17/7/06.

UNIT – II

FOURIER SERIES

Fourier Series representation of periodic signal :-

If we calculate the response of time-variant system for non-sinusoidal input, if it is periodic signal, we may use Fourier series for analysis, it is two types.

1. Trignometric Fourier series
2. Complex exponential Fourier series.

If it is aperiodic (or) non-periodic signal, we may use Fourier transform for analysis.

Trignometric Fourier Series :-

It is used for analysis of non-sinusoidal signals. If it is periodic signal, it can be written as weighted sum of infinite sinusoidal cosinusoidal of frequencies that are integral multiples of frequency of the given signal, added with D.C. term.

Mathematically,

If $g(t) = g(t \pm T) \ \forall t$, then

$$g(t) = a_0 + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \ldots\ldots + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \ldots\ldots$$

$$\boxed{g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)}$$

'T' is the timeperiod of the given signal in sec.

$\omega \rightarrow$ freq of given Signal in Radian/sec.

$$\omega = \frac{2\pi}{T} \text{ (rad/sec)}.$$

a_0, a_n, b_n are Fourier coefficients.

ORTHOGONAL FUNCTIONS :-

If $g_1(t)$ and $g_2(t)$ are periodic Signals with period 'T', then the integration of product of $g_1(t)$ and $g_2(t)$ w.r.t time over the interval $-T/2$ to $T/2$ is zero, then the two signals $g_1(t)$ & $g_2(t)$ are orthogonal functions.

Required definite integral formulae :-

$$\int_0^T \cos(n\omega t)\,dt = \int_0^T \sin(n\omega t)\,dt = 0.$$

$$\int_0^T \cos(m\omega t)\sin(n\omega t)\,dt = 0 \quad \forall\ m \& n.$$

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n. \end{cases}$$

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \end{cases}$$

EULER'S FORMULAE :-

If $g(t)$ is a periodic non-sinusoidal signal, it can be represented by using trignometric Fourier series over the interval $-\frac{T}{2} \leq t \leq \frac{T}{2}$ is given by

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

where

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} g(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} g(t) \sin(n\omega t)\, dt.$$

a_0, a_n, b_n are called as Euler's formulae.

Expressions for Fourier Series coefficients (a_0, b_n, a_n)

If $g(t)$ satisfies the property

$g(t) = g(t \pm T)$ for all 'T', then it is periodic signal. It can be represented by the trignometric Fourier series as

$$g(t) = a_0 + a_1 \cos(\omega t) + \dots + a_n \cos(n\omega t) + \dots$$
$$+ b_1 \sin(\omega t) + \dots + b_n \sin(n\omega t) + \dots \quad \text{--- (1)}$$

Integrate both sides of the above equation w.r.t 't' over the interval $-T/2$ to $T/2$, we get

$$\int_{-T/2}^{T/2} g(t)\,dt = \int_{-T/2}^{T/2} \Big[a_0 + a_1\cos(\omega t) + \dots + a_n\cos(n\omega t) + \dots + b_1\sin(\omega t) + \dots + b_n\sin(n\omega t) + \dots\Big]dt$$

$$= \int_{-T/2}^{T/2} a_0\,dt + \int_{-T/2}^{T/2} a_1\cos(\omega t)dt + \dots\dots + \int_{-T/2}^{T/2} a_n\cos(n\omega t)dt$$

$$+ \dots + \int_{-T/2}^{T/2} b_1\sin(\omega t)\,dt + \dots + \int_{-T/2}^{T/2} b_n\sin(n\omega t)dt + \dots\dots$$

$$= a_0 \Big|_{-T/2}^{T/2} + 0 + \dots + 0 + \dots + 0 + \dots + 0 + \dots$$

$$= a_0\left(\frac{T}{2} + \frac{T}{2}\right) = a_0 T$$

$$\Rightarrow \boxed{a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt}$$

Expression for 'a_n' :-

Multiply with $\cos(n\omega t)$ on both sides of eq.(1) and integrate w.r.t time over the interval $\frac{-T}{2}$ to $\frac{T}{2}$, we get

$$\int_{-T/2}^{T/2} g(t)\cos(n\omega t)dt = \int_{-T/2}^{T/2}\Big[a_0 + a_1\cos\omega t + \dots + a_n\cos(n\omega t) + \dots + b_1\sin\omega t + \dots + b_n\sin(n\omega t) + \dots\Big]\cos(n\omega t)\,dt$$

$$= \int_{-T/2}^{T/2} a_0 \cos(n\omega t)\,dt + \int_{-T/2}^{T/2} a_1 \cos(\omega t)\cos(n\omega t)\,dt + \cdots$$

$$+ \int_{-T/2}^{T/2} a_n [\cos(n\omega t)]^2 dt + \cdots + \int_{-T/2}^{T/2} b_1 \cos(n\omega t)\sin(\omega t)\,dt$$

$$+ \cdots + \int_{-T/2}^{T/2} b_n \sin(n\omega t)\cos(n\omega t)\,dt + \cdots$$

$$= 0 + 0 + \cdots + \int_{-T/2}^{T/2} a_n (\cos(n\omega t))^2 dt + \cdots + 0 + \cdots + 0 +$$

$$= a_n \left(\frac{T}{2}\right)$$

$$\Rightarrow \int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt = a_n \left(\frac{T}{2}\right)$$

$$\Rightarrow \boxed{a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt}$$

Expression for "b_n" :-

Multiply eq. 1 with $\sin(n\omega t)$ on both sides and integrate w.r.t time over the interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$, we get.

$$\int_{-T/2}^{T/2} g(t)\sin(n\omega t)\,dt = \int_{-T/2}^{T/2} \Big[a_0 + a_1 \cos\omega t + \cdots + a_n \cos(n\omega t) + \cdots + b_1 \sin\omega t + \cdots + b_n \sin(n\omega t) + \cdots\Big] (\sin(n\omega t))\,dt$$

$$= \int_{-T/2}^{T/2} a_0 \sin(n\omega t)\,dt + \int_{-T/2}^{T/2} a_1 \cos(\omega t)\sin(n\omega t)\,dt + \cdots$$

$$\cdots + \int_{-T/2}^{T/2} a_n \cos(n\omega t)\sin(n\omega t)\,dt + \cdots + \int_{-T/2}^{T/2} b_1 \sin(\omega t)\sin(n\omega t)\,dt$$

$$+ \cdots + \int_{-T/2}^{T/2} b_n (\sin n\omega t)^2\,dt + \cdots$$

$$= 0 + 0 + \cdots + 0 + \cdots \int_{-T/2}^{T/2} b_n (\sin n\omega t)^2\,dt + \cdots$$

$$= b_n \left(\frac{T}{2}\right)$$

$$\Rightarrow \boxed{b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\sin(n\omega t)\,dt}$$

NOTE :-

If we represent the coefficients as

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\sin(n\omega t)\,dt$$, then the representation is

$$g(t) = a_0 + 2\left[\sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)\right]$$

I. Representation of ~~8~~ Fourier Series for Symmetric property of periodic signals :-

1. ~~Even~~ Symmetry Periodic Signal.

If the periodic signal satisfies the following condition,

$$g(t) = g(-t) \quad \forall t ,$$ then it is called even signal and it satisfies even symmetry.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt = \frac{1}{T}\int_{-T/2}^{0} g(t)\,dt + \frac{1}{T}\int_{0}^{T/2} g(t)\,dt$$

Put $-t = \lambda$; $-dt = d\lambda$; $g(-t) = g(\lambda)$

$g(-\lambda) = g(t)$

$$\Rightarrow \quad a_0 = \frac{1}{T}\int_{-T/2}^{0} g(-\lambda)\,-d\lambda + \frac{1}{T}\int_{0}^{T/2} g(t)\,dt$$

$$= \frac{1}{T}\int_{0}^{T/2} g(t)\,dt + \frac{1}{T}\int_{0}^{T/2} g(t)\,dt$$

$$= \frac{2}{T}\int_{0}^{T/2} g(t)\,dt .$$

$\therefore$ For even symmetry,

$$\boxed{a_0 = \frac{2}{T}\int_{0}^{T/2} g(t)\,dt .}$$

$$\therefore a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt$$

$$\Rightarrow \boxed{a_n = \frac{4}{T}\int_{0}^{T/2} g(t)\cos(n\omega t)\,dt}$$

$$\therefore b_n = \frac{2}{T}\int_{0}^{T/2} g(t)\sin(n\omega t)\,dt = 0 \quad \therefore \boxed{b_n = 0}$$

- For an even symmetry,

$\therefore$

$$\boxed{g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t)}$$

Note:-
Fourier Series representation of even periodic signals containing only cosine terms.

~~odd~~ Symmetry :-

If a periodic signal $g(t)$, satisfies the condition $g(t) = -g(-t)$ is said to be odd periodic signal.

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt \qquad \because \int_{-a}^{a} g(x)\,dx = 0 \text{ where } g(x) = -g(-x)$$

$$\therefore a_0 = \underline{\underline{0}}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt = 0 \quad [\because g(t)\cos(n\omega t) \text{ is odd signal}]$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\,\sin(n\omega t)\,dt$$

$$= \frac{4}{T}\int_{0}^{T/2} g(t)\,\sin(n\omega t)\,dt \quad [\because g(t)\sin(n\omega t) \text{ is even signal}]$$

$$\therefore \boxed{g(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega t)}$$

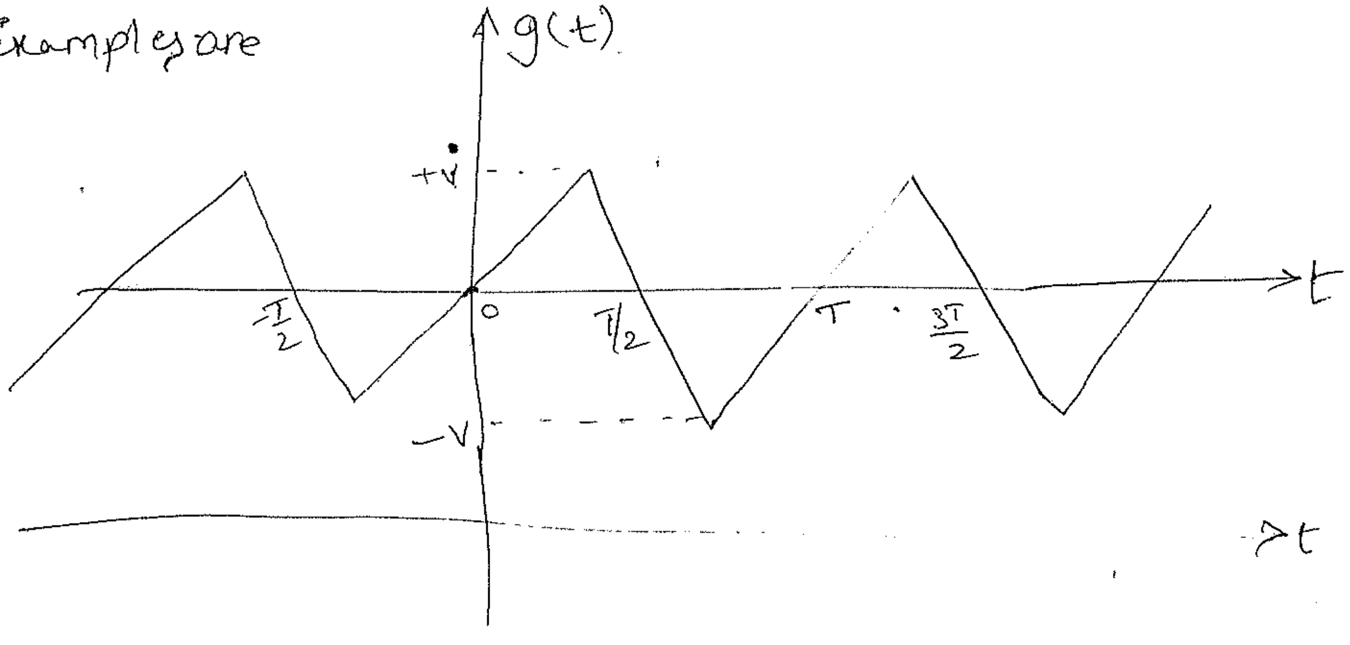
Note:- Fourier Series representation of odd periodic Signal contains only Sine terms.

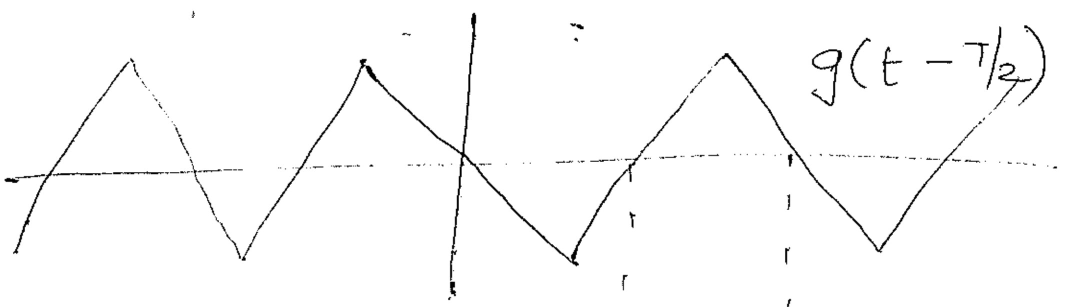
Half-wave Symmetry:-

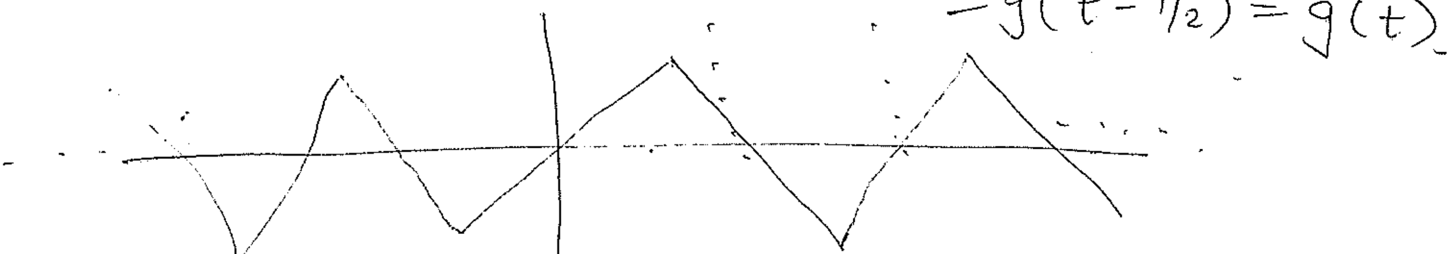
A periodic Signal $g(t)$ satisfies the Condition,

$$g(t) = -g(t \pm T/2),$$ then it is Said to be half-wave Symmetry.

Examples are







For half-wave sym.,

$a_0 = 0$

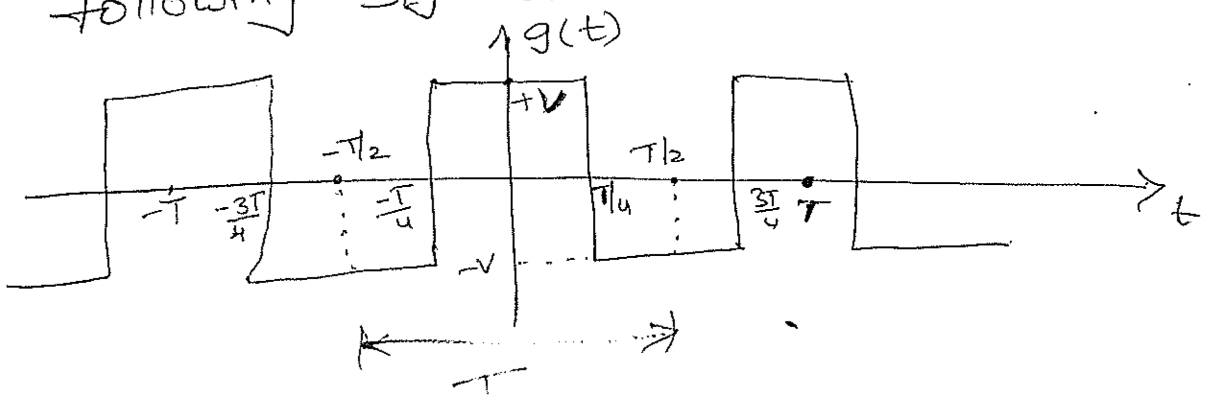
$a_n = b_n = 0$ for n is even

If n is odd, then

$$a_n = \frac{4}{T}\int_0^{T/2} g(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} g(t)\sin(n\omega t)\,dt.$$

* Determine trignometric Fourier Series of the following Signals.



→ $g(t)$ satisfies periodicity with period 'T' and it also satisfies even symmetry property.

$$g(t) = g(-t)$$

Fourier series rep. of $g(t)$ is :-

$$g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t).$$

$$g(t) = \begin{cases} -V \ ; & -T/2 \le t \le -T/4 \\ +V \ ; & -\frac{T}{4} \le t \le T/4 \\ -V \ ; & \frac{T}{4} \le t \le T/2. \end{cases}$$

(or)

$$g(t) = \begin{cases} +V \ ; & 0 \le t \le T/4 \\ -V \ ; & T/4 \le t \le 3T/4 \\ +V \ ; & 3T/4 \le t \le T. \end{cases}$$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, dt.$$

$$= \frac{1}{T}\left[\int_{-T/2}^{-T/4} (-V)\, dt + \int_{-T/4}^{T/4} V\, dt + \int_{T/4}^{T/2} (-V)\, dt \right]$$

$$= \frac{1}{T}\left[(-V)\left[\int_{-T/2}^{-T/4} dt - \int_{-T/4}^{T/4} dt + \int_{T/4}^{T/2} dt \right] \right]$$

$$= \frac{-V}{T}\left[\frac{-T}{4} + \frac{T}{2} - \frac{T}{4} - \frac{T}{4} + \frac{T}{2} - \frac{T}{4} \right]$$

$$= \underline{\underline{0}}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t)\cos(n\omega t)\,dt$$

$$= \frac{2}{T}\left[\int_{-T/2}^{-T/4}(-V)\cos(n\omega t)\,dt + \int_{-T/4}^{T/4}(+V)\cos(n\omega t)\,dt + \int_{T/4}^{T/2}(-V)\cos(n\omega t)\,dt\right]$$

$$= \frac{2V}{T}\left[-\int_{-T/2}^{-T/4}\cos(n\omega t)\,dt + \int_{-T/4}^{T/4}\cos(n\omega t)\,dt - \int_{T/4}^{T/2}\cos(n\omega t)\,dt\right]$$

$$= \frac{2V}{T}\left[-\frac{\sin(n\omega t)}{n\omega}\Bigg|_{-T/2}^{-T/4} + \frac{\sin(n\omega t)}{n\omega}\Bigg|_{-T/4}^{T/4} - \frac{\sin(n\omega t)}{n\omega}\Bigg|_{T/4}^{T/2}\right]$$

$$= \frac{2V}{n\omega T}\left[-\sin\left(n\omega \times \frac{-T}{4}\right) + \sin\left(n\omega \times \frac{-T}{2}\right) + \sin\left(n\omega \times \frac{T}{4}\right) - \sin\left(n\omega \times \frac{-T}{4}\right) - \sin\left(\frac{n\omega T}{2}\right) + \sin\left(n\omega \frac{T}{4}\right)\right]$$

$$= \frac{2V}{n\omega T}\left[\sin\left(n\frac{2\pi}{T}\times\frac{T}{4}\right) - \sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right) + \sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right) + \sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right) - \sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{2}\right) + \sin\left(n\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\right)\right]$$

$$= \frac{2V}{n\omega T}\left[\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) + \sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) + \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{2V}{n\cdot\frac{2\pi}{T}\cdot T}\left[4\sin\left(\frac{n\pi}{2}\right)\right]$$

$$\boxed{\sin(n\pi) = 0}$$

$$= \frac{4V}{n\pi}\sin\left(n\frac{\pi}{2}\right)$$

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & \text{for } n = 1, 5, 9, 13, \ldots \\ -1 & \text{for } n = 3, 7, 11, 15, \ldots \\ 0 & \text{for } n = \text{even.} \end{cases}$$

$\therefore \quad a_n = \frac{4V}{n\pi}(1) \quad \text{for } n = 1, 5, 9, \ldots$

$a_n = \frac{-4V}{n\pi} \quad \text{for } n = 3, 7, 11, 15, \ldots$

As $g(t)$ is even function, $b_n = \frac{2}{T}\int_{-T/2}^{T/2} g(t) \sin n\omega t \, dt$

($g(t)\sin(n\omega t)$ is odd function) $= \underline{\underline{0}}$.

$\therefore$ Fourier Series is

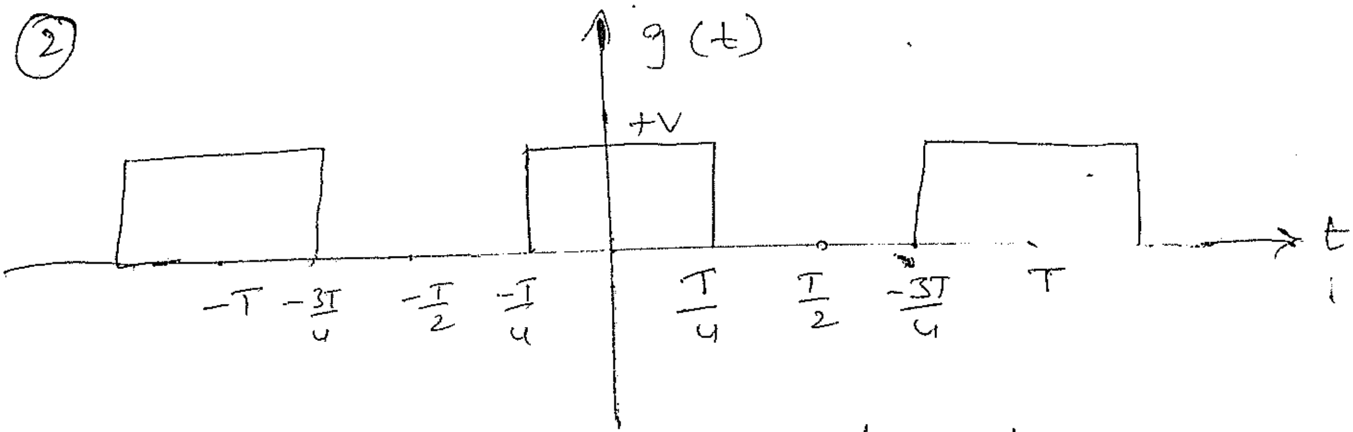
$$g(t) = \sum_{n=1}^{\infty} \frac{4V}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\omega t)$$

$$= a_1 \cos(\omega t) + a_2 \cos(2\omega t) + a_3 \cos(3\omega t) + \ldots$$

$$= \frac{4V}{\pi}\cos\omega t \mp \frac{4V}{3\pi}\cos(3\omega t) + \frac{4V}{5\pi}\cos(5\omega t) - \ldots$$

$$= \frac{4V}{\pi}\left[\cos(\omega t) - \frac{1}{3}\cos(3\omega t) + \frac{\cos(5\omega t)}{5} - \frac{\cos(7\omega t)}{7} + \ldots\right]$$

(2)



Train of rectangular pulses

$$\rightarrow \quad g(t) = \begin{cases} 0 & ; \ -T/2 \leq t \leq -T/4 \\ V & ; \ -T/4 \leq t \leq T/4 \\ 0 & ; \ T/4 \leq t \leq T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt = \frac{1}{T}\left[\int_{-T/2}^{-T/4} (0)\,dt + \int_{-T/4}^{T/4} V\,dt + \int_{T/4}^{T/2} 0\,dt\right]$$

$$= \frac{1}{T}\left[V\left(\frac{T}{4} + \frac{T}{4}\right)\right]$$

$$= \frac{V}{2} \,//$$

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} V \cos(n\omega t)\,dt$$

$$= \frac{2V}{T} \left. \frac{\sin(n\omega t)}{n\omega} \right|_{-T/4}^{T/4}$$

$$= \frac{2V}{n\omega T}\left(\sin\left(n \cdot \frac{2\pi}{T} \cdot \frac{T}{4}\right) + \sin\left(n \cdot \frac{2\pi}{T} \cdot \frac{T}{4}\right)\right)$$

$$= \frac{2V}{n\omega T}\left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4V}{n(2\pi)} \sin\frac{n\pi}{2}$$

$$= \frac{2V}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$b_n = 0$ Since it satisfies even Symmetry

$$\therefore\ g(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t)$$

$$= \frac{V}{2} + \sum_{n=1}^{\infty} \frac{2V}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\omega t)$$

$$= \frac{V}{2} + \frac{2V}{\pi}\left[\cos\omega t - \frac{1}{3}\cos(3\omega t) + \frac{\cos(5\omega t)}{5} \ldots\ldots\right]$$

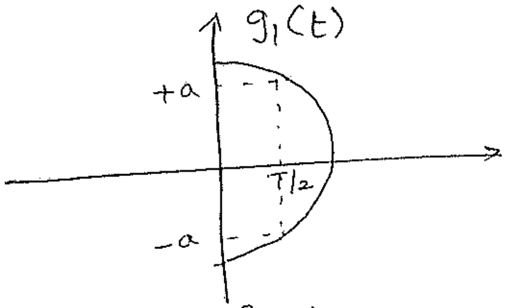
(8m)

Existance of Fourier Series (or) Dirichlet's Conditions:

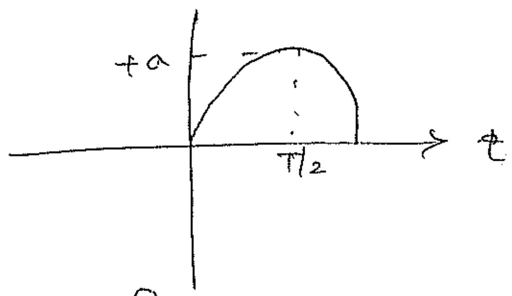
If $g(t)$ is a periodic signal, and it has the period of 'T',

① The signal is a single-valued function of time with in a duration 'T'.

Ex:-



fig(a)



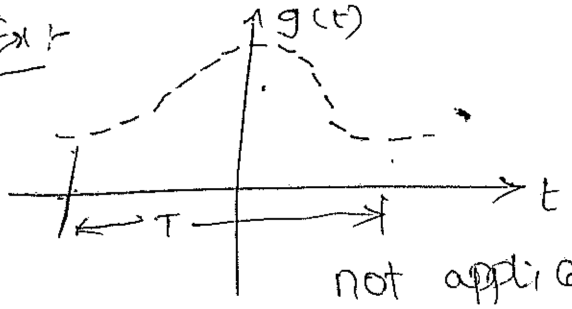
fig(b)

fig(a) cannot be represented by using Fourier Series because it has two values at $t = T/2$

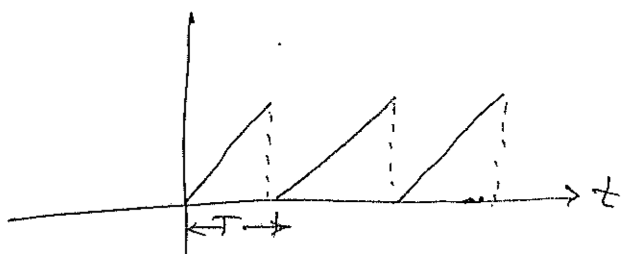
fig (b) can be represented as Fourier Series because it has only one value at $t = T/2$.

② The signal has utmost finite number of discontinuities within the interval 'T'.

Ex:-



not applicable



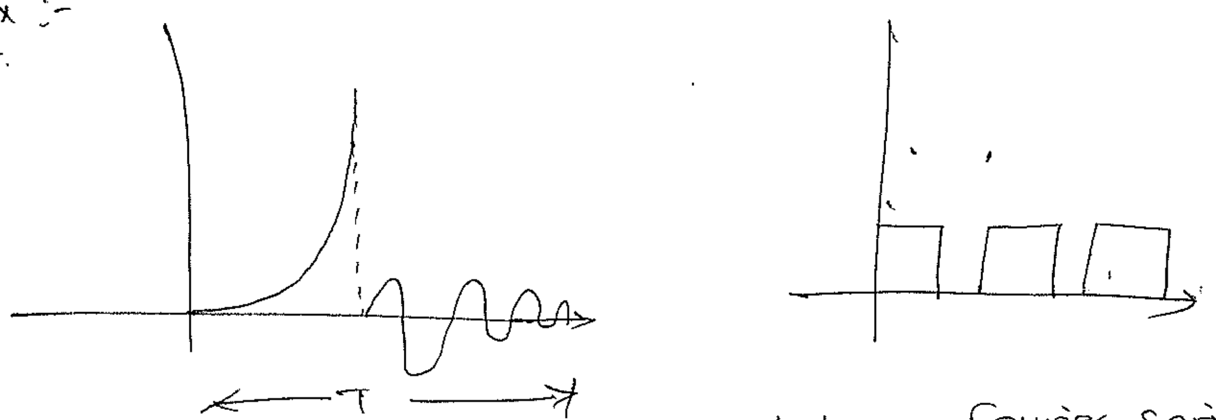
applicable

fig (a) cannot be represented as Fourier Series as it doesnot have finite no. of discontinuity.

fig (b) can be represented because it has finite no. of discontinuities

③ The signal has finite no. of maxima and minima within the duration 'T'.

Ex :-

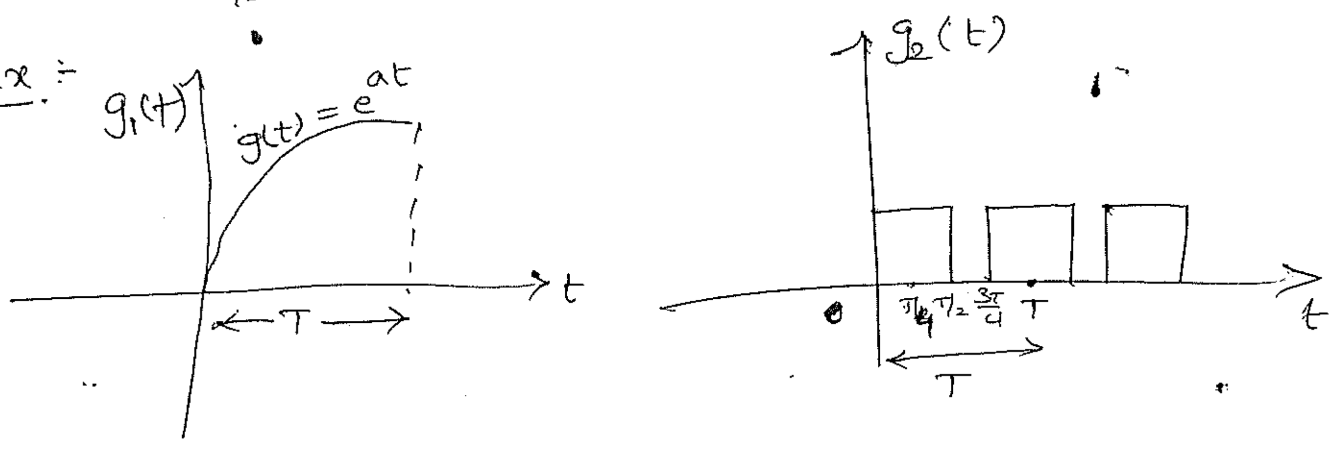


fig(a) cannot be represented as Fourier series, b/c it has infinite no. of max. & min. within the duration 'T'.

④ The signal is absolutely integrable with in the interval 'T'.

i.e; $\int_{-T/2}^{T/2} |g(t)|\,dt < \infty.$

Ex :-



$g_1(t) = e^{at}$ is not absolutely integrable

$$\int_{-\infty}^{\infty} |g_1(t)|\,dt = \int_{-\infty}^{\infty} |e^{at}|\,dt = \frac{e^{at}}{a}\Big|_{-\infty}^{\infty}$$

$$= \frac{e^{\infty} - e^{-\infty}}{a} = \underline{\infty}$$

$g_2(t)$ is absolutely integrable.

$$\int_{-\infty}^{\infty} |g(t)|\,dt = \int_{0}^{T/4} |V|\,dt + \int_{T/4}^{3T/4} 0\,dt + \int_{3T/4}^{T} |V|\,dt$$

$$= V\left(\frac{T}{4}\right) + V\left(\frac{T}{4}\right) = \frac{VT}{2}$$

II Exponential Fourier Series for periodic Signal

An arbitrary periodic function, $g(t)$, which can be represented by a linear combinations of exponential signals, in the duration of $\frac{-T}{2} \le t \le \frac{T}{2}$.

$$g(t) = c_0 + c_1 e^{j\omega t} + c_2 e^{j2\omega t} + c_3 e^{j3\omega t} + \dots + c_n e^{jn\omega t} + \dots$$
$$+ c_{-1} e^{-j\omega t} + c_{-2} e^{-j2\omega t} + c_{-3} e^{-j3\omega t} + \dots + c_{-n} e^{-jn\omega t} + \dots$$

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}}$$

Expressions for exponential Fourier Series Coefficients:

Expression for c_0 :-

Apply integration on both sides of the above integration in the duration $-T/2$ to $T/2$ we get

$$\int_{-T/2}^{T/2} e^{jn\omega t}\, dt = 0 \qquad e^{jn\pi} = (-1)^n$$

$$e^{-jn\pi} = (-1)^n$$

$$\int_{-T/2}^{T/2} e^{jm\omega t} \cdot e^{-jn\omega t}\, dt = \begin{cases} T & \text{for } m=n \\ 0 & \text{for } m\neq n \end{cases}$$

$$\int_{-T/2}^{T/2} g(t)\,dt = \int_{-T/2}^{T/2} c_0\, dt + \int_{-T/2}^{T/2} c_1 e^{j\omega t}\, dt + \dots + \int_{-T/2}^{T/2} c_n e^{jn\omega t}\, dt + \dots$$

$$+ \int_{-T/2}^{T/2} c_{-1} e^{-j\omega t}\, dt + \dots + \int_{-T/2}^{T/2} e^{-jn\omega t}\, dt + \dots$$

$$= \int_{-T/2}^{T/2} c_0\, dt + 0 + \dots + 0 + \dots + 0 + \dots + 0 + \dots$$

$$= c_0 (T)$$

$$\Rightarrow \boxed{c_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt}$$

Expression for c_n :-

Multiply both sides of above eqn by $e^{-jn\omega t}$ and integrate w.r.t time over the interval $\frac{-T}{2} \leq t \leq \frac{T}{2}$, we get..

$$\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt = \int_{-T/2}^{T/2} c_0\, e^{-jn\omega t}\, dt + \int_{-T/2}^{T/2} c_1\, e^{j\omega t}\, e^{-jn\omega t}\, dt + \dots$$

$$+ \int_{-T/2}^{T/2} c_n\, e^{jn\omega t}\, e^{-jn\omega t}\, dt + \int_{-T/2}^{T/2} c_{-1}\, e^{-j\omega t} \cdot e^{-jn\omega t}$$

$$+ \dots + \int^{T/2} c_{-n}\, e^{-jn\omega t} \cdot e^{-jn\omega t} \cdot dt + \dots$$

$$= 0 + 0 + \dots + c_n (T) + \dots + 0 + \dots$$

$$\Rightarrow \boxed{c_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt}$$

$\therefore$ The periodic function $g(t)$ which can be represented by using exponential fourier series is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t}$$

where

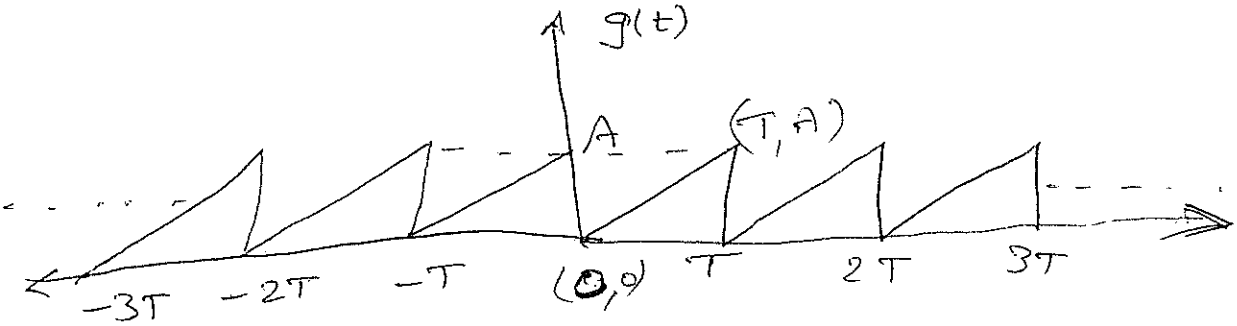
$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt$$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}.$$

where $c_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt.$

$$g(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega t}$$

* Determine exponential Fourier Series of the following Sawtooth waveform.



→ The given Signal Satisfies the periodicity property.

$g(t)$ = eq. of the points $(0,0)$ & (T,A)

$$y - y_1 = m(x - a)$$

$$\Rightarrow \quad y - 0 = \frac{A}{T}(x - 0)$$

$$\Rightarrow \quad y = \frac{A}{T}x$$

$$\therefore \quad g(t) = \frac{A}{T}t \quad ; \quad 0 \le t \le T$$

The exponential Fourier Series expansion is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}$$

where

$$c_n = \frac{1}{T}\int_0^T g(t)\, e^{-jn\omega t}\, dt$$

$$= \frac{1}{T}\int_0^T \frac{A}{T} t\, e^{-jn\omega t}\, dt$$

$$= \frac{A}{T^2}\int_0^T t\, e^{-jn\omega t}\, dt.$$

$$= \frac{A}{T^2}\left[t\,\frac{e^{-jn\omega t}}{-jn\omega} - 1 \times \frac{e^{-jn\omega t}}{(-jn\omega)^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[-T\,\frac{e^{-jn2\pi}}{jn\omega} \mp \frac{e^{-jn2\pi}}{(jn\omega)^2} + \frac{e^0}{(jn\omega)^2}\right]$$

$$= \frac{A}{T^2}\left[\frac{-e^{-j2n\pi}}{jn2\pi}T^2 + \frac{e^{-j2n\pi}}{4\pi^2 n^2}T^2 - \frac{T^2}{n^2 4\pi^2}\right]$$

$$= A\left[\frac{-1}{j2n\pi} + \frac{1}{4\pi^2 n^2} - \frac{1}{4n^2\pi^2}\right]$$

$$= \frac{-A}{j2n\pi} \;//\; = \underline{\underline{\frac{Aj}{2\pi n}}}$$

$$\boxed{\begin{aligned} e^{-j2\pi n} &= \cos(-2\pi n) + j\sin(-2\pi n) \\ &= \cos 2\pi n \\ &= \underline{\underline{1}} \end{aligned}}$$

$$\therefore\; g(t) = \sum_{n=-\infty}^{\infty} \frac{jA}{2\pi n} e^{jn\omega t}$$

$$c_n = \frac{jA}{2\pi n}, \quad n \neq 0.$$

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt$$

$$= \frac{1}{T}\int_0^T \frac{At}{T}\,dt$$

$$= \frac{A}{T^2}\cdot \frac{t^2}{2}\Big|_0^T$$

$$= \frac{A}{T^2}\,\frac{T^2}{2} = \frac{A}{2} \;//$$

$$g(t) = c_0 + c_1 e^{j\omega t} + c_2 e^{2j\omega t} + \cdots$$
$$\cdots + c_{-1} e^{-j\omega t} + c_{-2} e^{-2j\omega t} + \cdots$$

$$\therefore\ g(t) = \frac{A}{2} + \frac{jA}{2\pi} e^{j\omega t} + \frac{jA}{4\pi} e^{2j\omega t} + \dots$$

$$+ \frac{jA}{2\pi} e^{-j\omega t} + \dots\dots$$

Relation b/w Trignometric & Exponential Fourier Series

The trignometric fourier series representation of periodic signal $g(t)$ is given by

$$g(t) = a_0 + 2\left[\sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)\right]$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, dt$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t) \cos(n\omega t)\, dt$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t) \sin(n\omega t)\, dt$$

we know that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore\ g(t) = a_0 + 2\left[\sum_{n=1}^{\infty} a_n\left[\frac{e^{jn\omega t} + e^{-jn\omega t}}{2}\right] + \sum_{n=1}^{\infty} b_n\left[\frac{e^{jn\omega t} - e^{-jn\omega t}}{2j}\right]\right]$$

$$= a_0 + 2\left[\sum_{n=1}^{\infty} e^{jn\omega t}\left(\frac{a_n}{2} + \frac{b_n}{2j}\right) + e^{-jn\omega t}\left(\frac{a_n}{2} - \frac{b_n}{2j}\right)\right]$$

$$= a_0 + \sum_{n=1}^{\infty} \left[(a_n - jb_n) e^{jn\omega t} + (a_n + jb_n) e^{-jn\omega t} \right]$$

where

$$\left. \begin{array}{l} c_n = a_n - jb_n \ ; \ c_0 = a_0 \\ c_{-n} = a_n + jb_n \end{array} \right\} \longrightarrow (1)$$

then

$$g(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega t} + \sum_{n=1}^{\infty} c_{-n} e^{-jn\omega t}$$

$$\boxed{g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}}$$

(1) represents the trignometric coefficients of fourier series.

c_n & c_{-n} are complex conjugate to each other.

$$\therefore \quad \overline{c_n} = c_{-n} \quad \& \quad \overline{c_{-n}} = c_n$$

(or)

$$c_n^* = c_{-n} \quad \& \quad c_{-n}^* = c_n$$

$$|c_n| = \sqrt{a_n^2 + b_n^2} = |a_n - jb_n|$$

$$|c_{-n}| = \sqrt{a_n^2 + b_n^2} = |a_n + jb_n|$$

$$\therefore \quad |c_n| = |c_{-n}|$$

$$\angle c_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = -\tan^{-1}(b_n/a_n)$$

$$\angle c_{-n} = \tan^{-1}(b_n/a_n)$$

$$\therefore \quad \angle c_n = -\angle c_{-n} \quad \text{(or)} \quad \angle c_{-n} = -\angle c_{n}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt \quad \&$$

$$c_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} g(t)\, e^{jn\omega t}\, dt.$$

Compact (or) polar trignometric fourier Series :-

The trigonometric Fourier Series representation of periodic Signal $g(t)$ is

$$g(t) = a_0 + 2\left[\sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

where

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\,dt$$

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\cos n\omega t\,dt.$$

$$b_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\sin n\omega t\,dt.$$

$$g(t) = a_0 + 2\sqrt{a_n^2+b_n^2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{\sqrt{a_n^2+b_n^2}}\cos(n\omega t) + \frac{b_n}{\sqrt{a_n^2+b_n^2}}\sin(n\omega t)\right]$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$\Rightarrow \cos\phi_n = \frac{a_n}{\sqrt{a_n^2+b_n^2}}$$

$$\sin\phi_n = \frac{b_n}{\sqrt{a_n^2+b_n^2}} \qquad |c_n| = \sqrt{a_n^2+b_n^2}$$

$$\Rightarrow g(t) = a_0 + 2|c_n|\sum_{n=1}^{\infty}\left(\cos\phi_n \cos n\omega t + \sin(\phi_n)\sin(n\omega t)\right)$$

$$= a_0 + 2|c_n|\sum_{n=1}^{\infty}\cos(n\omega t - \phi_n)$$

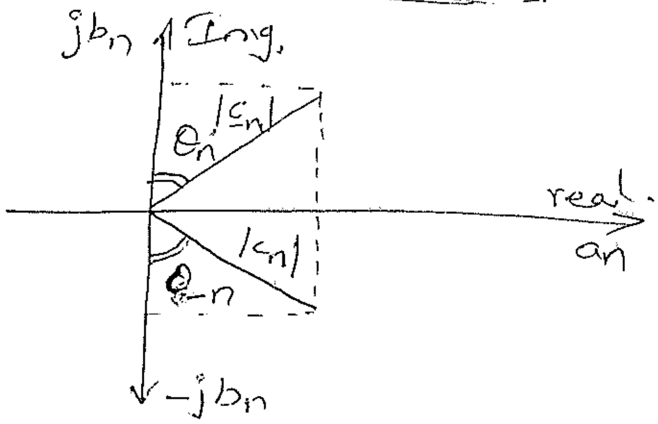
$$\boxed{g(t) = a_0 + D_n\sum_{n=1}^{\infty}\cos(n\omega t - \phi_n)}$$

where $\phi_n = \tan^{-1}(b_n/a_n)$

$$D_n = 2|c_n| = 2\sqrt{a_n^2+b_n^2}.$$

Complex Fourier Spectrum (or) Magnitude Fourier Spectrum

Here $\theta_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$



The Complex Fourier Series exp. of periodic signal $g(t)$ in the interval $-\frac{T}{2} \le t \le \frac{T}{2}$ is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}.$$

$$\Rightarrow \quad g(t) = c_0 + c_1 e^{j\omega t} + c_2 e^{2j\omega t} + \dots \dots$$
$$+ c_{-1} e^{-j\omega t} + c_{-2} e^{-2j\omega t} + \dots$$

This expansion of periodic signal having frequency components are $0, \pm\omega, \pm 2\omega, \pm 3\omega, \dots\dots$ and it has magnitude components $c_0, c_1, c_{-1}, c_2, c_{-2}, \dots\dots$ where

$$c_n = a_n - jb_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt$$

$$c_{-n} = a_n + jb_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{jn\omega t}\, dt$$

$$|c_n| = |c_{-n}| = \sqrt{a_n^2 + b_n^2} \quad \& \quad \angle c_n = -\angle c_{-n}$$

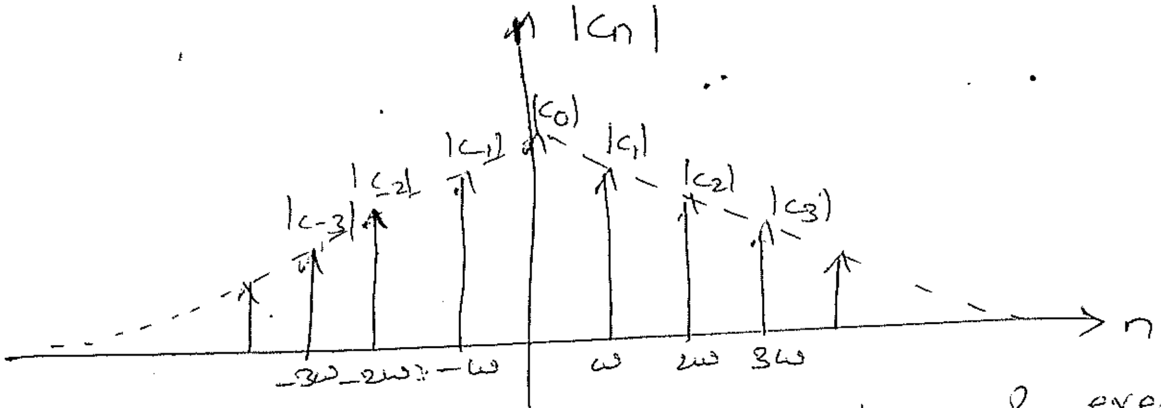
$$\therefore \quad c_n^* = c_{-n} \quad \& \quad c_{-n}^* = c_n$$

Magnitude Spectrum.

where $|c_n| = |c_{-n}|$

$\therefore$ It Satisfies the even Sym. property.

Def :- The magnitude spectrum is plotted b/w magnitudes & frequencies

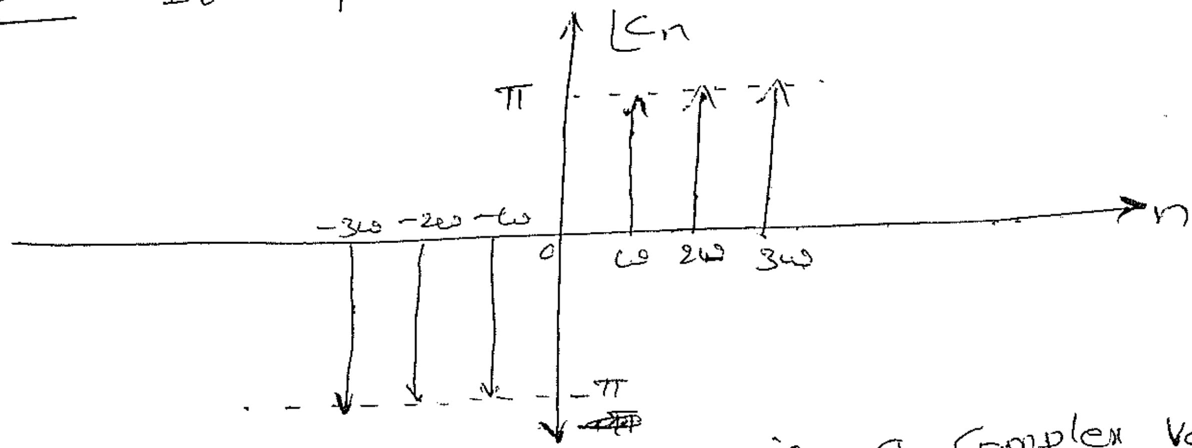


From this, the magnitude spectrum & even Symmetry w.r.t vertical axis passing through origin.

Phase Spectrum :-

The phase of fourier spectrum is odd sym.

Def :- It is plotted b/w phase & frequencies.



From this, in general C_n is a complex value then their phase spectrum is anti-symmetric w.r.t vertical axis passing though origin.

* The Fourier spectrum is simply called as Line Spectra.

(1) It is magnitude spectrum (or) frequency spectrum.

(2) phase spectrum.

Fourier Series Properties :—

① Periodic Power Spectrum (or) Parse value relation for fourier Series :-

(i) The avg. power of the periodic Signal $g(t)$ is

$$P = \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$

$$= \underset{T \to \infty}{Lt} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt$$ in the

duration $\frac{-T}{2} \leq t \leq \frac{T}{2}$.

☆ (ii) The exponential Fourier Series expansion of periodic Signal $g(t)$ in the duration $-\frac{T}{2} \leq t \leq \frac{T}{2}$ is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \text{ ——— ①}$$ where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{-jn\omega t} \, dt$$ and

$$c_{-n} = \frac{1}{T} \int_{-T/2}^{T/2} g(t) e^{jn\omega t} \, dt$$

Multiply $g(t)$ on both sides of the eq. ① & integrate w.r.t time in the duration $\frac{-T}{2} \leq t \leq \frac{T}{2}$, we get

$$\int_{-T/2}^{T/2} g(t) \, g(t) \, dt = \int_{-T/2}^{T/2} g(t) \sum_{n=1}^{\infty} c_n e^{jn\omega t} \, dt$$

$$\int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} c_n \int_{-T/2}^{T/2} g(t).e^{jn\omega t} dt$$

$$\Rightarrow \int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} c_n Tc_{-n}$$

$$\left[\because c_{-n}T = \int_{-T/2}^{T/2} g(t) e^{jn\omega t} dt\right]$$

$$= T \sum_{n=-\infty}^{\infty} c_n c_{-n}$$

$$\frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} c_n c_n^*$$

$$\therefore \boxed{P_{avg.} = \sum_{n=-\infty}^{\infty} |c_n|^2}$$

where $P_{avg.} = \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt.$

$$\therefore P_{avg.} = |c_0|^2 + |c_1|^2 + \cdots + |c_{-1}|^2 + |c_{-2}|^2 + \cdots \quad (2)$$

→ The avg. power of frequency component, $n\omega$ is $|c_n|^2$.

→ The avg. power of c_{-n} component is $|c_{-n}|^2$

$$\therefore |c_n|^2 = |c_{-n}|^2$$

∴ It satisfies. i.e; the power spectrum satisfi the even sym. property eq. (2) is known as parsvel's relation applied to fourier Series.

$$c_n = \frac{1}{1}\int_0^T A \sin(\pi t)\, e^{-jn\omega t}\, dt$$ and

$$= A\int_0^1 e^{(-jn\omega)t} \sin(\pi t)\, dt$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$c_n = A\left[\frac{e^{-jn\omega t}}{(-jn\omega)^2+\pi^2}\left(-jn\omega \sin(\pi t) - \pi\cos(\pi t)\right)\right]_0^1$$

$$= \frac{A}{\pi^2-(n\omega)^2}\left[e^{-jn\omega}(-jn\omega\sin\pi - \pi\cos\pi) + \pi\right]$$

$$= \frac{A}{\pi^2-(n\omega)^2}\left[\pi e^{-jn\omega} + \pi\right]$$

$$\omega = \frac{2\pi}{T} = 2\pi$$

$$\Rightarrow c_n = \frac{A\pi}{\pi^2 - 4\pi^2 n^2}\left(e^{-jn2\pi} + 1\right)$$

$$= \frac{2A}{\pi(1-4n^2)} \text{ for } -\infty \le \omega < \infty$$

$$c_0 = \frac{2A}{\pi}$$

$$\therefore g(t) = \dots + \frac{-2A}{15\pi}e^{-j2\omega t} + \frac{-2A}{3\pi}e^{-j\omega t} + \frac{2A}{\pi} - \frac{2A}{3\pi}e^{j\omega t} + \frac{-2A}{15\pi}e^{2j\omega t} - \dots$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt.$$ (Continuation on paper)

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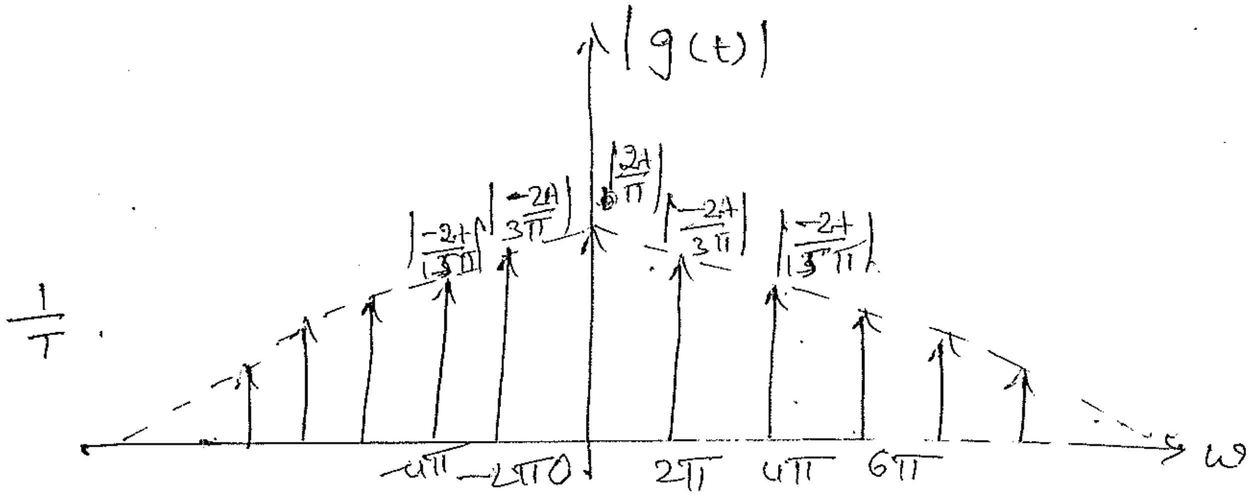
spectra
2

$$\int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} c_n \int^{T/2} g(t).e^{jn\omega_0 t} dt$$

$\Rightarrow$ for line Spectra, the above eqn exists.

Magnitude Spectrum



Phase Spectrum

where

$$c_n = \frac{-2A}{(4n^2-1)\pi} = a_n - jb_n = a_n - 0 = a_n$$

$$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$$

$$= 0, \pm\pi, \pm 2\pi, \cdots \pm n\pi$$

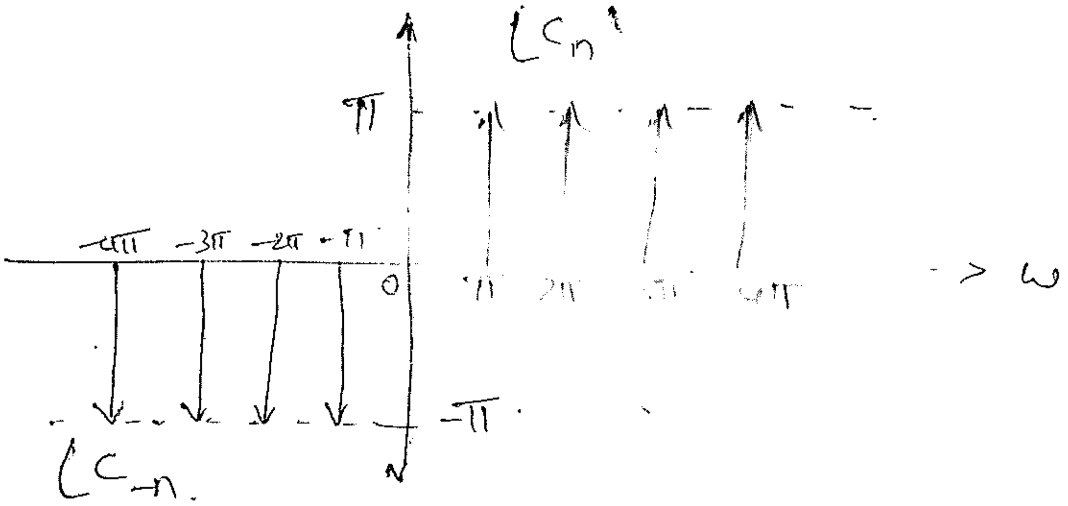
$\rightarrow$ π. $\therefore$ It obeys odd symmetry

$\leftrightarrow$ π

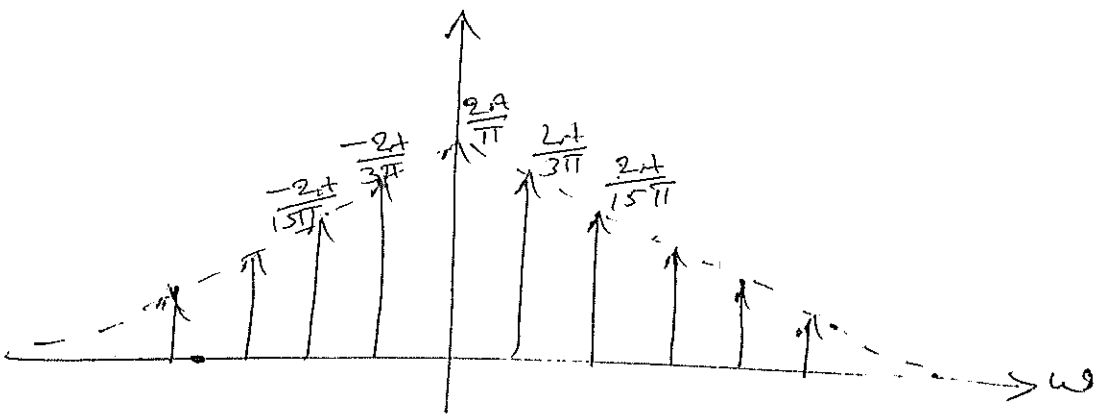
$\therefore$

the even sym. property

eq. ② is known as parsvel's relation applied to fourier Series.



Power Spectrum



$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt.$$ (continuation on paper)

$$\int_{-T/2}^{T/2} |g(t)|^2 dt = \sum^{\infty} c_n \int^{T/2} g(t).e^{jn\omega t} dt$$

$\Rightarrow$

$\frac{1}{T}$

where

$\leftrightarrow$ Tt

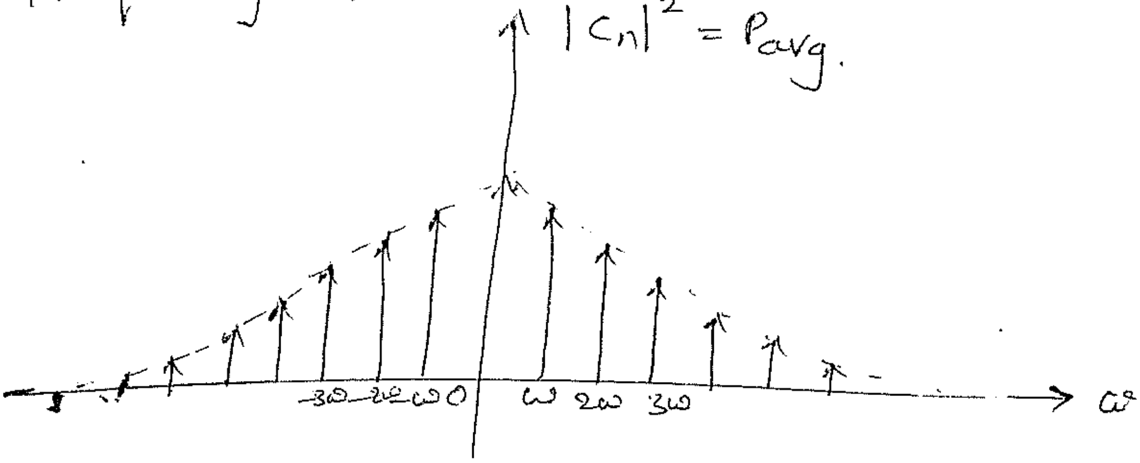
$\leftrightarrow$ Tt

The even sym. property

eq. (2) is known as parsvel's relation applied to fourier Series.

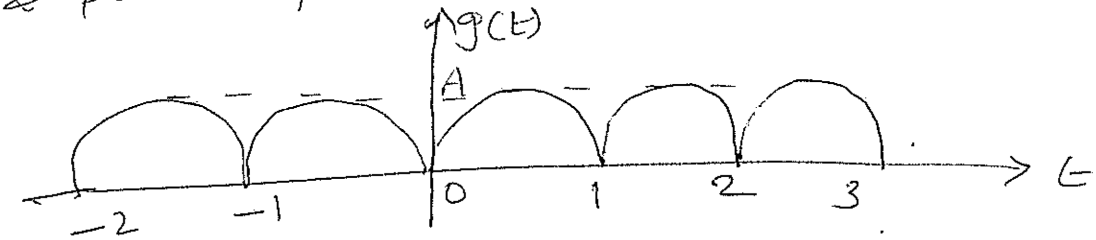
Def :-

The power spectrum is plotted b/w magnitude of the component square and frequency of components.

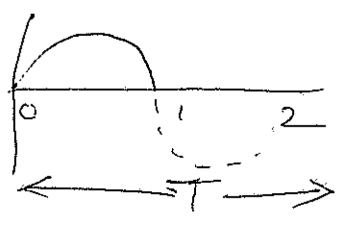


From this, the power spectrum is symmetric w.r.t vertical axis passing through origin.

* Expand the exp. fourier Series of full wave rectified Sine wave and also plot line Spectra & power Spectrum



Mathematical expression is



$g(t) = A\sin(\omega t)$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

$\therefore g(t) = A\sin(\pi t)$; $0 \le t \le$

The exp. fourier Series rep. of g(t) is

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t}$$

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t) e^{-jn\omega t} dt.$$ (Continuation on paper)

21/7/06

Representation of arbitrary signal by fourier series over entire interval.

The Complex fourier Series representation of periodic signal g(t) in the interval $-\frac{T}{2}$ to $\frac{T}{2}$ is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \quad -T/2 \le t \le T/2$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt$$

Here g(t) satisfies periodicity property $\forall t$

$$g(t) = g(t+T) \quad \forall t$$

$$g(t+T) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega(t+T)}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \cdot e^{jn\omega T}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \, e^{jn(2\pi)} \quad \because \left(\omega = \frac{2\pi}{T}\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} \, e^{j2\pi n}$$

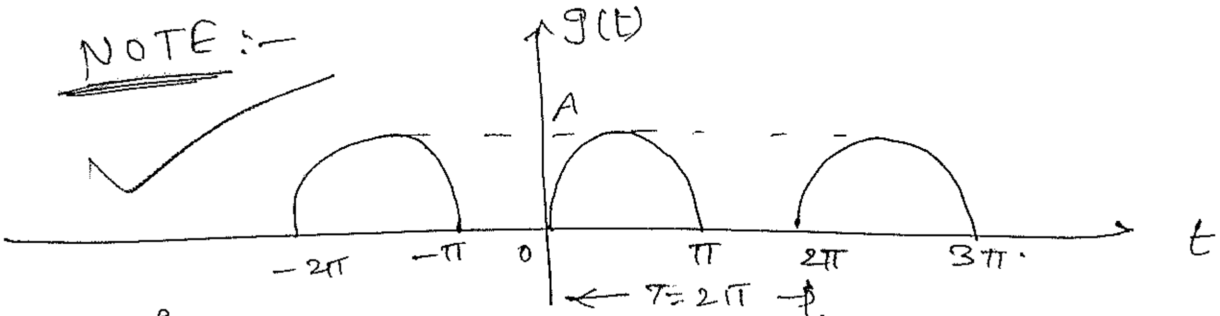
$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t} = g(t) \quad (\because e^{j2\pi n} = 1)$$

$\therefore$ The Fourier Series rep. of arbitrary Signal over the entire interval $-\infty \le t < \infty$ is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega t}, \text{ where}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g(t)\, e^{-jn\omega t}\, dt.$$

NOTE :-



This is half-wave rectified sine wave.

$T = 2\pi$.

If time axis is given interms of ω.

$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$.

$\cos(\omega_0 n t) = \cos(n t)$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt \; ; \; \omega = 1.$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} g(t)\cos(nt)\,dt \;\text{(or)}\; \frac{2}{2\pi}\int_{-\pi/2}^{\pi/2} g(t)\cos(n\omega t)\,d(\omega t)$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(nt)\,dt \;\text{(or)}\; \frac{2}{2\pi}\int_{-\pi}^{\pi} g(t)\sin(n\omega t)\,d(\omega t)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,e^{-jnt}\,dt \;\text{(or)}\; \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\,e^{-j\omega n t}\,d(\omega t)$$

21/7/06

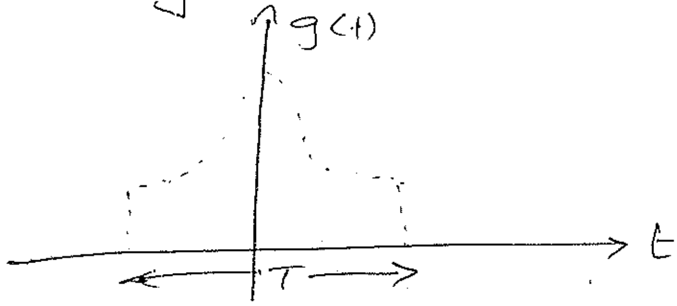
UNIT – III

1 ## FOURIER TRANSFORM

Fourier transform is obtained from fourier series. or Representation of arbitrary signal over entire interval $-\infty \le t \le \infty$ by Fourier transform

Derivation of Fourier transform from fourier series:

If $g(t)$ is non-periodic signal and it is represented graphically as shown in the figure.

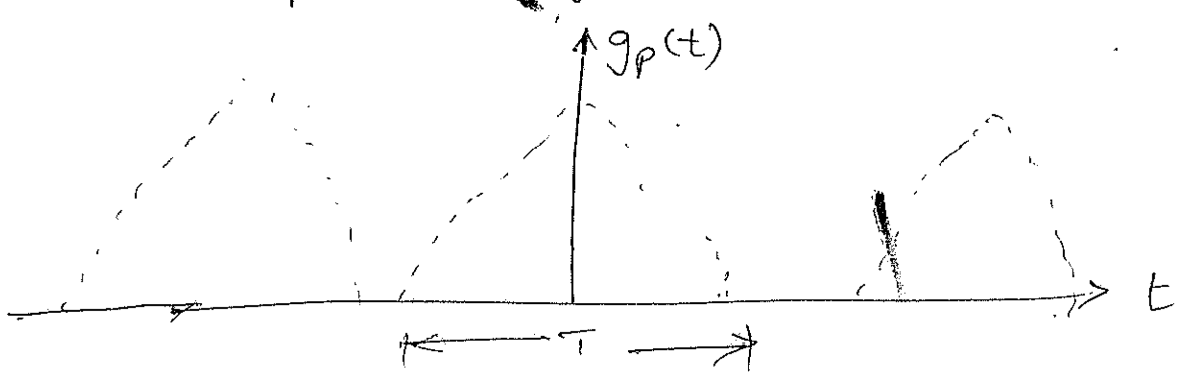


Definition for non-periodic signal is

$$g(t) = \lim_{T \to \infty} g_p(t)$$

where $g_p(t)$ is periodic signal with period T

$g_p(t)$ is constructed from $g(t)$. It is a periodic signal which contains one cycle of non-periodic $g(t)$ and is shown below.



we know the expansion of Complex Fourier Series of periodic Signal $g_p(t)$ is

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T} t}.$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} g_p(t)\, e^{-jn\frac{2\pi}{T}t}\, dt.$$

when $T \longrightarrow \infty$,

$$\Delta f = \frac{1}{T} \quad ; \quad f_n = \frac{n}{T}$$

$$G(f_n) = c_n T$$

$$g_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi f_n T} \quad \text{——} \; \otimes$$

$$= \sum_{n=-\infty}^{\infty} G(f_n) e^{j2\pi f_n T} \Delta f \quad \text{——} \; (1)$$

$$G(f_n) = \int_{-T/2}^{T/2} g_p(t) \cdot e^{-j2\pi f_n T} dt \quad \text{——} \; (2)$$

As we approach the duration of the pulse, $T \longrightarrow \infty$, in eq. (1), $g_p(t) \longrightarrow g(t)$ and the Summation changes to integration of the Continous time signal, $G(f)$ with Δf changed by df.

i.e;
$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df.$$

where f_n is discrete frequency that is changed to continous frequency 'f'.

As we approaches to $t \longrightarrow \infty$ in eq. (2), the discrete frequency f_n changes to Continous frequency f and the periodic signal $g_p(t)$ changes to non-periodic signal $g(t)$. We get

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt \quad —④$$

where $G(f)$ is the Fourier transform of the non-periodic signal, $g(t)$ and $g(t)$ is inverse Fourier transform of $G(f)$. Therefore $g(t)$ and $G(f)$ are Fourier-transform pairs.

i.e. $\boxed{\text{C.T.F.T of } g(t) = G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt}$

$$\boxed{\text{Inverse continous time F.T of } G(f) = g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi f t}\, df}$$

- For convienence, we represent the F.T symbol as

$$\rightleftharpoons \quad (\text{or}) \quad \longleftrightarrow$$

and the operation of F.T of function as

$$F[\;] \quad (\text{or}) \quad CTFT[\;] \quad (\text{or}) \quad FT[\;]$$

$$F[g(t)] = G(f) \; ; \quad g(t) \rightleftharpoons G(f).$$

- The inverse F.T operation is represented as

$$F^{-1}[\;] \quad (\text{or}) \quad I.C.T.F.T[\;]$$

$$F^{-1}[G(f)] = g(t).$$

NOTE :- If the Fourier transform operation is "in" angular frequency domain "ω"

$$\omega = \frac{2\pi}{T} = 2\pi f \quad (\because f = \tfrac{1}{T})$$

$$d\omega = 2\pi \cdot df.$$

These eqn's are in ω-domain.

$$\boxed{G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.}$$

$$g(t) = \int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, \frac{d\omega}{2\pi}$$

$$\Rightarrow \boxed{g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega.}$$

Continous-frequency spectrum (or) fourier-frequency Spectrum :-

The fourier transform of non-periodic Signal, g(t), is

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt \quad \text{--- (1)}$$

This means that the Fourier transform, transforms time-domain g(t) into frequency-domain Signal G(f)

If G(f) is Complex valued Signal, it has amplitude and phase.

$$G(f) = |G(f)|\, e^{j\angle G(f)}$$

$|G(f)|$ - Continous magnitude Spectrum

$\angle G(f)$ - Continous phase spectrum.

$$^{*}G(f) = \left[\int_{-\infty}^{\infty} g(t)\, e^{-j2\pi f t}\, dt\right]^{*}$$

If $g(t)$ is purely real and apply complex conjugate on both sides of above eq.n of ①,

$$^{*}[G(f)]^{*} = \left[\int_{-\infty}^{\infty} g(t)\, e^{-j2\pi ft}\, dt \right]^{*}$$

$$= \int_{-\infty}^{\infty} g^{*}(t)\, e^{j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} g(t)\, e^{-j2\pi(-f)t}\, dt. \quad (\because g(t) \text{ is purely real})$$

$$= \underline{G(-f)}.$$

$$\therefore \; ^{*}G(f) = G(-f)$$

$$|G(-f)| = |^{*}G(f)| = |G(f)|$$

$$\angle G(-f) = \angle ^{*}G(f) = -\angle G(f).$$

From this, the continous magnitude spectrum satisfies even symmetry property and the continous phase spectrum satisfies the odd symmetry property.

~~Hw~~

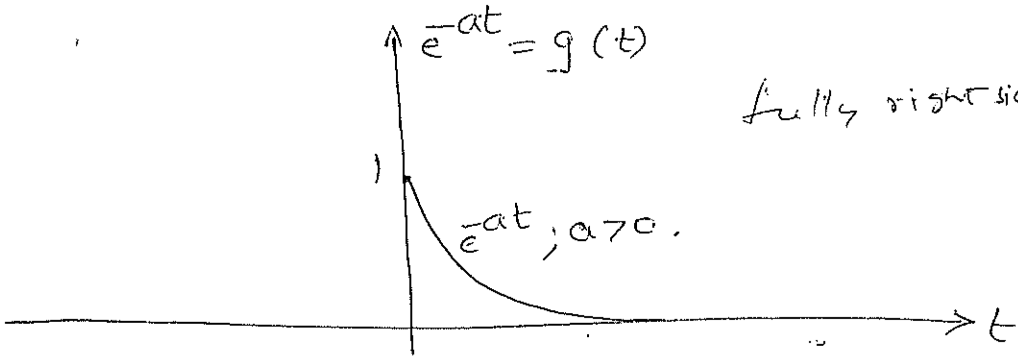
1. Expand the following signals by using ~~FT~~ ~~Exp~~

22/7/06.

<u>Fourier Transform of Standard continous time signals</u>

(i) Fourier transform of one sided exponential decaying pulse and its magnitude & phase spectrums :-

$$g(t) = e^{-at}\, u(t)\,;\; a>0.$$



fully right sided decaying spectrum,

Fourier transform of $g(t)$ is

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} [g(t)] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{0} e^{-at} (0) e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} (1) e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

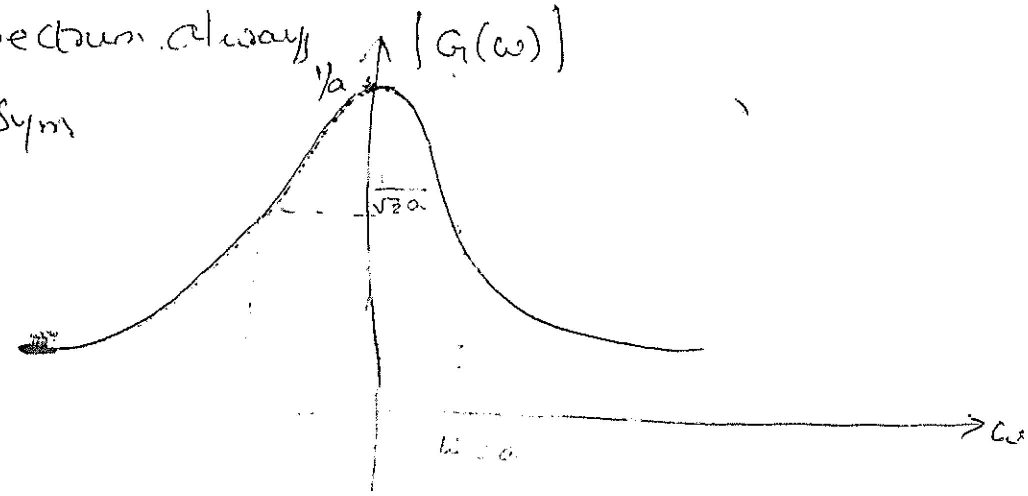
$$= 0 + \frac{1}{a+j\omega} = \frac{1}{a+j\omega}.$$

Magnitude.

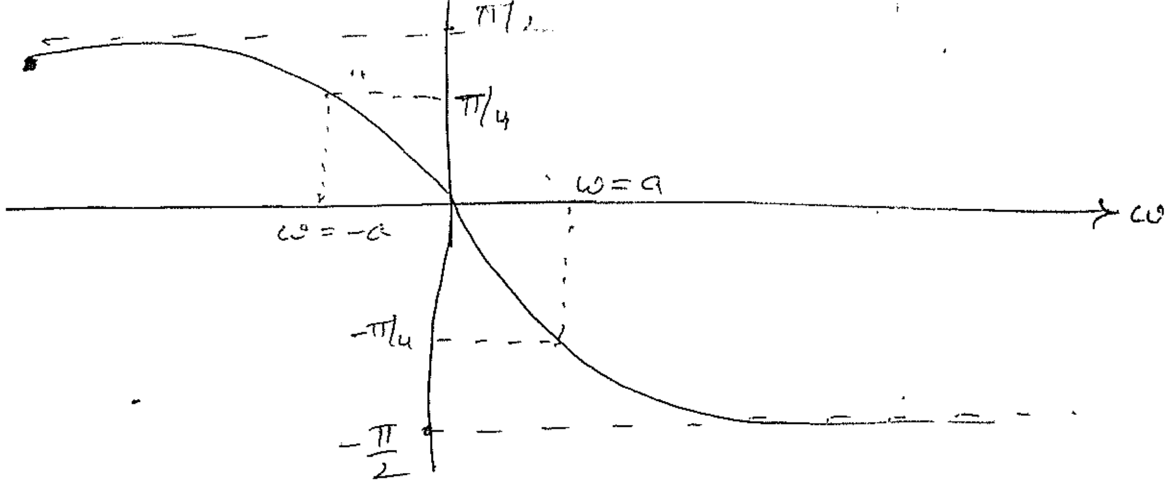
$$|G(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

Phase, $\angle G(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$

Magnitude Spectrum always Satisfy even sym

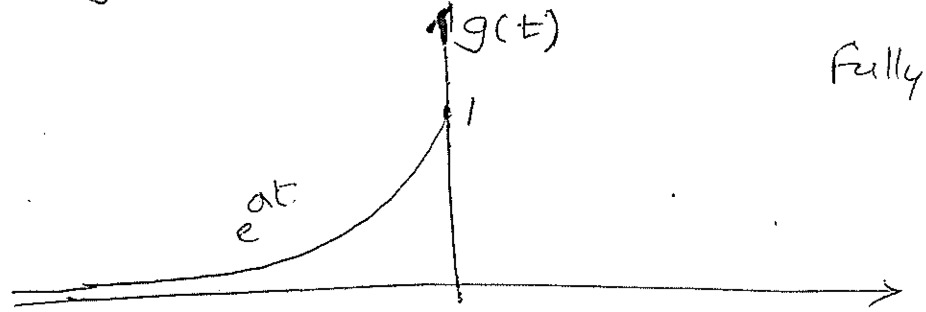


Phase Spectrum always satisfy odd sym.



② Fourier transform of one Sided rising exp. pulse :-

$$g(t) = e^{at}\, u(-t) \,;\; a > 0.$$



Fully left sided rising spectrum

F.T is $\qquad g(t) \longleftrightarrow G(\omega)$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}(1)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{at}(0)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt = \left. \frac{e^{(a-j\omega)t}}{a-j\omega} \right|_{-\infty}^{0}$$

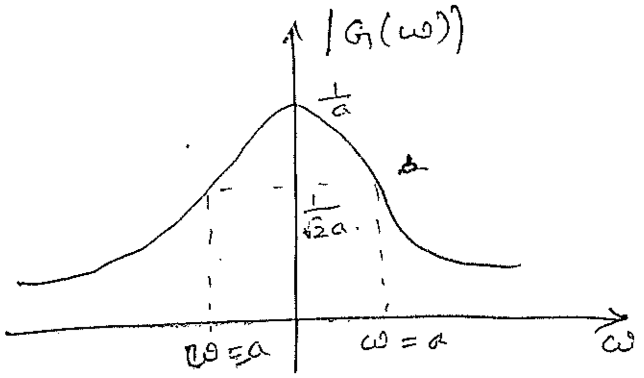
$$= \frac{1}{a-j\omega} - \frac{0}{a-j\omega} = \underline{\frac{1}{a-j\omega}}$$

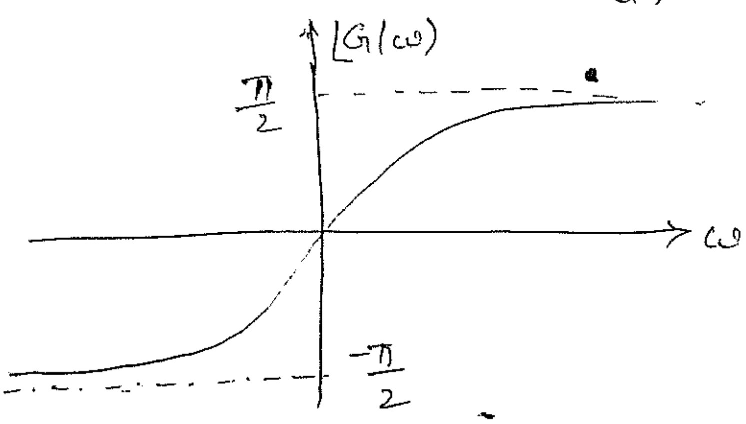
$$G(\omega) = \frac{1}{a-j\omega}$$

$$|G(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \; ; \quad \angle G(\omega) = -\tan^{-1}\left(\frac{-\omega}{a}\right)$$

$$= \tan^{-1}\left(\frac{\omega}{a}\right).$$

The graphical rep. is





③ <u>Fourier Transform of double exponential pulse</u>:-

$\cancel{g(t) = e^{-a|t|}}$

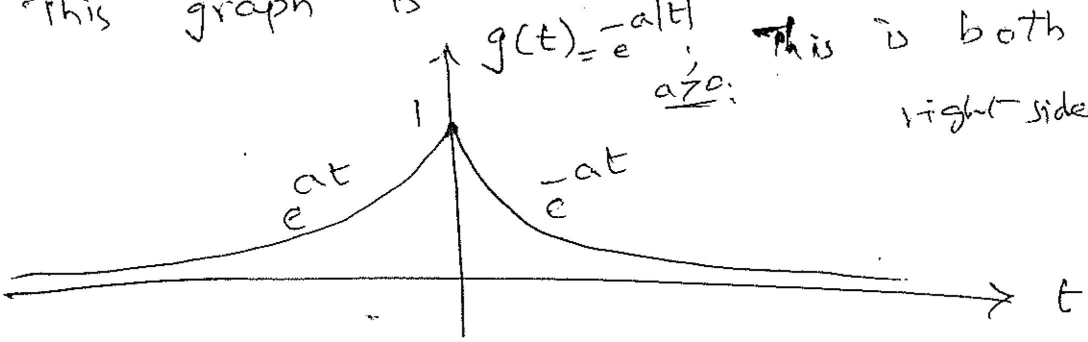
$$g(t) = e^{-a|t|} \; ; \; a > 0.$$

$$g(t) \quad e^{-a|t|} = \begin{cases} e^{-at} & ; t \geq 0 \\ e^{at} & ; t \leq 0. \end{cases}$$

$$= \begin{cases} e^{-at} u(t) & ; t \geq 0 \\ e^{at} u(-t) & ; t \leq 0. \end{cases}$$

$$g(t) = e^{-at} u(t) + e^{at} u(-t).$$

This graph is $g(t) = e^{-a|t|}$; $a>0$. This is both left & right sided pulse.



F.T is

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{at}\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

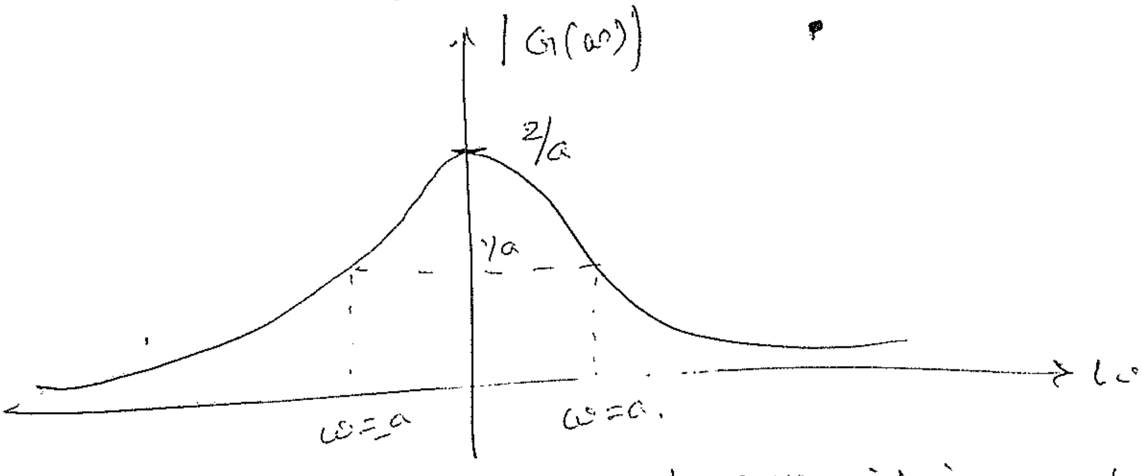
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}$$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$|G(\omega)| = \frac{2a}{a^2+\omega^2}; \quad \angle G(\omega) = \tan^{-1}(0) = \underline{\underline{0}}.$$

$\therefore$ Its magnitude spectrum is



& $\angle G(\omega) = \theta(\omega) = 0$ because it is a pure real value

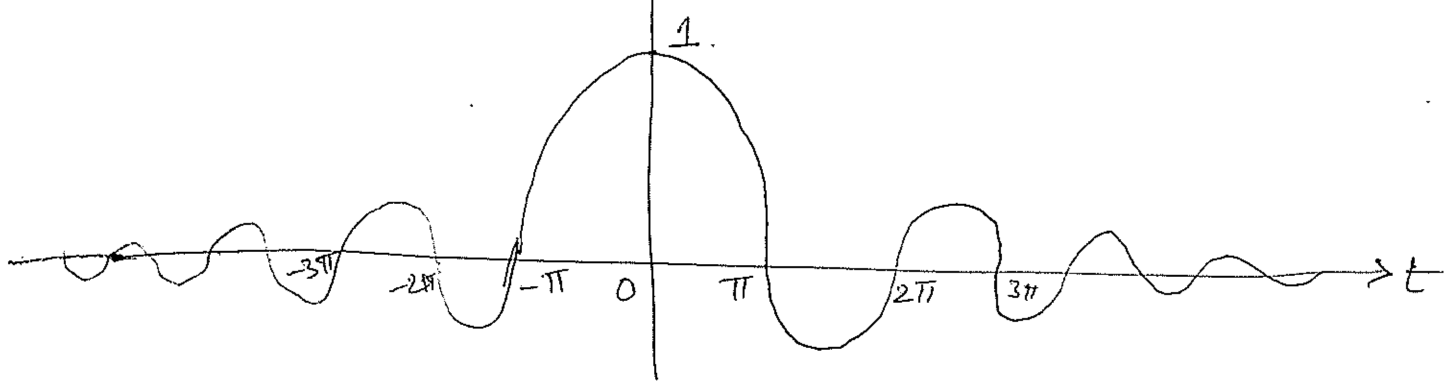
Sinc pulse (or) Sampled signal (or) interpolating Signal :-

It is denoted by $S_a(x)$ (or) $Sinc(x)$

It is defined mathematically as

$$Sa(x) = Sinc(x) = \frac{\sin(x)}{x}$$

$Sinc(x)$ (or) $Sa(x) = \sin x \times \frac{1}{x}$



- It has max. value of 1 at origin

$$Sa(0) = \underset{x \to 0}{Lt} \frac{\sin(x)}{x} = \underset{x \to 0}{Lt} \cos x = 1 \quad (\because \text{from L-hosp. rule})$$

- It has zero values at

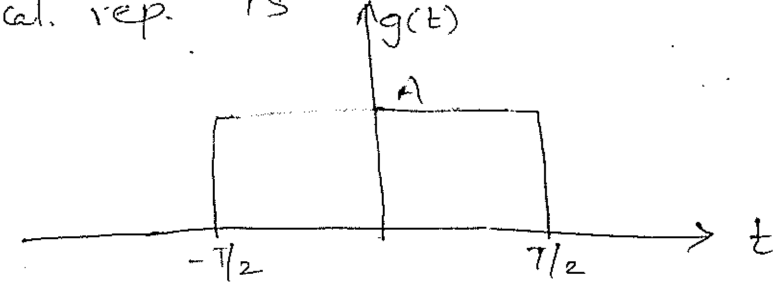
$\pm\pi, \pm 2\pi, \pm 3\pi, \ldots\ldots$

- It is an even function of x. So, it satisfies even symmetry.
- This function is sinusoidal oscillations followed by $\frac{1}{x}$ curve.

F.T of rectangular pulse :— (or) gate function :—

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} A \,;\ |t| \le \frac{T}{2} \text{ (or) } -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 \,;\ \text{else} \end{cases}$$

Graphical rep. is



F.T is

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-T/2}^{T/2} A\, e^{-j\omega t}\, dt$$

$$= A \cdot \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{T/2} = \frac{A}{-j\omega}\left(e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}}\right)$$

$$= \frac{A}{\omega}\left[\frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{j}\right]$$

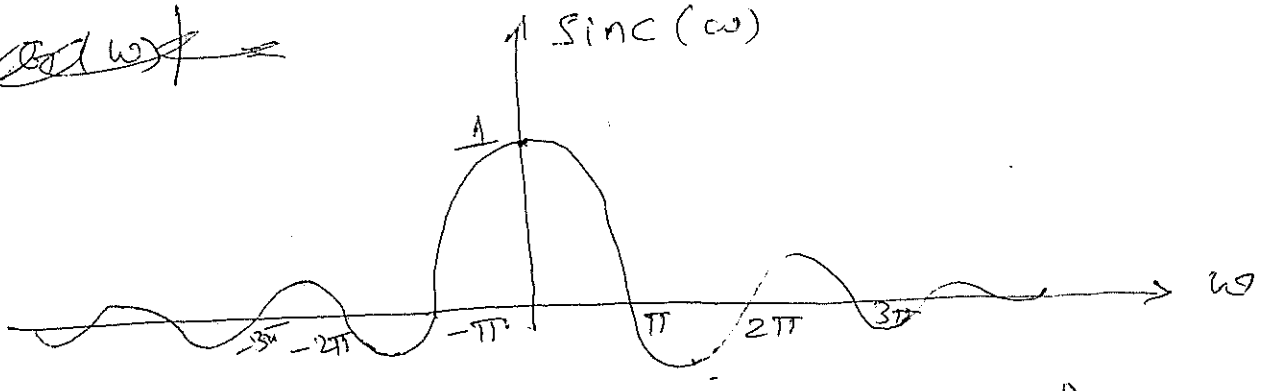
$$G(\omega) = \frac{2A}{\omega}\left(\frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{2j}\right)$$

$$= \frac{2A}{\omega} \sin\left(\frac{\omega T}{2}\right)$$

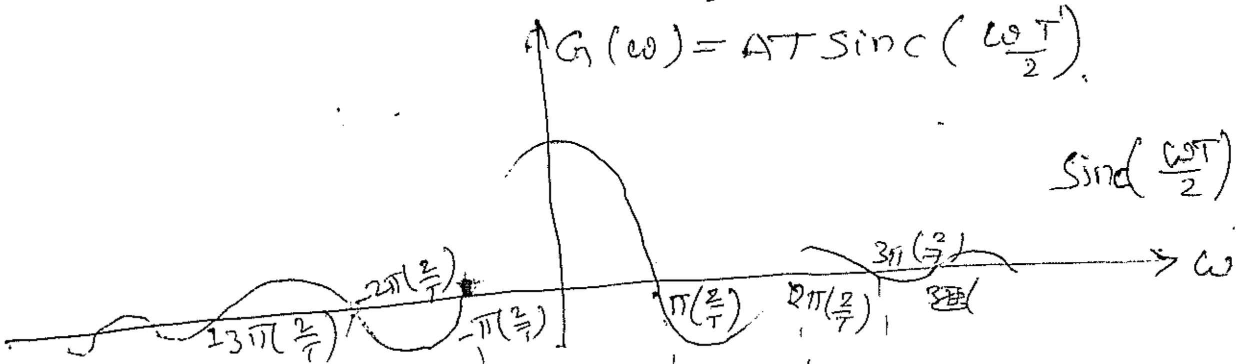
$$= AT \cdot \frac{\sin\left(\frac{\omega T}{2}\right)}{\frac{\omega T}{2}}$$

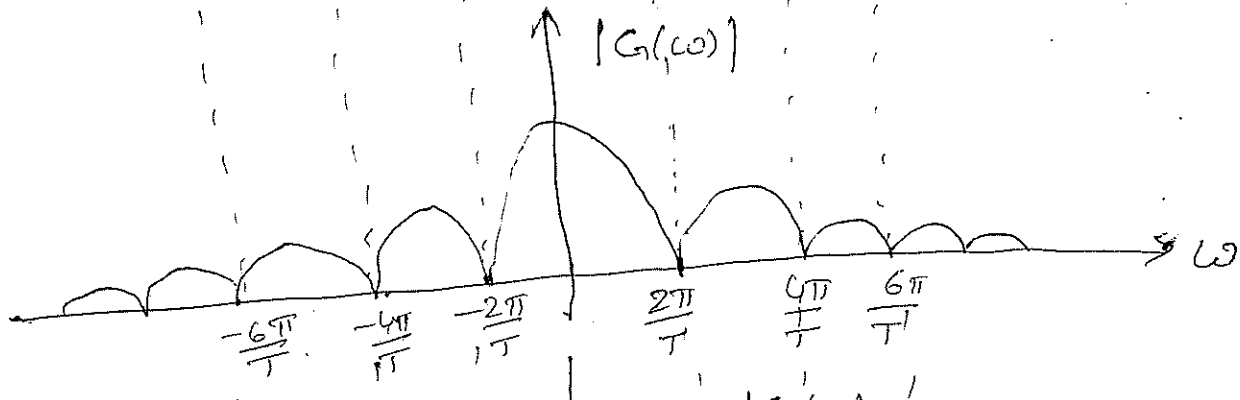
$$= AT \cdot \mathrm{sinc}\left(\frac{\omega T}{2}\right) \quad \left[\because \mathrm{sinc}(x) = \frac{\sin(x)}{x}\right]$$

~~$|G(\omega)|$~~ =

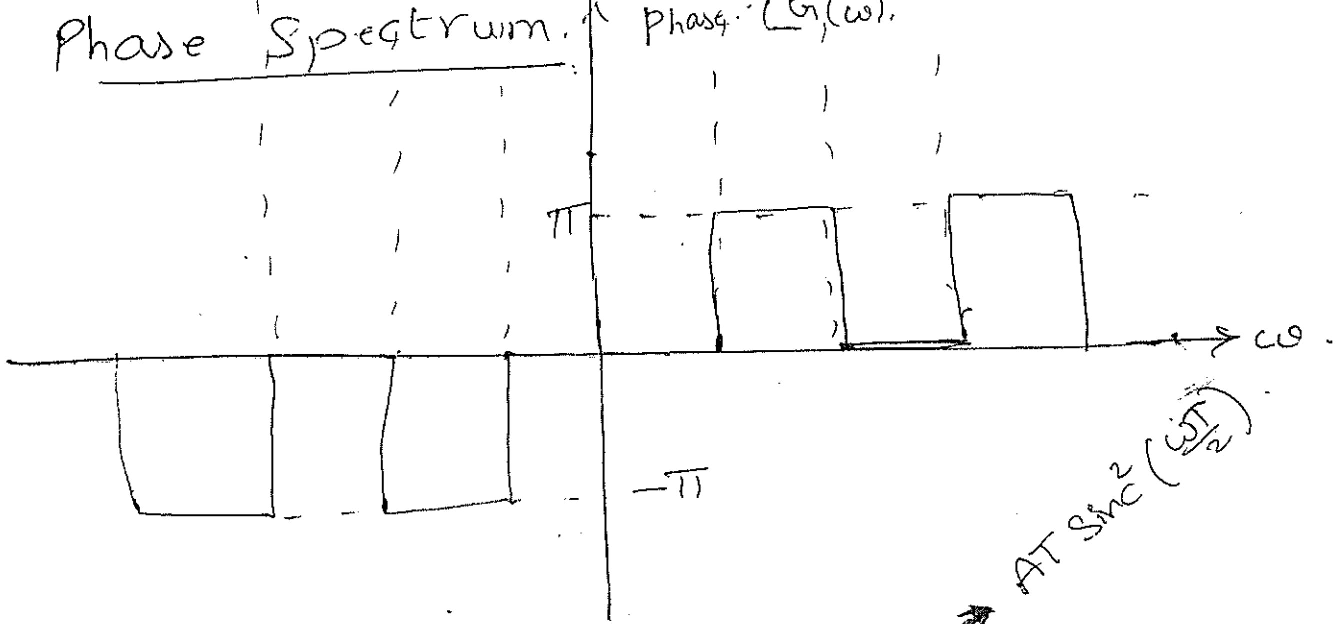


$$\mathrm{Sinc}\left(\frac{\omega T}{2}\right) = \mathrm{Sinc}\left(\frac{\omega}{2/T}\right)$$





Phase Spectrum.



$$A\Lambda\left(\frac{t}{2T}\right) \longleftrightarrow AT\,\mathrm{Sinc}^2\left(\frac{\omega T}{2}\right)$$

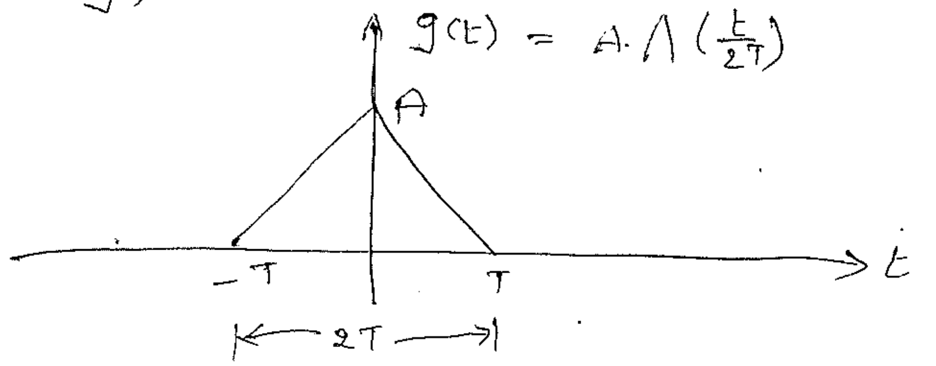
F.T of triangular pulse :-

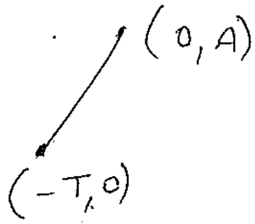
It is defined as

$$\Lambda\left(\frac{t}{2T}\right) = \begin{cases} 1 - \frac{|t|}{T} & ; \ |t| \le T \\ 0 & ; \ \text{else.} \end{cases}$$

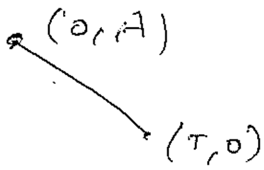
Now, $$g(t) = A \cdot \Lambda\left(\frac{t}{2T}\right) = \begin{cases} A\left(1 - \frac{|t|}{T}\right) & ; \ |t| \le T \\ 0 & ; \ \text{else.} \end{cases}$$

Graphically,





$$g(t) = \frac{A}{T}(t + T)$$
$$= A\left(1 + \frac{t}{T}\right) ; \ -T \le t \le 0$$



$$g(t) = \frac{-A}{T}(t - T)$$
$$= A\left(1 - \frac{t}{T}\right) ; \ 0 \le t \le T.$$

F.T is

$$G(f) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T}^{0} A\left(1 + \frac{t}{T}\right) e^{-j\omega t} dt + \int_{0}^{T} A\left(1 - \frac{t}{T}\right) e^{-j\omega t} dt$$

$$= A\left[\left(1 + \frac{t}{T}\right)\frac{e^{-j\omega t}}{-j\omega} + \frac{-1}{T}\,\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{-T}^{0}$$

$$+ A\left[\left(1 - \frac{t}{T}\right)\frac{e^{-j\omega t}}{-j\omega} - \left(\frac{-1}{T}\right)\frac{e^{-j\omega t}}{(-j\omega)^2}\right]_{0}^{T}$$

$$= A\left[\frac{-1}{j\omega} + \frac{1}{T\omega^2} - \left(\frac{1}{T\omega^2} e^{j\omega T}\right)\right] + A\left[\frac{-1}{T\omega^2} e^{-j\omega t} - \left(\frac{-1}{j\omega} - \frac{1}{T\omega^2}\right)\right]$$

$$= \cancel{-\frac{A}{j\omega}} + \frac{A}{T\omega^2} - \frac{A}{T\omega^2} e^{j\omega T} - \frac{A}{T\omega^2} e^{-j\omega T} + \cancel{\frac{A}{j\omega}} + \frac{A}{T\omega^2}$$

$$= \frac{2A}{T\omega^2} - \frac{A}{T\omega^2}\left(e^{j\omega T} + e^{-j\omega T}\right)$$

$$= \frac{2A}{T\omega^2} - \frac{A}{T\omega^2}\left(2\cos(\omega T)\right)$$

$$= \frac{2A}{T\omega^2}\left[1 - \cos\omega T\right] \quad \left(\because \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta\right)$$

$$= \frac{2A}{T\omega^2} \cdot \left(2\sin^2\frac{\omega T}{2}\right) \quad \left(\because 1 - \cos\theta = 2\sin^2\theta/2\right)$$

$$= \frac{4A}{T\omega^2} \sin^2\left(\frac{\omega T}{2}\right).$$

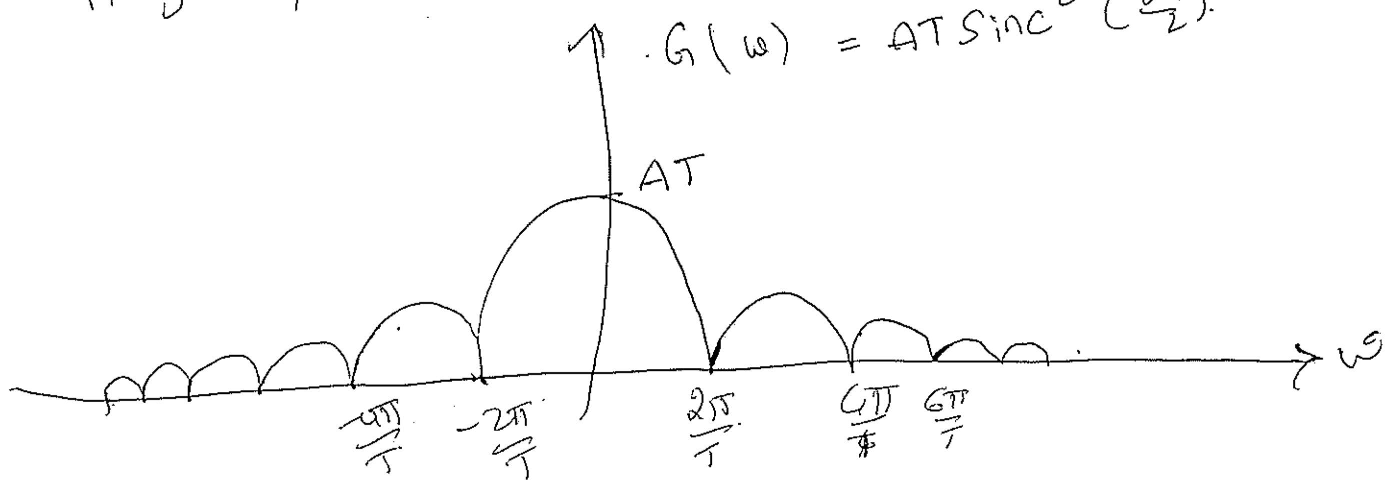
$$= AT\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right]^2$$

$$= AT \cdot \left[\mathrm{Sinc}\left(\frac{\omega T}{2}\right)\right]^2$$

$$= AT \cdot \mathrm{Sinc}^2\left(\frac{\omega T}{2}\right)$$

Mag. Spectrum is

$$G(\omega) = AT\,\mathrm{Sinc}^2\left(\frac{\omega T}{2}\right).$$

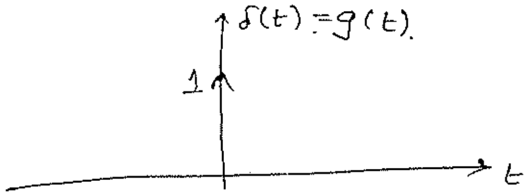


Their phase spectrum $\theta(\omega) = 0$ b/cause it ~~ka~~ is purely real & it has +ve amplitudes.

25/~~7~~/06

~~Fourier~~ Transform of unit impulse Sequence :-

$$g(t) = \delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

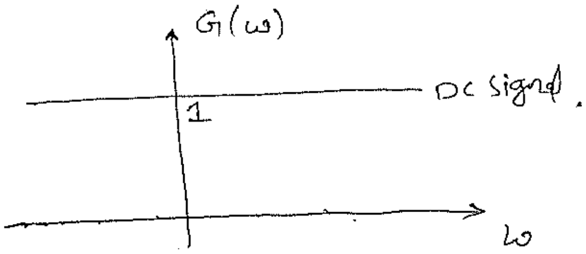
$$= \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt.$$

$$= \delta(t)\, e^{-j\omega t}\Big|_{t=0}$$

$$= \delta(0) = \underline{\underline{1}}.$$

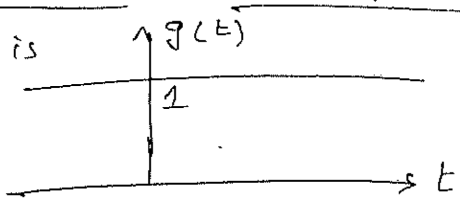
$\therefore$ $\boxed{\delta(t) \longleftrightarrow 1 \text{ (dc signal)}}$

The graphical representation is



F.T of DC Signal (or) Inverse F.T of $\delta(\omega)$:-

In time-domain, dc signal is



$$g(t) = 1 \quad \forall\, t$$

$$\delta(\omega) = \begin{cases} 1 & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0. \end{cases}$$

$$g(t) \longleftrightarrow G(\omega)$$

$$F^{-1}[G(\omega)] = g(t)$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\cdot e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\delta(\omega)\cdot e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\cdot \left[\delta(\omega)\cdot e^{j\omega t}\right]_{\omega=0}$$

$$\Rightarrow \quad F^{-1}[\delta(\omega)] = \frac{1}{2\pi}.$$

Apply F.T on both Sides.

$$\Rightarrow \quad F\left[F^{-1}(\delta(\omega))\right] = F\left[\frac{1}{2\pi}\right]$$

$$\Rightarrow \quad \delta(\omega) = F\left[\frac{1}{2\pi}\right]$$

$$\Rightarrow \quad 2\pi\,\delta(\omega) = F[1]$$

$$F[1] = 2\pi\,\delta(\omega) = \begin{cases} 2\pi & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0 \end{cases}$$

$$\boxed{1 \longleftrightarrow 2\pi\,\delta(\omega)}$$

From this, we conclude that F.T of unit impulse Signal is DC Signal and F.T of DC Signal is unit impulse Signal, with scaling factor 2π.

Inverse F.T of $\delta((\omega)-\omega_0)$:-

$$\delta(\omega-\omega_0) = \begin{cases} 1 & \text{for } \omega-\omega_0 = 0\ ;\ \omega=\omega_0 \\ 0 & \text{for } \omega-\omega_0 \neq 0 \therefore\ \omega \neq \omega_0. \end{cases}$$

$$\delta(\omega-\omega_0) \longleftrightarrow G(\omega)$$

$$F^{-1}[G(\omega)] = \delta(\omega-\omega_0)$$

§ $F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\delta(\omega-\omega_0)\, e^{j\omega t}\right] d\omega$$

$$= \frac{1}{2\pi}\left[\delta(\omega-\omega_0)\, e^{j\omega t}\right]_{\omega=\omega_0}$$

$$= \frac{1}{2\pi} e^{j\omega_0 t}$$

$$\therefore\ F^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi} e^{j\omega_0 t}$$

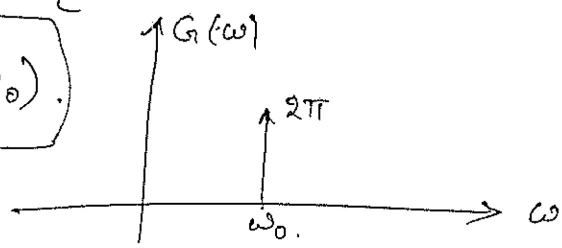
Apply F.T on both sides.

$$\Rightarrow \delta(\omega-\omega_0) = F\left[\frac{1}{2\pi} e^{j\omega_0 t}\right]$$

$$\Rightarrow 2\pi\,\delta(\omega-\omega_0) = F\left[e^{j\omega_0 t}\right]$$

$$\therefore\ F\left[e^{j\omega_0 t}\right] = 2\pi\,\delta(\omega-\omega_0) = \begin{cases} 2\pi & \text{for } \omega=\omega_0 \\ 0 & \text{for } \omega\neq\omega_0 \end{cases}$$

$$\therefore\ \boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)}$$



F.T of $e^{-j\omega_0 t}$ (or) I.F.T of $\delta(\omega+\omega_0)$:-

$$\delta(\omega+\omega_0) = \begin{cases} 1 & \text{for } \omega+\omega_0 = 0 \Rightarrow \omega = -\omega_0 \\ 0 & \text{for } \omega \neq -\omega_0 \end{cases}$$

$$\delta(\omega+\omega_0) \longleftrightarrow G(\omega)$$

~~But~~ $F^{-1}[G(\omega)] = \delta(\omega+\omega_0)$

$$F^{-1}[\delta(\omega+\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega+\omega_0)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\left[\delta(\omega+\omega_0)\, e^{j\omega t}\right]_{\omega=-\omega_0}$$

$$= \frac{1}{2\pi}\, e^{-j\omega_0 t}.$$

$$F^{-1}[\delta(\omega+\omega_0)] = \frac{1}{2\pi}\, e^{-j\omega_0 t}$$

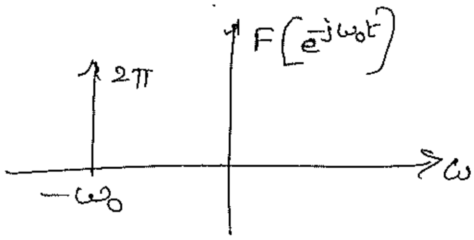
Apply F.T on both sides.

$$\Rightarrow \delta(\omega+\omega_0) = F\left[\frac{1}{2\pi}\, e^{-j\omega_0 t}\right]$$

$$\Rightarrow 2\pi\,\delta(\omega+\omega_0) = F\left[e^{-j\omega_0 t}\right]$$

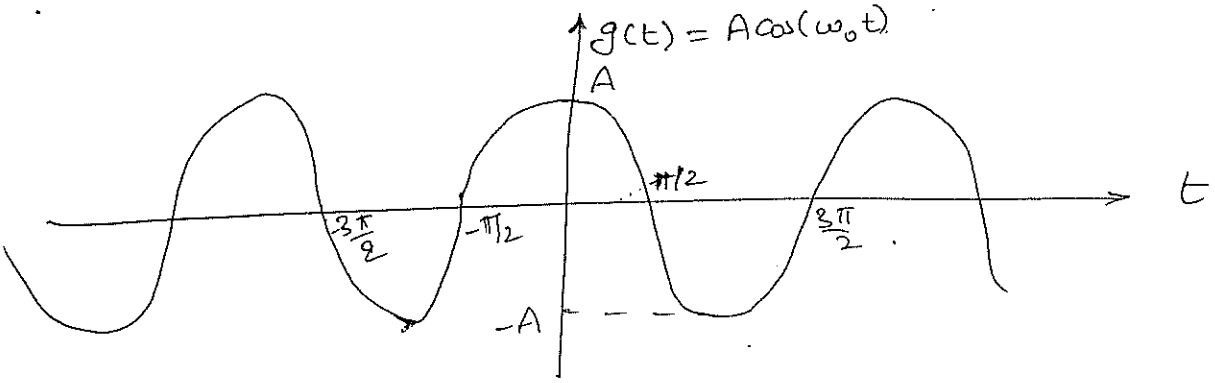
$$\Rightarrow F\left[e^{-j\omega_0 t}\right] = 2\pi\,\delta(\omega+\omega_0) = \begin{cases} 2\pi & \text{for } \omega = -\omega_0 \\ 0 & \text{for } \omega \neq \omega_0. \end{cases}$$

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$



F.T of cos function :-

$$g(t) = A\cos(\omega_0 t)\,; \quad \omega_0 = \frac{2\pi}{T_0}$$



$$g(t) \longleftrightarrow G(\omega).$$

$$g(t) = A\cos(\omega_0 t).$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= A\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$= \frac{A}{2} e^{j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t}.$$

$$g(t) \longleftrightarrow G(\omega).$$

$$G(\omega) = F\left[\frac{A}{2} e^{j\omega_0 t}\right] + F\left[\frac{A}{2} e^{-j\omega_0 t}\right]$$

$$= \frac{A}{2} \cdot F\left[e^{j\omega_0 t}\right] + \frac{A}{2} F\left[e^{-j\omega_0 t}\right]$$

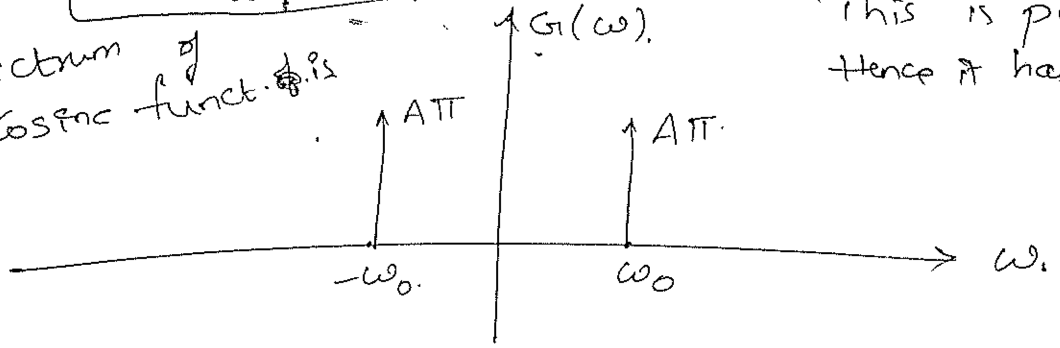
$$= \frac{A}{2}\left(2\pi\,\delta(\omega-\omega_0)\right) + \frac{A}{2}\left(2\pi\,\delta(\omega+\omega_0)\right)$$ [∵ from Previous Calculations]

∴ F.T of cos funct. is

$$\boxed{G(\omega) = A\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$

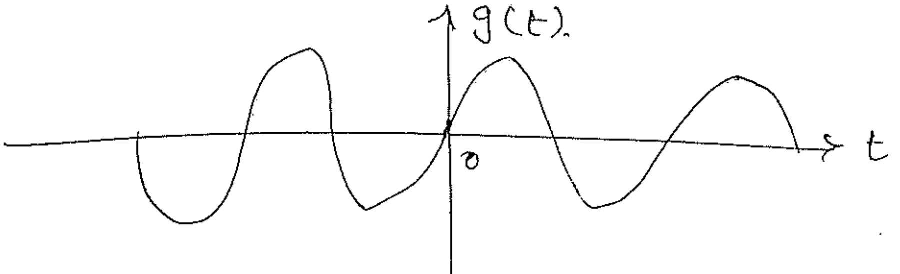
This is pure real value Hence it has no phase.

Spectrum of Cosine funct. is



F.T of Sine wave :-

$$g(t) = A\sin(\omega_0 t). \quad ; \quad \omega_0 = \frac{2\pi}{T_0}$$



$$g(t) = A\sin(\omega_0 t)$$

$$= A\left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] = \frac{A}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$$

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \cancel{\int_{-\infty}^{\infty}} F\left[\frac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right]$$

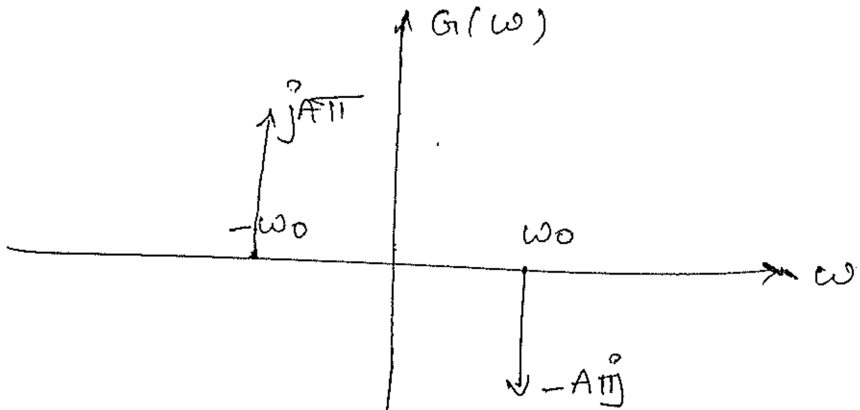
$$= \frac{A}{2j}\left(F\left[e^{j\omega_0 t}\right] - F\left[e^{-j\omega_0 t}\right]\right)$$

$$= \frac{-Aj}{2}\left[2\pi\,\delta(\omega-\omega_0) - 2\pi\,\delta(\omega+\omega_0)\right]$$

$$= A\pi j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

$\therefore$ F.T of sine-function is

$$\boxed{G(\omega) = A\pi j\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]}$$

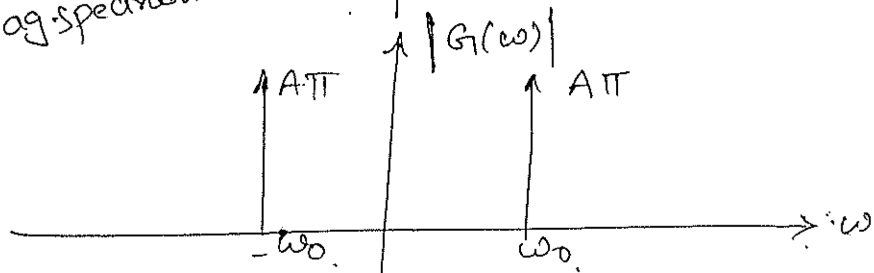


This is pure img. Hence it has phase.

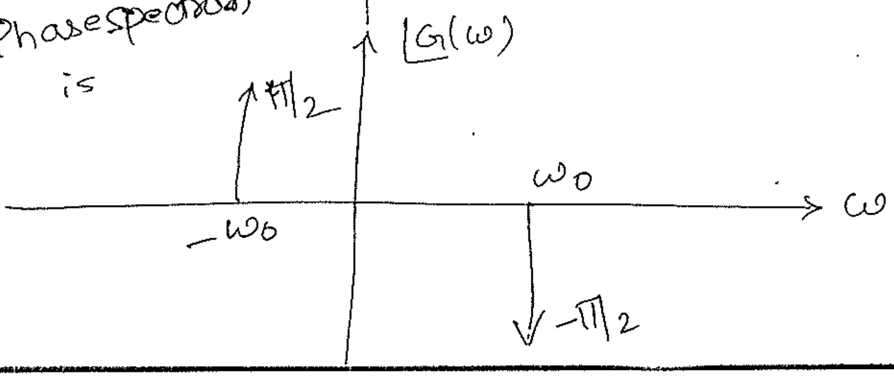
$\tan^{-1}(b/a) = \tan^{-1}(\infty) = \pi/2$.

$\tan^{-1}(-\infty) = -\pi/2$

Mag.spectrum is



Phasespectrum is



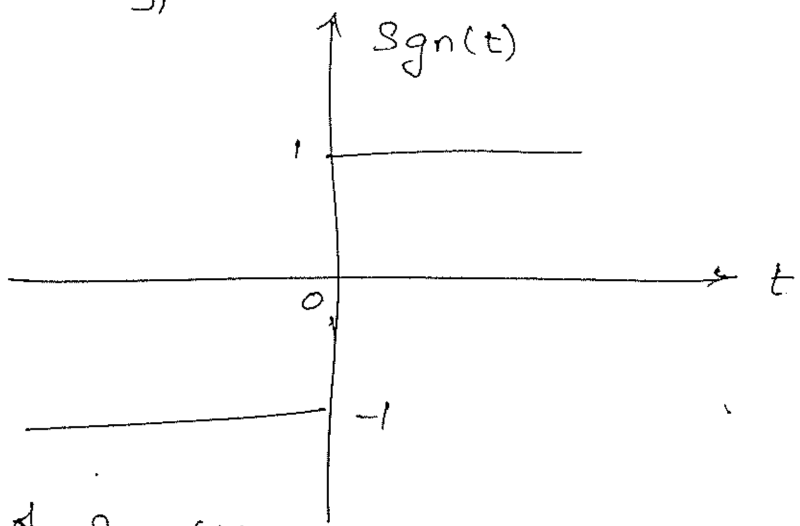
F.T of Signum function :-

It is defined by $Sgn(t) = \begin{cases} 1 & \text{for } t>0 \\ 0 & t=0 \\ -1 & t<0. \end{cases}$

for Convienency,

$$Sgn(t) = \begin{cases} \underset{a\to 0}{Lt}\, e^{-at}\, u(t) & ;\ \text{for } t>0 \\ 0 & ,\ \text{for } t=0 \\ \underset{a\to 0}{Lt} -e^{at}\, u(-t) & ;\ \text{for } t<0. \end{cases}$$

Graphically,



F.T of Sgn (t) $\longleftrightarrow$ $G(\omega)$.

$$G(\omega) = F\left[g(t)\right]$$

$$= \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} \underset{a\to 0}{Lt}\, e^{at} u(t)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} \underset{a\to 0}{Lt}\, e^{-at} u(t)\, e^{-j\omega t}\, dt.$$

$$= \underset{a\to 0}{Lt}\left[\int_{-\infty}^{0} - e^{at}(1)\, e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at}(1)\, e^{-j\omega t}\, dt\right]$$

$$= \underset{a\to 0}{Lt}\left[\int_{-\infty}^{0} - e^{(a-j\omega)t}\, dt + \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt\right]$$

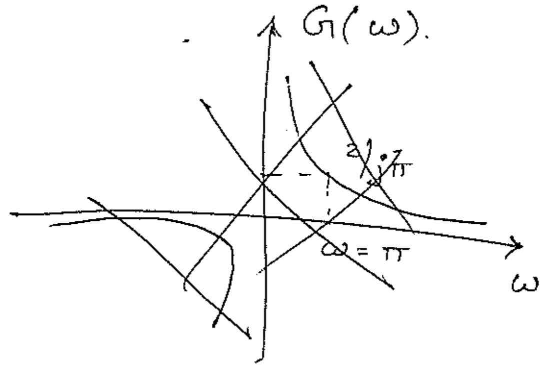
$$= \lim_{a\to 0}\left[\left.\frac{-e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} + \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_{0}^{\infty}\right]$$

$$= \lim_{a\to 0}\left[0 + \frac{-1}{a-j\omega} + \frac{+1}{a+j\omega}\right]$$

$$= \cancel{\lim_{a\to 0}}\ \frac{\cancel{a}+j\omega-\cancel{a}+j\omega}{\cancel{a^2}+\omega^2} = \cancel{\lim_{a\to 0}}\ \frac{j\omega}{a^2+\omega^2}$$

$$= \frac{j}{\omega}$$

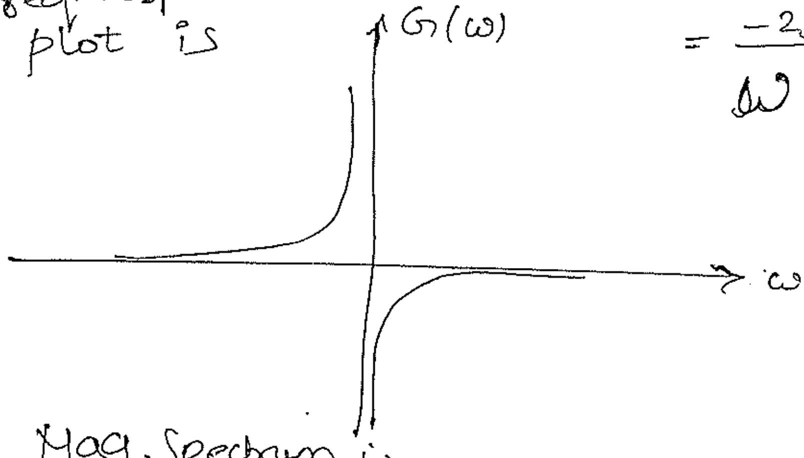
$$= \frac{-1}{-j\omega} + \frac{1}{j\omega}$$



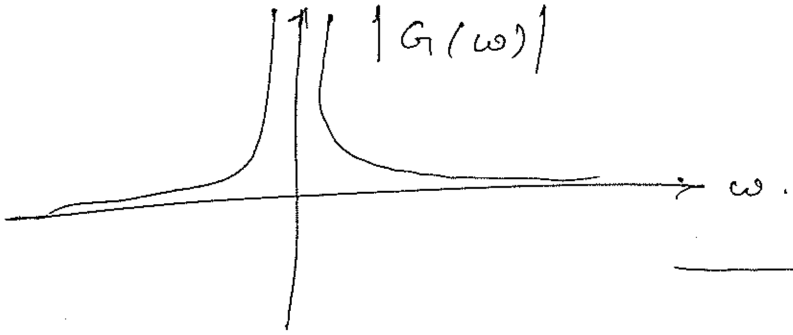
$$\therefore \quad G(\omega) = \frac{2}{j\omega}$$

$$= \frac{-2j}{\omega}$$

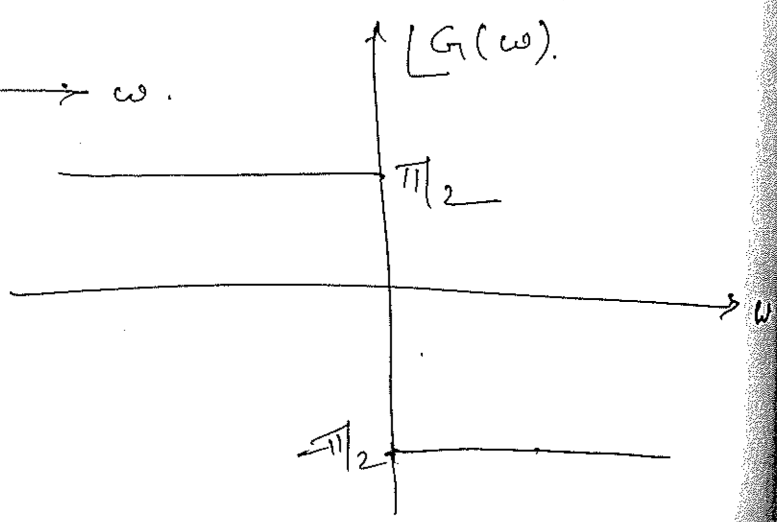
Freq. response plot is



Mag. Spectrum is



Phase Spectrum is.



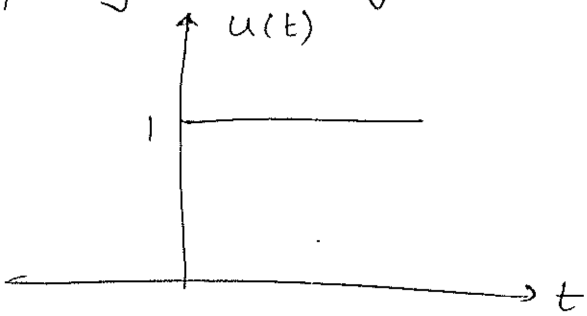
27/7/06.

F.T of unit step Signal :-

Unit step signal is defined by

$$u(t) = 1 \quad ; \quad t > 0$$
$$= 1/2 \quad ; \quad t = 0$$
$$= 0 \quad ; \quad t < 0$$

(or) $u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$



The relation b/w unit step signal & signum func. is

$$u(t) = \frac{1 + \text{Sgn}(t)}{2}$$

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$g(t) = u(t)$$

$$F[g(t)] = F[u(t)] = F\left[\tfrac{1}{2} + \tfrac{1}{2}\,\text{sgn}(t)\right]$$

F.T satisfies linearity property.

$$G(\omega) = F[u(t)] = \tfrac{1}{2} F[1] + \tfrac{1}{2} F[\text{sgn}(t)]$$

w.k.T $\quad 1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$

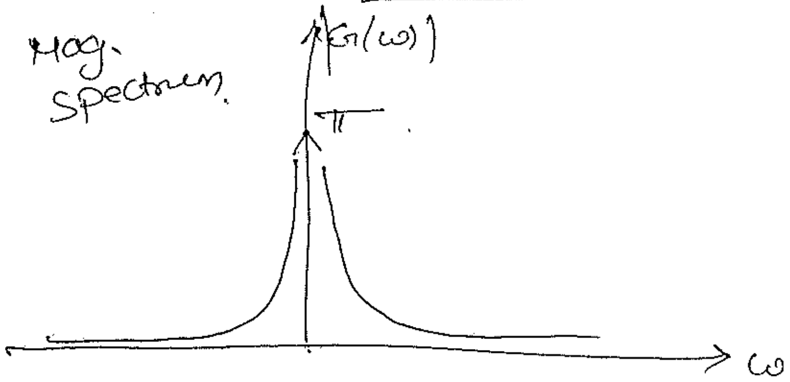
$$\text{Sgn}(t) \xleftrightarrow{F.T} 2/j\omega$$

$$G(\omega) = F[u(t)] = \tfrac{1}{2} \cdot 2\pi\,\delta(\omega) + \tfrac{1}{2} \cdot \frac{2}{j\omega}$$

$$= \pi\,\delta(\omega) + \frac{1}{j\omega}$$

$$\therefore \boxed{G(\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)}$$

Mag. Spectrum.



F.T of Continous time periodic signals :-

The Complex Fourier Series representation of periodic Signal $g(t)$ in the interval $-\frac{T}{2} < t < T/_2$ is

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\omega_0 nt} \quad ; \omega_0 = \frac{2\pi}{T_0}$$

where

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-j\omega nt}\, dt$$

Apply F.T on both sides of the above eq.①, we get

$$F[g(t)] = F\left[\sum_{n=-\infty}^{\infty} C_n e^{-j\omega_0 nt}\right]$$

$g(t) \longleftrightarrow G(\omega)$; From linearity property, we get

$$\Rightarrow \quad G(\omega) = \sum_{n=-\infty}^{\infty} C_n \, F\left[e^{-j\omega_0 nt}\right]$$

we know that $1 \longleftrightarrow 2\pi\,\delta(\omega)$

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

lly $e^{-j\omega_0 nt} \longleftrightarrow 2\pi\delta(\omega - n\omega_0)$

$$\therefore \quad G(\omega) = \sum_{n=-\infty}^{\infty} C_n \left[2\pi\,\delta(\omega - n\omega_0)\right]$$

$$\boxed{G(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\, \delta(\omega - n\omega_0)}$$

which is th [illegible] signal, $g(t)$.

where

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-j\omega_0 nt}\, dt.$$

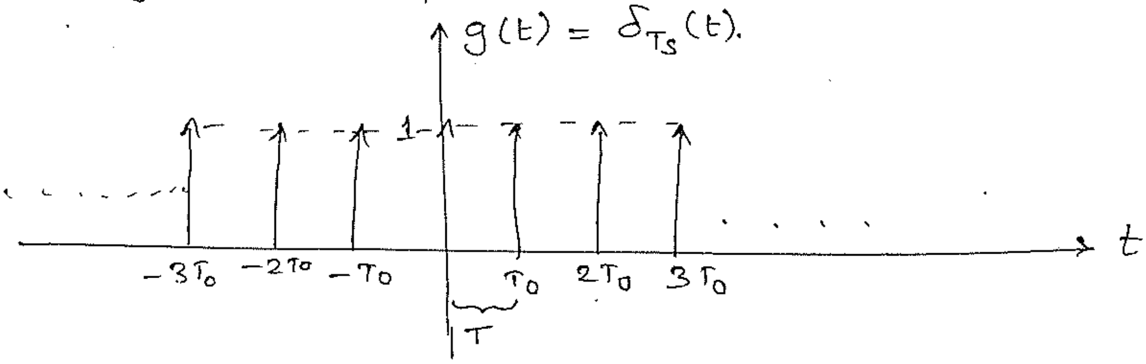
* Find the F.T of impulse train function (or) dirac
☆ Comb function.

→ The impulse train (or) dirac comb is defined by

$$g(t) = \delta_{T_s}(t) = \sum_{K=-\infty}^{\infty} \delta(t-KT_0)$$

$$= \dots + \delta(t+T_0) + \delta(t) + \delta(t-T_0) + \delta(t-2T_0) + \dots$$

Its graphical representation is



F.T of periodic signal, g(t) is

$$G(\omega) = 2\pi \sum_{n=-\infty}^{\infty} c_n \, \delta(\omega - n\omega_0).$$

where

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g(t) \, e^{-j\omega_0 n t} \, dt.$$

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sum_{K=-\infty}^{\infty} \delta(t-KT_0) \, e^{-j\omega_0 n t} \, dt.$$

$$= \frac{1}{T_0} \times 1 \quad \left(\because \text{for } K = \frac{t}{T_0}, \text{ the value of } \delta(t-KT_0) = \delta(0) = 1 \right)$$

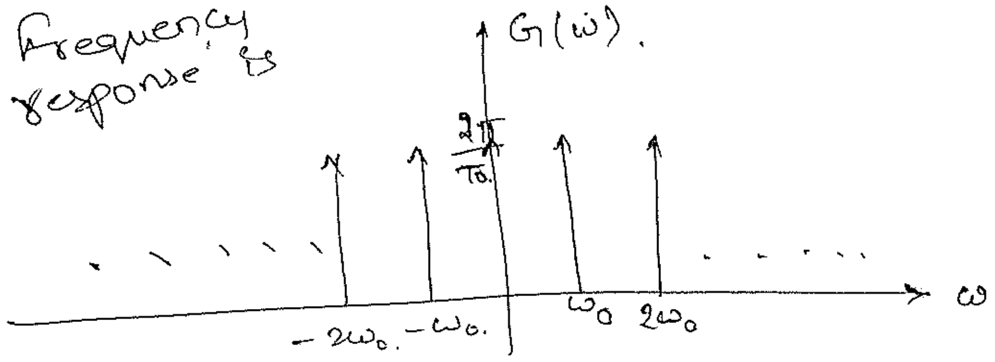
$$= \frac{1}{T_0}.$$

$$G(\omega) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T_0} \, \delta(\omega - n\omega_0)$$

$$\Rightarrow \boxed{G(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)}$$

$$= \frac{2\pi}{T_0} \left[\dots + \delta(\omega + \omega_0) + \delta(\omega) + \delta(\omega - \omega_0) + \dots \right].$$

Frequency response is



Phase = 0
∴ it is a pure real value.

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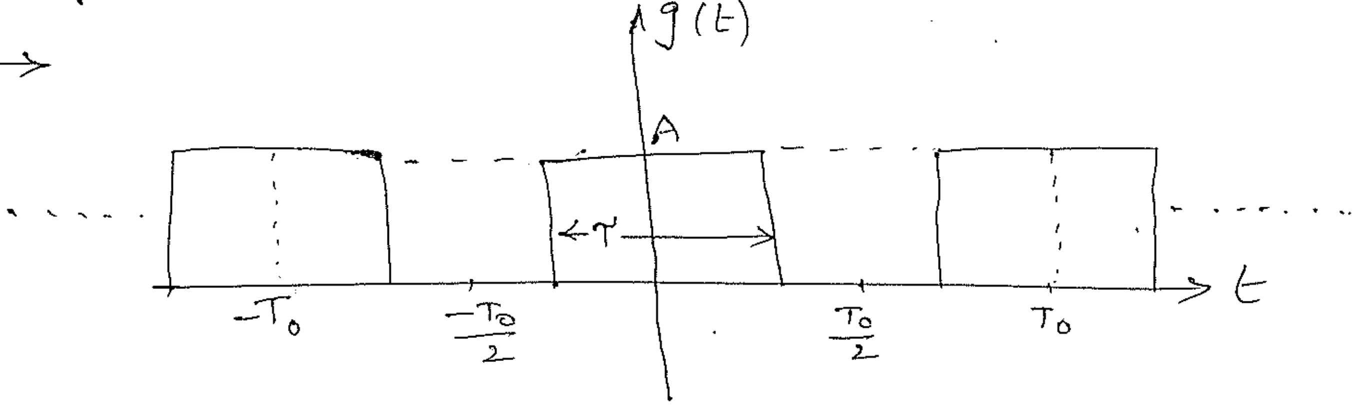
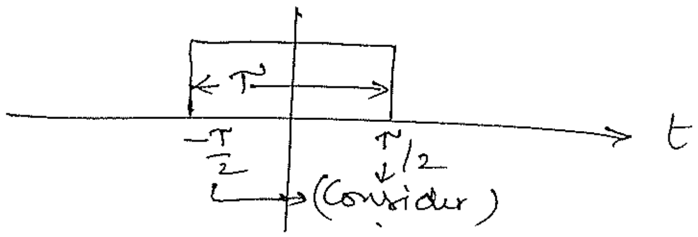
* 16m. ~~Find the~~
Expand by using Complex Fourier Series for the following square wave signal and also find FT of this signal.



→ Consider the period $\frac{-T_0}{2} \leq t \leq \frac{T_0}{2}$.

$\therefore \; T = T_0.$



$$g(t) = \begin{cases} 0 & ; \; -\frac{T_0}{2} \leq t \leq -\tau/2 \\ A & ; \; -\frac{\tau}{2} \leq t \leq \tau/2 \\ 0 & ; \; \tau/2 \leq t \leq T_0/2 \end{cases}$$

The F-Series expansion of g(t) is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega_0 n t} \quad , \; \omega_0 = \frac{2\pi}{T_0}$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-j\omega_0 n t}\, dt.$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A\, e^{-j\omega_0 n t}\, dt$$

$$= \frac{A}{T_0}\, \frac{e^{-j\omega_0 n t}}{-j\omega_0 n}\Bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{-j\omega_0 n T_0}\left[\quad - e^{+j\omega_0 n \frac{\tau}{2}}\right]$$

$$= \frac{A\times 2}{T_0\omega_0 n}\left[\frac{e^{j\omega_0 n \frac{\tau}{2}} - e^{-j\omega_0 n \frac{\tau}{2}}}{2j}\right]$$

$$= \frac{2A}{T_0\omega_0 n}\,\sin\left(\frac{n\omega_0\tau}{2}\right)$$

$$= \frac{\tau A \sin(n\omega_0\tau/2)}{T_0\,(n\omega_0\tau/2)}$$

$$= \boxed{\frac{A\tau}{T_0}\,\mathrm{sinc}\,(n\omega_0\tau/2)}$$

$$\left(\because \frac{\sin x}{x} = \mathrm{sinc}(x)\right)$$

$\therefore$ The F-series expansion of $g(t)$ is

$$g(t) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_0}\,\mathrm{sinc}(n\omega_0\tau/2)\, e^{j\omega_0 n t}$$

(i). F.T of Periodic Signal, $g(t)$ is

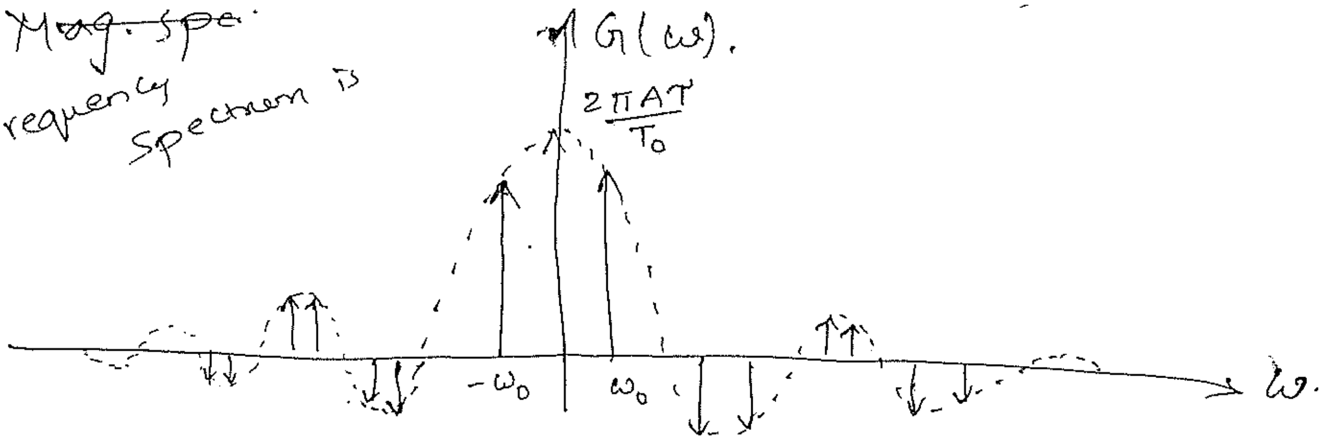
$$G(\omega) = 2\pi \sum_{n=-\infty}^{\infty} C_n\, \delta(\omega - n\omega_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_0}\,\mathrm{sinc}(n\omega_0\tau/2)\,\delta(\omega - n\omega_0)$$

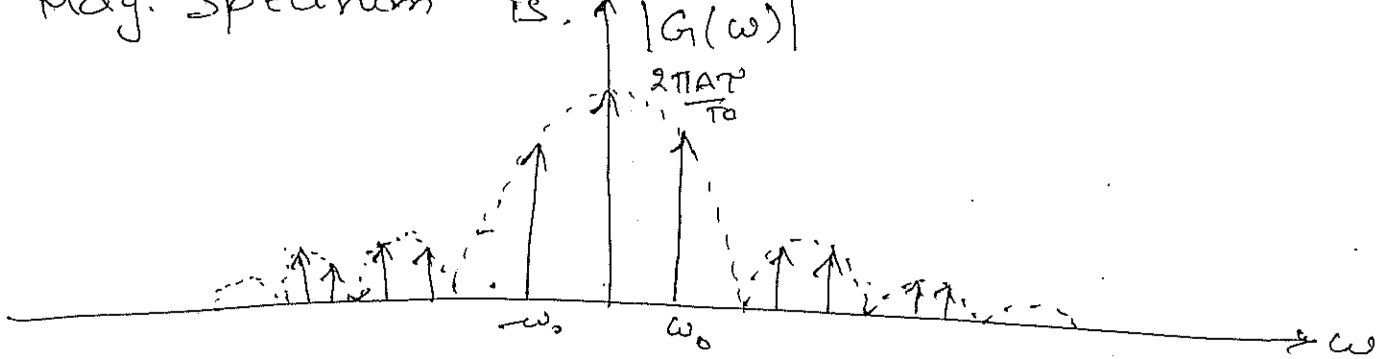
$G(\omega)$ Contains Pulses ~~at~~ of $\delta(\omega - n\omega_0)$ with the

Scaling factor of $2\pi A\tau$ sinc$(n\omega_0\tau/2)$

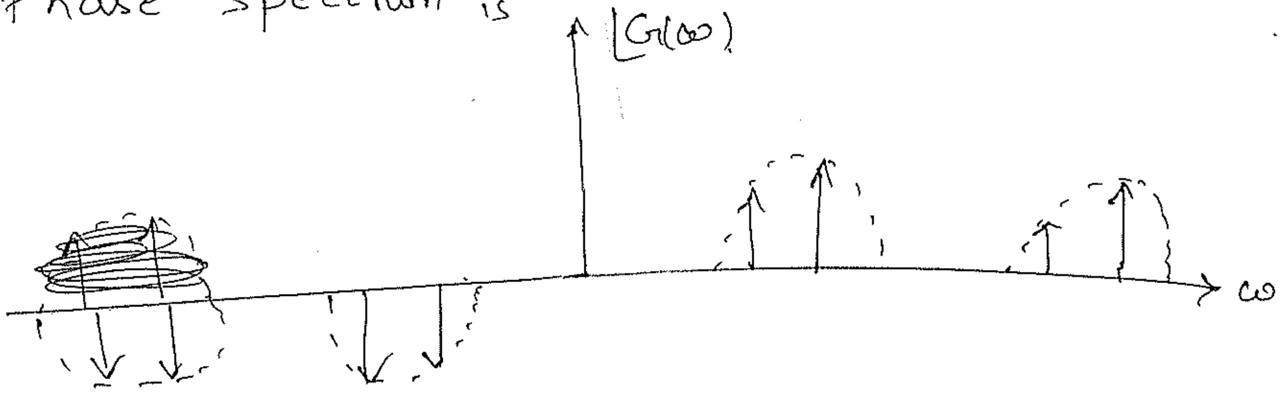
~~Mag. spe.~~ Frequency Spectrum is



Mag. Spectrum is



Phase spectrum is



PROPERTIES OF FOURIER TRANSFORMS :-

① Linearity property :- Let

$$g(t) \xleftrightarrow{F.T} G(\omega),$$

$$g_1(t) \longleftrightarrow G_1(\omega),$$

$$g_2(t) \longleftrightarrow G_2(\omega),$$

then $a_1 g_1(t) + a_2 g_2(t) \longleftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega)$.

i.e; $F\left[a_1 g_1(t) + a_2 g_2(t)\right] = a_1 F\left[g_1(t)\right] + a_2 F\left[g_2(t)\right]$

Pf :- L.H.S $= F\left[a_1 g_1(t) + a_2 g_2(t)\right]$

$$= \int_{-\infty}^{\infty} \left(a_1 g_1(t) + a_2 g_2(t)\right) e^{-j\omega t}\, dt \quad \left[\because F[g(t)] = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt\right]$$

$$= a_1 \int_{-\infty}^{\infty} g_1(t) e^{-j\omega t} dt + a_2 \int_{-\infty}^{\infty} g_2(t) e^{-j\omega t} dt$$

$$= a_1 G_1(\omega) + a_2 G_2(\omega) = \text{R.H.S}$$

Here F-Transform satisfies Superposition principle, it states that F.T of weighted sum of signals is equivalent to the weighted sum of F.T to each of individual signals.

where a_1, a_2 are arbitrary constants.

lly $a_1 g_1(t) + a_2 g_2(t) + \cdots + a_n g_n(t) \longleftrightarrow a_1 G_1(\omega) + a_2 G_2(\omega) + \cdots + a_n G_n(\omega)$

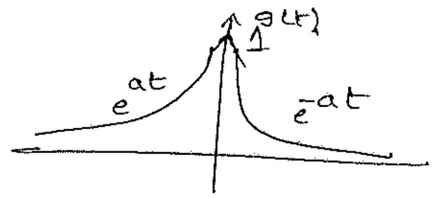
Ex :- Find the F.T of the foll. Signals. Use linearity property of F.T only.

1. Double exponential pulse.

$\longrightarrow$ $g(t) = e^{-a|t|} = e^{-at} u(t) + e^{at} u(-t)$

$F[g(t)] = F\left[e^{-a|t|}\right]$

$= F\left[e^{-at} u(t) + e^{at} u(-t)\right]$

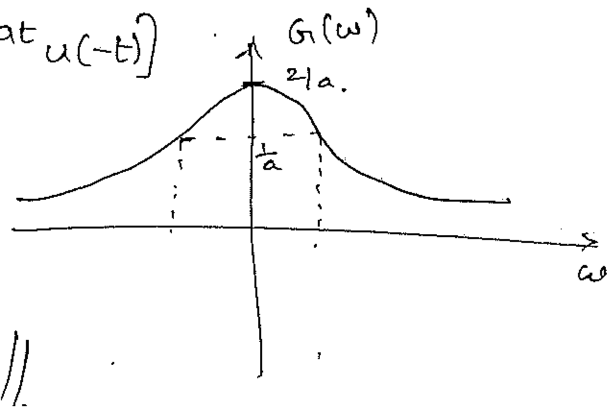


By using linearity of F.T,

$= F\left[e^{-at} u(t)\right] + F\left[e^{at} u(-t)\right]$

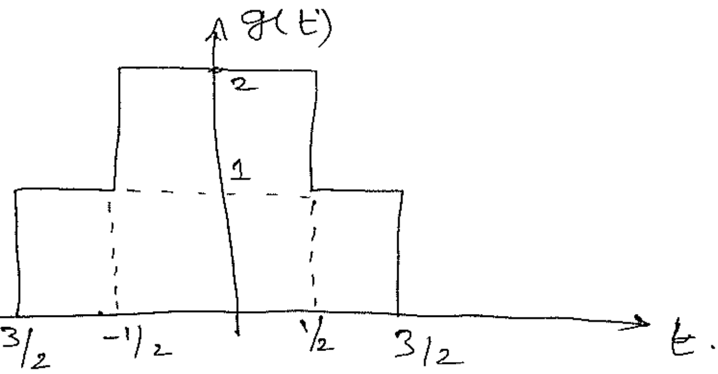
w.k.T $e^{-at} u(t) \longleftrightarrow \dfrac{1}{a+j\omega}$

$e^{at} u(-t) \longleftrightarrow \dfrac{1}{a-j\omega}$



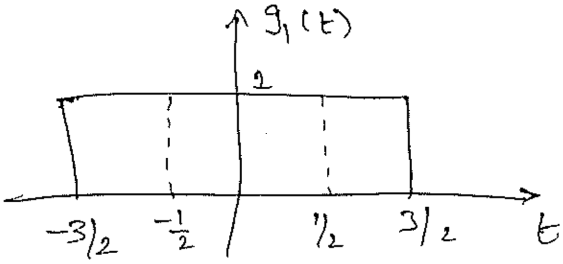
$\therefore$ $G(\omega) = \dfrac{1}{a+j\omega} + \dfrac{1}{a-j\omega} = \dfrac{2a}{a^2+\omega^2}$ //

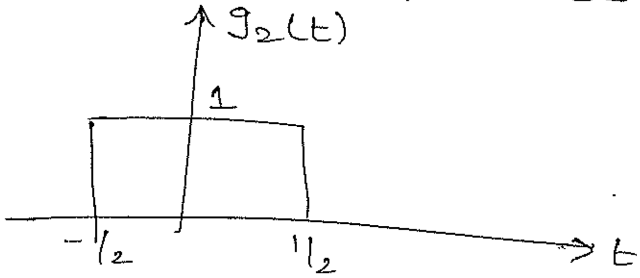
(2) The given pulse is



$$g(t) = 1 \; ; \; \frac{-3}{2} < t \le \frac{-1}{2}$$
$$= 2 \; ; \; \frac{-1}{2} \le t \le \frac{1}{2}$$
$$= 1 \; ; \; \frac{1}{2} \le t \le \frac{3}{2}.$$

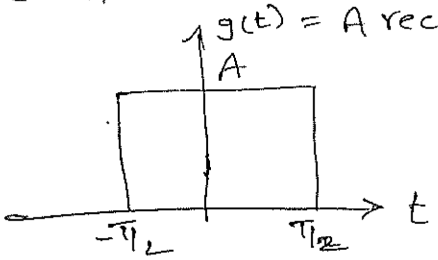
The given pulse is the sum of these two pulses.





we know that

$$g(t) = A\,\mathrm{rect}\left(\frac{t}{T}\right)$$



$$\longleftrightarrow \quad AT\,\mathrm{Sinc}\left(\frac{\omega T}{2}\right).$$

i.e; $A\,\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT\,\mathrm{Sinc}\left(\frac{\omega T}{2}\right).$

$$g_1(t) = 1.\mathrm{rect}\left(\frac{t}{3}\right) \longleftrightarrow 1\times 3\,\mathrm{sinc}\left(\frac{\omega\times 3}{2}\right)$$

$$g_2(t) = 1.\mathrm{rect}\left(\frac{t}{1}\right) \longleftrightarrow 1\times 1\,\mathrm{Sinc}\left(\frac{\omega\times 1}{2}\right)$$

$$g(t) = g_1(t) + g_2(t)$$

$$F[g(t)] = F[g_1(t) + g_2(t)]$$

from F.T linearity property,

$$= F[g_1(t)] + F[g_2(t)]$$

$$G(\omega) = 3\,\mathrm{sinc}\left(\frac{3\omega}{2}\right) + \mathrm{sinc}\left(\frac{\omega}{2}\right)$$

By Analytical method :-

$$g(t) = 1 \; ; \quad -3/2 \le t \le -1/2$$
$$= 2 \; ; \quad -1/2 \le t \le 1/2$$
$$= 1 \; ; \quad 1/2 \le t \le 3/2$$

$$g(t) \longleftrightarrow G(\omega)$$

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega_0 t}\, dt$$

$$= \int_{-3/2}^{-1/2} 1 \cdot e^{-j\omega t}\, dt + \int_{-1/2}^{1/2} 2\, e^{-j\omega t}\, dt + \int_{1/2}^{3/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-3/2}^{-1/2} + 2 \cdot \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-1/2}^{1/2} + \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{1/2}^{3/2}$$

$$= \frac{j}{\omega}\left[e^{\frac{j\omega}{2}} - e^{\frac{3j\omega}{2}} \right] + \frac{2j}{\omega}\left[e^{-\frac{j\omega}{2}} - e^{\frac{j\omega}{2}} \right] + \frac{j}{\omega}\left[e^{-\frac{3j\omega}{2}} - e^{-\frac{j\omega}{2}} \right]$$

$$= \frac{j}{\omega}\left[\cancel{e^{\frac{j\omega}{2}}} - \cancel{e^{-\frac{j\omega}{2}}} - e^{\frac{3j\omega}{2}} + e^{-\frac{3j\omega}{2}} + \cancel{2e^{-\frac{j\omega}{2}}} - \cancel{2e^{\frac{j\omega}{2}}} \right]$$

$$= \frac{j}{\omega}\left[e^{-\frac{3j\omega}{2}} - e^{\frac{3j\omega}{2}} \right] + \frac{j}{\omega}\left(e^{-\frac{j\omega}{2}} - e^{\frac{j\omega}{2}} \right)$$

$$= \frac{2}{\omega}\left[\frac{e^{\frac{3j\omega}{2}} - e^{-\frac{3j\omega}{2}}}{2j} \right] + \frac{2}{\omega}\left[\frac{e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}}{2j} \right]$$

$$= \frac{2}{\omega}\left[\sin\left(\frac{\omega}{2}\right) + \sin\left(\frac{3\omega}{2}\right) \right]$$

$$= \frac{\sin(\omega/2)}{\omega/2} + 3 \times \frac{\sin(3\omega/2)}{3\omega/2}$$

$$= \text{sinc}\left(\frac{\omega}{2}\right) + 3\, \text{sinc}\left(\frac{3\omega}{2}\right)$$

(2) Time Scaling property :- If $g(t) \xleftrightarrow{FT} G(\omega)$, then

$$g(at) \longleftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$$

Pf :- case (i) :- for $a > 0$, i.e., a is +ve value.

$$G(\omega) = F[g(t)] = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$$

$$F[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt \quad \text{Put } at = \tau \Rightarrow a\,dt = d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) e^{-j\frac{\omega}{a}\tau} \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} g(t) e^{-j\left(\frac{\omega}{a}\right)t} dt$$

$$= \frac{1}{a} G\left(\frac{\omega}{a}\right).$$

case (ii) :- for $a < 0$ i.e., a is a -ve value.

$$F[g(at)] = \int_{-\infty}^{\infty} g(at) e^{-j\omega t} dt \qquad \text{Put } at = \tau, \; a\,dt = d\tau$$

$t \to -\infty \Rightarrow \tau \to \infty$

$t \to \infty \Rightarrow \tau \to -\infty$

$$= \int_{\infty}^{-\infty} g(\tau) e^{-j\frac{\omega}{a}\tau} \frac{d\tau}{a}$$

$$= \frac{-1}{a} \int_{-\infty}^{\infty} g(\tau) e^{-j\frac{\omega}{a}\tau} d\tau$$

$$= \frac{-1}{a} G\left(\frac{\omega}{a}\right) //$$

$$\therefore F[g(at)] = \frac{1}{a} G\left(\frac{\omega}{a}\right) \; ; \; a > 0$$

$$\frac{-1}{a} G\left(\frac{\omega}{a}\right) \; ; \; a < 0,$$

Hence in general, $g(at) \xleftrightarrow{FT} \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$.

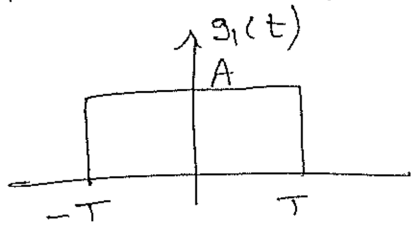
Significance of time scaling property :-

The signal $g(at)$ represents $g(t)$ signal compressed by a factor 'a', $G(\frac{\omega}{a})$ represents $G(\omega)$ expanded by a factor 'a', $a > 1$.

- Time scaling property states that compression version of signal in time domain is equivalent to the expanded of their frequency spectrum by a same factor, (or) vice-versa.

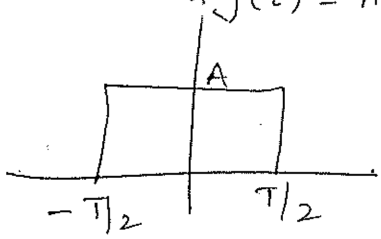
Ex:- Find the F.T of foll. signals by using time-scaling properties,

→



→ Representation of given signal is $g_1(t) = A\,\text{rect}(\frac{t}{2T})$.

$g(t) = A\,\text{rect}(\frac{t}{T}) \longleftrightarrow AT\,\text{sinc}(\frac{\omega T}{2}) = G(\omega)$



$g_1(t) = g(\frac{t}{2}) = A\,\text{rect}(\frac{t/2}{T}) = A\,\text{rect}(\frac{t}{2T})$

$g(t) \longleftrightarrow G(\omega)$

$g(at) \longleftrightarrow \frac{1}{a} G(\frac{\omega}{a})$

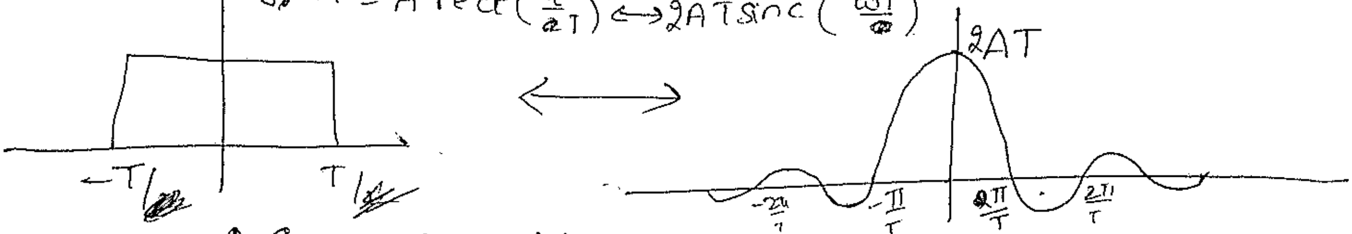
$g(\frac{1}{2}t) \longleftrightarrow \frac{1}{1/2} G(\frac{\omega}{1/2})$

$\Rightarrow g(\frac{t}{2}) \longleftrightarrow 2G(2\omega)$

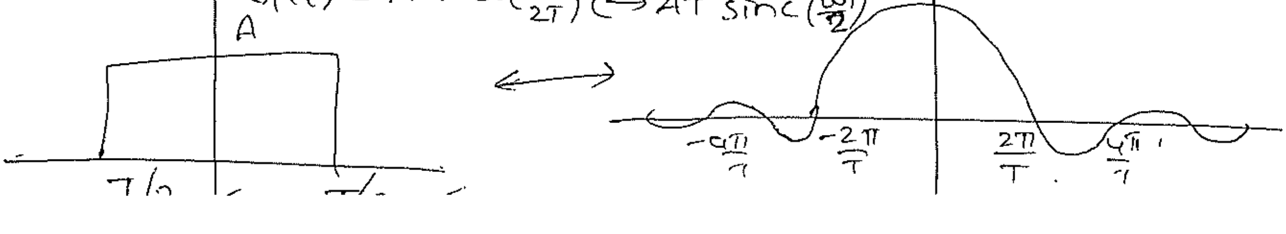
$g(\frac{t}{2}) \longleftrightarrow 2\,AT\,\text{Sinc}(\frac{2\omega T}{2})$

$g_1(t) = g(\frac{t}{2}) \longleftrightarrow 2\,AT\,\text{Sinc}(\omega T)$

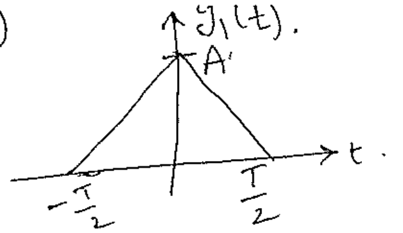
$g_1(t) = A\,\text{rect}(\frac{t}{2T}) \longleftrightarrow 2AT\,\text{sinc}(\omega T)$



$g_1(t) = A\,\text{rect}(\frac{t}{2T}) \longleftrightarrow AT\,\text{sinc}(\frac{\omega T}{2})$



(2)

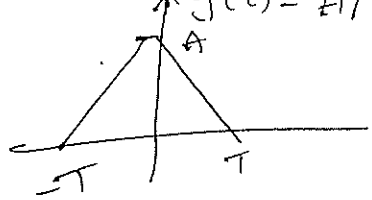


Sol:-

$$g_1(t) = A\Lambda\left(\frac{t}{T}\right)$$

$$g_1(t) = g(2t) = A\Lambda\left(\frac{2t}{2T}\right)$$
$$= A\Lambda\left(\frac{t}{T}\right)$$

$g(t) = A\Lambda\left(\frac{t}{2T}\right)$



$$g(t) = A\Lambda\left(\frac{t}{2T}\right) \longleftrightarrow AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right) = G(\omega).$$

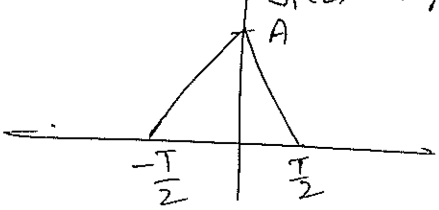
$$g(t) \longleftrightarrow G(\omega)$$

$$g(at) \longleftrightarrow \frac{1}{a} G\left(\frac{\omega}{a}\right)$$

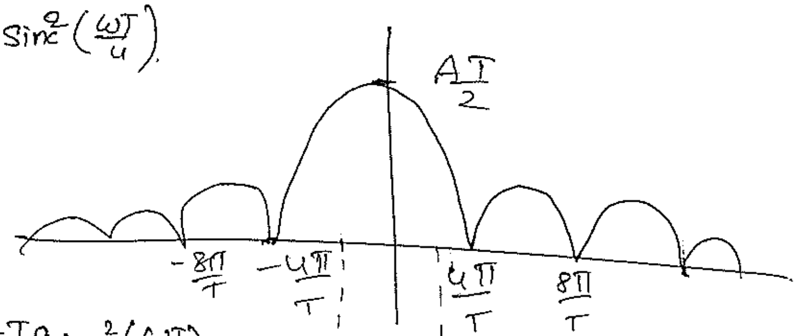
$$g(2t) \longleftrightarrow \frac{1}{2}\, G\left(\frac{\omega}{2}\right)$$

$$g_1(t) = g(2t) \longleftrightarrow \frac{1}{2}\, AT\,\text{sinc}^2\left(\frac{\omega T}{4}\right).$$

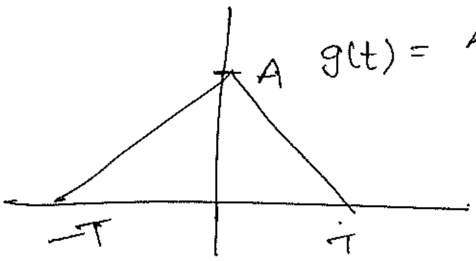
$g_1(t) = A\Lambda\left(\frac{t}{T}\right) \longleftrightarrow \frac{AT}{2}\,\text{sinc}^2\left(\frac{\omega T}{4}\right)$



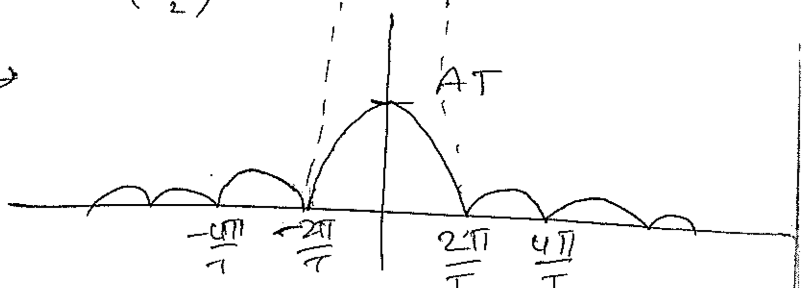
$\longleftrightarrow$



$g(t) = A\Lambda\left(\frac{t}{2T}\right) \longleftrightarrow AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$



$\longleftrightarrow$



(3) <u>Duality property</u> (or) <u>Symmetry property</u> :-

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Duality (or) Symmetry Property :-

Let $g(t) \xleftrightarrow{FT} G(\omega)$, then $G(t) \longleftrightarrow 2\pi\, g(-\omega)$.

Pf:-

$$G(\omega) = F[g(t)] = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.$$

lly

$$F^{-1}[G(\omega)] = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

$$g(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) \cdot e^{-j\omega t}\, d\omega.$$

Interchanging 't' & 'ω', we get

$$g(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(t)\, e^{-j\omega t}\, dt.$$

$$2\pi\, g(-\omega) = \int_{-\infty}^{\infty} G(t) \cdot e^{-j\omega t}\, dt$$

$$2\pi\, g(-\omega) = F[G(t)]$$

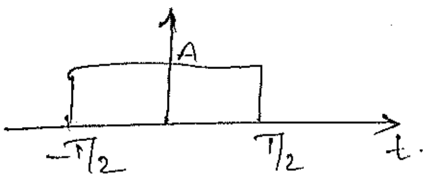
$$\therefore \quad \boxed{G(t) \longleftrightarrow 2\pi\, g(-\omega)}$$

From this, we say that the F.T of gate function is sinc func. & F.T of sinc func. is gate function.

Ex:-

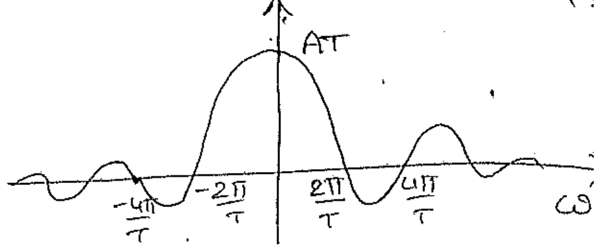
1. F.T of gate function.

$\rightarrow$ $g(t) = A\,\text{rect}\left(\frac{t}{T}\right)$



$\longleftrightarrow$

$G(\omega) = AT\text{sinc}(\frac{\omega T}{2})$



$$g(t) \longleftrightarrow G(\omega)$$

$$A\,\text{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT\,\text{sinc}\left(\frac{\omega T}{2}\right)$$

$$G(t) \longleftrightarrow 2\pi\, g(-\omega)$$

$$\cancel{AT\,\mathrm{Sinc}\left(\frac{-\omega T}{2}\right) \longleftrightarrow 2\pi\, A\,\mathrm{rect}\left(\frac{Tt}{2}\right)}$$

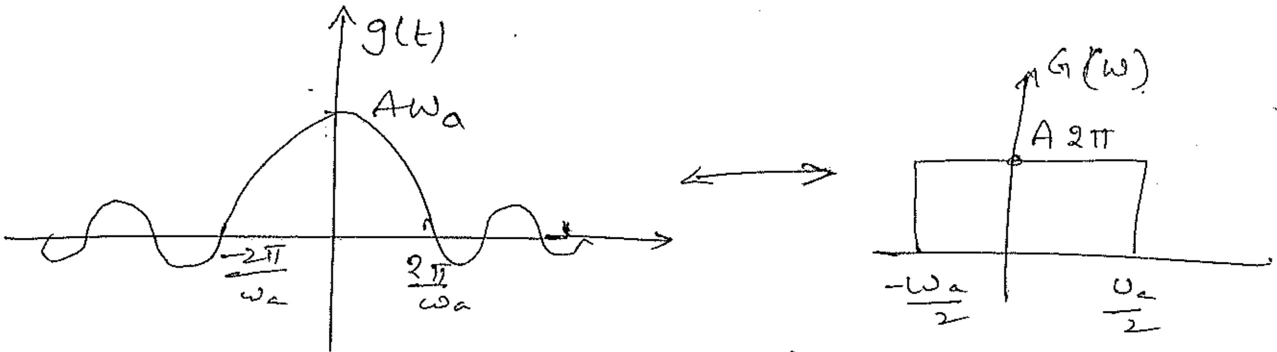
$$AT\,\mathrm{Sinc}\left(\frac{Tt}{2}\right) \Longleftrightarrow 2\pi A\,\mathrm{rect}\left(\frac{-\omega}{T}\right)$$

$$AT\,\mathrm{Sinc}\left(\frac{Tt}{2}\right) \longleftrightarrow 2\pi A\,\mathrm{rect}\left(\frac{\omega}{T}\right) \quad (\because \text{gate func is even func.})$$

when $T = \omega_a$

$$A\omega_a\,\mathrm{sinc}\left(\frac{\omega_a t}{2}\right) \longleftrightarrow 2\pi A\,\mathrm{rect}\left(\frac{\omega}{\omega_a}\right)$$

$$\mathrm{sinc}\left(\frac{\omega_a t}{2}\right) \longleftrightarrow \frac{2\pi}{\omega_a}\,\mathrm{rect}\left(\frac{\omega}{\omega_a}\right).$$



NOTE :- If $g(t)$ is even function of 't', i.e; $g(-t) = g(t)$

then $g(-\omega) = g(\omega)$, then

$$G(t) \longleftrightarrow 2\pi\, g(-\omega).$$

$$\boxed{G(t) \longleftrightarrow 2\pi\, g(\omega)}$$

- It is a perfect symmetric property

6. **Time shifting property** :-

(i) Time delay :-

let $g(t) \xleftrightarrow{FT} G(\omega)$

$$g(t - t_0) \longleftrightarrow e^{-j\omega t_0}\, G(\omega).$$

Pf :

$$G(\omega) = F[g(t)] = \int_{-\infty}^{\infty} [g(t)]\, e^{-j\omega t}\, dt$$

$$F[g(t-t_0)] = \int_{-\infty}^{\infty} g(t-t_0)\, e^{-j\omega t}\, dt$$

Put $k = t - t_0 \Rightarrow dt = dk$

$$= \int_{-\infty}^{\infty} g(k) \cdot e^{-j\omega(k+t_0)}\, dk$$

$$= \int_{-\infty}^{\infty} g(k)\, e^{-j\omega k} \cdot e^{-j\omega t_0} \cdot dk$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} g(k) \cdot e^{-j\omega k}\, dk$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt = e^{-j\omega t_0} \cdot G(\omega)$$

Case (ii) :- let $g(t) \leftrightarrow G(\omega)$, then

$$g(t+t_0) \leftrightarrow e^{j\omega t_0}\, G(\omega)$$

$$G(\omega) = F(g(t)) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$$

$$F[g(t+t_0)] = \int_{-\infty}^{\infty} g(t+t_0)\, e^{-j\omega t}\, dt$$

Put $t + t_0 = K$ $\quad = \int_{-\infty}^{\infty} g(t+t_0)\, e^{t(-j\omega)}\, dt$

$dt = dK$ $\quad = \int_{-\infty}^{\infty} g(K) \cdot e^{-j\omega(K-t_0)}\, dK$

$$= e^{j\omega K_0} \int_{-\infty}^{\infty} g(K)\, e^{-j\omega K}\, dK$$

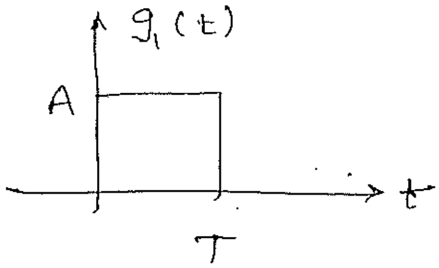
$$= e^{j\omega K_0} \cdot G(\omega)$$

NOTE:- If the Signal is shifted to right by t_0 units in time domain, then their frequency Spectrum multiplied by a factor $e^{-j\omega t_0}$.

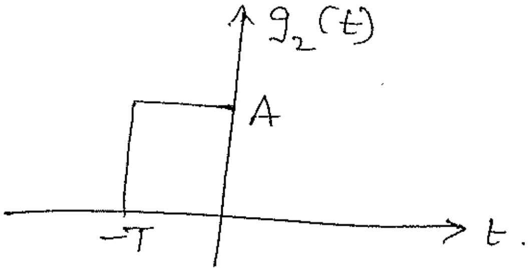
i.e; their magnitude Spectrum doesn't change & phase spectrum changed by a factor $(-\omega t_0)$.

Ex:- Find the F.T of all the foll. Signals,

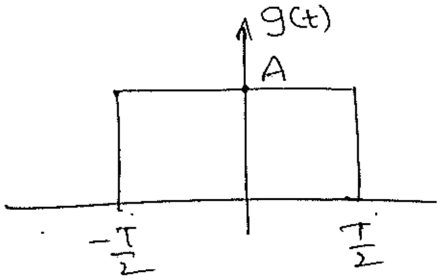
(i)



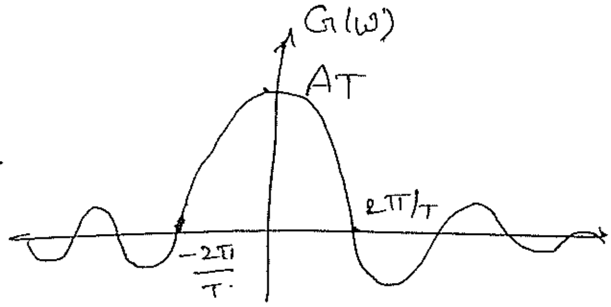
(ii)



(i) we know that



$\longleftrightarrow$



$$g(t) \longleftrightarrow G(\omega)$$

$$A \text{ rect}\left(\frac{t}{T}\right) \longleftrightarrow AT \text{ Sinc}\left(\frac{\omega T}{2}\right).$$

1)
$$g_1(t) = g\left(t - \frac{T}{2}\right)$$

$$g(t - t_0) \longleftrightarrow e^{-j\omega t_0} G(\omega)$$

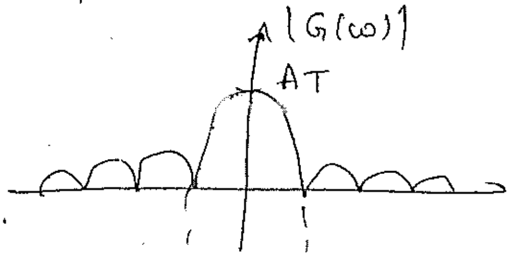
$$A \text{ rect}\left(\frac{t - T/2}{T}\right) \longleftrightarrow e^{-j\omega\frac{T}{2}} AT \text{ sinc}\left(\frac{\omega T}{2}\right)$$

2)
$$g_2(t) = g\left(t + \frac{T}{2}\right)$$

$$g(t + t_0) \longleftrightarrow e^{j\omega t_0} G(\omega).$$

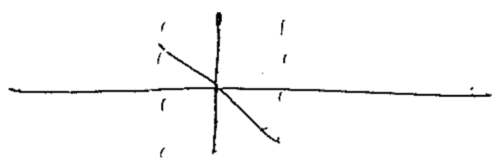
$$A \text{ rect}\left(\frac{t + T/2}{T}\right) \longleftrightarrow e^{j\omega\frac{T}{2}} AT \text{ sinc}\left(\frac{\omega T}{2}\right).$$

For both cases, mag. spectrum is same

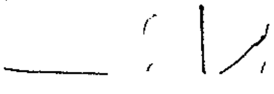


phase spectrum

For 1st one



For 2nd one.



* Find the F.T of the foll. signals by using properties.

1. $g(t) = \frac{1}{1+j2\pi t}$

2. $g(t) = \frac{2}{1+t^2}$

→ ① $g(t) = \frac{1}{1+j2\pi t}$

$g(t) \longleftrightarrow G(\omega)$

By using duality Property, $G(t) \longleftrightarrow 2\pi g(-\omega)$.

w.k.T $\underset{g(t)}{e^{-at}u(t)} \longleftrightarrow \underset{G(\omega)}{\frac{1}{a+j\omega}}$.

By duality property, $G(t) \longleftrightarrow 2\pi g(-\omega)$

$$\frac{1}{a+jt} \longleftrightarrow 2\pi e^{a\omega} u(-\omega).$$

Put $a=1$,

$$\underset{g(t)}{\frac{1}{1+jt}} \longleftrightarrow 2\pi e^{\omega} u(-\omega).$$

Here $\frac{1}{1+jt} = g(t)$

$\therefore g(2\pi t) = \frac{1}{1+j2\pi t} \longleftrightarrow \frac{1}{2\pi} 2\pi e^{\omega/2\pi} u\left(-\frac{\omega}{2\pi}\right)$.

$$\left[\because g(at) = \frac{1}{a} G\left(\frac{\omega}{a}\right)\right]$$

$$\frac{1}{1+j2\pi t} \longleftrightarrow e^{f} u(-f)$$

② $g(t) = \frac{2}{1+t^2}$.

w.k.T $g(t) \longleftrightarrow G(\omega)$

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}.$$

$$G(t) \longleftrightarrow 2\pi \; g(-\omega).$$

$$\frac{2a}{a^2+t^2} \longleftrightarrow 2\pi \cdot e^{-a|-\omega|}$$

$$\frac{2a}{a^2+t^2} \longleftrightarrow 2\pi \cdot e^{-a|\omega|}$$

when $a = 1$,

$$\frac{2}{1+t^2} \longleftrightarrow 2\pi \; e^{-|\omega|}$$

Frequency shifting Property (or) Frequency Modulation theorem (or) Frequency Translation ~~Property~~ Theorem :-

Case (i):

Frequency Delay.

Let $g(t) \xleftrightarrow{FT} G(\omega)$, then

$$e^{j\omega_c t} g(t) \longleftrightarrow G(\omega - \omega_c).$$

Pf :-

$$F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t) \, e^{-j\omega t} \, dt$$

$$F\left[g(t).e^{j\omega_c t}\right] = \int_{-\infty}^{\infty} g(t) \, e^{j\omega_c t} . e^{-j\omega t} \, dt$$

$$= \int_{-\infty}^{\infty} g(t) \, e^{-j(\omega - \omega_c) t} \, dt$$

$$= G(\omega - \omega_c)$$

Hence Proved.

Conclusion:-

The Signal $g(t)$ multiplied by a factor $e^{j\omega_c t}$ in time-domain Corresponding their freq. spectrum delay by ω_c units to the right in freq. domain. This is also known as frequency modulation (or) freq. translation theorem.

Case (ii) :- Frequency advance.

Let $g(t) \longleftrightarrow G(\omega)$, then

$$e^{-j\omega_c t} \cdot g(t) \longleftrightarrow G(\omega+\omega_c)$$

Pf :- $F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$

$$F[e^{-j\omega_c t}\, g(t)] = \int_{-\infty}^{\infty} g(t)\, e^{-j(\omega+\omega_c)t}\, dt$$

$$= G(\omega+\omega_c)$$

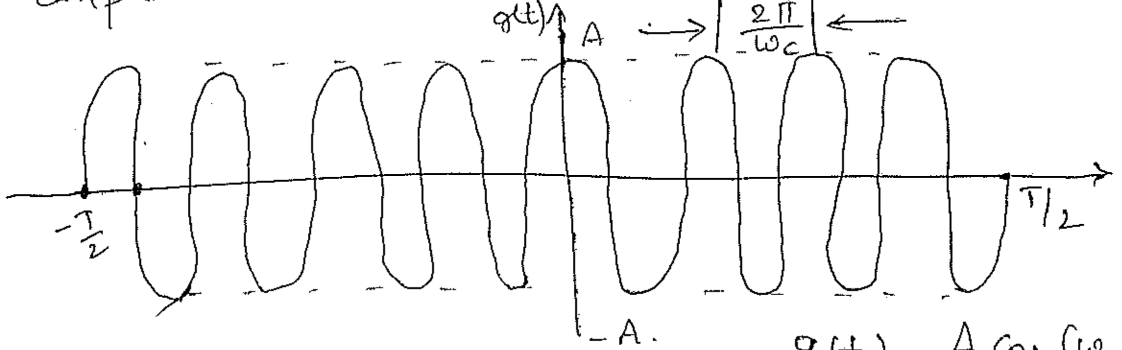
Hence Proved.

Conclusion:-

The Signal $g(t)$ multiplied by a factor $e^{-j\omega_c t}$ in time-domain Corresponding their freq. spectrum advance by ω_c unit to left in freq. domain.

Ex ① Radio freq. pulse (Rf - Pulse) :-

The Rf - Pulse as shown in fig contains Cosinusoidal oscillations with frequency ω_c and has amplitude 'A' in the duration $-T/2$ to $T/2$.



$g(t) = A\cos(\omega_c t)\, \mathrm{rect}\left(\frac{t}{T}\right)$

$$g(t) = A\cos(\omega_c t)\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$g(t) = \frac{A}{2} e^{j\omega_c t}\,\mathrm{rect}\left(\frac{t}{T}\right) + \frac{A}{2} e^{-j\omega_c t}\,\mathrm{rect}\left(\frac{t}{T}\right)$$

$$F[g(t)] = F[\quad ,\quad] \qquad \left\{\because \cos\theta = \frac{e^{\theta}+}{2}\right.$$

By Linearity Prop. of FT, we get

$$= \frac{1}{2} F\left[e^{j\omega_c t} A\,\mathrm{rect}\left(\frac{t}{T}\right)\right] + \frac{1}{2} F\left[A\,\mathrm{rect}\left(\frac{t}{T}\right) e^{-j\omega_c t}\right]$$

$G(\omega)$

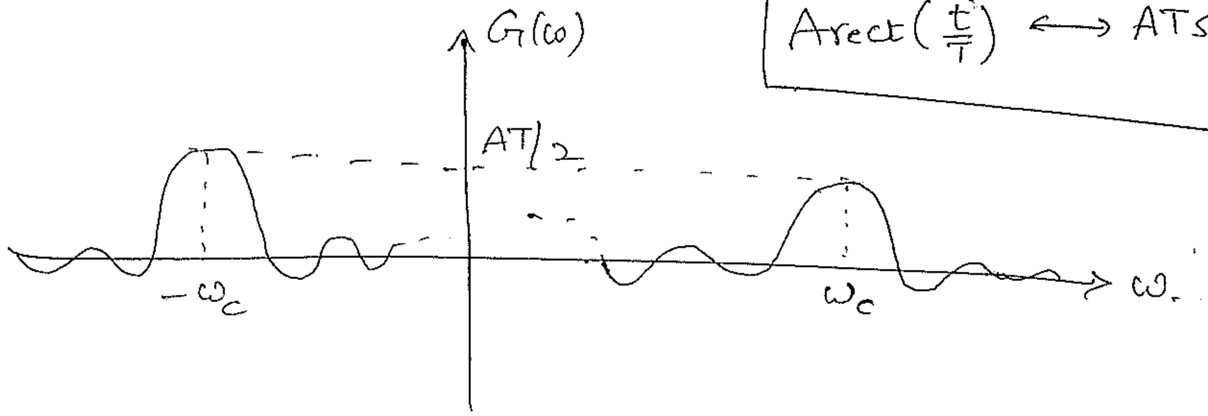
$$= \frac{AT}{2}\,\mathrm{Sinc}\left(\frac{(\omega-\omega_c)T}{2}\right) + \frac{AT}{2}\,\mathrm{sinc}\left(\frac{(\omega+\omega_c)T}{2}\right)$$

$g(t) \longleftrightarrow G(\omega)$

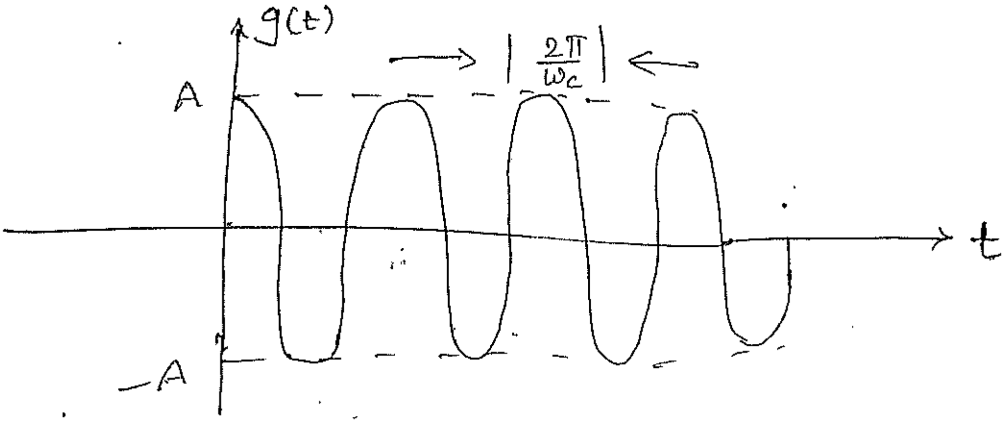
$e^{j\omega_c t} g(t) \longleftrightarrow G(\omega-\omega_c)$

$e^{-j\omega_c t} g(t) \longleftrightarrow G(\omega+\omega_c)$

$A\,\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT\,\mathrm{sinc}\left(\frac{\omega T}{2}\right.$



(2)



$$g(t) = A\cos(\omega_c t)\, u(t)$$

$$g(t) = \frac{A}{2} e^{j\omega_c t} u(t) + \frac{A}{2} e^{-j\omega_c t} u(t)$$

$$F[g(t)] = F(\quad ,\quad)$$

By linearity property of F.T, we get.

$$G(\omega) = \frac{A}{2} F\left[e^{j\omega_c t} u(t)\right] + \frac{A}{2} F\left[e^{-j\omega_c t} u(t)\right]$$

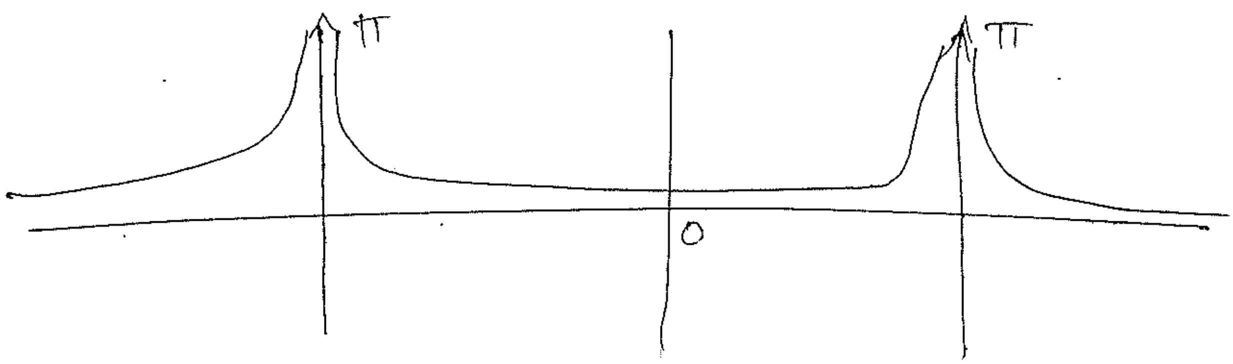
$$g(t) \longleftrightarrow G(\omega)$$

$$e^{j\omega_c t} g(t) \longleftrightarrow G(\omega - \omega_c)$$

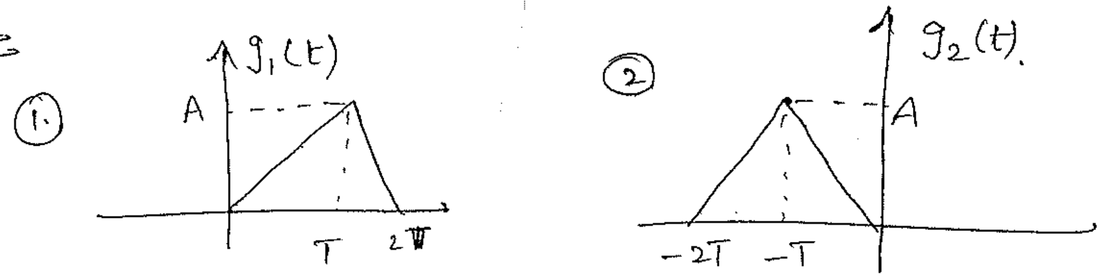
$$e^{-j\omega_c t} g(t) \longleftrightarrow G(\omega + \omega_c)$$

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\,\delta(\omega)$$

$$G(\omega) = \frac{A}{2}\left[\frac{1}{j(\omega-\omega_c)} + \pi\,\delta(\omega-\omega_c)\right] + \frac{A}{2}\left[\frac{1}{j(\omega+\omega_c)} + \pi\,\delta(\omega+\omega_c)\right]$$



H.W



③ $g(t) = A\sin(\omega_c t)\,u(t)$

④ $g(t) = A\sin(\omega_c t)\ \mathrm{rect}\left(\frac{t}{T}\right)$.

Time Differentiation Property :-

Let $g(t) \longleftrightarrow G(\omega)$, then

$$\frac{d}{dt}(g(t)) \longleftrightarrow j\omega\, G(\omega) \quad \text{and}$$

$$\frac{d^n}{dt^n}(g(t)) \longleftrightarrow (j\omega)^n G(\omega)$$

Pf :- $g(t) = F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$

$$\frac{d}{dt}(g(t)) = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, \frac{d}{dt}\left(e^{j\omega t}\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) . j\omega . e^{j\omega t}\, d\omega$$

$$= F^{-1}\left[j\omega\, G(\omega)\right]$$

$$F\left[\frac{d}{dt}(g(t))\right] = F\left[F^{-1}(j\omega\, G(\omega))\right]$$

$$= j\omega . G(\omega).$$

$$F\left[\frac{d}{dt}(g(t))\right] = j\omega\, G(\omega)$$

Ily $\quad F\left[\frac{d^2}{dt^2}(g(t))\right] = (j\omega)^2\, G(\omega)$

$\therefore \quad F\left[\frac{d^n}{dt^n}[g(t)]\right] = (j\omega)^n\, G(\omega)$

Frequency differentiation Property :-

Let $g(t) \longleftrightarrow G(\omega)$, then

$-jt\, g(t) \longleftrightarrow \frac{d}{d\omega}[G(\omega)]$ and

$(-jt)^n g(t) \longleftrightarrow \frac{d^n}{d\omega^n}[G(\omega)]$.

Pf :-

$$G(\omega) = F[g(t)] = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt.$$

$$\frac{d}{d\omega}[G(\omega)] = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt\right]$$

$$= \int_{-\infty}^{\infty} g(t) \frac{d}{d\omega}\left(e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{\infty} -jt\; g(t)\, e^{-j\omega t}\, dt$$

$$= F[-jt.g(t)]$$

$\therefore\ -jt\, g(t) \longleftrightarrow \frac{d}{d\omega}[G(\omega)]$

$|||^{ly}$ $(-jt)^2 g(t) \longleftrightarrow \frac{d^2}{d\omega^2}[G(\omega)]$

$(-jt)^n g(t) \longleftrightarrow \frac{d^n}{d\omega^n}[G(\omega)]$

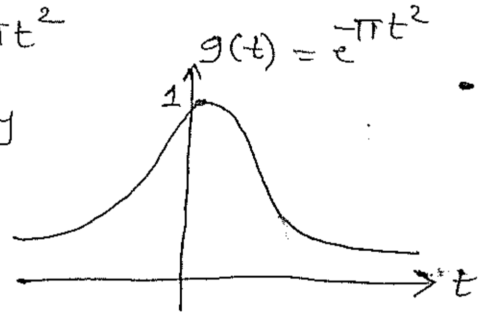
Conclusion :-

This property says that. when we differentiate the given signal, The lowest frequency components of Signal are get attenuated and highest frequency Component of signal are amplified. So it is called differentiator in time-domain and it is called high-pass filter in frequency domain.

5/8/06. Ex :- Guassian Pulse :-

It is defined by $g(t) = e^{-\pi t^2}$

Graphically it is represented by



$$g(t) = e^{-\pi t^2}$$

Apply diff. on both sides w.r.t 't',

$$\frac{d}{dt}[g(t)] = \frac{d}{dt}[e^{-\pi t^2}]$$

$$= -2\pi t\, e^{-\pi t^2}$$

w.k.T

$$g(t) \longleftrightarrow G(\omega)$$

$$\frac{d}{dt}(g(t)) \longleftrightarrow j\omega . G(\omega) \quad [\because \text{from time-diff prop.}]$$

$$-jt\; g(t) \longleftrightarrow \frac{d}{d\omega} G(\omega)$$

Apply FT for bothsides of

$$F\left[\frac{d}{dt}[g(t)]\right] = F\left[-2\pi t\, e^{-\pi t^2}\right]$$

$$= 2\pi\, F\left[-t e^{-\pi t^2}\right],$$

W.K.T

$$-jt\, g(t) \longleftrightarrow \frac{d}{d\omega}(G(\omega))$$

$$-t\, g(t) \longleftrightarrow -j \frac{d}{d\omega}[G(\omega)]$$

Hence $F\left[\frac{d}{dt}[g(t)]\right] = 2\pi\left[-j\frac{d}{d\omega}[G(\omega)]\right]$

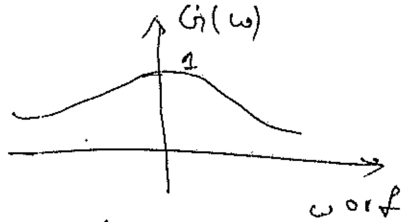
$$\Rightarrow j\omega\, G(\omega) = 2\pi . \left(-j . \frac{d}{d\omega}[G(\omega)]\right)$$

$$\Rightarrow \frac{-\omega}{2\pi} d\omega = \frac{1}{G(\omega)} . d[G(\omega)]$$

Integrating on both sides, we get

$$\int \frac{-\omega}{2\pi} d\omega = \int \frac{1}{G(\omega)} . d[G(\omega)]$$

$$\Rightarrow \quad \frac{-\omega^2}{4\pi} = \ln\left[G(\omega)\right]$$



$$\Rightarrow \quad G(\omega) = e^{-\omega^2/4\pi}$$

$$= e^{-(2\pi f)^2/4\pi} = e^{-\pi f^2}$$

$$\therefore \quad \boxed{e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi} \text{ (or) } e^{-\pi f^2}}$$

(2) $g_1(t) = e^{-\pi t^2/\tau^2}$

$\longrightarrow$ w.k.T

$$g(t) = e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi} \text{ (or) } e^{-\pi f^2}.$$

From time-scaling property,

$$g(at) \Longleftrightarrow \frac{1}{|a|} G\left(\frac{\omega}{a}\right)$$

$$g_1(t) = e^{-\pi(t^2/\tau^2)} = g\left(\frac{t}{\tau}\right)$$

$$g\left(\frac{t}{\tau}\right) \longleftrightarrow \frac{1}{|1/\tau|} G\left(\frac{\omega}{1/\tau}\right)$$

$$e^{-\pi t^2/\tau^2} \longleftrightarrow |\tau| \, e^{-\omega^2/4\pi\tau^2}$$

(or)

$$|\tau| \, e^{-\pi f^2 \tau^2}$$

(3) $g_2(t) = e^{-t^2}$

$$\longrightarrow \text{wkT} \quad g(t) = e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi} \text{ (or) } e^{-\pi f^2}.$$

$$g_2(t) = e^{-t^2} = e^{-\frac{\pi t^2}{(\sqrt{\pi})^2}} = e^{-\pi\left(\frac{t}{\sqrt{\pi}}\right)^2} = g\left(\frac{t}{\sqrt{\pi}}\right).$$

$$e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi}$$

$$e^{-\pi\left(\frac{t}{\sqrt{\pi}}\right)^2} \longleftrightarrow e^{-(\omega\sqrt{\pi})^2/4\pi}$$

$$\longleftrightarrow e^{-\omega^2/4}$$

//

(3) $g_3(t) = e^{-t^2/\tau^2}$

$\longrightarrow$ w.k.t $g(t) = e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi}$

$$g_3(t) = e^{-t^2/\tau^2} = e^{-\pi\left(\frac{t}{\sqrt{\pi}\tau}\right)^2} = g\left(\frac{t}{\tau\sqrt{\pi}}\right)$$

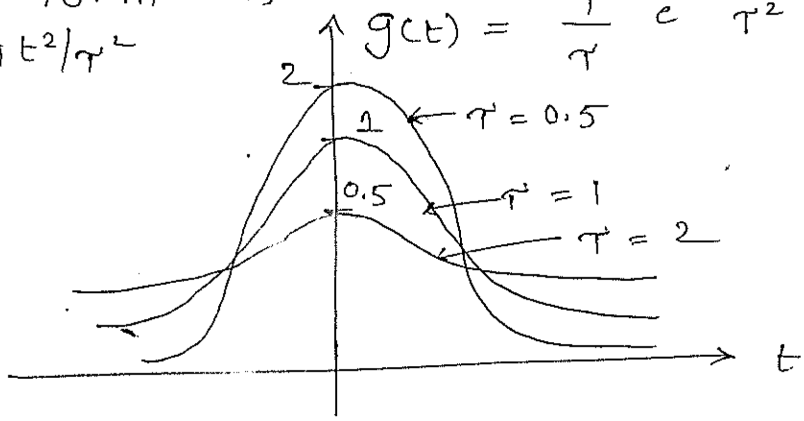
$$g(t) \longleftrightarrow e^{-\omega^2/4\pi}$$

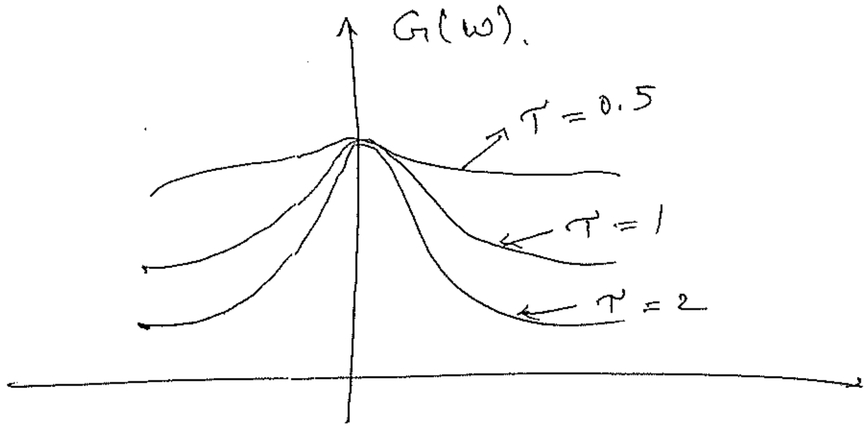
$$g\left(\frac{t}{\tau\sqrt{\pi}}\right) \longleftrightarrow e^{-(\omega\tau\sqrt{\pi})^2/4\pi}$$

$$\longleftrightarrow e^{-\omega^2\tau^2/4}$$

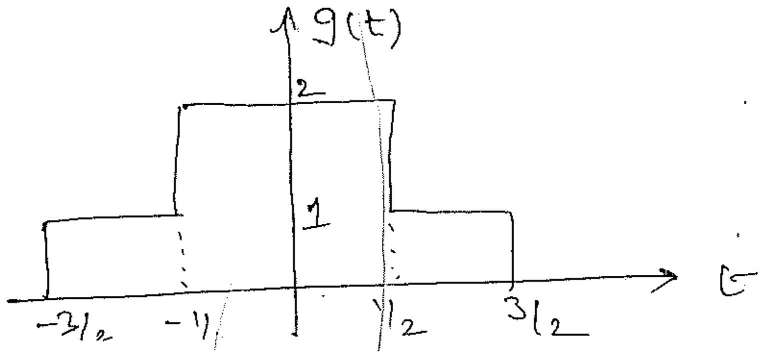
General form is $g(t) = \frac{1}{\tau} e^{-\pi \frac{t^2}{\tau^2}}$

of $g(t) = \frac{1}{\tau} e^{-\pi t^2/\tau^2}$





* Find the fourier transform of the foll. signals by using time diff. & time shifting properties only.



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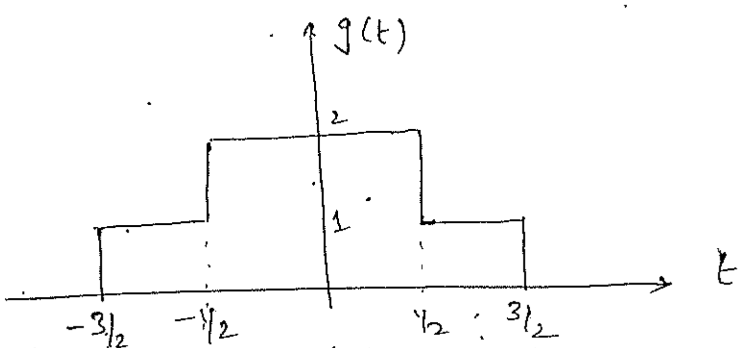
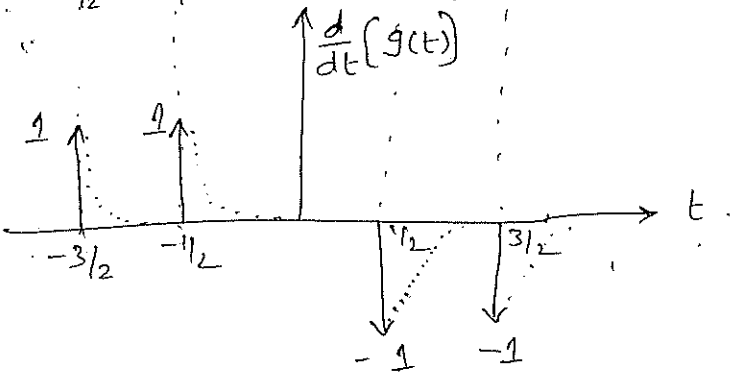




$$\frac{d}{dt}[g(t)] = 1.\ \delta(t+3/2) + 1.\ \delta(t+1/2) + (-1)\delta(t-1/2) + (-1)\,\delta(t-3/2)$$

$$F\left[\frac{d}{dt}[g(t)]\right] = F\left(\delta(t+3/2)\right) + F\left(\delta(t+1/2)\right) - F\left[\delta(t-1/2)\right] - F\left[\delta(t-3/2)\right]$$

$$g(t) \longleftrightarrow G(\omega)$$

$$g(t-t_0) \longleftrightarrow e^{-j\omega t_0} G(\omega)$$

$$g(t+t_0) \longleftrightarrow e^{j\omega t_0} G(\omega)$$

$$\delta(t-t_0) \longleftrightarrow e^{-j\omega t_0} \times 1 \qquad (\because F[\delta(t)] = 1)$$

$$\delta(t+t_0) \longleftrightarrow e^{j\omega t_0} \times 1$$

$$\frac{d}{dt}(g(t)) \longleftrightarrow j\omega\, G(\omega).$$

$$j\omega\, G(\omega) = e^{j\omega 3/2} + e^{j\omega/2} - e^{-j\omega/2} - e^{-3j\omega/2}$$

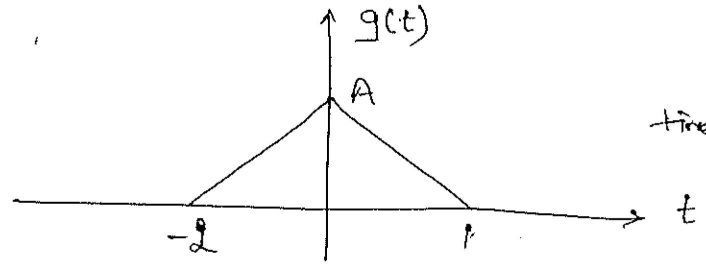
$$G(\omega) = \frac{e^{3j\omega/2} - e^{-3j\omega/2}}{j\omega} + \frac{e^{j\omega/2} - e^{-j\omega/2}}{j\omega}$$

$$= \frac{2}{\omega}\left[\sin(3\omega/2) + \sin\left(\frac{\omega}{2}\right)\right]$$

$$= 3 \times \frac{\sin(3\omega/2)}{3\omega/2} + \frac{\sin(\omega/2)}{\omega/2}$$

$$= 3\,\mathrm{Sinc}(3\omega/2) + \mathrm{Sinc}(\omega/2)$$

$$\therefore\ G(\omega) = \mathrm{Sinc}(\omega/2) + 3\,\mathrm{Sinc}(3\omega/2) \ //$$

* 

Find F.T by using ~~time~~ shifting & ~~time~~ diff prop. only.

$(-2, 0)\ (0, A)$
$x_1\ y_1\quad x_2\ y_2$

$y = \frac{A}{+2}(x+2)$

$\frac{A}{2}(t+2)$

$(0, A)\ (1, 0)$
$x_1\ y_1\quad x_2\ y_2$

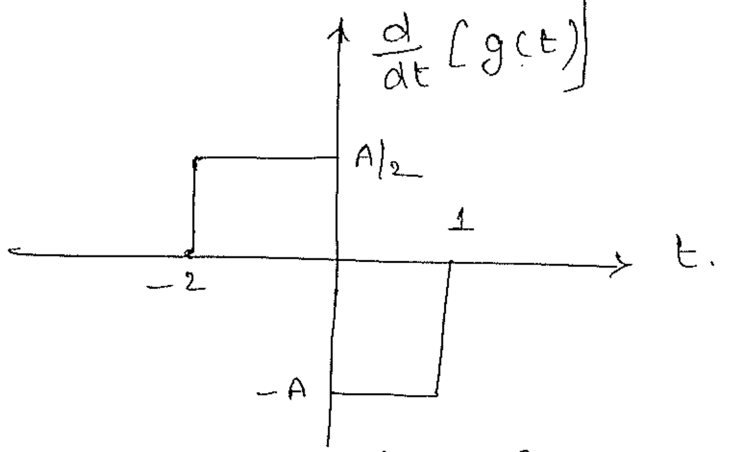
$y - A = \frac{-A}{1}(x)$

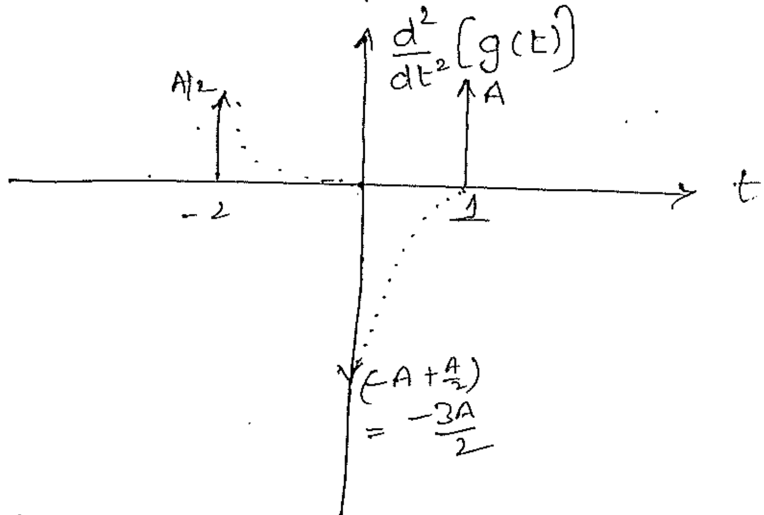
$-At + A$

$= -A(t-1)$

$$g(t) = \begin{cases} 0 & t < -2 \\ \frac{A}{2}[t+2] & -2 \le t \le 0 \\ -A[t-1] & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$

$$\frac{d}{dt}[g(t)] = \begin{cases} 0 & t < -2 \\ \frac{A}{2} & -2 \le t \le 0 \\ -A & 0 \le t \le 1 \\ 0 & t > 1 \end{cases}$$





$$\frac{d^2}{dt^2}[g(t)] = \frac{A}{2}\delta(t+2) - \frac{3A}{2}\delta(t) + A\,\delta(t-1)$$

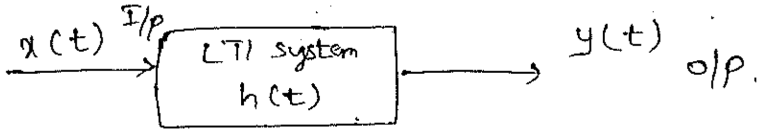
Apply F.T on both sides, we get

$$(j\omega)^2 G(\omega) = \frac{A}{2}e^{2j\omega} - \frac{3A}{2} + A e^{-j\omega}$$

$$\Rightarrow \quad G(\omega) = \frac{1}{\omega^2}\left[\frac{3A}{2} - \frac{A}{2}e^{2j\omega} - A e^{-j\omega}\right]$$

Convolution integral Definition :-

If the time domain continuous time signal $x(t)$ is given to the I/p of an LTI system and it has unit sample response $h(t)$, the response of the system $y(t)$ is as shown in the figure.



$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

(or)

$$y(t) = \int_{-\infty}^{\infty} h(t)\, x(t-\tau)\, d\tau = h(t) * x(t)$$

Time Convolution Theorem (or) Convolution in time-domain :-

If $g(t) \longleftrightarrow G(\omega)$, then

$g_1(t) \longleftrightarrow G_1(\omega)$, $g_2(t) \longleftrightarrow G_2(\omega)$, then

$$g_1(t) * g_2(t) \longleftrightarrow G_1(\omega)\, G_2(\omega).$$

(or)

$$\int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\, d\tau \longleftrightarrow G_1(\omega).\, G_2(\omega)$$

Pf :- $F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t).\, e^{-j\omega t}\, dt$

$$F[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} [g_1(t) * g_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\, d\tau \right] e^{-j\omega t}\, dt.$$

($\because$ from Convolution integral Theorem)

$$= \int_{-\infty}^{\infty} g_1(\tau) \left[\int_{-\infty}^{\infty} g_2(t-\tau) e^{-j\omega t} dt \right] d\tau$$

w.k.T

$$g(t-t_0) \longleftrightarrow e^{-j\omega t_0} G(\omega)$$

$$g_2(t-\tau) \longleftrightarrow e^{-j\omega\tau} G_2(\omega)$$

$$= \int_{-\infty}^{\infty} g_1(t) \; F[g_2(t-\tau)] \, d\tau$$

$$= \int_{-\infty}^{\infty} g_1(t) \, e^{-j\omega\tau} G_2(\omega) \, d\tau$$

$$= G_2(\omega) \int_{-\infty}^{\infty} g_1(t) \, e^{-j\omega\tau} d\tau = G_2(\omega) . F[g_1(\tau)]$$

$$= G_2(\omega) . G_1(\omega)$$

$$\therefore \quad \boxed{F[g_1(t) * g_2(t)] = G_1(\omega) . G_2(\omega)}$$

Conclusion :-

Time Convolution theorem states that Convolution b/w two signals in time domain be equivalent to their spectras multiplied in freq. domain

Frequency Convolution theorem (or) Multiplication in time domain :-

If $g(t) \longleftrightarrow G(\omega)$, $g_1(t) \longleftrightarrow G_1(\omega)$, $g_2(t) \longleftrightarrow G_2(\omega)$

then $g_1(t) . g_2(t) \longleftrightarrow \frac{1}{2\pi} G_1(\omega) * G_2(\omega)$

(or)

$$g_1(t) . g_2(t) \longleftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} G_1(\lambda) . G_2(\omega - \lambda) d\lambda$$

Pf :-

$$F^{-1}[G(\omega)] = g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

$$F^{-1}[G_1(\lambda)] = g_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\lambda)\, e^{j\lambda t}\, d\lambda.$$

$$F[g_1(t)\, g_2(t)] = \int_{-\infty}^{\infty} [g_1(t)\, g_2(t)]\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\lambda)\, e^{j\lambda t}\, d\lambda\ g_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\lambda)\int_{-\infty}^{\infty} g_2(t)\, e^{j\lambda t}\, e^{-j\omega t}\, dt \cdot d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\lambda)\left[\int_{-\infty}^{\infty} g_2(t)\, e^{-j(\omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\lambda)\, G_2(\omega-\lambda)\, d\lambda$$

$$\boxed{F_1(g_1(t)\cdot g_2(t)) = \frac{1}{2\pi}\, G_1(\omega) * G_2(\omega).}$$

Conclusion :- Frequency convolution theorem states that multiplication of two signals in time domain be equivalent to their frequency spectras convolved in frequency domain and by a scaling factor ($1/2\pi$).

Integration in time domain :-

If $g(t) \longleftrightarrow G(\omega)$, then,

$$\int_{-\infty}^{t} g(\tau)\, d\tau \longleftrightarrow \frac{G(\omega)}{j\omega} + \pi G(0)\, \delta(\omega).$$

Pf :-

$$g(t) * u(t) \longleftrightarrow \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} g(\tau) \times 1 \, d\tau \qquad u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \cancel{\text{otherwise}}\ t < 0 \end{cases}$$

$$g(t) * u(t) = \int_{-\infty}^{t} g(\tau)\, d\tau \qquad u(t-\tau) = \begin{cases} 1 & \text{for } t-\tau \geq 0,\ \tau \leq t \\ 0 & \tau > t \end{cases}$$

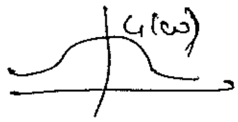
$$F\left[\int_{-\infty}^{t} g(\tau)\, d\tau\right] = F\left[g(t) * u(t)\right]$$

$$= F\left[g(t)\right] \cdot F\left[u(t)\right] \qquad g_1(t) * g_2(t) \longleftrightarrow G_1(\omega).\, G_2(\omega)$$

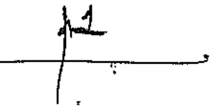
$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi \delta(\omega)$$

$$= G(\omega)\left[\frac{1}{j\omega} + \pi \delta(\omega)\right]$$

$$= \frac{G(\omega)}{j\omega} + \pi\, G(\omega)\, \delta(\omega)$$



$$= \frac{G(\omega)}{j\omega} + \pi \cdot G(0) \cdot \delta(\omega)$$



NOTE:- If $G(0) = 0$,

then $\int_{-\infty}^{t} g(\tau)\, d\tau \longleftrightarrow \frac{G(\omega)}{j\omega}$

Conclusion:-

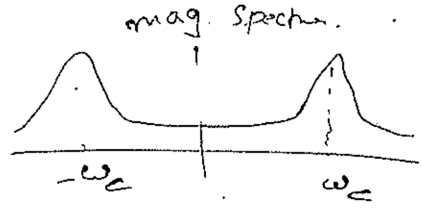
This propety states that when we integrate the given signal in time domain; the lowest freq. components of signals are amplified and highest freq. " ~~are~~ " " " get attenuated So it is Called integrator in time domain and low pass filter in frequency domain.

H.W

1. $g(t) = t e^{-at} u(t)$

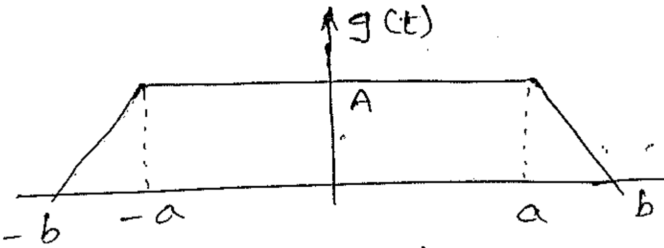
2. $g(t) = t e^{-a|t|}$

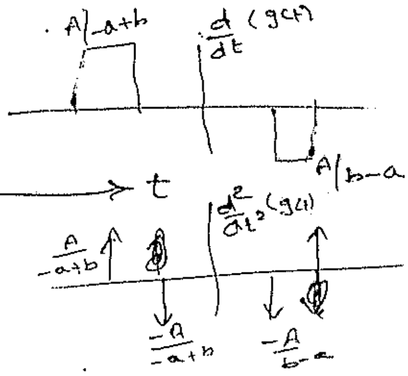
3. $g(t) = e^{-at} \cos(\omega_c t)\, u(t)$



4. $g(t) = e^{-at} \sin(\omega_c t)\, u(t)$

5.





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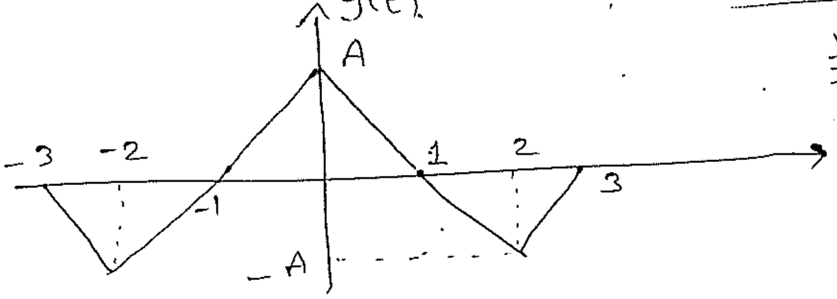

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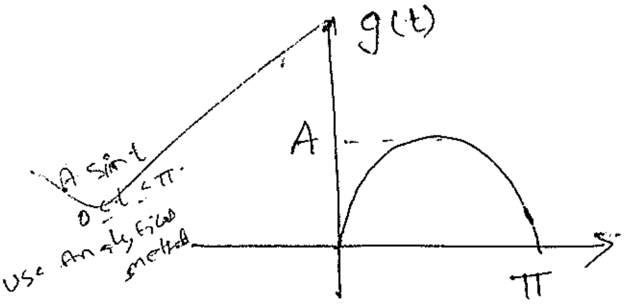

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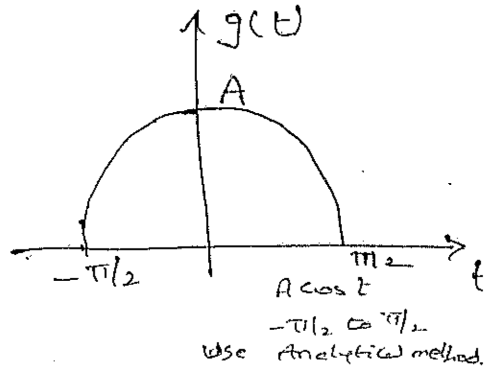


8/8/06.

Time reversal property :-

If $g(t) \longleftrightarrow G(\omega)$, then $g(-t) \longleftrightarrow G(-\omega)$.

Pf :- $F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt$

$$F[g(-t)] = \int_{-\infty}^{\infty} g(-t) e^{-j\omega t} dt$$

Put $-t = k$
$-dt = dk$.

$$= \int_{\infty}^{-\infty} g(k) e^{j\omega k} (-dk)$$

$$= \int_{-\infty}^{\infty} g(k) e^{-j\omega k} dk$$

$$= \int_{-\infty}^{\infty} g(t) e^{-j(-\omega) t} dt = G(-\omega)$$

Conclusion:- when a signal is folded in time domain, their corresponding spectrum is also folded in frequency domain.

Complex - Conjugate Sym. Property :-

If $g(t) \xleftrightarrow{F.T} G(\omega)$, then

$$g^*(t) \longleftrightarrow G^*(-\omega)$$

(or)

$$g^*(-t) \longleftrightarrow G^*(\omega)$$

Pf :- $F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$

(i) $F[g^*(t)] = \int_{-\infty}^{\infty} g^*(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \left[g(t)\, e^{j\omega t}\right]^* dt$$

$$= \left[\int_{-\infty}^{\infty} g(t)\, e^{j\omega t}\, dt\right]^*$$

$$= \left[\int_{-\infty}^{\infty} g(t)\, e^{-j(-\omega)t}\, dt\right]^*$$

$$= \left[G(-\omega)\right]^*$$

$$\therefore \boxed{F[g^*(t)] = G^*(-\omega)}$$

(ii) $F[g^*(-t)] = \int_{-\infty}^{\infty} g^*(-t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \left(g(-t)\, e^{j\omega t}\right)^* dt$$

$$= \left(\int_{-\infty}^{\infty} g(-t)\, e^{j\omega t}\, dt \right)^{*}$$

Put $-t = k$

$-dt = dk$

$$= \left[\int_{\infty}^{-\infty} g(k)\, e^{-j\omega k}\, -dk \right]^{*}$$

$$= \left(\int_{-\infty}^{\infty} g(k)\, e^{-j\omega k}\, dk \right)^{*}$$

$$= \left(\int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt \right)^{*} = \left[G(\omega) \right]^{*}$$

$$\therefore \boxed{F[g^{*}(-t)] = G^{*}(\omega)}$$

Standard Universal Definitions :-

If $g(t)$ is complex valued, then $g(t) = g_r(t) + j\, g_i(t)$

$$g^{*}(t) = g_r(t) - j\, g_i(t) \quad \text{—— (2)}$$

(1) + (2),

$g_r(t) = \text{real}\,(g(t))$
$g_i(t) = \text{img}\,(g(t))$

$$g(t) + g^{*}(t) = 2\, g_r(t)$$

$$\boxed{g_r(t) = \frac{1}{2}\left[g(t) + g^{*}(t) \right]}$$

(1) − (2),

$$g(t) - g^{*}(t) = 2j\, g_i(t)$$

$$\boxed{g_i(t) = \frac{1}{2j}\left[g(t) - g^{*}(t) \right]}$$

- If $g(t)$ is real valued signal, it can be decomposed into even & odd parts as

$$g(t) = g_e(t) + g_o(t) \quad \text{—— (1)}$$

$$g(-t) = g_e(-t) + g_o(-t) = g_e(t) - g_o(t) \quad \text{—— (2)}$$

① + ② $\Rightarrow$ $g(t) + g(-t) = 2 g_e(t)$

$$g_e(t) = \frac{1}{2}[g(t) + g(-t)]$$

① − ② $\Rightarrow$ $g(t) - g(-t) = 2 g_o(t)$

$$g_o(t) = \frac{1}{2}[g(t) - g(-t)]$$

where $g_e(t) = \text{Even}[g(t)]$; $g_o(t) = \text{odd}[g(t)]$

- If $g(t)$ is Complex valued fn, then

$$g_e(t) = \frac{1}{2}[g(t) + g^*(-t)]$$

$$g_o(t) = \frac{1}{2}[g(t) - g^*(-t)]$$

- $G(\omega) = G_R(\omega) + j\, G_I(\omega)$,

$$G(\omega) = G_e(\omega) + G_o(\omega).$$

where

$$G_e(\omega) = \frac{1}{2}[G(\omega) + G^*(-\omega)]$$

$$G_o(\omega) = \frac{1}{2}[G(\omega) - G^*(-\omega)]$$

Prove the following properties :-

If $g(t) \longleftrightarrow G(\omega)$, then

(i) $Re[g(t)] = g_r(t) \longleftrightarrow \frac{1}{2}[G(\omega) + G^*(-\omega)] = G_e(\omega)$

(ii) $j\, Img[g(t)] = j\, g_i(t) \longleftrightarrow \frac{1}{2}[G(\omega) - G^*(-\omega)] = G_o(\omega)$.

(iii) $g_e(t) = \frac{1}{2}[g(t) + g^*(-t)] \longleftrightarrow G_R(\omega) = Re[G(\omega)]$

(iv) $g_o(t) = \frac{1}{2}[g(t) - g^*(-t)] \longleftrightarrow j G_I(\omega) = j\, Im(G(\omega))$

Pf :- (i) $\quad g_r(t) = \frac{1}{2}[g(t) + g^*(t)]$

$$F[g_r(t)] = F[\tfrac{1}{2}(g(t) + g^*(t))]$$

By linear property of F.T, w.k.

$$= \tfrac{1}{2} F[g(t)] + \tfrac{1}{2} \cdot F[g^*(t)]$$

w.k.T $\quad g^*(t) \longleftrightarrow G^*(-\omega)$

$$= \tfrac{1}{2} G(\omega) + \tfrac{1}{2} G^*(-\omega)$$

$$= \tfrac{1}{2}[G(\omega) + G^*(-\omega)] = G_e(\omega).$$

$$\therefore \quad \boxed{g_r(t) \longleftrightarrow G_e(\omega)}$$

(ii) $\quad j\, g_i(t) = \frac{1}{2}[g(t) - g^*(t)]$

$$F[j\, g_i(t)] = F(\tfrac{1}{2}(g(t) - g^*(t)))$$

$$= \tfrac{1}{2} F[g(t)] - \tfrac{1}{2} F[g^*(t)]$$

$$= \tfrac{1}{2} G(\omega) - \tfrac{1}{2} G^*(-\omega)$$

$$= \tfrac{1}{2}[G(\omega) - G^*(-\omega)]$$

$$= G_o(\omega)$$

$$\therefore \quad \boxed{j\, g_i(t) \longleftrightarrow G_o(\omega)}$$

(iii) $\quad g_e(t) = \frac{1}{2}[g(t) + g^*(-t)]$

$$F[g_e(t)] = F[\tfrac{1}{2}(g(t) + g^*(-t))]$$

$$= \tfrac{1}{2} F[g(t)] + \tfrac{1}{2} \cdot F[g^*(-t)]$$

$$= \tfrac{1}{2} \cdot G(\omega) + \tfrac{1}{2} \cdot G^*(\omega).$$

($\because$ From time ~~reversal~~ Property)

$$= \frac{1}{2}\left[G(\omega) + G^*(-\omega)\right]$$

$$F\left[g_e(t)\right] = \frac{1}{2}\left(G_R(\omega) + jG_I(\omega) + G_R(\omega) - jG_I(\omega)\right)$$

$$= G_R(\omega) = Re\left[G(\omega)\right] \qquad \left[\because G(\omega) = G_R(\omega) + jG_I(\omega)\right]$$

$$\boxed{g_e(t) \longleftrightarrow G_R(\omega)}$$

(iv) $$g_o(t) = \frac{1}{2}\left[g(t) - g^*(-t)\right]$$

$$F\left[g_o(t)\right] = \frac{1}{2} F\left[g(t)\right] - \frac{1}{2} F\left[g^*(-t)\right]$$

$$= \frac{1}{2} G(\omega) - \frac{1}{2} G^*(\omega)$$

$$= \frac{1}{2}\left(G(\omega) - G^*(\omega)\right)$$

$$= \frac{1}{2}\left[G_R(\omega) + jG_I(\omega) - \left(G_R(\omega) - jG_I(\omega)\right)\right]$$

$$= jG_I(\omega) = j\, Img\left[G(\omega)\right]$$

$$\boxed{g_o(t) \longleftrightarrow jG_I(\omega)}$$

★★ Rayleigh's Energy theorem (or) Parseval's energy theorem (or) Parsevel's theorem for non-periodic Signals :—

If $g(t) \longleftrightarrow G(\omega)$, and

$g_1(t) \longleftrightarrow G_1(\omega)$

$g_2(t) \longleftrightarrow G_2(\omega)$, then

$$\int_{-\infty}^{\infty} g_1(t)\, g_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega) \cdot G_2^*(\omega)\, d\omega.$$

If $g_1(t) = g_2(t) = g(t)$, then

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega.$$

Proof :- L.H.S

$$= \int_{-\infty}^{\infty} g_1(t)\, g_2^*(t)\, dt \; ; \quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega$$

$$g_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\, e^{j\omega t}\, d\omega$$

$$= \int_{-\infty}^{\infty} g_1(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(\omega)\, e^{j\omega t}\, d\omega\right]^* dt$$

$$= \int_{-\infty}^{\infty} g_1(t) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2^*(\omega)\, e^{-j\omega t}\, d\omega \cdot dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2^*(\omega) \left[\int_{-\infty}^{\infty} g_1(t)\, e^{-j\omega t}\, dt\right] d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2^*(\omega)\, F[g_1(t)]\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_2^*(\omega) \cdot G_1(\omega)\, d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G_1(\omega) \cdot G_2^*(\omega)\, d\omega = \text{R.H.S.}$$

If $g_1(t) = g_2(t) = g(t)$,

$$\text{L.H.S} = \int_{-\infty}^{\infty} g(t)\, g^*(t)\, dt = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} g(t) \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega\right]^* dt$$

$$= \int_{-\infty}^{\infty} g(t)\, \frac{1}{2\pi}\int_{-\infty}^{\infty} G^*(\omega)\, e^{-j\omega t}\, d\omega \cdot dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \left[\int_{-\infty}^{\infty} g(t) e^{-j\omega t} dt \right] d(\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \; F[g(t)] \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G^*(\omega) \; G(\omega) \, d\omega .$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega .$$

where the signal $g(t)$ energy in time-domain is

$$E = \int_{-\infty}^{\infty} |g(t)|^2 dt .$$

where the signal $g(t)$ energy in frequency domain is

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega .$$

$|G(\omega)|^2$ is amplitude square of Fourier transform of $g(t)$ signal. and it is known as energy spectral density. Its unit is J/Hz. and it is represented as $\Psi_g(\omega) = |G(\omega)|^2$. and

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_g(\omega) \, d\omega .$$

Area under $g(t)$ signal :-

If $g(t) \longleftrightarrow G(\omega)$, then

$$\int_{-\infty}^{\infty} g(t) \, dt = G(0) = G(\omega)\Big|_{\omega = 0}$$

Area under $g(t)$ signal be equivalent to its F.T. value at $\omega = 0$

Proof :- W.K.T $F[g(t)] = G(\omega) = \int_{-\infty}^{\infty} g(t)\, e^{-j\omega t}\, dt$

Subst. $\omega = 0$.

$$G(\omega)\Big|_{\omega=0} = F[g(t)] = G(0) = \int_{-\infty}^{\infty} g(t)\, dt.$$

$$\boxed{G(0) = \int_{-\infty}^{\infty} g(t)\, dt}$$

Ex :- Sinc pulse.

Find the area of $\text{Sinc}\left(\frac{\omega_m t}{2}\right)$.

$\rightarrow$ W.K.T $A\,\text{sinc}\left(\frac{\omega_m t}{2}\right) \longleftrightarrow \frac{2\pi A}{\omega_m}\, \text{rect}\left(\frac{\omega}{\omega_m}\right)$.

Ily $\text{Sinc}\left(\frac{\omega_m t}{2}\right) \longleftrightarrow \frac{2\pi}{\omega_m}\, \text{rect}\left(\frac{\omega}{\omega_m}\right)$.

Area of ~~sinc Pulse~~ signal is.

$$\int_{-\infty}^{\infty} g(t)\, dt = G(\omega)\Big|_{\omega=0}$$

$$\int_{-\infty}^{\infty} \text{Sinc}\left(\frac{\omega_m t}{2}\right) = \frac{2\pi}{\omega_m}\, \text{rect}\left(\frac{0}{\omega_m}\right)$$

$$= \frac{2\pi}{\omega_m} //$$

Area under $F[\omega]$:-

If $g(t) \longleftrightarrow G(\omega)$, then $\int_{-\infty}^{\infty} G[\omega]\, d\omega = 2\pi\, g(0)$.

Area under (F.T of g(t) signal) $G(\omega)$ be equivalent to the signal at $t=0$ in time domain by a scaling factor, 2π.

Proof :- $F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{j\omega t}\, d\omega.$

Put $t=0$, Then

$$g(t)\Big|_{t=0} = g(0) = F^{-1}[G(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} G(\omega)\, d\omega = 2\pi\, g(0).$$

Ex :- Exponential pulse.

Find the area under $\frac{1}{a+j\omega}$.

$\longrightarrow$ $\omega \cdot k \cdot T$

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega}$$

$$\int_{-\infty}^{\infty} G(\omega)\, d\omega = 2\pi\, g(0)$$

$$\int_{-\infty}^{\infty} \frac{1}{a+j\omega}\, d\omega = 2\pi e^{-at} u(t)\Big|_{t=0}$$

$$= 2\pi e^{-a(0)} u(0)$$

$$= \underline{2\pi}$$

* Calculate area under $\frac{1}{a^2+\omega^2}$.

$\longrightarrow$ $\omega \cdot k \cdot T$

$$e^{-a|t|} \longleftrightarrow \frac{2a}{a^2+\omega^2}.$$

$$\int_{-\infty}^{\infty} G(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2}\, d\omega$$

$$= 2\pi\, g(0)$$

$$= 2\pi\, e^{-a(0)}$$

$$= 2\pi.$$

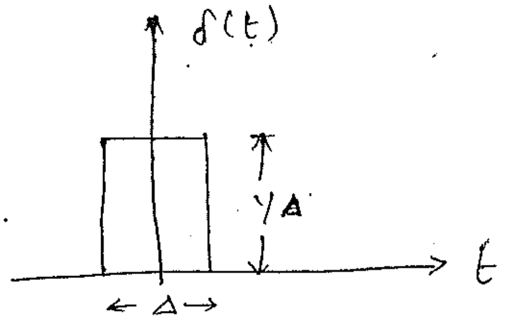
$$\therefore \int_{-\infty}^{\infty} \frac{1}{a^2+\omega^2}\, d\omega = \frac{1}{2a} \int_{-\infty}^{\infty} \frac{2a}{a^2+\omega^2}\, d\omega = \frac{2\pi}{2a} = \frac{\pi}{a} \,//$$

Properties and applications of unit impulse fn. (or) dirac-delta fn. (or) unit sample fn.

(or)

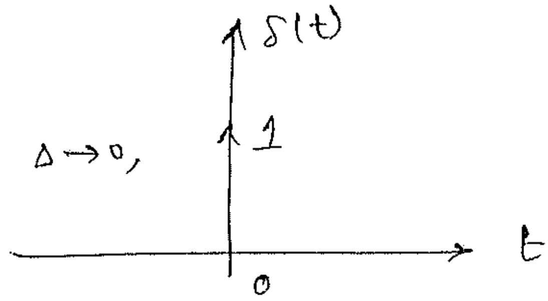
F.T. involving on unit impulse fn :—

Consider a pulse occuring at $t=0$ of height $\left(\frac{1}{\Delta}\right)$ and duration 'Δ' as shown in figure.



As we let $\Delta \to 0$, the area of pulse remains one unit and it occurs $t=0$. This is known as unit impulse fn. (or) unit sample fn. (or) dirac-delta fn. and it is represented mathematically as

$$\delta(t) = \begin{cases} 1 & \text{for } t=0 \\ 0 & \text{for } t \neq 0 \end{cases} \quad \text{as } \Delta \to 0,$$



PROPERTIES :-

1. Area under unit impulse fn. is unity.

i.e; $$\int_{-\infty}^{\infty} \delta(t)\, dt = \delta(t)\Big|_{t=0} = \delta(0) = 1$$

2. The integral of product of $\delta(t)$ and other time domain fn, $g(t)$ i.e; continous at $t=0$.

i.e; $$\int_{-\infty}^{\infty} \delta(t) \cdot g(t)\, dt = g(t)\,\delta(t)\Big|_{t=0} = g(0)\,\delta(0) = g(0).$$

3. Shifting property :-

$$\int_{-\infty}^{\infty} g(t)\,\delta(t-t_0)\, dt = g(t)\,\delta(t-t_0)\Big|_{t=t_0} = g(t_0).$$

which is used to Sperate the specific component of the given signal.

4. $g(t) \cdot \delta(t) = g(t)\,\delta(t)\Big|_{t=0} = g(0)$.

$g(t)\,\delta(t-t_0) = g(t)\,\delta(t)\Big|_{t=t_0} = g(t_0)$.

5. <u>Scaling</u> <u>property</u> :-

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Proof :- w.k.t $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$.

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\infty}^{\infty} \delta(ka)\,a\,dk \qquad \text{Put } t = ka,\ dt = a\,dk$$

$$= \begin{cases} \int_{-\infty}^{\infty} \delta(ka)\,a.dk & \text{if } a>0 \\ \int_{\infty}^{-\infty} \delta(ka)\,a.dk & \text{if } a<0. \end{cases}$$

If $a<0$,

$$\int_{-\infty}^{\infty} \delta(t)\,dt = -\int_{-\infty}^{\infty} \delta(ka)\,a.dk.$$

$\therefore$ In general,

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \int_{-\infty}^{\infty} |a|\,\delta(ka)\,dk$$

$$= \int_{-\infty}^{\infty} |a|\,\delta(at)\,dt$$

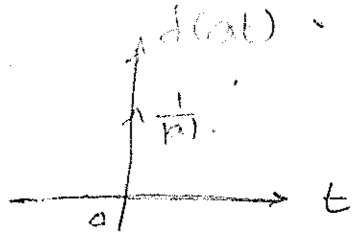
$$\Rightarrow \int_{-\infty}^{\infty} \delta(at) = \frac{1}{|a|}\int_{-\infty}^{\infty} \delta(t)\,dt$$

$$\therefore \quad \delta(t) = |a|\,\delta(ta)$$

$$\therefore \quad \boxed{\delta(at) = \frac{1}{|a|}\,\delta(t)}$$

~~For~~

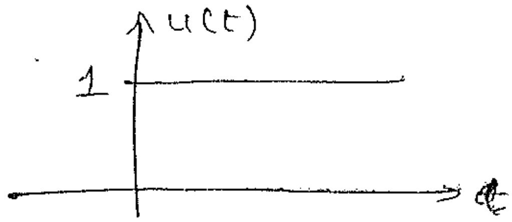
7. $$\int_{-\infty}^{\infty} |a|\,\delta(at)\,dt$$



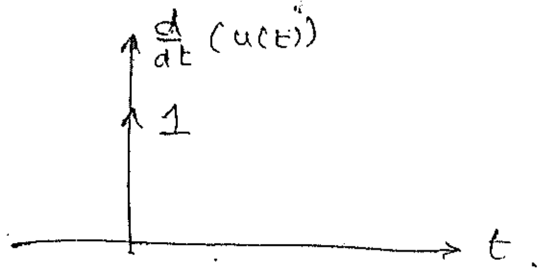
$$\int_{-\infty}^{\infty} \delta(at)\,dt = \frac{1}{|a|}$$

Relation b/w unit step and δ-function :-

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



$$\frac{d}{dt}(u(t)) = 1.$$



$$\therefore \quad \boxed{\frac{d}{dt}[u(t)] = \delta(t)}$$

8. **Replication (or) Image (or) Convolution identity :-**

$$\delta(t) * g(t) = g(t) * \delta(t) = g(t)$$

~~Any~~ time-domain fn. convolve with unit impulse fn. which gives the same function.

Pf :- $g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\,d\tau$

$|||^{ly}$ $\delta(t) * g(t) = \int_{-\infty}^{\infty} \delta(\tau)\, g(t-\tau)\,d\tau$

As $\delta(\tau)$ is finite at only $\tau = 0$, $= \delta(\tau)\, g(t-\tau)\Big|_{\tau=0}$

$$= g(t).$$

$$g(t) * \delta(t) = \int_{-\infty}^{\infty} g(\tau)\, \delta(t-\tau)\, d\tau$$

$$= g(\tau)\, \delta(t-\tau)\Big|_{t=\tau}$$

$$= \underline{\underline{g(t)}}.$$

$$\therefore \boxed{\delta(t) * g(t) = g(t) * \delta(t) = g(t)}$$

9. $$\int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

Pf :- w.k.T $\delta(t) * u(t) = \int_{-\infty}^{\infty} \delta(\tau)\, u(t-\tau)\, d\tau$

$$= \cancel{\delta(\tau)\, u(t-\tau)}\Big|_{\cancel{\tau=0}}$$

$$u(t-\tau) = \begin{cases} 1 & \text{for } t-\tau \geq 0 \quad \tau \leq t \\ 0 & \text{for } \tau > t \end{cases}$$

$$\Rightarrow \delta(t) * u(t) = \int_{-\infty}^{\infty} \delta(\tau)\, u(t-\tau)\, d\tau$$

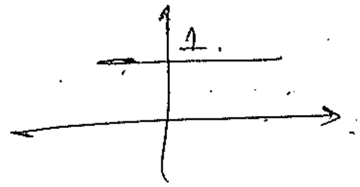
$$u(t) = \int_{-\infty}^{t} \delta(\tau) \times 1\, d\tau$$

$$\therefore \boxed{u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau}$$

Applications :-

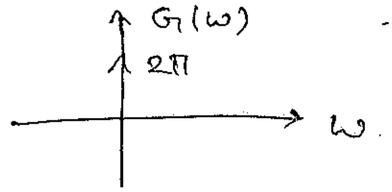
1. F.T of Impulse function.

$$\delta(t) \longleftrightarrow 1.$$



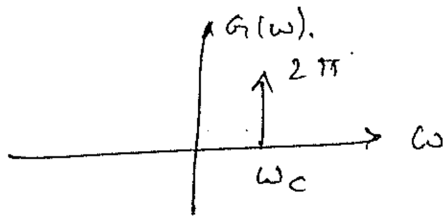
2. D.C. signal.

$$1 \longleftrightarrow 2\pi\,\delta(\omega).$$

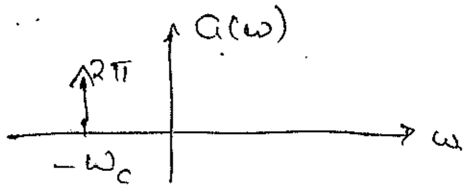


3. Exponential Series.

$$e^{j\omega_c t} \longleftrightarrow 2\pi\;\delta(\omega-\omega_c)$$

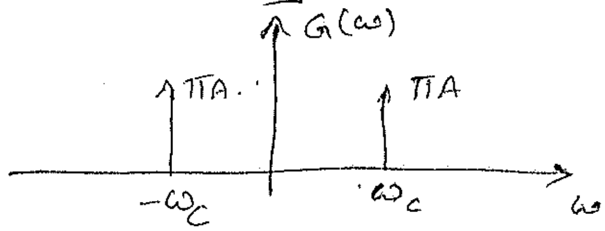


$$e^{-j\omega_c t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_c)$$

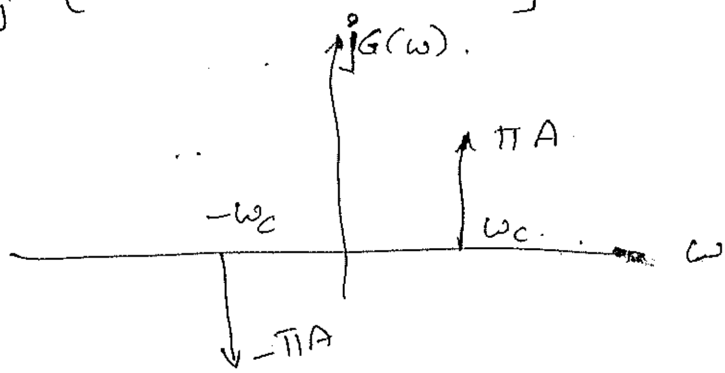


4. Sinusoidal Signal.

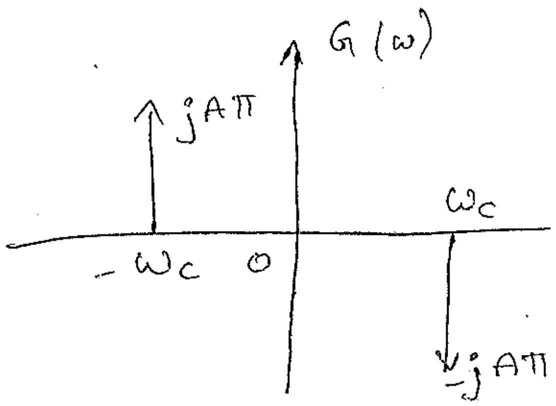
$$A\cos(\omega_c t) \longleftrightarrow \pi A\left[\delta(\omega+\omega_c)+\delta(\omega-\omega_c)\right]$$



5. $$A\sin(\omega_c t) \longleftrightarrow \frac{\pi A}{j}\left[\delta(\omega-\omega_c)-\delta(\omega+\omega_c)\right].$$



(or)



* Evaluate the following.

(i) $\int_{-\infty}^{\infty} \delta(t) \cos(2t)dt$ (ii) $\int_{-\infty}^{\infty} \delta(t-5) e^{-4t} dt$ (iii) $\int_{-\infty}^{\infty} \delta(4t) dt$

→ (i) $\int_{-\infty}^{\infty} \delta(t) \cos(2t).dt$

$$= \delta(t)\cos 2t\Big|_{t=0} = \cos 0 = 1.$$

(ii) $\int_{-\infty}^{\infty} \delta(t-5) e^{-4t} dt = \delta(t-5) e^{-4t}\Big|_{t=5} = e^{-20}$

(iii) $\int_{-\infty}^{\infty} \delta(4t)\, dt = \frac{1}{4}$ //

* Find energy of foll. sinc pulses. ⊛

(i) $\text{Sinc}\left(\frac{\omega_m t}{2}\right)$ (ii) $\text{Sinc}(2\omega_m t)$ Question Paper (iii) $\text{Sinc}\left(\frac{t}{2}\right)$ (iv) $\text{Sinc}(t)$

→ (i) $\text{Sinc}\left(\frac{\omega_m t}{2}\right)$

W.K.T

$$\cancel{At\ \text{Sinc}\left(\frac{\omega_m t}{2}\right) \longleftrightarrow}$$

$$A \cdot \text{rect}\left(\frac{t}{T}\right) \longleftrightarrow AT \sin\left(\frac{\omega T}{2}\right)$$

$$G(t) \longleftrightarrow 2\pi\, g(-\omega).$$

$$AT\, \text{Sinc}\left(\frac{tT}{2}\right) \longleftrightarrow 2\pi A\, \text{rect}\left(\frac{-\omega}{T}\right)$$

$T = \omega_m$

$$A\omega_m \text{Sinc}\left(\frac{t\omega_m}{2}\right) \longleftrightarrow 2\pi A\, \text{rect}\left(\frac{-\omega}{\omega_m}\right).$$

$$g(t) = \text{Sinc}\left(\frac{\omega_m t}{2}\right) \longleftrightarrow \frac{2\pi}{\omega_m} \text{rect}\left(\frac{-\omega}{\omega_m}\right).$$

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \cancel{G(\omega)\big|_{\omega=0}}\ \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{2\pi}{\omega_m}\,\mathrm{rect}(\omega/\omega_m)\;\cancel{\mathrm{Sinc}\left(\frac{\omega_m t}{2}\right)}\right|^2 d\omega = \frac{1}{2\pi}\left(\frac{2\pi}{\omega_m}\right)^2\int_{-\omega_m/2}^{\omega_m/2} 1\,d\omega \quad \cancel{\frac{2\pi}{\omega_m}\,\mathrm{rect}\left(\frac{\omega}{\omega_m}\right)\Big|_{\omega=0}}$$

$$= \frac{2\pi}{\omega_m}$$

(ii) $\mathrm{Sinc}(2\omega_m t)$ $g(\omega t)$

$g(at)$

$$\mathrm{Sinc}(2\omega_m t) \longleftrightarrow \frac{1}{4}\times\frac{2\pi}{\omega_m}\times\mathrm{rect}\left(\frac{\omega/4}{\omega_m}\right)$$

$$\longleftrightarrow \frac{\pi}{2\omega_m}\,\mathrm{rect}\left[\frac{\omega}{4\omega_m}\right]$$

Energy is
$$\int_{-\infty}^{\infty}\frac{1}{2\pi}\left|\frac{\pi}{2\omega_m}\,\mathrm{rect}\left(\frac{\omega}{4\omega_m}\right)\;\cancel{\mathrm{Sinc}(2\omega_m t)}\right|^2 d\omega = \frac{1}{2\pi}\left(\frac{\pi}{2\omega_m}\right)^2\int_{-2\omega_m}^{2\omega_m}1\,d\omega = \frac{\pi}{2\omega_m}$$

$$= \frac{\pi}{8\omega_m^2}(4\omega_m)$$

$$= \frac{\pi}{2\omega_m}$$

(iii) $$\mathrm{Sinc}\left(\frac{t}{2}\right) = g\left(\frac{t}{\omega_m}\right)$$

$$\mathrm{Sinc}(t/2) \longleftrightarrow \frac{1}{1/\omega_m}\,\frac{2\pi}{\omega_m}\,\mathrm{rect}\left[\frac{\omega/1/\omega_m}{\omega_m}\right]$$

$$\longleftrightarrow 2\pi\,\mathrm{rect}(\omega)$$

$$\int_{-\infty}^{\infty}(g(t))^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}|G(\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\cdot 4\pi^2\int_{-\infty}^{\infty}(\mathrm{rect}(\omega))^2\,dt$$

$$= 2\pi\int_{-1/2}^{1/2}1\,d\omega$$

$$= 2\pi(1) = 2\pi$$

(iv) $\text{Sinc}(t)$

$$\text{Sinc}(t) = g\left(\frac{2t}{\omega_m}\right).$$

$$\text{Sinc}(t) \longleftrightarrow \frac{\omega_m}{2} \cdot \frac{2\pi}{\omega_m} \, \text{rect}\left[\frac{\omega/\cancel{2}\omega_m}{\omega_m}\right]$$

$$\longleftrightarrow \pi \, \text{rect}\left[\frac{\omega}{2}\right]$$

$$\int_{-\infty}^{\infty} g(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \cdot \pi^2 \int_{-\infty}^{\infty} \left(\text{rect}\left[\frac{\omega}{2}\right]\right)^2 d\omega.$$

$$= \frac{\pi}{2} \int_{-1}^{1} 1 \cdot d\omega$$

$$= \frac{\pi}{2}(1+1) = \underline{\underline{\pi}}$$

* Find inverse F.T of $e^{-k\omega^2}$.

$\longrightarrow$ w.k.T $\quad e^{-\pi t^2} \longleftrightarrow e^{-\omega^2/4\pi} = e^{-\left(\omega/\sqrt{4\pi}\right)^2}.$

$$e^{\cancel{-\sqrt{4\pi}\pi t^2}} \longleftrightarrow \cancel{e^{-\omega^2}}$$

$$e^{\cancel{-\pi(\sqrt{4\pi}t)^2}} \cancel{\longleftrightarrow} \cancel{e^{-\omega^2} \frac{1}{\sqrt{4\pi}}} \times e$$

$$g\left(\frac{t}{\sqrt{4\pi}}\right) = e^{-\pi\left(\frac{t}{\sqrt{4\pi}}\right)^2} \longleftrightarrow \frac{1}{1/\sqrt{4\pi}} G(\omega\sqrt{4\pi})$$

$$= \sqrt{4\pi}\, G(\omega\sqrt{4\pi})$$

$$\Rightarrow \quad e^{-\pi \frac{t^2}{4\pi}} \longleftrightarrow \sqrt{4\pi} \cdot e^{-\left(\frac{\omega\sqrt{4\pi}}{\sqrt{4\pi}}\right)^2}$$

$$e^{-\cancel{\pi} \frac{t^2}{4}} \longleftrightarrow \sqrt{4\pi} \cdot e^{-\omega^2}$$

$$e^{-\frac{\pi}{4} \frac{t^2}{k}} \longleftrightarrow \sqrt{4\pi}\,\sqrt{k}\; e^{-(\omega\sqrt{k})^2}$$

$$e^{-\pi/4 \left(\frac{t}{\sqrt{k}}\right)^2} \longleftrightarrow \sqrt{4\pi k}\; e^{-\omega^2 k}.$$

$$\frac{1}{\sqrt{4\pi k}}\; e^{-\frac{\pi}{4}\left(\frac{t}{\sqrt{k}}\right)^2} \longleftrightarrow e^{-k\omega^2}.$$

$$\therefore \quad F^{-1}\left[e^{-k\omega^2}\right] = \frac{1}{\sqrt{4\pi k}}\; e^{-\frac{\pi t^2}{4k}} \quad //$$

* Find I.F.T of $\exp\left[-k\omega^2 + j\omega t_0\right]$ $\quad t_0$ units advanced

$\rightarrow$ $F^{-1}\left(e^{-k\omega^2} \cdot e^{j\omega t_0}\right)$ $\quad$ Indirect Ex.

$$\frac{1}{\sqrt{4\pi k}}\; e^{\frac{-\pi(t+t_0)^2}{4k}} \longleftrightarrow e^{j\omega t_0} \cdot e^{-k\omega^2}.$$

* I.F.T of $e^{-k\omega^2 - j\omega t_0}$

$F^{-1}\left[e^{-k\omega^2} \cdot e^{-j\omega t_0}\right]$ $\quad t_0$ units delayed.

$$\frac{1}{\sqrt{4\pi k}}\; e^{\frac{-\pi(t-t_0)^2}{4k}} \longleftrightarrow e^{-j\omega t_0} \cdot e^{-k\omega^2}$$

* Calculate F.T of $g(t) = \exp(-4t + j5t) \cdot u(t)$.

→ $e^{-4t+j5t} \cdot u(t)$.

$= e^{-4t} \cdot u(t) \cdot e^{j5t}$

W.K.T.

$$e^{-at}\, u(t) \longleftrightarrow \frac{1}{a+j\omega}.$$

A scaling factor of e^{j5t} is multiplied. i.e; in F.T 5 units is delayed.

$$e^{j5t}\, e^{-4t}\, u(t) \longleftrightarrow \frac{1}{4+j(\omega-5)}$$

UNIT-II

Properties of Continous time Fourier Series :-

Fourier Series :-

Def^n

F.S. rep. of periodic signal $x(t)$ is

$x(t)$

F.S. Coeff. = $_k$ $x($ $^{-j\omega kt}$ dt

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\omega nt}$$

where $c_n = \frac{1}{T} \int_T g(t)\, e^{-j\omega nt}\, dt.$

$$x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega kt}$$

$$a_k = \frac{1}{T} \int_T x_1(t)\, e^{-j\omega kt}\, dt$$

$$x_2(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega kt}$$

$$b_k = \frac{1}{T} \int_T x_2(t)\, e^{-j\omega kt}\, dt$$

$\therefore$ $x(t) \xleftrightarrow{F.S} c_k$

$x_1(t) \longleftrightarrow a_k$

$x_2(t) \longleftrightarrow b_k$

Properties :-

(1) Linearity property :-

If $x(t) \xleftrightarrow{F.S.} c_K$

$x_1(t) \leftrightarrow a_K$ and

$x_2(t) \leftrightarrow b_K$, then

$$a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 a_k + a_2 b_K$$

Pf :- F.S. Coeff. of $x(t)$ is c_K

where $c_K = \frac{1}{T}\int_T x(t)\, e^{-j\omega k t}\, dt.$

F.S. coeff. of $a_1 x_1(t) + a_2 x_2(t)$ is

$$= \frac{1}{T}\int_T \left[a_1 x_1(t) + a_2 x_2(t)\right] e^{-j\omega k t}\, dt$$

$$= a_1 \frac{1}{T}\int_T x_1(t)\, e^{-j\omega k t}\, dt + a_2 \frac{1}{T}\int_T x_2(t) e^{-j\omega k t}\, dt$$

$$= a_1 a_K + a_2 b_K.$$

$\therefore$ $\boxed{a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1 a_K + a_2 b_K}$

(2) Timeshifting property :-

Delay: If $x(t) \leftrightarrow c_K$, then

~~$x_0(t) \leftrightarrow c_K$ and~~

~~$x_0(t)$~~

$$x(t - t_0) \xleftrightarrow{F.S.} e^{-j\omega k t_0} c_K \quad (\text{or}) \quad e^{-j\frac{2\pi}{T} k t_0} c_K ;$$

$$\omega = \frac{2\pi}{T}.$$

Pf:- case(i):- Time delay property.

F.S. Coeff of $x(t) = C_K = \frac{1}{T}\int_T x(t)\, e^{-j\omega kt}\, dt$

" $x(t-t_0) = \frac{1}{T}\int_T x(t-t_0)\, .\, e^{-j\omega kt}\, dt.$

Put $t - t_0 = \lambda$

$\Rightarrow dt = d\lambda$

$= \frac{1}{T}\int_T x(\lambda)\, e^{-j\omega k(\lambda+t_0)}\, d\lambda$

$= \frac{1}{T}\int_T x(\lambda)\, e^{-j\omega k\lambda} . e^{-j\omega k t_0}\, d\lambda$

$= e^{-j\omega k t_0} . \frac{1}{T}\int_T x(\lambda)\, e^{-j\omega k\lambda}\, d\lambda$

$= e^{-j\omega k t_0}\, C_K.$

$\therefore$ $\boxed{x(t-t_0) \longleftrightarrow e^{-j\omega k t_0}\, C_K}$

Case(ii):- Time advance property.

If $x(t) \Leftrightarrow C_K$, then

$x(t+t_0) \longleftrightarrow e^{j\omega k t_0}\, C_K$ (or) $e^{j\frac{2\pi}{T}k t_0}\, C_K \because \omega = \frac{2\pi}{T}.$

Pf:- F.S. Coeff. of $x(t) = \frac{1}{T}\int_T x(t)\, e^{-j\omega kt}\, dt$

$x(t+t_0) = \frac{1}{T}\int_T x(t+t_0)\, e^{-j\omega kt}\, dt$

Put $t+t_0 = \lambda$

$\Rightarrow dt = d\lambda$

$= \frac{1}{T}\int_T x(\lambda)\, e^{-j\omega k(\lambda - t_0)}\, d\lambda$

$= \frac{1}{T}\int_T x(\lambda)\, e^{-j\omega k\lambda} . e^{j\omega k t_0}\, d\lambda$

$= e^{j\omega k t_0} . C_K$

(3) Case (i) :-

Frequency delay property :-

If $x(t) \longleftrightarrow C_k$, then

$$e^{j\omega m t} x(t) \longleftrightarrow C_{k-m}$$

Pf :- F.S. Coeff. of $x(t) = C_k = \frac{1}{T}\int_T x(t) \cdot e^{-j\omega k t} dt$.

" " $e^{j\omega m t} x(t)$

$$= \frac{1}{T}\int_T x(t) e^{j\omega m t} \cdot e^{-j\omega k t} dt$$

$$= \frac{1}{T}\int_T x(t) e^{-j\omega t(k-m)} dt$$

$$= \underline{\underline{C_{k-m}}}$$

Case (ii) :-

Frequency advance property :-

If $x(t) \longleftrightarrow C_k$, then

$$e^{-j\omega m t} x(t) \longleftrightarrow C_{k+m}$$

Pf :- F.S. Coeff. of $e^{-j\omega m t} x(t) = \frac{1}{T}\int_T x(t) e^{-j\omega m t} \cdot e^{-j\omega k t} dt$

$$= \frac{1}{T}\int_T x(t) e^{-j\omega t(k+m)} dt$$

$$= \underline{\underline{C_{k+m}}}.$$

(4) Time reverse property :-

If $x(t) \longleftrightarrow C_k$, then $x(-t) \xleftrightarrow{F.S} C_{-k}$.

Pf :- F.S. representation of $x(t)$ is

$$\sum_{k=-\infty}^{\infty} C_k e^{j\omega k t}$$

$x(-t)$ F.S. Repr. is $\sum_{k=-\infty}^{\infty} C_k e^{-j\omega k t}$

Put $-k = \lambda$.

i.e; $x(-t) = \sum_{\lambda=\infty}^{-\infty} C_{-\lambda} e^{j\omega\lambda t}$

$= \sum_{k=\infty}^{-\infty} C_{-k} e^{j\omega k t}$ (Replace λ by

$\therefore$ F.S. Coeff. of $x(-t)$ is C_{-k}.

~~i.e;~~

i.e;

$$\boxed{x(-t) \longleftrightarrow C_{-k}}$$

(5) Periodic Convolution in t-domain

Convolution b/w two periodic Signals is kno as "periodic Convolution."

$*$ - Linear convl

$\circledast$ - Periodic convl

Stat :- If $x(t) \Leftrightarrow C_k$

$x_1(t) \leftrightarrow a_k$

$x_2(t) \Leftrightarrow b_k$, then

$$x_1(t) \circledast x_2(t) \longleftrightarrow T\, a_k b_k$$

or $$\int_T x_1(\tau)\, x_2(t-\tau)\, d\tau \longleftrightarrow T\, a_k b_k.$$

Pf :- F.S. Coeff. of $x(t) = C_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega kt}\, dt$

" " $x_1(t) \circledast x_2(t) = \frac{1}{T}\int_T \left(\int x_1(\tau)\, x_2(t-\tau)\, d\tau\right) e^{-j\omega kt}\, dt$

$$= \int_T x_1(\tau) \left[\frac{1}{T}\int_T x_2(t-\tau)\, e^{-j\omega kt}\, dt\right] d\tau$$

$$= \int_T x_1(\tau)\ \text{F.S. Coeff.}\left[x_2(t-\tau)\right] d\tau$$

$x(t-t_0) \longleftrightarrow e^{-j\omega k t_0}\, C_k$

$x_2(t-\tau) \longleftrightarrow e^{-j\omega k\tau}\, b_k$

$$= \int_T x_1(\tau)\, e^{-j\omega k\tau}\, b_k\, d\tau$$

$$= b_k T \cdot \frac{1}{T}\int_T x_1(\tau)\, e^{-j\omega k\tau}\, d\tau$$

$$= b_k T \cdot a_k = \underline{T a_k \cdot b_k}$$

(6) Multiplication in time-domain :-

If $x(t) \longleftrightarrow C_k$

$x_1(t) \longleftrightarrow a_k$

$x_2(t) \longleftrightarrow b_k$, then

$$x_1(t)\, x_2(t) \longleftrightarrow \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} \quad \text{(or)} \quad a_k * b_k$$

Pf :- F.S. Coeff. of $x(t) = C_k = \frac{1}{T}\int_T x(t)\, e^{-j\omega kt}\, dt$

" " $x_1(t)\, x_2(t) = \frac{1}{T}\int_T x_1(t)\, \underline{x_2(t)}\, e^{-j\omega kt}\, dt$. (1)

W.K.T

$$x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega k t}$$

$$= \sum_{l=-\infty}^{\infty} a_l e^{j\omega l t} \quad \text{k replaced by l}$$

From ①,

F.S. coeff of $x_1(t)\, x_2(t)$ $= \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} a_l\, x_2(t)\, e^{j\omega l t} e^{-j\omega k t} dt$

$$= \sum_{l=-\infty}^{\infty} a_l \frac{1}{T}\int_T x_2(t)\, e^{-j\omega(k-l)t} dt$$

$$= \sum_{l=-\infty}^{\infty} a_l\, b_{k-l} \quad \text{(from ③ Property)}$$

$$= \underline{\underline{a_k * b_k}}$$

⑦ Complex Conjugate property :-

If $x(t) \longleftrightarrow c_k$, then $x^*(t) \longleftrightarrow c^*_{-k}$

<u>Pf</u> :- F.S. rep. of $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega k t}$

" " $x^*(t) = \left[\sum_{k=-\infty}^{\infty} c_k e^{j\omega k t} \right]^*$

$$= \sum_{k=-\infty}^{\infty} c_k^* e^{-j\omega k t}$$

Put $-k = \lambda$ $= \sum_{\lambda=\infty}^{-\infty} c^*_{-\lambda} e^{j\omega \lambda t}$

$\therefore \quad x^*(t) = \sum_{k=-\infty}^{\infty} c^*_{-k} e^{-j\omega k t}$

Hence. F.S. coeff. of $x^*(t)$ is c^*_{-k};

i.e; $x^*(t) \longleftrightarrow c^*_{-k}$ //

NOTE :- $x^*(-t) \longleftrightarrow c_k^*$

Pf :-

$$x(-t) \longleftrightarrow c_{-k}$$

$$x^*(t) \longleftrightarrow c_{-k}^*$$

$$x^*(-t) \longleftrightarrow c_{-(-k)}^*$$

$$\longleftrightarrow c_k^*$$

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⑧ Differentiation Property :-

If $x(t) \longleftrightarrow c_k$, then

$$\frac{d}{dt}[x(t)] \longleftrightarrow j\omega k c_k \quad \text{(or)} \quad j\frac{2\pi}{T} k c_k.$$

Pf :- F.S. rep. of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega kt}$$

$$\frac{d}{dt}[x(t)] = \sum_{k=-\infty}^{\infty} c_k \frac{d}{dt}\left[e^{j\omega kt}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \, j\omega k \, e^{j\omega kt}$$

$$= \sum_{k=-\infty}^{\infty} (c_k j\omega k) \, e^{j\omega kt}$$

$\therefore$ F.S. Coeff. of $\frac{d}{dt}[x(t)]$ is $j\omega k c_k$

i.e;

$$\frac{d}{dt}[x(t)] \longleftrightarrow j\omega k c_k.$$

Parseval's relation for periodic signals :- (or) Parseval's relation for power signals :-

Statement :-

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad \text{(or)}$$

$$\frac{1}{T}\int_T |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

Pf :- Refer unit - II .