UNIT-II

FOURIER SERIES AND FOURIER TRANSFORM

Fourier Series:- Frequency domain representation of any continuous time periodic function.

(or)

Fourier series is infinite series of periodic function.

Periodic signals:- A continuous-time signal $x(t)$ to be periodic if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \quad \text{all } t$$

The fundamental period T_0 of $x(t)$ is the smallest positive value of T ~~for which~~ and $1/T_0 = f_0$ is referred to as the fundamental frequency.

Two basic examples of periodic signals are the real sinusoidal signal.

$$x(t) = \cos(\omega_0 t + \phi)$$

and the complex exponential signal

$$x(t) = e^{j\omega_0 t}$$

Where $\omega_0 = 2\pi/T_0 = 2\pi f_0$ is called the fundamental angular frequency.

Types of fourier series

1.) Complex exponential fourier series

2.) Trigonometric form of fourier series

3.) Harmonic form fourier series.

Complex exponential fourier series:-

The complex exponential fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}} \qquad \omega_0 = \frac{2\pi}{T_0}$$

Where C_k are known as the complex fourier coefficients and are given by

$$\boxed{C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt}$$

where $\int_{T_0}$ denotes the integral over any one period and 0 to T_0 or $-T_0/2$ to $T_0/2$ is commonly used for the integration.

Sub $k=0$ in C_k, we have

$$\boxed{C_0 = \frac{1}{T_0}\int_{T_0} x(t)\cdot dt}$$

which indicates that C_0 equals the average value of $x(t)$ over a period. When $x(t)$ is real, then from C_k it follows that

$$\boxed{C_{-k} = C_k^*}$$

where the asterisk indicates the complex conjugate.

Trigonometric Fourier series:-

The trigonometric fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\omega_0 t + b_k \sin k\omega_0 t) \quad \omega_0 = \frac{2\pi}{T_0}$$

where a_k and b_k are the fourier coefficients given by

$$a_k = \frac{2}{T_0}\int_{T_0} x(t) \cos k\omega_0 t \, dt$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t) \sin k\omega_0 t \, dt$$

The coefficients a_k and b_k are the complex fourier coefficients C_k are related by

$$\frac{a_0}{2} = C_0, \quad a_k = C_k + C_{-k}, \quad b_k = j(C_k - C_{-k}).$$

$$C_k = \frac{1}{2}(a_k - jb_k), \quad C_{-k} = \frac{1}{2}(a_k + jb_k)$$

When $x(t)$ is real, then a_k and b_k are real

$$a_k = 2\,\mathrm{Re}[C_k], \quad b_k = -2\,\mathrm{Im}[C_k]$$

Harmonic Form Fourier Series:-

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos(k\omega_0 t - \theta_k) \quad \omega_0 = \frac{2\pi}{T_0}$$

$$C_0 = \frac{a_0}{2} \quad C_k = \sqrt{a_k^2 + b_k^2} \quad \theta_k = \tan^{-1}\frac{b_k}{a_k}$$

Convergence of Fourier Series:

It is known that a periodic signal $x(t)$ has a fourier series representation if it satisfies the following Dirichlet conditions:

1.) $x(t)$ is absolutely integrable over any period, that is,

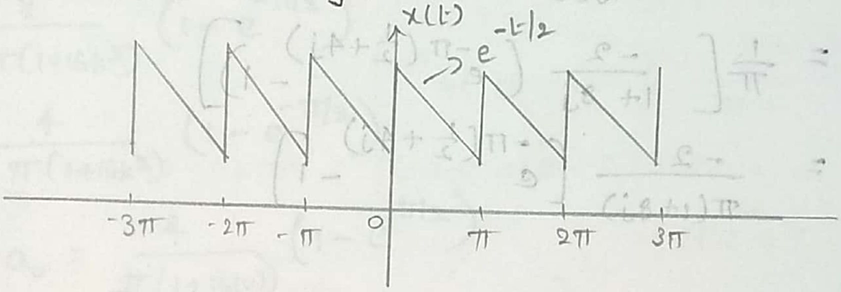
$$\int_{T_0} |x(t)|\,dt < \infty$$

2.) $x(t)$ has a finite number of maxima and minima within any finite interval of t.

3.) $x(t)$ has a finite number of discontinuities within any finite interval of t, and each of these discontinuities is finite.

Note that the Dirichlet conditions are sufficient but not necessary conditions for the Fourier series representation.

1) Find the ~~four~~ complex exponential fourier series representation of $x(t)$ as shown in figure below.



W.K.T,

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad T_0 = \pi$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\,dt.$$

$$C_k = \frac{1}{\pi}\int_0^{\pi} e^{-t/2} \cdot e^{-jk2t}\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} e^{-t/2 - jk2t}\,dt \Rightarrow \frac{1}{\pi}\int_0^{\pi} e^{-t\left(\frac{1}{2}+2jk\right)}\,dt$$

$$= \frac{1}{\pi}\left[\frac{e^{-t\left(\frac{1}{2}+2jk\right)}}{-\left(\frac{1}{2}+2jk\right)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{\left(\frac{1+4jk}{2}\right)}\left(e^{-\pi\left(\frac{1}{2}+2jk\right)} - e^{0}\right)\right]$$

$$C_K = \frac{1}{\pi}\left[-\frac{2}{1+4jK}\left(e^{-\pi\left(\frac{1}{2}+2jK\right)} - 1\right)\right]$$

$K=0$

$$C_0 = \frac{1}{\pi}\left[\frac{-2}{1+0}\left(e^{-\pi\left(\frac{1}{2}+2j(0)\right)} - 1\right)\right]$$

$$= \frac{1}{\pi}\left[-2\left(e^{-\pi/2} - 1\right)\right] \qquad e^{-\pi/2} = 0.208$$

$$= \frac{1}{\pi}\left[-2(0.208-1)\right] = \frac{1}{\pi}\left[-2(-0.792)\right]$$

$$= \frac{1.584}{3.14} = 0.5044$$

$K=1$

$$C_1 = \frac{1}{\pi}\left[\frac{-2}{1+4j(1)}\left(e^{-\pi\left(\frac{1}{2}+2j(1)\right)} - 1\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{1+4j}\left(e^{-\pi\left(\frac{1}{2}+2j\right)} - 1\right)\right] = \frac{-2}{\pi(1+4j)}\left[e^{-\pi\left(\frac{1}{2}+2j\right)} - 1\right]$$

$K=2$

$$C_2 = \frac{1}{\pi}\left[\frac{-2}{1+4j(2)}\left(e^{-\pi\left(\frac{1}{2}+2j(2)\right)} - 1\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{1+8j}\left(e^{-\pi\left(\frac{1}{2}+4j\right)} - 1\right)\right]$$

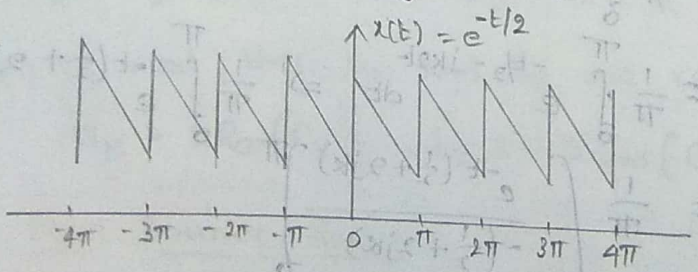
$$= \frac{-2}{\pi(1+8j)}\left[e^{-\pi\left(\frac{1}{2}+4j\right)} - 1\right]$$

$k=3$

$$C_3 = \frac{1}{\pi}\left[\frac{-2}{1+4j(3)}\left(e^{-\pi\left(\frac{1}{2}+2j(3)\right)} - 1\right)\right]$$

$$= \frac{-2}{\pi(1+12j)}\left(e^{-\pi\left(\frac{1}{2}+6j\right)} - 1\right)$$

2.) Find the fourier series representation of $x(t)$ as shown in figure below [Trigonometric fourier series]



$$x(t) = \frac{a_0}{2} + \sum_{K=1}^{\infty}\left[a_K \cos K\omega_0 t + b_K \sin K\omega_0 t\right] dt$$

$$a_K = \frac{2}{T_0}\int_{T_0} x(t)\cos K\omega_0 t \, dt$$

$$b_K = \frac{2}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$a_K = \frac{2}{T_0}\int_{T_0} x(t)\cos k\omega_0 t\, dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2kt\, dt$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/2}}{(-\frac{1}{2})^2+(2k)^2}\left[-\frac{1}{2}\cos 2kt + 2k\sin 2kt\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{e^{-\pi/2}}{\frac{1}{4}+4k^2}\left[-\frac{1}{2}\cos 2\pi k + 2k\sin 2\pi k\right] - \frac{e^0}{\frac{1}{4}+4k^2}\left[-\frac{1}{2}\cos 0 + 2k\sin 0\right]\right]$$

$$= \frac{2}{\pi}\left[\frac{4}{1+16k^2}\right]\left[e^{-\pi/2}\left(-\frac{1}{2}(1)+0\right) - 1\left(-\frac{1}{2}+0\right)\right]$$

$$= \frac{8}{\pi(1+16k^2)}\left[-\frac{e^{-\pi/2}}{2}+\frac{1}{2}\right]$$

$$= \frac{8}{2\pi(1+16k^2)}\left(1-e^{-\pi/2}\right)$$

$$a_K = \frac{4}{\pi(1+16k^2)}\left(1-e^{-\pi/2}\right)$$

$k=0$, $$a_0 = \frac{4}{\pi(1+16(0))}\left(1-e^{-\pi/2}\right)$$

$$= \frac{4}{3.14}(1-0.2080)$$

$$= 1.2738\,(0.792)$$

$$a_0 = 1.0088$$

$k=1$,
$$a_1 = \frac{4}{\pi(1+16)}\left[1-e^{-\pi/2}\right]$$

$$= \frac{4}{3.14(17)}\left[1-0.2080\right]$$

$$= 0.0749\,(0.792) = 0.0593$$

$k=2$,
$$a_2 = \frac{4}{\pi(1+16(4))}\left[1-e^{-\pi/2}\right]$$

$$= \frac{4}{3.14(65)}\left[1-0.2080\right]$$

$$= 0.01959\,(0.792)$$

$$= 0.0155$$

$$b_k = \frac{2}{T_0}\int_{T_0} x(t)\sin k\omega_0 t\, dt.$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2kt\, dt.$$

$$= \frac{2}{\pi}\left[\frac{e^{-t/2}}{(-\frac{1}{2})^2+(2k)^2}\left[-\frac{1}{2}\sin 2kt - 2k\cos 2kt\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{e^{-\pi/2}}{\frac{1}{4}+4k^2}\left(-\frac{1}{2}\sin 2k\pi - 2k\cos 2k\pi\right) - \frac{e^0}{\frac{1}{4}+4k^2}\left(-\frac{1}{2}\sin 0 - 2k\cos 0\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{4}{1+16k^2}\right)\left[e^{-\pi/2}(0-2k) - (-2k)\right]$$

$$= \frac{2}{\pi}\left(\frac{4}{1+16k^2}\right)\left[e^{-\pi/2}(-2k) + 2k\right]$$

$$= \frac{8}{\pi(1+16k^2)}\left[2k - 2k e^{-\pi/2}\right]$$

$k=0$,

$$b_0 = \frac{8}{\pi(1)}\left[2(0) - 2(0)e^{-\pi/2}\right] = 0$$

$k=1$,

$$b_1 = \frac{8}{17\pi}\left[2(1) - 2(1)e^{-\pi/2}\right]$$

$$= 0.1498\left[2 - 2(0.2080)\right] = 0.1498(1.584)$$

$$= 0.2373.$$

$k=2$,

$$b_2 = \frac{8}{\pi(1+16(4))}\left[2(2) - 2(2)e^{-\pi/2}\right]$$

$$= \frac{8}{65\pi}\left[4 - 4e^{-\pi/2}\right] = \frac{32}{65\pi}\left[1 - 0.2080\right]$$

$$= 0.1567(0.792)$$

$$b_2 = 0.1241$$

Find the complex exponential fourier series of $\cos(\omega_0 t)$.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad (1)$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_0 = \frac{1}{T_0}\int_{T_0} x(t)\, dt$$

$x(t) = \cos\omega_0 t$

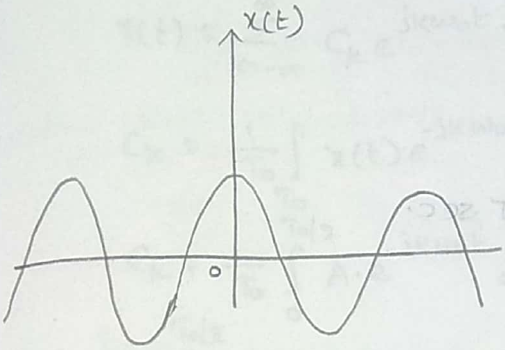
$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

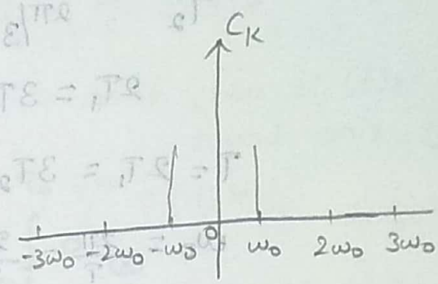
$$\cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t}) + \frac{1}{2}(e^{-j\omega_0 t}) \quad \text{— (2)}$$

$$\dots C_{-2} e^{-2j\omega_0 t} + C_{-1} e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{2j\omega_0 t} + \dots C_k e^{jk\omega_0 t}$$

Comparing

$$C_1 = \frac{1}{2}, \quad C_{-1} = \frac{1}{2}, \quad C_0 = C_2 = C_{-2} = C_3 = C_{-3} = 0$$





frequency components - ω_0 & $-\omega_0$.

$\omega_0 = 2\pi F_0$

$\omega_0 = 2\pi / T_0$

$T_0 = 2\pi / \omega_0$

$\sin(\omega_0 t)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega_0 k t}$$

$$C_k = \int_{T_0} x(t) e^{-jk\omega_0 t}\, dt.$$

$$x(t) = \sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

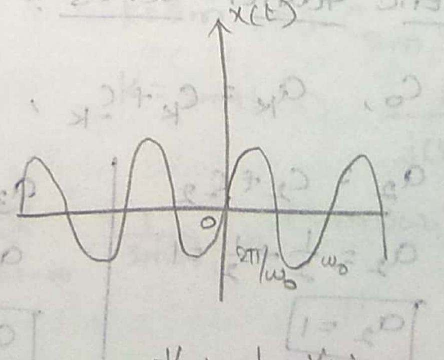
$$= \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

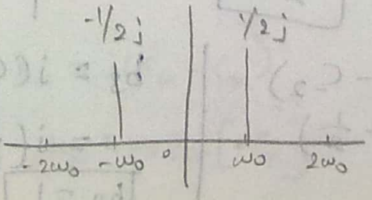
$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$





Find the Complex exponential fourier series & trigonometric fourier series of $x(t) = \cos 2t + \sin 3t$.

Given, $x(t) = \cos 2t + \sin 3t$

Time period of signal $\cos 2t$ is

$$T_1 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2} = \pi \text{ sec.}$$

Time period of signal $\sin 3t$ is

$$T_2 = \frac{2\pi}{\omega_0} = \frac{2\pi}{3}$$

$$\frac{T_1}{T_2} = \frac{\pi}{2\pi/3} = \frac{3}{2}$$

$$2T_1 = 3T_2$$

$$T = 2T_1 = 3T_2 = 2\pi \text{ sec.}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$x(t) = \cos 2t + \sin 3t$$

$$= \frac{e^{j2t} + e^{-j2t}}{2} + \frac{e^{j3t} - e^{-j3t}}{2j}$$

$$= \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t} + \frac{1}{2j}e^{j3t} - \frac{1}{2j}e^{-j3t}$$

$$= -\frac{1}{2j}e^{-j3t} + \frac{1}{2}e^{-j2t} + \frac{1}{2}e^{j2t} + \frac{1}{2j}e^{j3t}$$

$$= -\frac{1}{2j}e^{j3(1)t} + \frac{1}{2}e^{-j2(1)t} + \frac{1}{2}e^{j2(1)t} + \frac{1}{2j}e^{j3(1)t}$$

$$C_{-3}e^{-j3(\omega_0)t} + C_{-2}e^{-j2(\omega_0)t} + C_2 e^{j2(\omega_0)t} + C_3 e^{j3(\omega_0)t}$$

$$C_{-3} = \frac{-1}{2j}, \quad C_{-2} = \frac{1}{2}, \quad C_2 = \frac{1}{2}, \quad C_3 = \frac{1}{2j}, \quad C_0 = C_1 = C_{-1} \cdots = 0$$

Trigonometric fourier series :-

$$\frac{a_0}{2} = C_0, \quad a_K = C_K + C_{-K}, \quad b_K = j(C_K - C_{-K})$$

$\boxed{a_0 = 0}$

$a_2 = C_2 + C_{-2}$

$a_2 = \frac{1}{2} + \frac{1}{2} = 1$

$\boxed{a_2 = 1}$

$a_3 = C_3 + C_{-3}$

$a_3 = \frac{1}{2j} - \frac{1}{2j}$

$\boxed{a_3 = 0}$

$b_2 = j(C_2 - C_{-2})$

$= j\left(\frac{1}{2} - \frac{1}{2}\right)$

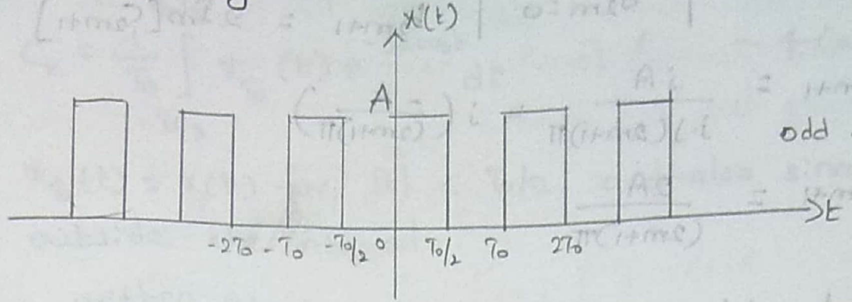
$\boxed{b_2 = 0}$

$b_3 = j(C_3 - C_{-3})$

$= j\left(\frac{1}{2j} + \frac{1}{2j}\right) = j\left(\frac{2}{2j}\right)$

$\boxed{b_3 = 1}$

Find the complex exponential fourier series & Trigonometric fourier series for given $x(t)$.



odd signal.

$$x(t) = \begin{cases} A, & 0 \le t \le T_0/2 \\ 0, & T_0/2 \le t \le T_0 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t}\, dt$$

$$C_k = \frac{1}{T_0}\int_0^{T_0/2} A \cdot e^{-jk\omega_0 t} \cdot dt + \int_{T_0/2}^{T_0} 0 \cdot$$

$$C_0 = \frac{1}{T_0}\int_0^{T_0/2} A \cdot dt$$

$$= \frac{A}{T_0}\left[T\right]_0^{T_0/2}$$

$$C_0 = \frac{A}{T_0} \times \frac{T_0}{2} = \frac{A}{2}$$

$$C_k = \frac{1}{T_0}\int_0^{T_0/2} A \cdot e^{-jk\omega_0 t}\, dt$$

$$= \frac{A}{T_0}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^{T_0/2}$$

$$= \frac{A}{T_0}\left[\frac{e^{-jk\omega_0 (T_0/2)}}{-jk\omega_0} - \frac{1}{-jk\omega_0}\right]$$

$$= \frac{A}{jk\omega_0 T_0}\left[1 - e^{-jk\omega_0 T_0/2}\right]$$

$$= \frac{A}{jk2\pi}\left[1 - e^{-jk\pi}\right]$$

$$= \frac{A}{jk2\pi}\left[1 - (-1)^k\right]$$

$e^{-jk\pi}$

$k=0 \Rightarrow 1$

$k=1 \Rightarrow e^{-j\pi} = \cos\pi - j\sin\pi$
$= -1$

$k=2 \Rightarrow e^{-2j\pi} \Rightarrow \cos 2\pi - j\sin 2\pi$
$= 1-0 = 1$

$k=3 \Rightarrow e^{-3j\pi} \Rightarrow \cos 3\pi - j\sin 3\pi$
$= -1$

$\vdots$ $(-1)^k$

$k = 2m, \Rightarrow C_{2m} = 0$

$$k = 2m+1 \Rightarrow \frac{A}{j(2m+1)\pi}$$

$$x(t) = A/2 + \frac{A}{j\pi}\sum_{m=-\infty}^{\infty} \frac{1}{2m+1} e^{j(2m+1)\omega_0 t}$$

Trigonometric fourier series

$$\frac{a_0}{2} = C_0, \quad a_k = C_k + C_{-k}, \quad b_k = j(C_k - C_{-k})$$

$$a_k = 2Re\left[C_k\right], \quad b_k = -2Im\left[C_k\right]$$

$$\frac{a_0}{2} = C_0 = A/2 \quad \Big| \quad a_{2m} = 0, \; b_{2m} = 0 \quad \Big| \quad a_{2m+1} = 2\,\mathrm{Re}\{C_{2m+1}\} = 0, \quad b_{2m+1} = 2\,\mathrm{Im}[C_{2m+1}]$$

$$C_{2m+1} = \frac{jA}{j \cdot j(2m+1)\pi} = j\left(\frac{-A}{(2m+1)\pi}\right)$$

$$b_{2m+1} = \frac{2A}{(2m+1)\pi}$$

Even and odd signals:- If a periodic signal $x(t)$ is even, then $b_K = 0$ and its fourier series contains only cosine terms.

$$x(t) = \frac{a_0}{2} + \sum_{K=1}^{\infty} a_K \cos K\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

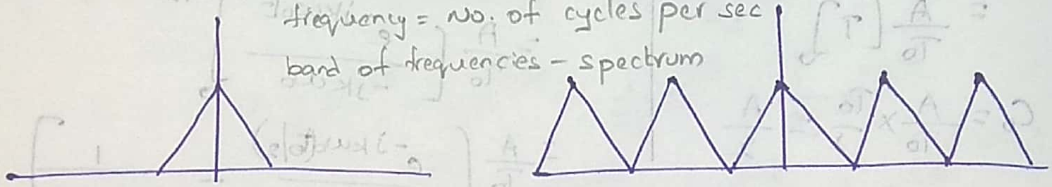
If $x(t)$ is odd, then $a_K = 0$ and its fourier series contains only sine terms

$$x(t) = \frac{a_0}{2} + \sum_{K=1}^{\infty} b_K \sin K\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}.$$

Fourier Transform:- Frequency domain representation of any periodic & aperiodic signals.

frequency = no. of cycles per sec

band of frequencies - spectrum



Aperiodic   periodic

We consider aperiodic signal we assume that the aperiodic signal is periodic with time $t = \infty$

$$T_0 \to \infty \cong \text{Aperiodic signal.}$$

From Fourier series to fourier transform:-

let $x(t)$ be a nonperiodic signal of finite duration that is

$$x(t) = 0, \quad |t| > T_1 \qquad -1$$

Let $x_{T_0}(t)$ be a periodic signal formed by repeating $x(t)$ with fundamental period T_0. If we let $T_0 \to \infty$, we have

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t) \qquad -2$$

The complex exponential fourier series of $x_{T_0}(t)$ is given by

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0} \quad - 3$$

Where $C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt \quad - 4(a)$

Since $x_{T_0}(t) = x(t)$ for $|t| < T_0/2$ and also since $x(t) = 0$ outside the interval

Can be written as

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \quad - 4(b)$$

Let us define $X(\omega)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad - 5$$

Then from (eq 4(b)) the complex fourier coefficients C_k can be expressed as

$$C_k = \frac{1}{T_0} X(k\omega_0) \quad - 6$$

Substituting eq-6 in eq-3, we have

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T_0} X(k\omega_0) e^{jk\omega_0 t}$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\omega_0) e^{jk\omega_0 t} \omega_0 \quad - 7$$

As $T_0 \to \infty$, $\omega_0 = 2\pi/T_0$ becomes infinitesimal $(\omega_0 \to 0)$

Thus, let $\omega_0 = \Delta\omega$. Then eq-7 becomes

$$x_{T_0}(t) \xrightarrow[T_0 \to \infty]{} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \Delta\omega \quad - 8$$

$$\therefore x(t) = \lim_{T_0 \to \infty} x_{T_0}(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t} \Delta\omega \quad - 9$$

The sum on the right hand side of eq-9 can be viewed as the area under the ~~fo~~ function $X(\omega) e^{j\omega t}$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad - 10$$

Fourier transform pair :- The function $X(\omega)$ defined by Eqn is called the fourier transform of $x(t)$, and eq- 3 defines the inverse fourier transform of $X(\omega)$. Symbolically they are denoted by

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = F^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad -12$$

and we say that $x(t) = X(\omega)$ form a fourier transform pair denoted by

$$x(t) \longleftrightarrow X(\omega) \quad -13$$

Fourier Spectra:-

The fourier transform $X(\omega)$ of $x(t)$ is, in general, complex and it can be expressed as

$$X(\omega) = |X(\omega)| e^{j\phi(\omega)} \quad -14$$

By analogy, with the terminology used for the complex fourier coefficients of a periodic signal $x(t)$, the fourier transform $X(\omega)$ of nonperiodic signal $x(t)$ is the frequency domain specification of $x(t)$ and is referred to as the spectrum (or fourier spectrum) of $x(t)$. The quantity $|X(\omega)|$ is called the magnitude spectrum of $x(t)$, and $\phi(\omega)$ is called the phase spectrum of $x(t)$.

Fourier series versus fourier transform:

	continuous time	Discrete time
Periodic	Fourier series	Discrete fourier transform
Aperiodic	fourier transform	Discrete fourier transform.

→ Fourier series for continuous - time periodic signals → discrete spectra.

→ Fourier transform for continuous - time aperiodic signals → continuous spectra.

If $x(t)$ is a real signal, then from eq-11 we get

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt \quad - 15$$

Then it follows that

$$X(-\omega) = X(\omega) \quad - 16(a)$$

$$|X(-\omega)| = |X(\omega)| \quad \phi(-\omega) = -\phi(\omega) \rightarrow 16(b).$$

Convergence of fourier transforms :-

Just as in the case of periodic signals, the sufficient conditions for the convergence of $X(\omega)$ are the following (again referred to as the dirichlet condition)

1.) $x(t)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad - 17$$

2.) $x(t)$ has a finite number of maxima and minima within any finite interval.

3.) $x(t)$ has a finite number of discontinuities within any finite interval, and each of these discontinuous is finite.

Although the above dirichlet conditions guarantee the existence of the fourier transform for a signal, if impulse functions are permitted in the transform, signals which do not satisfy those condition can have fourier transform.

Connection between the fourier transform and the laplace transform:

Eq-11 defines the fourier transform of $x(t)$ as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad - 18$$

The bilateral laplace transform of $x(t)$, as defined in eq(4.3) is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad - 19$$

Comparing Eq-18 and eq-19, we see that the fourier transform is a special case of the laplace transform in which $s = j\omega$, that is

$$|X(s)|_{s=j\omega} = F\{x(t)\} \quad - 20$$

Setting $s = \sigma + j\omega$ in eq-19, we have

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} \left[x(t)\, e^{-\sigma t}\right] dt$$

$$X(\sigma + j\omega) = F\left\{x(t)\, e^{-\sigma t}\right\} \quad - 21$$

Amplitude and phase spectra of a periodic signal :-

Let the complex fourier coefficients C_K in eq-20 be expressed as

$$C_K = |C_K|\, e^{j\phi_K}$$

A plot of $|C_K|$ versus the angular frequency ω is called the amplitude spectrum of the periodic signal $x(t)$, and a plot of ϕ_K versus ω is called the phase spectrum of $x(t)$. Since the index k assumes only integers, the amplitude and phase spectra are not continuous curves but appear only at the discrete frequencies $k\omega_0$ they are therefore referred to as discrete frequency spectra or line spectra.

For a real periodic signal $x(t)$ we have $C_{-K} = C_K^*$. Thus,

$$|C_{-K}| = |C_K|, \quad \phi_{-K} = \phi_K$$

-> Spectrum is set of or group of frequencies

-> The combination of amplitude and phase spectrums is called as frequency response.

Find the fourier transform of $x(t) = \delta(t)$

$$\delta(t) = \begin{cases} 1 \; ; \text{ for } t = 0 \\ 0 \; , \text{ otherwise} \end{cases}$$

$$X(\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \quad \cancel{\Rightarrow \int_{-\infty} \dots + \int_{t=0}^{\infty} 1 \cdot e^{-j\omega t} dt \neq 1}$$

$$= \int_{t=-\infty}^{\infty} \delta(t) \cdot e^{-j\omega t} dt$$

$$= \delta(t)\, e^{-j\omega t} \Big|_{t=0}$$

$$= 1 \cdot e^{-j\omega(0)}$$

$$= 1$$

$$\delta(t) \xleftrightarrow{F.T.} 1$$

$$\cancel{= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = \int_{\delta(t)=0}^{\infty} 1 \cdot e^{-j\omega t} dt = e^{-j\omega(0)} = 1}$$

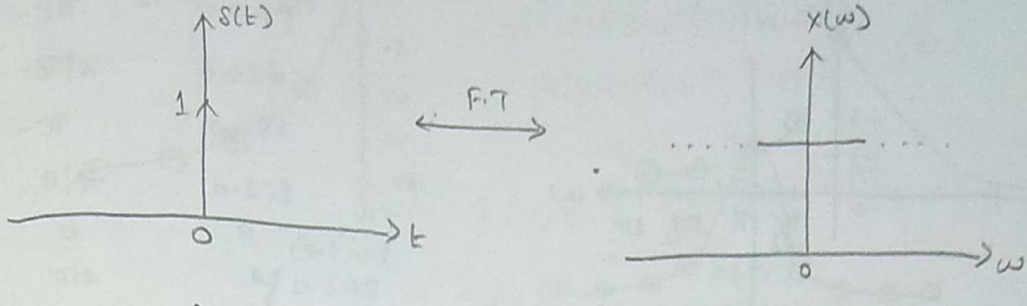
$$\Rightarrow \int_{-\infty}^{0} + \int_{t=0}^{\infty} 1 \cdot e^{-j\omega t} dt = 1$$

$$F\{\delta(t)\} = 1$$

$$\Rightarrow \delta(t) \xleftrightarrow{F.T.} 1$$

$X(\omega) = 1$

$F\{\delta(t)\} = 1$ (or) $\delta(t) \xleftrightarrow{F.T} 1$



Find the fourier transform of $x(t) = e^{-t}u(t)$.

$$u(t) = \begin{cases} 1 \;; & t \geq 0 \\ 0 \;; & t < 0 \end{cases}$$

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} \dots \times + \int_{0}^{\infty} 1 \cdot e^{-t} e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt \Rightarrow \left[\frac{e^{-t(1+j\omega)}}{-(1+j\omega)} \right]_0^{\infty}$$

$$= \frac{-1}{1+j\omega} \left[e^{-\infty} - e^{0} \right]$$

$$= \frac{+1}{1+j\omega}(+1) = \frac{1}{1+j\omega}$$

$$X(\omega) = \frac{1}{1+j\omega}$$

$|a+jb| = \sqrt{a^2+b^2}$ $\qquad \dfrac{1}{1+j\omega} \times \dfrac{1-j\omega}{1-j\omega} = \dfrac{1-j\omega}{1+\omega^2}$

$$X(\omega) = \frac{1}{1+\omega^2} + j\, \frac{-\omega}{1+\omega^2}$$

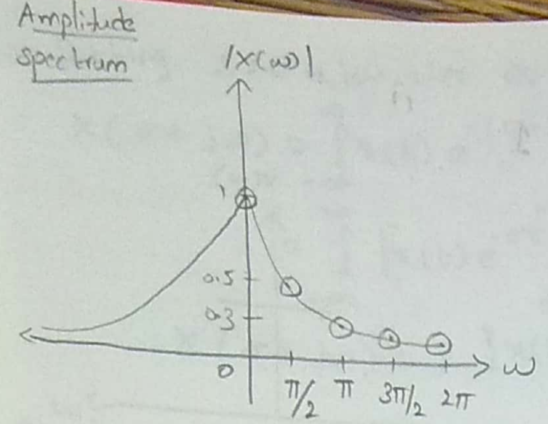
$$|X(\omega)| = \sqrt{\left(\frac{1}{1+\omega^2}\right)^2 + \frac{\omega^2}{(1+\omega^2)^2}} = \sqrt{\frac{1+\omega^2}{(1+\omega^2)^2}}$$

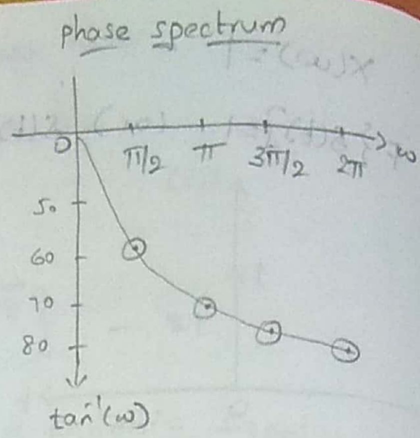
$$= \frac{1}{\sqrt{1+\omega^2}}$$

$$\angle X(\omega) = \phi(\omega) = \tan^{-1}\left(\frac{Im}{R}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{\omega}}{1}\right)$$

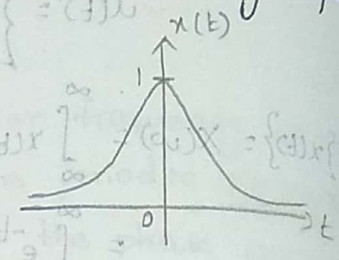
$$= \tan^{-1}\left(\frac{1}{\omega}\right)$$





Find the F.T of $x(t) = e^{-|t|}$ and also find frequency response

$$e^{-|t|} = \begin{cases} e^{-t}, & t \geq 0 \\ e^{t}, & t < 0 \end{cases}$$



$$F\{x(t)\} = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{0} x(t)\, e^{-j\omega t}\, dt + \int_{t=0}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t(1-j\omega)}\, dt + \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt$$

$$= \left[\frac{e^{t(1-j\omega)}}{1-j\omega}\right]_{-\infty}^{0} + \left[\frac{e^{-t(1+j\omega)}}{-(1+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{1-j\omega}\left[e^{0} - e^{-\infty}\right] + \frac{1}{1+j\omega}\left[-e^{-\infty} + e^{0}\right]$$

$$= \frac{1}{1-j\omega}\left[1-0\right] + \frac{1}{1+j\omega}\left[-0+1\right]$$

$$= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \Rightarrow \frac{1+j\omega+1-j\omega}{1+\omega^2} = \frac{2}{1+\omega^2}$$

$$X(\omega) = \frac{2}{1+\omega^2}$$

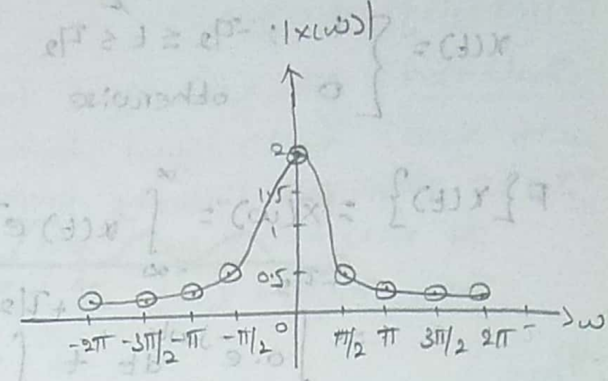
$$e^{-|t|} \overset{F.T}{\longleftrightarrow} \frac{2}{1+\omega^2}.$$

(or)

$$F\{e^{-|t|}\} = X(\omega) = \frac{2}{1+\omega^2}.$$

ω	$\lvert X(\omega)\rvert = \frac{2}{1+\omega^2}$
-2π	0.049
$-3\pi/2$	0.086
$-\pi$	0.184
$-\pi/2$	0.578
0	2
$\pi/2$	0.578
π	0.184
$3\pi/2$	0.086
2π	0.049

Magnitude spectrum:

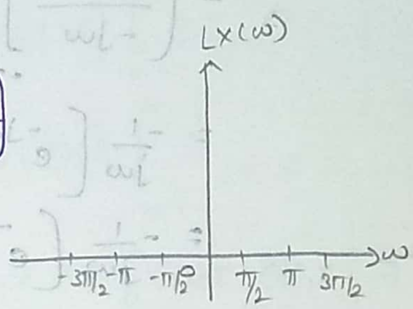


phase spectrum

$$\angle X(\omega) = \tan^{-1}\left[\frac{\text{Im part of } X(\omega)}{\text{Real part of } X(\omega)}\right]$$

$$= \tan^{-1}\left[\frac{0}{2/(1+\omega^2)}\right]$$

$$= 0.$$

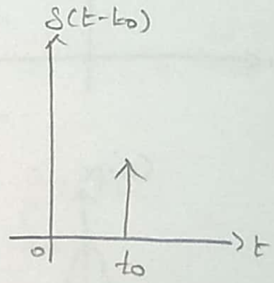


Find the fourier transform of $\delta(t-t_0)$

$$\delta(t-t_0) = \begin{cases} 1 \ ; & \text{at } t-t_0=0 \\ 0 \ ; & \text{otherwise.} \end{cases}$$

$t-t_0=0$

$t=t_0$



$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= 1 \cdot e^{-j\omega t} \Big/_{t=t_0}$$

$$= e^{-j\omega t_0}$$

$$\delta(t) \xleftrightarrow{F.T} 1$$

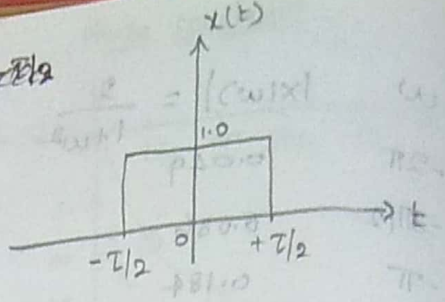
$$\delta(t-t_0) \xleftrightarrow{F.T} e^{-j\omega t_0}$$

$$\delta(t+t_0) = \begin{cases} 1 ; & t=-t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = 1 \cdot e^{-j\omega t} \Big/_{t=-t_0} = e^{j\omega t_0}$$

$$\delta(t+t_0) \xleftrightarrow{F.T} e^{j\omega t_0}$$

Find the F.T of $x(t) = \begin{cases} \cancel{1; |t| < T/2} \\ \end{cases}$



$$x(t) = \begin{cases} 1 & ; -\tau/2 \le t \le \tau/2 \\ 0 & \text{otherwise} \end{cases}$$

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{-\tau/2} \underbrace{0 \cdot e^{-j\omega t}\, dt}_{0} + \int_{-\tau/2}^{+\tau/2} 1 \cdot e^{-j\omega t}\, dt + \int_{+\tau/2}^{\infty} \underbrace{0 \cdot e^{-j\omega t}\, dt}_{0}$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega(\tau/2)} - e^{-j\omega(-\tau/2)}\right]$$

$$= -\frac{1}{j\omega}\left[e^{-\frac{j\omega\tau}{2}} - e^{\frac{j\omega\tau}{2}}\right]$$

$$= \frac{e^{\frac{j\omega\tau}{2}} - e^{-\frac{j\omega\tau}{2}}}{j\omega}$$

$$= \frac{2}{2}\left(\frac{e^{\frac{j\omega\tau}{2}} - e^{-\frac{j\omega\tau}{2}}}{j\omega}\right)$$

$$= \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}$$

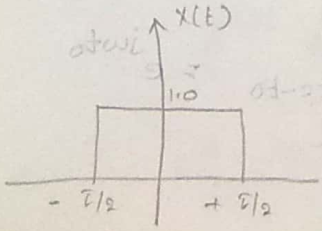
$$= \frac{2(\tau/2)}{\omega(\tau/2)} \sin\left(\frac{\omega\tau}{2}\right)$$

$$= \tau \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\omega\tau/2\right)}$$

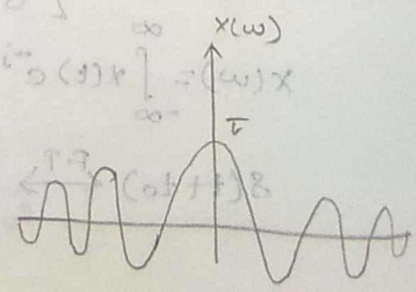
$$X(\omega) = \tau\, \mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \qquad \frac{\sin\theta}{\theta} = \mathrm{sinc}\,\theta$$

sub $\tau = 0$

Time domain $x(t)$ — frequency domain $X(\omega)$



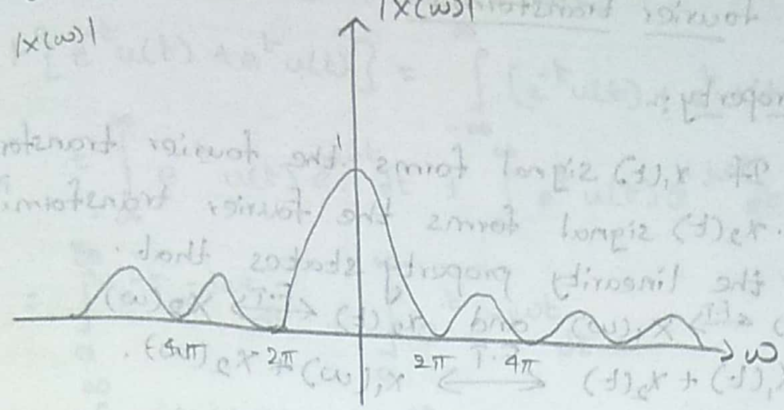
F.T with $\longleftrightarrow$ sinc function



Gate function (or)
Rectangular function

Magnitude spectrum

$|X(\omega)|$



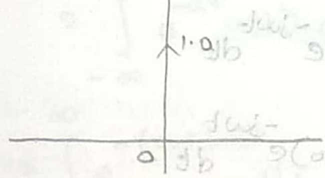
Phase spectrum

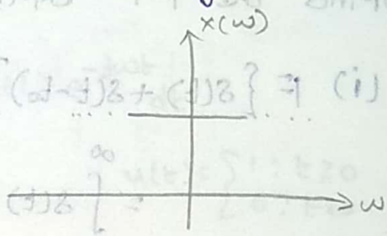
$\angle X(\omega)$

Time domain

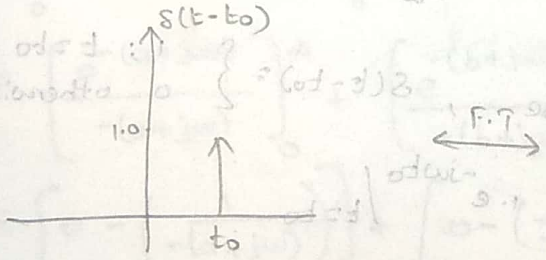
Frequency domain

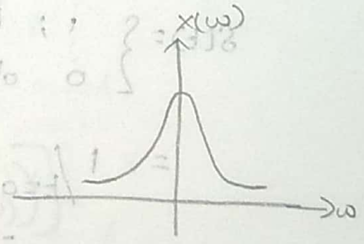
1)



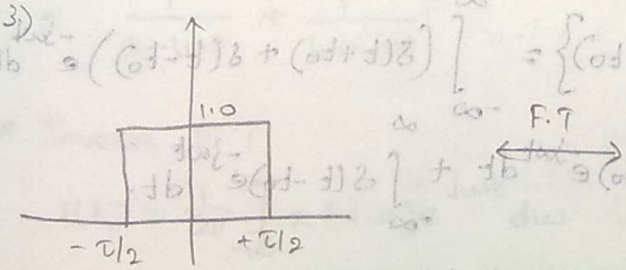


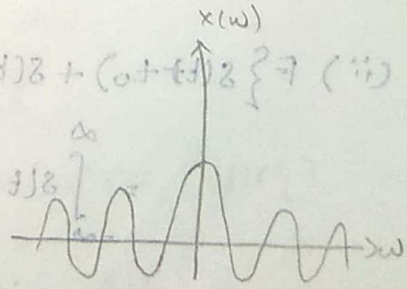
2)





3)





Properties of fourier transforms

1) Linearity property:-

Statement:- If $x_1(t)$ signal forms the fourier transform pair with $X_1(\omega)$. $x_2(t)$ signal forms the fourier transform pair with $X_2(\omega)$ then the linearity property states that.

If $x_1(t) \xleftrightarrow{F.T} X_1(\omega)$ and $x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

Then, $x_1(t) + x_2(t) \xleftrightarrow{F.T} X_1(\omega) + X_2(\omega)$.

Proof:-

$$F\{x_1(t) + x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) + x_2(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \left[x_1(t) e^{-j\omega t} + x_2(t) e^{-j\omega t}\right] dt$$

$$= \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$F\{x_1(t) + x_2(t)\} = X_1(\omega) + X_2(\omega)$$

$$\therefore\ x_1(t) + x_2(t) \xleftrightarrow{F.T} X_1(\omega) + X_2(\omega).$$

1) Find the F.T of (i) $\delta(t) + \delta(t-t_0)$ (ii) $\delta(t+t_0) + \delta(t-t_0)$

(i) $$F\{\delta(t) + \delta(t-t_0)\} = \int_{-\infty}^{\infty} (\delta(t) + \delta(t-t_0)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j\omega t} dt.$$

$$\delta(t) = \begin{cases} 1 ; & t = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \delta(t-t_0) = \begin{cases} 1 ; & t = t_0 \\ 0 & \text{otherwise} \end{cases}$$

$$= 1\Big/_{t=0} + 1 \cdot e^{-j\omega t_0}\Big/_{t=t_0}$$

$$= 1 + e^{-j\omega t_0}.$$

(ii) $$F\{\delta(t+t_0) + \delta(t-t_0)\} = \int_{-\infty}^{\infty} (\delta(t+t_0) + \delta(t-t_0)) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} \delta(t+t_0) e^{-j\omega t} dt + \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j\omega t} dt.$$

$$= e^{-j\omega t_0} + e^{j\omega t_0}$$

2) Find the F.T of $e^{-t}u(t) + e^{t}u(t)$

$$F\{e^{-t}u(t)+e^{t}u(t)\} = \int_{-\infty}^{\infty} \left(e^{-t}u(t)+e^{t}u(t)\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} e^{t}u(t)e^{-j\omega t}\,dt$$

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$= \int_{0}^{\infty} e^{-t}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{t}e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-(1+j\omega)t}\,dt + \int_{0}^{\infty} e^{(1-j\omega)t}\,dt$$

$$= \left[\frac{e^{-(1+j\omega)t}}{-(1+j\omega)}\right]_0^{\infty} + \left[\frac{e^{(1-j\omega)t}}{1-j\omega}\right]_0^{\infty}$$

$$= \left[0-\left(\frac{-1}{1+j\omega}\right)\right] + \left[0-\frac{1}{1-j\omega}\right]$$

$$= \frac{1}{1+j\omega} - \frac{1}{1-j\omega}.$$

3.) F.T of $e^{-at}u(t) + e^{-bt}u(t)$

$$F\{e^{-at}u(t)+e^{-bt}u(t)\} = \int_{-\infty}^{\infty}\left(e^{-at}u(t)+e^{-bt}u(t)\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} e^{-bt}u(t)e^{-j\omega t}\,dt$$

$$u(t) = \begin{cases} 1: & t \geq 0 \\ 0; & t < 0 \end{cases}$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt + \int_{0}^{\infty} e^{-(b+j\omega)t}\,dt$$

$$= \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} + \left[\frac{e^{-(b+j\omega)t}}{-(b+j\omega)}\right]_0^{\infty}$$

$$= \left[0-\left(\frac{1}{-(a+j\omega)}\right)\right] + \left[0-\left(\frac{1}{-(b+j\omega)}\right)\right]$$

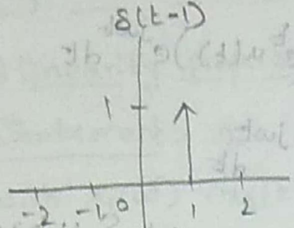
$$= \frac{1}{a+j\omega} + \frac{1}{b+j\omega}$$

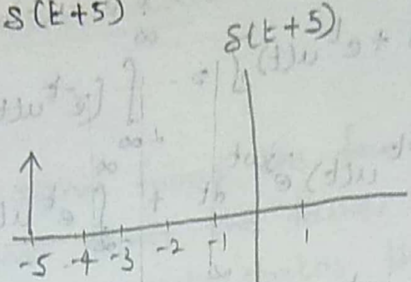
* Inverse F.T

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

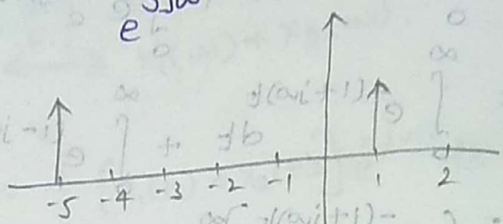
$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

4) F.T of $\delta(t-1) + \delta(t+5)$





$e^{-j\omega}$ $\qquad$ $e^{5j\omega}$



$$= e^{-j\omega} + e^{5j\omega}$$

2) Time shifting property :-

Statement:- The time shifting property states that if a signal $x(t)$ is shifted by to sec, the spectrum is modified by a linear phase shift of slope $-\omega t_0$, i.e.,

$$\text{If} \quad x(t) \xleftrightarrow{F.T} X(\omega)$$

$$\text{then} \quad x(t-t_0) \xleftrightarrow{F.T} X(\omega) \cdot e^{-j\omega t_0}$$

Proof:- $F\{x(t-t_0)\} = \int_{t=-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$

Let $t - t_0 = \lambda \Rightarrow \cancel{t = \lambda + t_0}$ $\quad$ Limits:-

$\Rightarrow t = \lambda + t_0$

L.L $\Rightarrow t \to -\infty$ then $\lambda \to -\infty$

$dt = d\lambda$ $\quad$ U.L $\Rightarrow t \to \infty$ then $\lambda \to +\infty$

$$= \int_{\lambda=-\infty}^{+\infty} x(\lambda)\, e^{-j\omega(\lambda + t_0)}\, d\lambda$$

$$= \int_{\lambda=-\infty}^{+\infty} x(\lambda) \cdot e^{-j\omega\lambda} \cdot e^{-j\omega t_0}\, d\lambda$$

$$= e^{-j\omega t_0} \int_{\lambda=-\infty}^{+\infty} x(\lambda) \cdot e^{-j\omega\lambda}\, d\lambda$$

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$F\{x(\lambda)\} = e^{-j\omega t_0}\, X(\omega).$$

$$\therefore \boxed{x(t-t_0) \xleftrightarrow{F.T} e^{-j\omega t_0} \cdot X(\omega)}$$

similarly, $\boxed{x(t+t_0) \xleftrightarrow{F.T} e^{j\omega t_0} \cdot X(\omega)}$

1.) Find the fourier transform of $\delta(t)$ and by using time shifting property.

(i) Find the F.T of $\delta(t-1)$

(ii) Find the F.T of $\delta(t+5)$

(iii) Find the F.T of $2\delta(t-1) + 3\delta(t+5)$.

Sol:- W.K.T,

$$\delta(t) \xleftrightarrow{F.T} 1$$

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$x(t-t_0) \xleftrightarrow{F.T} X(\omega) \cdot e^{-j\omega t_0}$$

$$\delta(t-1) \xleftrightarrow{F.T} 1 \cdot e^{-j\omega(1)}$$

$$\therefore \delta(t-1) \xleftrightarrow{F.T} e^{-j\omega}$$

(ii) $\delta(t-(-5)) \xleftrightarrow{F.T} e^{-j\omega(-5)}$

$$\delta(t+5) \xleftrightarrow{F.T} e^{5j\omega}$$

(iii) $2\delta(t-1) + 3\delta(t+5) \xleftrightarrow{F.T} 2e^{-j\omega} + 3e^{5j\omega}$

3.) Time Scaling property :-

Statement :- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Proof:- $F\{x(at)\} = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$

let $at = \lambda$

$\Rightarrow t = \frac{\lambda}{a}$

$\Rightarrow dt = \frac{1}{a} d\lambda$

Limits :-

U.L If $t \to \infty$ then $\lambda = at \to \infty$

$t \to -\infty$ then $\lambda \to -\infty$

Now the integration becomes

$$= \int_{\lambda=-\infty}^{+\infty} x(\lambda) \cdot e^{-j\omega\left(\frac{\lambda}{a}\right)} \cdot \frac{1}{a} d\lambda$$

$$= \frac{1}{a} \int_{\lambda=-\infty}^{\infty} x(\lambda) \cdot e^{-j\left(\frac{\omega}{a}\right)\lambda} d\lambda$$

change λ to t.

$$= \frac{1}{a} \int_{t=-\infty}^{\infty} x(t) \cdot e^{-j\left(\frac{\omega}{a}\right)t} dt$$

$$= \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

$$\therefore x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$a > 0$, $a < 0$

$$x(-at) \xleftrightarrow{F.T} -\frac{1}{a} X\left(\frac{\omega}{a}\right)$$

Find the F.T of $e^{-t} \cdot u(t)$ and by using time scaling property find the F.T of (i) $e^{-at} \cdot u(t)$ (ii) $e^{-5t} u(t)$ (iii) $e^{3t} u(t)$.

Sol:- W.K.T

$$\underbrace{e^{-t} u(t)}_{x(t)} \xleftrightarrow{F.T} \underbrace{\frac{1}{1+j\omega}}_{X(\omega)}$$

(i) $e^{-at} \cdot u(t) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

$$\xleftrightarrow{F.T} \frac{1}{|a|} \cdot \frac{1}{1+j\left(\frac{\omega}{a}\right)}$$

$$\xleftrightarrow{F.T} \frac{1}{a} \cdot \frac{a}{a+j\omega}$$

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}.$$

(ii) $e^{-5t} \cdot u(t) \xleftrightarrow{F.T} \frac{1}{5} \cdot \frac{1}{1+j\left(\frac{\omega}{5}\right)}$

$$e^{-5t} \cdot u(t) \xleftrightarrow{F.T} \frac{1}{\cancel{5}} \cdot \frac{\cancel{5}}{5+j\omega}$$

$$e^{-5t} \cdot u(t) \xleftrightarrow{F.T} \frac{1}{5+j\omega}.$$

(iii) $e^{3t} u(t) \xleftrightarrow{F.T} \frac{1}{-3} \cdot \frac{1}{1+j\left(\frac{\omega}{-3}\right)}$

$$e^{3t} u(t) \xleftrightarrow{F.T} \frac{1}{-3+j\omega}.$$

4) Duality Property :-

Statement :- If $x(t) \xleftrightarrow{F.T} X(\omega)$

Then $X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

Proof :- We know that the inverse fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} X(\omega) e^{+j\omega t}\, d\omega$$

$$\Rightarrow 2\pi\, x(t) = \int_{-\infty}^{+\infty} X(\omega) e^{+j\omega t}\, d\omega .$$

Swape the variables t & $\omega \Rightarrow t = -\omega$

$$\Rightarrow 2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t}\, dt \quad \text{— ①}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \text{— ②}$$

from ① & ② $\boxed{X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)}$

Find the F.T of $x(t) = 1$ using duality property.

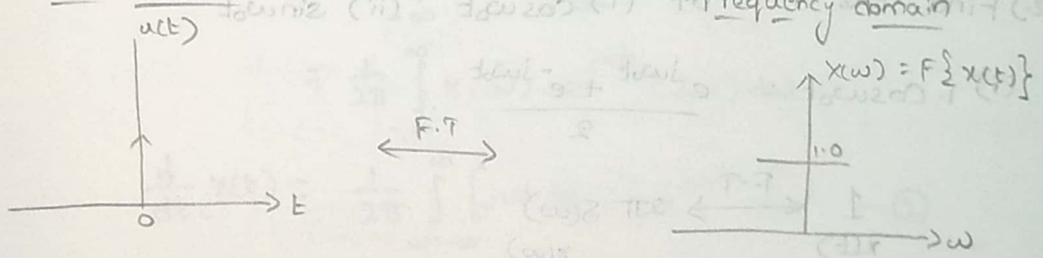
$$\underset{x(t)}{\delta(t)} \xleftrightarrow{F.T} \underset{X(\omega)}{1}$$

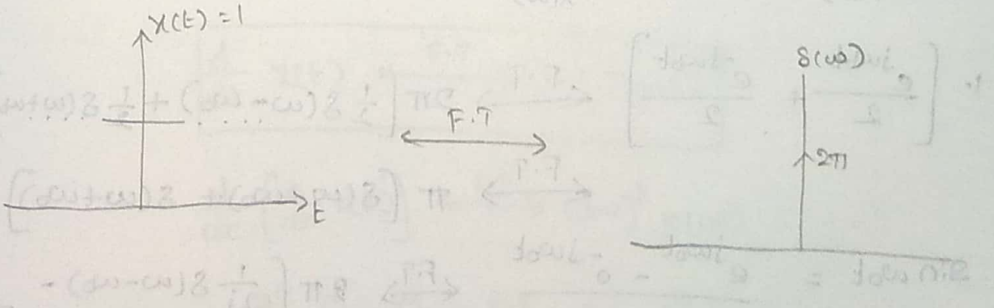
Duality $X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

$$1 \xleftrightarrow{F.T} 2\pi\, \delta(-\omega)$$

$$\boxed{1 \xleftrightarrow{F.T} 2\pi\, \delta(\omega)}$$

Time domain — Frequency domain





5) Frequency shifting property:-

Statement :- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $x(t) \cdot e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0)$

Proof:- $F\{x(t) \cdot e^{j\omega_0 t}\} = \int_{-\infty}^{\infty} x(t) e^{j\omega_0 t} \cdot e^{-j\omega t} dt$

$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j(\omega-\omega_0)t} dt$

$= X(\omega - \omega_0)$

$$\boxed{x(t) \cdot e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega - \omega_0)}$$

1) Find the F.T of $e^{j\omega_0 t}$

F.T of $e^{-j\omega_0 t}$

F.T of $\{1 \cdot e^{j\omega_0 t}\} = ?$

$\underset{x(t)}{1} \xleftrightarrow{F.T} \underset{X(\omega)}{2\pi\,\delta(\omega)}$

$1 \cdot e^{j\omega_0 t} \xleftrightarrow{F.T} 2\pi\,\delta(\omega - \omega_0)$

F.T of $e^{-j\omega_0 t}$

$1 \cdot e^{j(-\omega_0)t} \xleftrightarrow{F.T} 2\pi\,\delta(\omega + \omega_0)$.

2) Find the F.T of (i) $\cos\omega_0 t$ (ii) $\sin\omega_0 t$

(i) $1 \cdot \cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$\underset{x(t)}{1} \xleftrightarrow{F.T} \underset{X(\omega)}{2\pi\,\delta(\omega)}$

$1 \cdot \left[\dfrac{e^{j\omega_0 t}}{2} + \dfrac{e^{-j\omega_0 t}}{2}\right] \xleftrightarrow{F.T} 2\pi\left[\frac{1}{2}\delta(\omega-\omega_0) + \frac{1}{2}\delta(\omega+\omega_0)\right]$

$\xleftrightarrow{F.T} \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$

(ii) $\sin\omega_0 t = \dfrac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \xleftrightarrow{F.T} 2\pi\left[\frac{1}{2j}\delta(\omega-\omega_0) - \frac{1}{2j}\delta(\omega+\omega_0)\right]$

$\sin\omega_0 t \xleftrightarrow{F.T} \dfrac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$

3.) Find the F.T of (i) $x(t)\cos\omega_0 t$ (ii) $x(t)\sin\omega_0 t$ in terms of $X(\omega)$.

(i) $x(t)\cos\omega_0 t = \frac{1}{2}x(t)e^{j\omega_0 t} + \frac{1}{2}x(t)e^{-j\omega_0 t} \xleftrightarrow{F.T}$

$\frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$

$$x(t)\cos\omega_0 t \xleftrightarrow{F.T} \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

(ii) $x(t)\sin\omega_0 t = \frac{1}{2j}x(t)e^{j\omega_0 t} - \frac{1}{2j}x(t)e^{-j\omega_0 t} \xleftrightarrow{F.T}$

$\frac{1}{2j}X(\omega-\omega_0) - \frac{1}{2j}X(\omega+\omega_0)$

$$x(t)\sin\omega_0 t \xleftrightarrow{F.T} \frac{1}{2j}\left[X(\omega-\omega_0)-X(\omega+\omega_0)\right]$$

6.) Differentiation in Time Domain Property:-

Statement:- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $\frac{d}{dt}x(t) \xleftrightarrow{F.T} j\omega\, X(\omega)$.

Proof:- The inverse fourier transform is

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega \quad \text{— ①}$$

differentiation on both sides.

$$\frac{d}{dt}x(t) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot\frac{d}{dt}\left[e^{j\omega t}\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t}(j\omega)\,d\omega$$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\left[j\omega\, X(\omega)\right]e^{j\omega t}\,d\omega \quad \text{— ②}$$

$$\therefore \boxed{\frac{d}{dt}x(t) \xleftrightarrow{F.T} j\omega\, X(\omega)}$$

$$\frac{d}{dt}\left[\frac{d}{dt}x(t)\right] \xleftrightarrow{F.T} (j\omega)^2 X(\omega)$$

$$\frac{d^n}{dt^n}\left[x(t)\right] \xleftrightarrow{F.T} (j\omega)^n X(\omega).$$

7.) Differentiation in frequency domain property:-

Statement:- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $t\,x(t) \xleftrightarrow{F.T} j\frac{d}{d\omega}X(\omega)$ (or)

$-jt\,x(t) \xleftrightarrow{F.T} \frac{d}{d\omega}X(\omega)$

Proof:- $X(\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$ — ①

differentiation on both sides w.r.t 'ω'

$$\frac{d}{d\omega}X(\omega) = \frac{d}{d\omega}\int_{-\infty}^{+\infty} \left[x(t)\, e^{-j\omega t}\, dt\right]$$

$$= \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}(e^{-j\omega t})\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t}\, dt$$

$$= -j\int_{-\infty}^{+\infty} \left[t\, x(t)\right] e^{-j\omega t}\, dt$$

$$\Rightarrow \frac{1}{-j}\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} t\cdot x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow j\frac{d}{d\omega}X(\omega) = \int_{-\infty}^{\infty} t\cdot x(t)\, e^{-j\omega t}\, dt$$

$$\therefore \boxed{t\cdot x(t) \xleftrightarrow{F.T} j\frac{d}{d\omega}X(\omega)}$$

1.) Find the F.T of $x(t) = te^{-t}u(t)$.

$$\underbrace{e^{-t}u(t)}_{x(t)} \xleftrightarrow{F.T} \underbrace{\frac{1}{1+j\omega}}_{X(\omega)}$$

$$t\left[e^{-t}u(t)\right] \xleftrightarrow{F.T} j\frac{d}{d\omega}\left[\frac{1}{1+j\omega}\right]$$

$$\xleftrightarrow{F.T} j\left(\frac{-1}{(1+j\omega)^2}\right)(j)$$

$$t(e^{-t}u(t)) \xleftrightarrow{F.T} \frac{1}{(1+j\omega)^2}$$

8) Conjugate Property:-

Statement:- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $\boxed{x^*(t) \xleftrightarrow{F.T} X^*(-\omega)}$

Proof:-
$$F\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t} dt$$

$$= \left[\int_{-\infty}^{\infty} x(t) e^{+j\omega t} dt\right]^*$$

$$\left[e^{-j\omega t}\right]^* = e^{j\omega t}$$

$$= \left[\int_{-\infty}^{\infty} x(t) e^{-j(-\omega)t} dt\right]^*$$

$$= \left[X(-\omega)\right]^* = X^*(-\omega)$$

$$\therefore \boxed{x^*(t) \xleftrightarrow{F.T} X^*(-\omega)}$$

9.) Parseval's theorem:-

Statement:- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$

Finding the energy in the time domain is same as the frequency domain.

Proof:-
$$\int_{-\infty}^{+\infty} x(t)\, x^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, X^*(-\omega)\, d\omega$$

since $x^*(t) \xleftrightarrow{F.T} X^*(-\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

1.) Find the energy of the signal $e^{-t}u(t)$. and also find the energy in the frequency domain.

$$x(t) \xleftrightarrow{F.T} X(\omega)$$

$$e^{-t}u(t) \xleftrightarrow{F.T} \frac{1}{1+j\omega}$$

Energy in the time domain is $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$E = \int_{-\infty}^{\infty} \left|\frac{1}{1+j\omega}\right|^2 d\omega$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{1+\omega^2}}\right)^2 d\omega$$

$$= \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega$$

$$= \left[\tan^{-1}(\omega)\right]_{-\infty}^{\infty} = \cancel{\tan} \frac{\pi}{\cancel{2}}$$

$$= \left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right)$$

$$= \frac{\pi}{2} - \frac{3\pi}{2} = -\pi$$

(or)

$$= \int_{-\infty}^{\infty} \left[e^{-t} u(t)\right]^2 dt$$

$$\int_0^{\infty} e^{-t^2} (1) + \int_{-\infty}^{0} 0$$

$$= \left[\frac{e^{-t^2}}{-2t}\right]_0^{\infty} = \frac{1}{2}$$

$$= \left[(\cdot) e^{-\infty} + \frac{e^{-2(0)}}{2}\right] = \frac{1}{2}$$

Find the F.T of $e^{-at} u(t)$ and also find F.T $e^{at} u(t)$

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}$$

$$x(t) \xleftrightarrow{F.T} x^*(\omega) = x(-\omega)$$

$$e^{-at} u(t) \xleftrightarrow{F.T} \frac{1}{a-j\omega}.$$

$$e^{at} u(t) \xleftrightarrow{F.T} \frac{1}{a+j\omega}.$$

Inverse fourier transform:-

1) Find the inverse F.T of $x(\omega) = 1$

I.F.T $\quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega = \frac{1}{2\pi} \left[\frac{e^{j\omega t}}{jt}\right]_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi} [\infty - 0] \quad \text{Not defined}$$

$$\delta(t) \xleftrightarrow{F.T} 1$$

$$1 \xleftrightarrow{F.T} 2\pi\, \delta(\omega)$$

$$F.T[1] = \int_{-\infty}^{\infty} e^{-j\omega t} dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\infty}^{\infty} = [\infty - 0]$$

Not defined

2) Find the I.F.T of $x(\omega) = 2\pi\, \delta(\omega)$

I.F.T $\quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\, \delta(\omega) e^{j\omega t} d\omega$$

$$= \int_{-\infty}^{\infty} \delta(\omega) e^{-j(\omega) t} d\omega$$

$$x(t) = e^{-j(0)t} \Big/_{\omega=0}$$

$$= 1$$

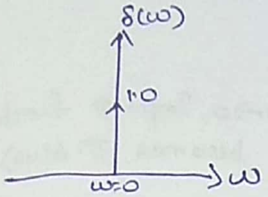
$X(\omega) = 2\pi\,\delta(\omega)$

Sol:- Inverse F.T is $x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} X(\omega) e^{j\omega t}\, d\omega$

$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} 2\pi\,\delta(\omega) e^{+j\omega t}\, d\omega\right]$

$= \frac{2\pi}{2\pi}\left(\int_{-\infty}^{\infty} \delta(\omega) e^{+j\omega t}\, d\omega\right)$

Here we know, $\delta(\omega) = \begin{cases} 1 & , \omega = 0 \\ 0 & , \omega \neq 0 \end{cases}$



$= \int_{-\infty}^{0^-} \delta(\omega) e^{j\omega t}\, d\omega + \int_{\omega=0} \delta(\omega) e^{j\omega t}\, d\omega + \int_{\omega=0^+}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega$

$= 0 + \delta(\omega) \cdot e^{j\omega t} / _{\omega=0} + 0$

$= \delta(0) \cdot e^{j(0)t}$

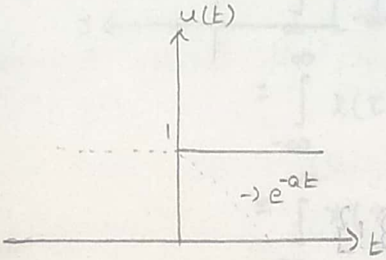
$= (1)(1) = 1$

$\therefore \boxed{F^{-1}\{2\pi\,\delta(\omega)\} = 1}$

(or) $\boxed{1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)}$

Find the F.T of $x(t) = u(t)$.

Inverse F.T is $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t}\, d\omega$



$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$X(\omega) = \int_0^{\infty} 1 \cdot e^{j\omega t}\, dt$

$= \left[\frac{e^{j\omega t}}{j\omega}\right]_0^{\infty}$

$= -\frac{1}{j\omega}$

$\cancel{u(t) = e^{-at} \cdot u(t)}$

$u(t) = \underset{a \to 0}{Lt}\ e^{-at} u(t)$

Apply F.T on both sides

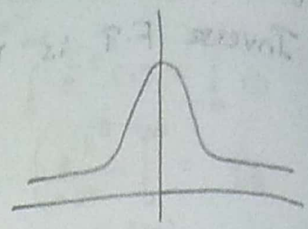
$F\{u(t)\} = F\{\underset{a \to 0}{Lt}\ e^{-at} \cdot u(t)\}$

$= \underset{a \to 0}{Lt}\ F\{e^{-at} \cdot u(t)\}$

$= \underset{a \to 0}{Lt}\ \frac{1}{a + j\omega}$

$= \underset{a \to 0}{Lt}\left[\frac{1}{a + j\omega} \times \frac{a - j\omega}{a - j\omega}\right]$

$$= \underset{a \to 0}{Lt} \frac{a - j\omega}{a^2 + \omega^2}$$



$$= \underset{a \to 0}{Lt} \left[\frac{a}{a^2+\omega^2} - j \frac{\omega}{a^2+\omega^2} \right]$$

$$= \underset{a \to 0}{Lt} \left[\frac{a}{a^2+\omega^2} \right] - j \frac{\omega}{0+\omega^2}$$

$$= \underset{a \to 0}{Lt} \left(\frac{a}{a^2+\omega^2} \right) - \frac{j}{\omega}$$

$$\int_{-\infty}^{+\infty} \frac{a}{a^2+\omega^2} = \tan^{-1}(\omega/a) = \pi$$

as $\omega \to 0$, Impulse function then π becomes $\pi \cdot \delta(\omega)$

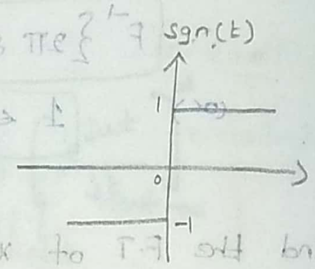
$$= \pi + \frac{1}{j\omega}.$$

$$\therefore \boxed{F\{u(t)\} = \pi\, \delta(\omega) + \frac{1}{j\omega}}$$

$$1 \xleftrightarrow{F.T} 2\pi\, \delta(\omega)$$

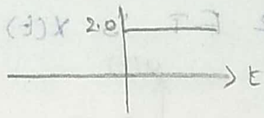
2.) $F\{sgn(t)\}$

Sol:-
$$sgn(t) = \begin{cases} 1 & \text{if } t > 0 : t > 0^+ \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 : t < 0^- \end{cases}$$



$$\boxed{sgn(t) = 2u(t) - 1}$$

$$sgn(t) + 1 = 2u(t)$$



Apply F.T on both sides

$$F\{sgn(t)\} = F\{2u(t) - 1\}$$

$$= F\{2u(t)\} - F\{1\}$$

$$= 2F\{u(t)\} - F\{1\}$$

$$= 2\left[\pi\, \delta(\omega) + \frac{1}{j\omega}\right] - 2\pi\, \delta(\omega)$$

$$= 2\pi\, \delta(\omega) + \frac{2}{j\omega} - 2\pi\, \delta(\omega)$$

$$= \frac{2}{j\omega}$$

$$\therefore F\{sgn(t)\} = \frac{2}{j\omega}$$

(or) $\boxed{sgn(t) \xleftrightarrow{F.T} \frac{2}{j\omega}}$

Integration in time domain property :

Statement :- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.T} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Proof:- $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\,d\omega$

$$\int_{-\infty}^{t} x(t)\,dt = \int_{-\infty}^{t}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\,dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\int_{-\infty}^{t} e^{j\omega t}\,dt\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\frac{e^{j\omega t}}{j\omega}\right)_{-\infty}^{t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{X(\omega)}{j\omega}\right)e^{j\omega t}\,d\omega$$

$$F\left[\int_{-\infty}^{t} x(t)\,dt\right] = \frac{1}{j\omega}X(\omega)$$

$$\int_{-\infty}^{t} x(t)\,dt \xleftrightarrow{F.T} \frac{1}{j\omega}X(\omega)$$

Proof:- $\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$

$$F\left\{\int_{-\infty}^{t} x(\tau)\,d\tau\right\} = F\left\{\int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau\right\}$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau\right]e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} u(t-\tau)\,e^{-j\omega t}\,dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\cdot F\{u(t-\tau)\}\,d\tau$$

$x(t-t_0) \longleftrightarrow X(\omega)\cdot e^{-j\omega t_0}$

$$= \int_{-\infty}^{\infty} x(\tau)\cdot\left(\frac{1}{j\omega} + \pi\,\delta(\omega)\right)e^{-j\omega\tau}\,d\tau$$

$$= \left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]\int_{-\infty}^{\infty} x(\tau)\cdot e^{-j\omega\tau}\,d\tau$$

$$= \left[\frac{1}{j\omega} + \pi\,\delta(\omega)\right]X(\omega)$$

$$= \frac{X(\omega)}{j\omega} + \pi X(\omega)\cdot\delta(\omega)$$

$X(\omega)\cdot\delta(\omega) = X(0)\cdot\delta(\omega)$

$$= \frac{X(\omega)}{j\omega} + \pi X(0)\cdot\delta(\omega)$$

$$\boxed{\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T} \frac{X(\omega)}{j\omega} + \pi X(0)\cdot\delta(\omega)}$$

Integration in frequency domain property of fourier transform

Statement:- If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $\frac{x(t)}{t} \xleftrightarrow{F.T} \int_{\omega}^{\infty} X(\omega)\, d\omega$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-\omega t}\, dt$$

$$\int_{\omega}^{\infty} X(\omega)\, d\omega = \int_{\omega}^{\infty} \left[\int_{-\infty}^{\infty} x(t)\, e^{-\omega t}\, dt \right] d\omega$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-\omega t}}{-t} \right]_{\omega}^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{e^{-\infty t}}{-t} - \frac{e^{-\omega t}}{-t} \right) dt$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t}\, e^{-\omega t}\, dt$$

$$\int_{\omega}^{\infty} X(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{x(t)}{t}\, e^{-\omega t}\, dt$$

$$\therefore \boxed{\frac{x(t)}{t} \xleftrightarrow{F.T} \int_{\omega}^{\infty} X(\omega)\, d\omega}$$

Convolution in Time domain property of fourier transform

Statement:-

If $x_1(t) \xleftrightarrow{F.T} X_1(\omega)$

and $x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

then

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \xleftrightarrow{F.T} X_1(\omega) \cdot X_2(\omega)$$

Proof:-

$$F\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right] d\tau$$

$$F\{x_2(t-\tau)\} = \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt$$

$$= X_2(\omega)\, e^{-j\omega\tau}$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, X_2(\omega)\, e^{-j\omega\tau}\, d\tau$$

$$= X_2(\omega) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= X_2(\omega) \cdot X_1(\omega)$$

$$= X_1(\omega) \cdot X_2(\omega)$$

$$x_1(t) * x_2(t) \xleftrightarrow{F.T} X_1(\omega) \cdot X_2(\omega)$$

Convolution in frequency domain property of fourier transform (or) Multiplication in time domain.

Statement:- If $x_1(t) \xleftrightarrow{F.T} X_1(\omega)$

and $x_2(t) \xleftrightarrow{F.T} X_2(\omega)$

then $x_1(t) \cdot x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$

Proof:- $$F\{x_1(t) \cdot x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) \cdot x_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(\lambda)\, e^{j\lambda t}\, d\lambda\right] e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(\lambda)\, d\lambda\, e^{-j(\omega-\lambda)t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(t)\, e^{-j(\omega-\lambda)t}\, dt \int_{-\infty}^{\infty} X_2(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\omega-\lambda)\, X_2(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

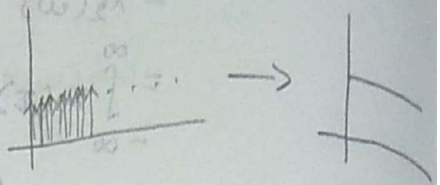
$$\therefore x_1(t) \cdot x_2(t) \xleftrightarrow{F.T} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

→ Convolution in time domain leads to multiplication in frequency domain.

→ Convolution in frequency domain leads to multiplication in time domain.

Find the F.T of $u(t)$ using integration in time domain property.

Hint:- $u(t) = \int \underbrace{\delta(t)}_{x(t)}$



$\delta(t) \xleftrightarrow{F.T} 1$

Here $x(t) = \delta(t)$ & $X(\omega) = 1$

$$\int \delta(t)\,dt = u(t) \xleftrightarrow{F.T} \frac{1}{j\omega} + \pi X(0) \cdot \delta(\omega)$$

$$\xleftrightarrow{F.T} \frac{1}{j\omega} + \pi \delta(\omega)$$

$$\delta(t) = \frac{d}{dt} u(t)$$

$\delta(t) - \int \delta(t)$

$u(t) - \int u(t)$

$tri(t)$

$\frac{d}{dt}$ tri $\to$ Gate function

Hilbert Transform

* Fourier, Laplace, and Z-transforms change from the time-domain representation of a signal to the frequency-domain representation of the signal.
* The resulting two signals are equivalent representations of the same signal in terms of time or frequency
* In contrast, the Hilbert transform does not involve a change of domain, unlike many other transforms
* Strictly speaking, the Hilbert transform is not a transform in this sense

-> First, the result of a Hilbert transform is not equivalent to the original signal, rather it is a completely different signal.

-> Second, the Hilbert transform does not involve a domain change, i.e., the Hilbert transform of a signal $x(t)$ is another signal denoted by $\hat{x}(t)$ in the same domain (i.e., time domain)

* The Hilbert transform of a signal $x(t)$ is a signal $\hat{x}(t)$ whose frequency components lag the frequency components of $x(t)$ by 90°.

* A delay of $\pi/2$ at all frequencies

-> $e^{j2\pi f_0 t}$ will become $e^{j2\pi f_0 t} = -je^{j2\pi f_0 t}$

-> $e^{-j2\pi f_0 t}$ will become $e^{-j(2\pi f_0 t - \pi/2)} = je^{-j2\pi f_0 t}$

* $\hat{x}(t)$ has exactly the same frequency components present in $x(t)$ with the same amplitude except there is a 90° phase delay.

* At positive frequencies, the spectrum of the signal is multiplied by $-j$

* The Hilbert transform of $x(t) = A\cos(2\pi f_0 t + \theta)$ is $A\cos(2\pi f_0 t + \theta - 90^\circ) = A\sin(2\pi f_0 t + \theta)$

* At negative frequencies, it is multiplied by $+j$

→ This is equivalent to saying that the spectrum (Fourier transform) of the signal is multiplied by $-j\,\mathrm{sgn}(f)$.

* Assume that $x(t)$ is real and has no DC component: $X(f)|_{f=0} = 0$, then

$x(t) \xleftrightarrow{f.t} X(\omega)$

$$F\left[\hat{x}(t)\right] = -j\,\mathrm{sgn}(f)\, X(f) \qquad F\{\hat{x}(t)\} \xleftrightarrow{F.T} X(\omega)\left(-j\,\mathrm{sgn}(\omega)\right)$$

$$F^{-1}\left[-j\,\mathrm{sgn}(f)\right] = \frac{1}{\pi t}$$

$$\hat{x}(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, d\tau$$

→ The operation of the Hilbert transform is equivalent to a convolution, i.e., filtering.

* Obviously performing the Hilbert transform on a signal is equivalent to a $90°$ phase shift in all its frequency components.

* Therefore, the only change that the Hilbert transform performs on a signal is changing its phase.

* The amplitude of the frequency components of the signal do not change by performing the Hilbert-transform.

* On the other hand, since performing the Hilbert transform changes cosines into sines, the Hilbert transform $\hat{x}(t)$ of a signal $x(t)$ is orthogonal to $x(t)$.

* Also, since the Hilbert transform introduces a $90°$ phase shift, carrying it out twice causes a $180°$ phase shift, which can cause a sign reversal of the original signal.

* Evenness and Oddness

* The Hilbert transform of an even signal is odd, and the Hilbert transform of an odd signal is even

→ proof

* If $x(t)$ is even, then $X(f)$ is a real and even function

* Therefore, $-j\,\mathrm{sgn}(f)\, X(f)$ is an imaginary and odd function.

* Hence, its inverse Fourier transform $\hat{x}(t)$ will be odd

* If $x(t)$ is odd, then $X(f)$ is imaginary and odd

* Thus $-j\,\mathrm{sgn}(f)\, X(f)$ is real and even

* Therefore, $\hat{x}(t)$ is even

* Sign Reversal

* Applying the Hilbert - transform operation to a signal twice causes a sign reversal of the signal, i.e.,

$$\hat{\hat{x}}(t) = -x(t)$$

-> proof

$$F[\hat{\hat{x}}(t)] = [-j\,sgn(f)]^2 X(f)$$

$$F[\hat{\hat{x}}(t)] = -X(f)$$

* $X(f)$ does not contain any impulses at the origin

Fourier transform of periodic signals:

Consider a continuous periodic signal $x(t)$ having frequency ω_0 then fourier series of $x(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \quad \text{--- (1)}$$

Apply F.T on both sides

$$F\{x(t)\} = F\left\{\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}\right\}$$

If $x(t) \xleftrightarrow{F.T} X(\omega)$

then $x(t)e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega-\omega_0)$ [from frequency shifting property]

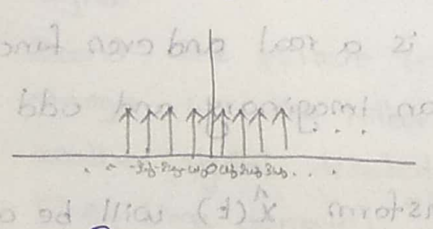
$$X(\omega) = \sum_{k=-\infty}^{\infty} C_k \, 2\pi\, \delta(\omega - k\omega_0)$$

Since $F\{C_k\} = 2\pi \cdot C_k \cdot \delta(\omega)$ $\quad [C_k = 1 \cdot C_k]$

By using frequency shifting property

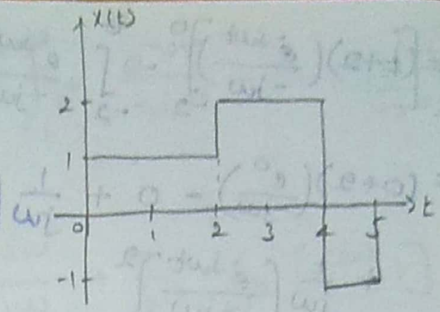
$$F\left\{\sum C_k e^{jk\omega_0 t}\right\} = 2\pi \sum_{k=-\infty}^{\infty} C_k \cdot \delta(\omega - k\omega_0)$$

complex exponential fourier series coefficient.



Fourier series	Fourier transform
$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ where $C_k = \frac{1}{T_0}\int_{T_0} x(t) e^{-jk\omega_0 t} dt$	Inverse F.T $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

1.) Find the fourier transform of



Sol:- $x(t) = \begin{cases} 1 & , 0 \le t \le 2 \\ 2 & , 2 \le t \le 4 \\ -1 & , 4 \le t \le 5 \end{cases}$

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} 0 \cdot e^{-j\omega t} dt + \int_{0}^{2} 1 \cdot e^{-j\omega t} dt + \int_{2}^{4} 2 \cdot e^{-j\omega t} dt + \int_{4}^{5} -1 \cdot e^{-j\omega t} dt + \int_{5}^{\infty} 0 \cdot e^{-j\omega t} dt.$$

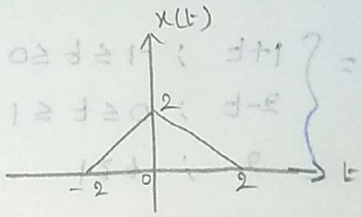
$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^2 + 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_2^4 + (-1)\left[\frac{e^{-j\omega t}}{-j\omega}\right]_4^5$$

$$= \frac{-e^{-j2\omega} + 1 - 2e^{-j4\omega} + 2e^{-j2\omega} + e^{-j5\omega} - e^{-j4\omega}}{j\omega}$$

$$= \frac{1 + e^{-j2\omega} - 3e^{-j4\omega} + e^{-j5\omega}}{j\omega}$$

$$X(\omega) = \frac{1 + e^{-j2\omega} - 3e^{-j4\omega} + e^{-j5\omega}}{j\omega}$$

2.) Find the F.T of



$x(t) = \begin{cases} 2+t & , t \le 0 \\ 2-t & , t \ge 0 \end{cases}$

$$F.T\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{-2} x(t)\, e^{-j\omega t} dt + \int_{-2}^{0} x(t)\, e^{-j\omega t} dt + \int_{0}^{2} x(t)\, e^{-j\omega t} dt + \int_{2}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$= \int_{-\infty}^{-2} 0 \cdot e^{-j\omega t} dt + \int_{-2}^{0} (t+2)\, e^{-j\omega t} dt + \int_{0}^{2} (2-t)\, e^{-j\omega t} dt + \int_{2}^{\infty} 0 \cdot e^{-j\omega t} dt$$

$$= \int_{-2}^{0} (t+2)\, e^{-j\omega t} dt + \int_{0}^{2} (2-t)\, e^{-j\omega t} dt$$

$$= (t+2)\int_{-2}^{0} e^{-j\omega t} dt - \int_{-2}^{0} \frac{d}{dt}(t+2)\left[\int_{-2}^{0} e^{-j\omega t} dt\right] dt + (2-t)\int_{0}^{2} e^{-j\omega t} dt - \int_{0}^{2} \frac{d}{dt}(2-t)\left[\int_{0}^{2} e^{-j\omega t} dt\right] dt$$

$$= \left[(t+2)\left(\frac{e^{-j\omega t}}{-j\omega}\right)\right]_{-2}^{0} - \int_{-2}^{0} \frac{e^{-j\omega t}}{-j\omega}\, dt + \left[(2-t)\left[\frac{e^{-j\omega t}}{-j\omega}\right]\right]_{0}^{2} - \int_{0}^{2} \frac{e^{-j\omega t}}{j\omega}$$

$$= (0+2)\left(\frac{e^{0}}{-j\omega}\right) - 0 + \frac{1}{j\omega}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-2}^{0} + 0 - \left(\frac{2}{-j\omega}\right)(e^{0})$$

$$+ \frac{1}{j\omega}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{2}$$

$$= -\frac{2}{j\omega} - \frac{1}{j^2\omega^2}\left[e^{0} - e^{2j\omega}\right] + \frac{2}{j\omega} - \frac{1}{j^2\omega^2}\left[e^{-2j\omega} - e^{0}\right]$$

$$= -\frac{2}{j\omega} + \frac{1}{\omega^2}\left[1 - e^{2j\omega}\right] + \frac{2}{j\omega} + \frac{e^{-2j\omega}}{\omega^2} - \frac{1}{\omega^2}$$

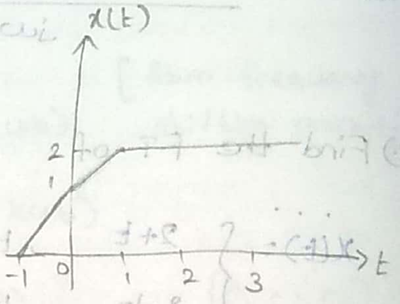
$$= -\cancel{\frac{2}{j\omega}} + \cancel{\frac{1}{\omega^2}} - \frac{e^{2j\omega}}{\omega^2} + \cancel{\frac{2}{j\omega}} + \frac{e^{-2j\omega}}{\omega^2} - \cancel{\frac{1}{\omega^2}}$$

$$= \frac{e^{-2j\omega}}{\omega^2} - \frac{e^{2j\omega}}{\omega^2} = \frac{1}{\omega^2}\left[e^{-2j\omega} - e^{2j\omega}\right] = -\frac{1}{\omega^2}\left[e^{2j\omega} - e^{-2j\omega}\right]$$

$$= -\frac{2}{\omega^2}\left(\frac{e^{2j\omega} - e^{-2j\omega}}{2}\right) = -\frac{2j}{\omega^2}\left[\frac{e^{2j\omega} - e^{-2j\omega}}{2j}\right]$$

$$= \frac{-2j}{\omega^2}\sin 2\omega = \frac{-2j}{\omega}$$

3.) Find the F.T of



$$x(t) = \begin{cases} 1+t & ; -1 \le t \le 0 \\ 2-t & ; 0 \le t \le 1 \\ 2 & ; t \ge 1 \end{cases}$$

$$X(\omega) = \int_{-1}^{0} (1+t)e^{-j\omega t}\, dt + \int_{0}^{1} (2-t)e^{-j\omega t}\, dt + \int_{1}^{\infty} 2e^{-j\omega t}\, dt$$

$$= (1+t)\int_{-1}^{0} e^{-j\omega t}\, dt - \int_{-1}^{0}\left[\frac{d}{dt}(1+t)\int_{-1}^{0} e^{-j\omega t}\, dt\right] dt + (2-t)\int_{0}^{1} e^{-j\omega t}\, dt$$

$$- \int_{0}^{1}\left[\frac{d}{dt}(2-t)\int_{0}^{1} e^{-j\omega t}\, dt\right] dt + 2\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{1}^{\infty}$$

$$= \left[(1+t)\left(\frac{e^{-j\omega t}}{-j\omega}\right)\right]_{-1}^{0} - \int_{-1}^{0} \frac{e^{-j\omega t}}{-j\omega}\, dt + \left[(2-t)\left(\frac{e^{-j\omega t}}{-j\omega}\right)\right]_{0}^{1}$$

$$- \int_{0}^{1} (-1)\, \frac{e^{-j\omega t}}{-j\omega}\, dt - \frac{2}{j\omega}\left[0 - e^{-j\omega}\right]$$

$$= (1+0)\left(\frac{e^{0}}{-j\omega}\right) - 0 + \frac{1}{j\omega}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-1}^{0} + \frac{e^{-j\omega}}{-j\omega} - (2-0)\left(\frac{e^{0}}{-j\omega}\right)$$

$$-\frac{1}{j\omega}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{0}^{1} + \frac{2}{j\omega}e^{-j\omega}$$

$$= -\frac{1}{j\omega} + \frac{1}{\omega^2}\left[e^{0} - e^{j\omega}\right] - \frac{e^{-j\omega}}{j\omega} + \frac{2}{j\omega} - \frac{1}{\omega^2}\left[e^{-j\omega} - 1\right]$$

$$+ \frac{2}{j\omega}e^{-j\omega}$$

$$= -\frac{1}{j\omega} + \frac{1}{\omega^2} - \frac{e^{j\omega}}{\omega^2} - \frac{e^{-j\omega}}{j\omega} + \frac{2}{j\omega} - \frac{e^{-j\omega}}{\omega^2} + \frac{1}{\omega^2} + \frac{2}{j\omega}e^{-j\omega}$$

$$= \frac{2}{\omega^2} + \frac{1}{j\omega} + \frac{1}{j\omega}e^{-j\omega} - \frac{e^{j\omega}}{\omega^2} - \frac{e^{-j\omega}}{\omega^2}.$$