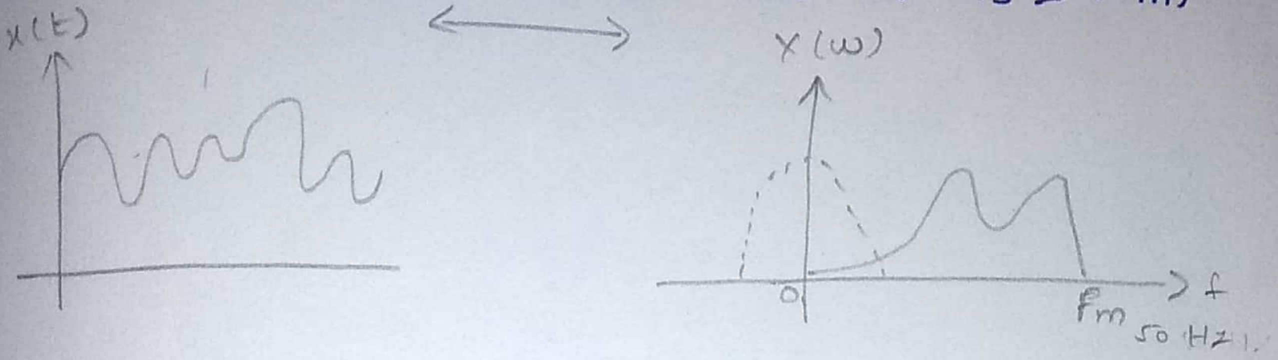
* The process of converting analog signal to discrete-time signal is called sampling
* The process of converting Discrete-time to Digital signal is called quantization.





Sampling Theorem :-

The sampling theorem states that f_m any band limited signal $x(t)$ with maximum frequency f_m can be represented into its samples taken at the rate of $f_s \geq 2 f_m$



$f_s \geq 2 f_m$

$\geq 2 \times 50$

≥ 100

$\frac{1}{f_s} = 0.01$

$T_s = 0.01$