UNIT-III

19/8/18

UNIT - 2

* Multiple Random Variables *

* The Joint Probability Distribution Function $[F_{X,Y}(x,y)]$:

Let $A = \{X \le x\}$ and $B = \{Y \le y\}$ are two events.

Then the joint probability distribution function of random variables X and Y is defined as

The Joint CDF of X & $Y = F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$\boxed{F_{X,Y}(x,y) = P(X \le x \cap Y \le y)}$$

For discrete random variables, let $X = \{x_1, x_2, \ldots x_N\}$ and $Y = \{y_1, y_2, \ldots y_M\}$, the joint CDF of X and Y is

$$\boxed{F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_m)}$$

For 'n' number of random variables, X_n, $n = 1, 2, 3 \ldots N$ the joint CDF is defined as

$$\boxed{F_{X_1, X_2, X_3 \ldots X_N}(x_1, x_2, x_3 \ldots x_N) = P(X_1 \le x_1, X_2 \le x_2, \ldots X_N \le x_N)}$$

* Properties of Joint Distribution Function:

1. $F_{X,Y}(-\infty, -\infty) = 0$

Proof: The joint CDF of X & $Y = F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$= P(X \le x \cap Y \le y)$$

$$F_{X,Y}(-\infty, -\infty) = P(X \le -\infty \cap Y \le -\infty)$$

$$X \le -\infty = \phi \quad Y \le -\infty = \phi$$

$$F_{X,Y}(-\infty, -\infty) = P(\phi \cap \phi)$$

$$= P(\phi) \qquad \because \phi \cap \phi = \phi, \; P(\phi) = 0$$

$$F_{X,Y}(-\infty, -\infty) = 0$$

2. $F_{X,Y}(x, -\infty) = 0$

Proof:

The joint CDF of X & Y is defined as,

$$F_{X,Y}(x,y) \triangleq P(X \leq x \cap Y \leq y)$$

$$F_{X,Y}(x, -\infty) = P(X \leq x \cap Y \leq -\infty)$$

$$X \leq x = X \quad ; \quad Y \leq -\infty = \phi$$

$$\therefore F_{X,Y}(x, -\infty) = P(X \cap \phi)$$ $\because A \cap \phi = \phi$, $P(\phi) = 0$

$$= P(\phi)$$

$$\therefore F_{X,Y}(x, -\infty) = 0$$

(3) $F_{X,Y}(-\infty, y) = 0$

The joint CDF of X & Y is defined as

$$F_{(X,Y)}(x,y) = P(X \leq x \cap Y \leq y)$$

$$F_{X,Y}(-\infty, y) = P(X \leq -\infty \cap Y \leq y)$$

$$X \leq -\infty = \phi \quad ; \quad Y \leq y = y$$

$$F_{(X,Y)}(-\infty, y) = P(\phi \cap y)$$

$$= P(\phi)$$

$$\therefore F_{X,Y}(-\infty, y) = 0$$

4. $F_{X,Y}(\infty, \infty) = 1$

Proof: The joint CDF of X & Y is defined as

$$F_{(X,Y)}(x,y) = P(X \leq x \cap Y \leq y)$$

$$F_{X,Y}(\infty, \infty) = P(X \leq \infty \cap Y \leq \infty)$$

$X \leq \infty = S$; $Y \leq \infty = S$

$F_{X,Y}(\infty,\infty) = P(S \cap S)$ $\quad \because S \cap S = S$

$= P(S)$ $\quad P(S) = 1$

$\therefore F_{X,Y}(\infty,\infty) = 1$.

5. The distribution function bounded b/w 0 and 1 i.e,

$0 \leq F_{X,Y}(x,y) \leq 1$

Proof: The range of distribution function is

$F_{X,Y}(-\infty,-\infty) \leq F_{X,Y}(x,y) \leq F_{X,Y}(\infty,\infty)$

We know $F_{X,Y}(-\infty,-\infty) = 0$ {from the above properties}

$F_{X,Y}(\infty,\infty) = 1$

$\therefore 0 \leq F_{X,Y}(x,y) \leq 1$.

i.e, distribution function is always non-negative or positive.

6. Marginal Distribution Functions of $X = F_X(x,\infty) = F_X(x)$

Marginal Distribution Function $Y = F_Y(\infty, y) = F_Y(y)$

Proof: The marginal distribution functions are defined as if the individual distribution functions are evaluated from joint distribution function then these functions are called as marginal distribution functions.

(i) The joint CDF of X & Y $= F_{X,Y}(x,y) = P(X \leq x, Y \leq y) = P(X \leq x \cap Y \leq y)$

$F_{X,Y}(x,\infty) = P(X \leq x \cap Y \leq \infty)$ $\quad Y \leq \infty = S$

$\because A \cap S = A$

$= P(X \leq x \cap S)$

$= P(X \leq x)$

$\boxed{F_{X,Y}(x,\infty) = F_X(x)}$

(ii) The joint CDF of X & Y is $F_{X,Y}(x,y) = P(X \le x \cap Y \le y)$

$$F_{X,Y}(\infty, y) = P(X \le \infty \cap Y \le y)$$

$$= P(S \cap Y \le y)$$

$$= P(Y \le y)$$

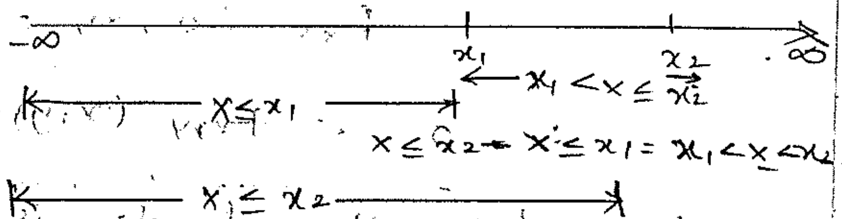
$$\therefore \boxed{F_{X,Y}(\infty, y) = F_Y(y)}$$

7. If $x_1 \le x_2$ & $y_1 \le y_2$ then $P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$

$$= F_{X,Y}(x_1, y_1) + F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$$

Proof: The joint CDF of X & $Y = F_{X,Y}(x,y) = P(X \le x, Y \le y) = P(X \le x \cap Y \le y)$

$(x_1 < X \le x_2, y_1 \le Y \le y_2) = ?$



$$(x_1 < X \le x_2, y_1 < Y \le y_2) = ((X \le x_2 - X \le x_1),\ y_1 < Y \le y_2)$$

$$= (X \le x_2, y_1 < Y \le y_2) - (X \le x_1, y_1 < Y \le y_2)$$

$$= (X \le x_2, Y \le y_2 - Y \le y_1) - (X \le x_1, Y \le y_2 - Y \le y_1)$$

$$= (X \le x_2, Y \le y_2) - (X \le x_2, Y \le y_1) - [(X \le x_1, Y \le y_2 - X \le x_1, Y \le y_1)]$$

$$= (X \le x_2, Y \le y_2) - (X \le x_2, Y \le y_1) - (X \le x_1, Y \le y_2) + (X \le x_1, Y \le y_1)$$

By taking probabilities, we get

$$P(x_1 < X \le x_2, y_1 < Y \le y_2) = P(X \le x_2, Y \le y_2) - P(X \le x_2, Y \le y_1) - P(X \le x_1, Y \le y_2) + P(X \le x_1, Y \le y_1)$$

$\because$ If A and B are mutually exclusive

$P(A \cup B) = P(A) + P(B)$

$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1).$$

$$\therefore P(x_1 < X \le x_2, y_1 < Y \le y_2) = F_{X,Y}(x_1, y_1) + F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$$

Hence proved.

8. The joint distribution function is a monotonic non-decreasing function of X.

*Example Problems:

1. Let the probabilities of joint sample space is as shown in table find joint distribution and marginal distributions as shown in table.

XY	(0,0) x_1, y_1	(1,2) $x_2\ y_2$	(2,3) $x_3\ y_3$	(3,2) x_4, y_4
P(x,y)	0.2 $P(x_1, y_1)$	0.3 $P(x_2, y_2)$	0.4 $P(x_3, y_3)$	0.1 $P(x_4, y_4)$

For discrete random variables, the JDF is defined by

$$F_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, u(x - x_n)\, u(y - y_m)$$

$$= \sum_{n=1}^{4} \sum_{m=1}^{4} P(x_n, y_m)\, u(x - x_n)\, u(y - y_n)$$

$$= P(x_1, y_1)\, u(x - x_1)\, u(y - y_1) + P(x_2, y_2)\, u(x - x_2)\, u(y - y_2) + P(x_3, y_3)\, u(x - x_3)\, u(y - y_3) + P(x_4, y_4)\, u(x - x_4)\, u(y - y_4)$$

$$\boxed{F_{X,Y}(x,y) = 0.2\, u(x)\, u(y) + 0.3\, u(x-1)\, u(y-2) + 0.4\, u(x-2)\, u(y-3) + 0.1\, u(x-3)\, u(y-2)}$$

Marginal distribution function of $X = F_X(x) = F_X(x, \infty)$

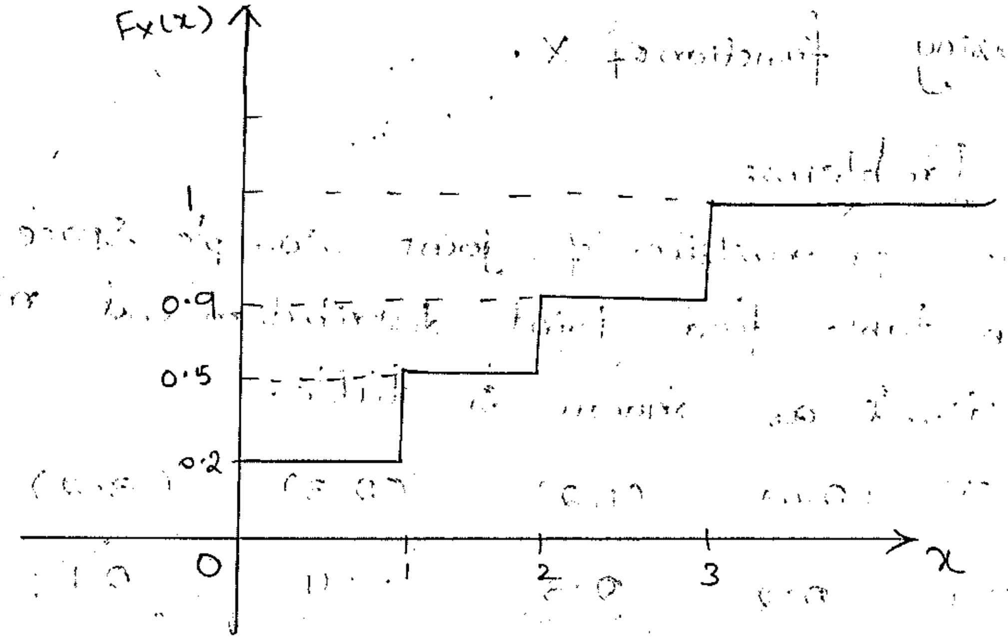
$= 0.2\,u(x) + 0.3\,u(x-1) + 0.4\,u(x-2) + 0.1\,u(x-3)$

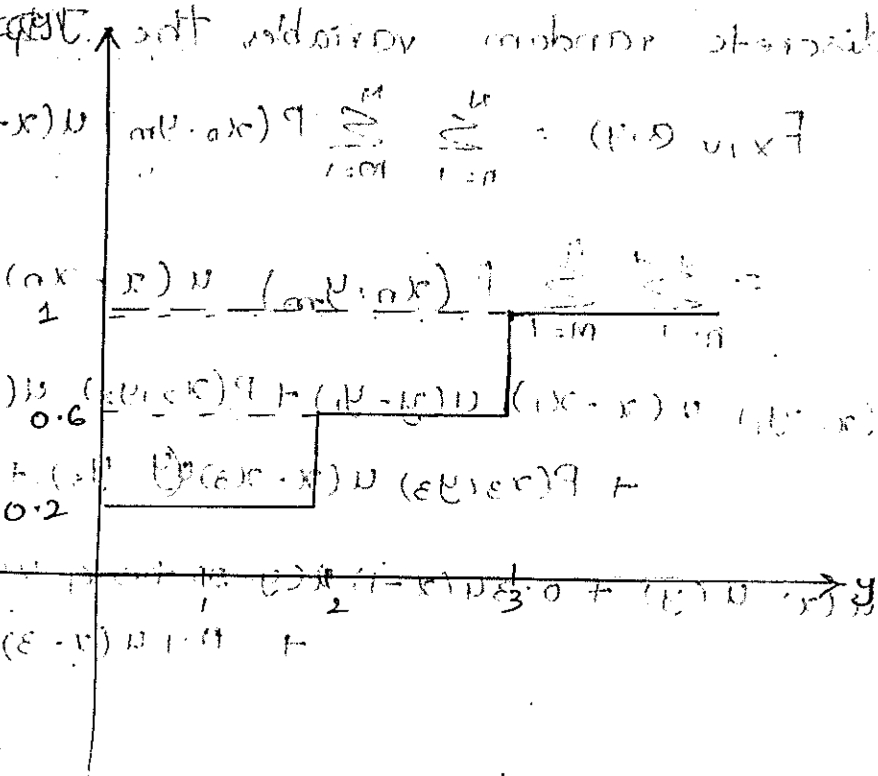
Marginal distribution function of $Y = F_Y(y) = F_X(\infty, y)$

$= 0.2\,u(y) + 0.3\,u(y-2) + 0.4\,u(y-3) + 0.1\,u(y-2)$

$= 0.2\,u(y) + 0.4\,u(y-3) + 0.4\,(u(y-2))$

Plots:





2. The probabilities of X and Y are shown in table. Find joint distribution and marginal distribution functions.

X/y	y_1 -1	y_2 0	y_3 1
x_1 0	3/18 $P(x_1,y_1)$	2/18 $P(x_1,y_2)$	3/18 $P(x_1,y_3)$
x_2 1	1/18 $P(x_2,y_1)$	3/18 $P(x_2,y_2)$	1/18 $P(x_2,y_3)$
x_3 2	2/18 $P(x_3,y_1)$	1/18 $P(x_3,y_2)$	2/18 $P(x_3,y_3)$

$$F_{X,Y}(x,y) = \sum_{n=1}^{3}\sum_{m=1}^{3} P(x_n, y_m)\,u(x-x_n)\,u(y-y_m)$$

$$= P(x_1,y_1)\,u(x-x_1)\,u(y-y_1) + P(x_2,y_2)u(x-x_2)u(y-y_2)$$
$$+P(x_3,y_3)\,u(x-x_3)\,u(y-y_3)$$
$$+P(x_1,y_2)\,u(x-x_1)\,u(y-y_2) + P(x_1,y_3)\,u(x-x_1)\,u(y-y_3)$$
$$+P(x_2,y_1)\,u(x-x_2)\,u(y-y_1) + P(x_2,y_3)\,u(x-x_2)\,u(x-x_3)$$
$$+P(x_3,y_1)\,u(x-x_3)\,u(y-y_1) + P(x_3,y_2)\,u(x-x_3)\,u(x-y_2)$$

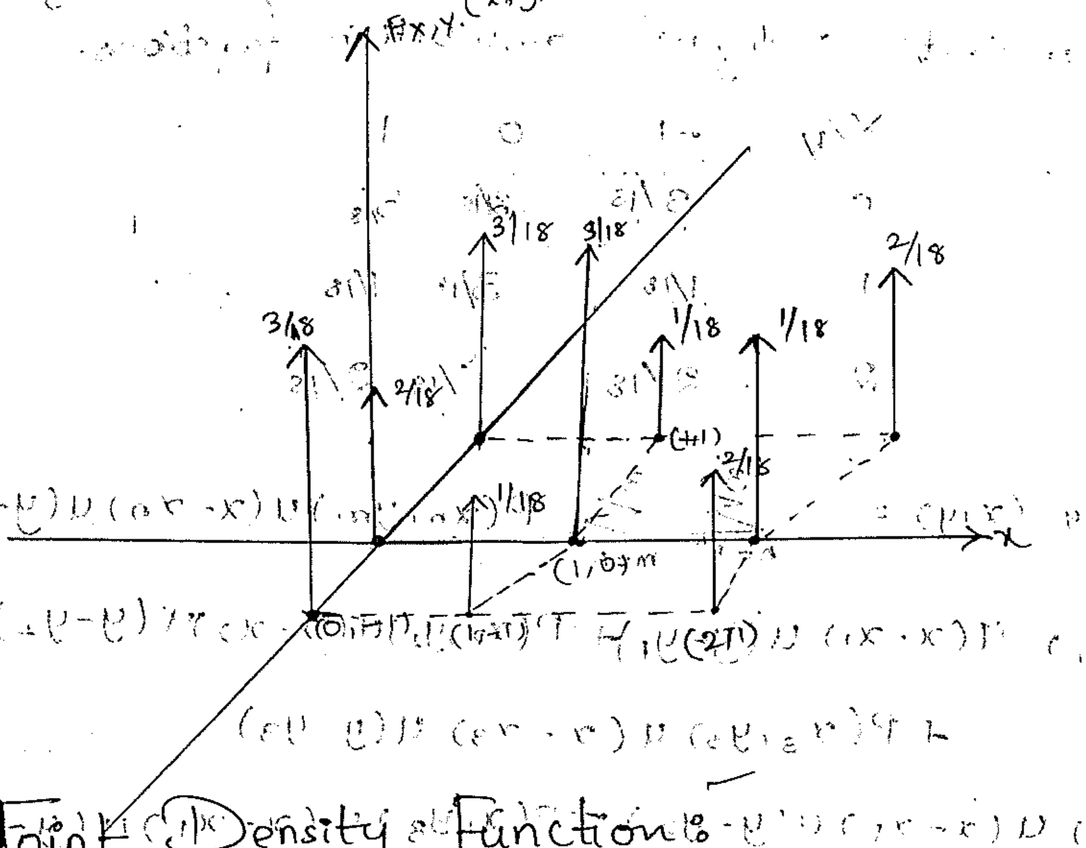
$$F_{X,Y}(x,y) = \frac{3}{18}u(x-0)u(y+1) + \frac{2}{18}u(x-0)\,u(y-\tfrac{3}{18}) + \frac{3}{18}u(x)\,u(y-1)$$
$$+\frac{1}{18}u(x-1)\,u(y+1) + \frac{3}{18}u(x-1)\,u(y) + \frac{1}{18}u(x-1)\,u(y-1)$$
$$+\frac{2}{18}u(x+2)\,u(y+1) + \frac{1}{18}u(x-2)\,u(y) + \frac{2}{18}u(x-2)u(y-1)$$

Marginal distribution function of $x = F_X(x) = F_X(x,\infty)$

$$= \frac{3}{18}u(x) + \frac{2}{18}u(x) + \frac{3}{18}u(x) + \frac{1}{18}u(x-1) + \frac{3}{18}u(x-1)$$
$$+\frac{13}{18}u(x-1) + \frac{2}{8}u(x-2) + \frac{1}{18}u(x-2) + \frac{2}{18}u(x+2)$$

$$F_X(x) = \frac{8}{18}u(x) + \frac{5}{18}u(x-1) + \frac{5}{18}u(x-2)$$

$$F_Y(y) = \frac{6}{18}u(y+1) + \frac{6}{18}u(y) + \frac{16}{18}u(y-1)$$



* Joint Density Function:

The joint probability density function of two random variables X and Y is defined as second derivative of joint distribution function. It is represented by

$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2 [F_{X,Y}(x,y)]}{\partial x\, \partial y}}$$

For discrete random variables $X = \{x_1, x_2, \ldots x_N\}$ and $Y = \{y_1, y_2, \ldots, y_N\}$ then

$$\boxed{f_{X,Y}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} P(x_n, y_m)\, \delta(x - x_n)\, \delta(y - y_n)}$$

For 'n' number of random variables $X_n = 1, 2, 3, \ldots N$ the ~~distribution~~ N-dimensional density function is defined as N-fold derivative of N-dimensional distribution function.

$$\boxed{f_{X_1, X_2, X_3 \ldots X_N}(x_1, x_2, x_3, \ldots x_N) = \frac{\delta^N [F_{X_1, X_2, X_3 \ldots X_N}(x_1, x_2, x_3 \ldots x_N)]}{\delta x_1\, \delta x_2\, \delta x_3 \ldots \delta x_N}}$$

* Distribution function in terms of Density function:

$$\boxed{F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy}$$

For 'N' no. of random variable $X_N = 1,2,3 \dots N$

$$\boxed{F_{X_1 X_2 X_3 \dots X_N}(x_1,x_2,x_3\dots x_N) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3}\dots\int_{-\infty}^{x_N} f_{X_1 X_2 X_3 \dots X_N}(x_1,x_2,x_3,\dots x_N)\,dx_1\,dx_2\,dx_3\dots dx_N}$$

* Properties of Joint density function:

1. Joint density function is non-negative value i.e., +ve

$f_{X,Y}(x,y) \geq 0$

__Proof:__ The joint PDF of X and Y is $= f_{X,Y}(x,y) = \frac{\partial^2[F_{X,Y}(x,y)]}{\partial x \partial y}$

We know $0 \leq F_{X,Y}(x,y) \leq 1$

i.e., $F_{X,Y}(x,y)$ is +ve value.

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) \geq 0$$

Hence $f_{X,Y}(x,y) = \frac{\partial^2[F_{X,Y}(x,y)]}{\partial x \partial y} \geq 0$

2. Area under JDF is unity i.e., $\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1}$

Proof: The joint PDF is $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}[F_{X,Y}(x,y)]$

Take L.H.S $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x \partial y}[F_{X,Y}(x,y)]\,dx\,dy$$

$$= \int_{\infty}^{\infty} \frac{d}{dy}\left[\int_{-\infty}^{\infty}\left[\frac{d}{dx} F_{X,Y}(x,y)\right]dx\right]dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} \left[\int_{-\infty}^{\infty} 1 \cdot d \left[F_{X,Y}(x,y) \right] \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} \left[F_{X,Y}(x,y) \Big|_{-\infty}^{\infty} \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} \left[\left[F_{X,Y}(\infty,y) - F_{X,Y}(-\infty,y) \right] \right] dy$$

$$= \int_{-\infty}^{\infty} \frac{d}{dy} \left[F_Y(y) - 0 \right] dy$$

$$= \int_{-\infty}^{\infty} 1 \cdot d\left[F_Y(y)\right]$$

$\because F_{X,Y}(\infty, y) = F_Y(y)$

$F_{X,Y}(-\infty, y) = 0$

$$= F_Y(y) \Big|_{-\infty}^{\infty}$$

$\because F_Y(-\infty) = 0; \; F_Y(\infty) = 1$

$$= F_Y(\infty) - F_Y(-\infty)$$

$$= 1 - 0$$

$$= 1.$$

$= R.H.S$

3. $$\int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy = F_{X,Y}(x,y)$$

Proof: The joint PDF of X & Y is $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\left[F_{X,Y}(x,y)\right]$

Consider L.H.S $= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy$

$$= \int_{-\infty}^{y} \int_{-\infty}^{x} \frac{\partial^2 \left[F_{X,Y}(x,y)\right]}{\partial x \partial y} dx\, dy$$

$$= \int_{-\infty}^{y} \frac{d}{dy} \left[\int_{-\infty}^{x} \frac{d}{\cancel{dx}} \left(F_{X,Y}(x,y)\, \cancel{dx} \right) \right] dy$$

$$= \int_{-\infty}^{y} \frac{d}{dy} \left[\int_{-\infty}^{x} 1 \cdot d\left[F_{X,Y}(x,y) \right] \right] dy$$

$$= \int_{-\infty}^{y} \frac{d}{dy} \left[F_{X,Y}(x,y) \Big|_{-\infty}^{x} \right] dy$$

$$= \int_{-\infty}^{y} \frac{d}{dy} \left(\left[F_{X,Y}(x,y) - F_{X,Y}(-\infty,y) \right] \right) dy$$

$$= \int_{-\infty}^{y} \frac{d}{dy} \left[F_{X,Y}(x,y) - 0 \right] dy$$

$$= \int_{-\infty}^{y} \frac{d}{\cancel{dy}} \left[F_{X,Y}(x,y) \right] \cancel{dy}$$

$$= \int_{-\infty}^{y} 1 \cdot d\left[F_{X,Y}(x,y) \right]$$

$$= F_{X,Y}(x,y) \Big|_{-\infty}^{y}$$

$$= F_{X,Y}(x,y) - F_{X,Y}(x,-\infty)$$

$$= F_{X,Y}(x,y) - 0$$

$$= F_{X,Y}(x,y)$$

= R.H.S.

4. $\int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy = P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$

<u>Proof:</u> The joint PDF of X,Y is $f_{X,Y}(x,y) = \dfrac{\partial^2 [F_{X,Y}(x,y)]}{\partial x \partial y}$

Consider L.H.S $= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} \frac{\partial^2}{\partial x \partial y}[f_{X,Y}(x,y)]\,dx\,dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy}\left[\left[\int_{x_1}^{x_2}\frac{d}{dx} F_{X,Y}(x,y)\,dx\right]\right] dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy}\left[\int_{x_1}^{x_2} d\,[F_{X,Y}(x,y)]\right] dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy}\left[F_{X,Y}(x,y)\Big|_{x_1}^{x_2}\right] dy$$

$$= \int_{y_1}^{y_2} \frac{d}{dy}\left[F_{X,Y}(x_2,y) - F_{X,Y}(x_1,y)\right] dy$$

$$= \int_{y_1}^{y_2} 1 \cdot d\left[F_{X,Y}(x_2,y) - F_{X,Y}(x_1,y)\right]$$

$$= F_{X,Y}(x_2,y)\Big|_{y_1}^{y_2} - F_{X,Y}(x_1,y)\Big|_{y_1}^{y_2}$$

$$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - [F_{X,Y}(x_1,y_2) - F_{X,Y}(x_1,y_1)$$

$$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_1,y_1)$$

$$= F_{X,Y}(x_1,y_1) + F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1)$$

∵ From Joint CDF function which is already proved

$$= P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$$

$= R.H.S$

Hence proved.

* Marginal Distribution Functions:

1. Marginal distribution function of 'x' $= F_{X,Y}(x,\infty)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

<u>Proof:</u> $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dy\, dx$

$$F_{X,Y}(x,\infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$$

∵ $F_X(x) = F_{X,Y}(x,\infty)$

|||ly $Y = F_Y(y) = F_{X,Y}(\infty,y) = \int_{-\infty}^{y} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$

* Marginal Density Function:

Marginal density function of 'X' $= f_X(x) = \frac{d}{dx}[F_X(x)]$

$$= \frac{\partial}{\partial x}[F_{X,Y}(x,\infty)] = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

<u>Proof:</u> The joint CDF of X & Y $= F_{X,Y}(x,y)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dx\, dy$$

Marginal CDF of $X = F_X(x) = F_X(x,\infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$

Apply $\frac{d}{dx}$ to both sides

$$\frac{d}{dx}[F_X(x)] = \frac{\partial}{\partial x} F_X(x,\infty) = \cancel{\frac{d}{dx}} \left[\cancel{\int_{-\infty}^{x}} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, \cancel{dx} \right]$$

$$\frac{d}{dx}[F_X(x)] = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad \therefore f_X(x) = \frac{d}{dx}[F_X(x)]$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

IIIly Marginal PDF of 'y' $= f_Y(y) = \frac{d}{dy}[F_Y(y)]$

$$= \frac{\partial}{\partial y} F_{X,Y}(\infty, y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

* Problems :

1. The joint density function of random variables X, Y is $f_{X,Y}(x,y) = \begin{cases} c(x^2+2y) ; & x = 0,1,2 \ \& \ y = 1,2,3,4 \\ 0 ; & \text{otherwise} \end{cases}$

(i) Find the value of constant C.

(ii) Probability of X = P(X=2, Y=3) (iii) P(X ≤ 1, Y ≥ 3)

(iv) Marginal Density functions of X and Y i.e $f_X(x)$ and $f_Y(y)$.

Sol:

x \ y	1	2	3	4	Total
0	2c	4c	6c	8c	20c
1	3c	5c	7c	9c	24c
2	6c	8c	10c	12c	36c
Total	11c	17c	23c	29c	80c

Total probability $= \sum_{\langle x \rangle} \sum_{\langle y \rangle} f_{X,Y}(x,y) = 1$

$80c = 1$ (From table)

(i) $\boxed{c = \frac{1}{80}}$

(ii) From table, $P(X=2, Y=3) = 10C = \frac{10}{80} = \frac{1}{8}$

$$P(X=2, Y=3) = \frac{1}{8}$$

(iii) $P(X \leq 1, Y \geq 3)$

$X \leq 1 \Rightarrow X = 0, 1$

$Y \geq 3 \Rightarrow Y = 3, 4$

$$= P(X=0, Y=3) + P(X=0, Y=4) + P(X=1, Y=3) + P(X=1, Y=4)$$

$$= 6C + 8C + 7C + 9C$$

$$= 30C$$

$$\therefore P(X \leq 1, Y \geq 3) = \frac{30}{80}$$

(iv) Marginal density function of 'X' $= f_X(x) = \begin{cases} 20C & ; x=0 \\ 24C & ; x=1 \\ 36C & ; x=2 \end{cases}$

$$\therefore f_X(x) = \begin{cases} \frac{20}{80} & ; x=0 \\ \frac{24}{80} & ; x=1 \\ \frac{36}{80} & ; x=2 \end{cases}$$

Marginal density function of 'y' $= f_Y(y) = \begin{cases} 11C & ; y=1 \\ 17C & ; y=2 \\ 23C & ; y=3 \\ 29C & ; y=4 \end{cases}$

$$\therefore f_Y(y) = \begin{cases} \frac{11}{80} & ; y=1 \\ \frac{17}{80} & ; y=2 \\ \frac{23}{80} & ; y=3 \\ \frac{29}{80} & ; y=4 \end{cases}$$

→ And also examine X and Y are independent variables or not.

⇒ If X and Y are independent variables then

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$$

The marginal density function of $X = f_X(x)$

$$f_X(x) = \sum_{\langle y \rangle} f_{X,Y}(x,y) \qquad \left(\because f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,d\right.$$

$$= \sum_{y=1}^{4} c(x^2+2y)$$

$$= cx^2 \sum_{y=1}^{4} 1 + 2c \sum_{y=1}^{4} y \qquad \left(\because \sum_{n=N_1}^{N_2} 1 = N_2 - N_1 + 1\right)$$

$$= cx^2(4-1+1) + 2c[1+2+3+4]$$

$$= 4cx^2 + 20c$$

$$= 2c(2x^2+10) \quad ; \ x = 0,1,2$$

$$\boxed{f_X(x) = 2c(2x^2+10) \ ; \ x = 0,1,2}$$

The marginal density function of $Y = f_Y(y)$

$$f_Y(y) = \sum_{\langle x \rangle} f_{X,Y}(x,y)$$

$$= \sum_{x=0}^{2} c(x^2+2y)$$

$$= c \sum_{x=0}^{2} x^2 + 2c \sum_{x=0}^{2} y(1)$$

$$= c \sum_{x=0}^{2} x^2 + 2cy \sum_{x=0}^{2} 1$$

$$= c[0^2+4+1] + 2cy[2-0+1]$$

$$\boxed{\therefore f_Y(y) = 5c + 6cy \ ; \ y = 1,2,3,4}$$

$$f_X(x)\cdot f_Y(y) = 2c(2x^2+10)\cdot c(5+6y)$$

$$f_{X,Y}(x,y) = c(x^2+2y)$$

$$\therefore f_X(x)\cdot f_Y(y) \neq f_{X,Y}(x,y)$$

Hence X and Y are dependent variables or not independent random variables.

→ And also find joint distribution function.

The joint CDF of X and Y = $F_{X,Y}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} P(x_n, y_m)\, u(x-x_n)\, u(y-y_m)$

$$F_{X,Y}(x,y) = \frac{2}{80}u(x)\,u(y-1) + \frac{4}{80}u(x)\,u(y-2) + \frac{6}{80}u(x)\,u(y-3)$$
$$+\frac{8}{80}u(x)\,u(y-4) + \frac{3}{80}u(x-1)\,u(y-1) + \frac{5}{80}u(x-1)\,u(y-2)$$
$$+\frac{7}{80}u(x-1)(y-3) + \frac{9}{80}u(x-1)\,u(y-4) +$$
$$+\frac{6}{80}u(x-2)\,u(y-1) + \frac{8}{80}u(x-2)\,u(y-2) + \frac{10}{80}u(x-2)\,u(y-3)$$
$$+\frac{12}{80}u(x-2)\,u(y-4)$$

→ Marginal distribution function of X and Y:

The CDF of 'X' = $F_X(x) = \frac{20}{80}u(x) + \frac{24}{80}u(x-1) + \frac{36}{80}u(x-2)$

The PDF of 'X' = $f_X(x) = \frac{20}{80}\delta(x) + \frac{24}{80}\delta(x-1) + \frac{36}{80}\delta(x-2)$

The CDF of 'Y' = $F_Y(y) = \frac{11}{80}u(y-1) + \frac{17}{80}u(y-2) + \frac{23}{80}u(y-3) + \frac{29}{80}u(y-4)$

The PDF of 'Y' = $f_Y(y) = \frac{11}{80}\delta(y-1) + \frac{17}{80}\delta(y-2) + \frac{23}{80}\delta(y-3) + \frac{29}{80}\delta(y-4)$

→ And also find $f_{Y/X}(y/1)$ & $f_{X/Y}(x/2)$.

The conditional P.D.F of $X = f_{X/Y}(x/y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$$f_{X/Y}(x/2) = \frac{f_{X,Y}(x,2)}{f_Y(2)}$$

$$f_Y(y) = f_Y(2) = \frac{17}{80}$$

$$f_{X,Y}(x,y) = c(x^2+2y)$$

$$f_{X,Y}(x,2) = c(x^2+4) = \frac{1}{80}(x^2+4) \quad ; x = 0,1,2$$

$$f_{X/Y}(x/2) = \frac{\frac{1}{80}(x^2+4)}{\frac{17}{80}} \quad ; x = 0,1,2$$

$$\boxed{\therefore f_{X/Y}(x,2) = \frac{1}{17}(x^2+4) \quad ; x = 0,1,2}$$

$$f_{Y/X}(y/1) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{f_X(1)}$$

$$f_X(1) = 24c = \frac{24}{80}$$

$$f_{X,Y}(x,y) = c(x^2+2y)$$

$$= c(1+2y)$$

$$= \frac{1}{80}(1+2y) \; ; y = 1,2,3,4$$

$$f_{Y/X}(y/1) = \frac{\frac{1}{\cancel{80}}(1+2y)}{\frac{24}{\cancel{80}}}$$

$$\boxed{\therefore f_{Y/X}(y/1) = \frac{1}{24}(1+2y) \quad ; y = 1,2,3,4}$$

$\rightarrow P(X=1, Y=2) = 5C = \frac{5}{80}$

* Conditional Joint Distribution Function:

The conditional joint distribution function of random variable X, given that Y is known is defined as

$$\boxed{\begin{aligned}&\text{The conditional CDF of `X' given that `Y'} = F_{X/Y}(x/y)\\ &= \frac{P(X \le x, Y \le y)}{P(Y \le y)} = \frac{F_{X,Y}(x,y)}{F_Y(y)} \; ; F_Y(y) \ne 0\end{aligned}}$$

Marginal CDF of `y' $= F_Y(y) = F_Y(\infty, y)$

$$\boxed{\begin{aligned}&\text{The conditional CDF of `Y' given that `X'} = F_{Y/X}(y/x)\\ &= \frac{P(X \le x, Y \le y)}{P(Y \le y)} = \frac{F_{X,Y}(x,y)}{F_X(x)} \; ; F_X(x) \ne 0\end{aligned}}$$

Marginal CDF of `X' $= F_X(x) = F_X(x, \infty)$

* Properties of Conditional Joint Distribution Function:

1. $F_{X/Y}(-\infty/y) = 0$ similarly $F_{Y/X}(-\infty/x) = 0$

Proof: We know $F_{X/Y}(x/y) = \frac{P(X \le x, Y \le y)}{P(Y \le y)}$

$$F_{X/Y}(-\infty/y) = \frac{P(X \le -\infty, Y \le y)}{P(Y \le y)}$$

$$= \frac{P(\phi \cap Y \le y)}{P(Y \le y)}$$

$$= \frac{P(\phi)}{P(Y \le y)} \quad \left\{ \begin{array}{l} \because X \le -\infty = \phi \\ P(\phi \cap A) = \phi \end{array} \right\}$$

$$= \frac{0}{P(Y \le y)}$$

$$\therefore F_{X/Y}(-\infty/y) = 0$$

2. $F_{X/Y}(\infty/y) = 1$ Similarly $F_{Y/X}(\infty/x) = 1$

Proof: $F_{X/Y}(\infty/y) = \dfrac{P(X \leq \infty, Y \leq y)}{P(Y \leq y)}$

$= \dfrac{P(S \cap Y \leq y)}{P(Y \leq y)}$ $\left\{ \begin{array}{l} x \leq \infty = S \\ \therefore S \cap A = S \end{array} \right\}$

$= \dfrac{P(Y \leq y)}{P(Y \leq y)}$

$\therefore F_{X/Y}(\infty/y) = 1$

3. $0 \leq F_{X/Y}(x/y) \leq 1$ Similarly $0 \leq F_{Y/X}(y/x) \leq 1$

Proof: We know $F_{X/Y}(-\infty/y) \leq F_{X/Y}(x/y) \leq F_{X/Y}(\infty/y)$

$0 \leq F_{X/Y}(x/y) \leq 1$

{From ① and ② properties}

* Conditional Joint Density Function:

The conditional density function of 'X' given that 'Y' is defined as

$$\boxed{\text{The conditional PDF is 'X' given 'Y'} = f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}; \ f_Y(y) \neq 0}$$

$$\text{Marginal PDF of 'Y'} = f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$\boxed{\text{The conditional PDF of Y given X} = f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}; \ f_X(x) \neq 0}$$

$$\text{Marginal PDF of X} = f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

*Properties of CJDF: conditional density

1. $f_{X/Y}(x/y) \geq 0$ is a non-negative or positive.

<u>Proof:</u> $f_{X/Y}(x/y) = \frac{d}{dx}\left[F_{X/Y}(x/y)\right]$

We know $0 \leq F_{X/Y}(x/y) < 1$

$$\therefore \frac{d}{dx}\left[F_{X/Y}(x/y)\right] \geq 0$$

Hence $f_{X/Y}(x/y) = \frac{d}{dx}\left[F_{X/Y}(x/y)\right] \geq 0$.

2. Area under density function is unity. i.e.,

$$\int_{-\infty}^{\infty} f_{X/Y}(x/y)\, dx = 1$$

<u>Proof:</u> Consider L.H.S $= \int_{-\infty}^{\infty} f_{X/Y}(x/y)\, dx$

We know $0 \leq F_{X/Y}(x/y) < 1$

$$f_{X/Y}(x/y) = \frac{d}{dx}\left[F_{X/Y}(x/y)\right]$$

$$= \int_{-\infty}^{\infty} \frac{d}{\cancel{dx}}\left[F_{X/Y}(x/y)\right] \cancel{dx}$$

$$= \int_{-\infty}^{\infty} 1 \cdot d\left[F_{X/Y}(x/y)\right]$$

$$= F_{X/Y}(x/y)\Big|_{-\infty}^{\infty}$$

$$= F_{X/Y}(\infty/y) - F_{X/Y}(-\infty/y)$$

$$= 1 - 0$$

$$= 1$$

$= R.H.S$

3. $\int_{-\infty}^{x} f_{X/Y}(x/y)\, dx = F_{X/Y}(x/y)$

Proof: Consider L.H.S $= \int_{-\infty}^{x} f_{X/Y}(x/y)\, dx$

We know $f_{X/Y}(x/y) = \frac{d}{dx}\left[F_{X/Y}(x/y)\right]$

$$= \int_{-\infty}^{x} \frac{d}{dx}\left[F_{X/Y}(x/y)\right] dx$$

$$= \int_{-\infty}^{x} 1 \cdot d\left[F_{X/Y}(x/y)\right]$$

$$= F_{X/Y}(x/y)\Big|_{-\infty}^{x}$$

$$= F_{X/Y}(x/y) - F_{X/Y}(-\infty/y)$$

$$= F_{X/Y}(x/y) - 0$$

$$= F_{X/Y}(x/y)$$

$$= R.H.S$$

4. $\int_{x_1}^{x_2} f_{X/Y}(x/y)\, dx = P(x_1 < X \leq x_2 / y)$

Proof: Consider L.H.S $= \int_{x_1}^{x_2} f_{X/Y}(x/y)\, dx$

We know $f_{X/Y}(x/y) = \frac{d}{dx}\left[F_{X/Y}(x/y)\right]$

$$= \int_{x_1}^{x_2} \frac{d}{dx} F_{X/Y}(x/y)\, dx$$

$\because P(x_1 < X \leq x_2/B)$
$= F_{X/Y}(x_2/B) - F_{X/Y}(x_1/B)$

$$= \int_{x_1}^{x_2} 1 \cdot d\, F_{X/Y}(x/y)$$

$$= F_{X/Y}(x_2/y) - F_{X/Y}(x_1/y)$$

$$= P(x_1 \leq X \leq x_2 / Y)$$

= R.H.S

* Point Conditioning:

The conditional Joint distrubution of random variable 'X' given that 'Y' at a specific value i.e. $y = Y$ is defined as

$$\boxed{F_{X/Y}(x/Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx}{f_Y(y)}}$$

The conditional joint density of random variable 'X' given that 'Y' at a specific value i.e. $Y=y$ is defined as

$$\boxed{f_{X/Y}(x/Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

Proof: Let the distribution function of random variable X given the event B is defined as

$$F_{X/B}(x/B) = \frac{P(X \leq x \cap B)}{P(B)}$$

Let us consider the event B for point conditioning

$$B = y - \Delta y < Y \leq y + \Delta y \quad \text{as} \quad \Delta y \to 0$$

where Δy is very small i.e.

$$B = Y = y$$

$$F_{X/B}(x/B) = \frac{P(X \leq x \cap y - \Delta y < Y \leq y + \Delta y)}{P(y - \Delta y < Y \leq \Delta y + y)}$$

$$= \frac{P(X \leq x, y - \Delta y < Y \leq y + \Delta y)}{P(y - \Delta y < Y \leq y + \Delta y)}$$

$$= \frac{\int_{y-\Delta y}^{y+\Delta y} \int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy}{\int_{y-\Delta y}^{y+\Delta y} f_Y(y)\,dy}$$

as $\Delta y \to 0$

$$= \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\, dx \int_{y-\Delta y}^{y+\Delta y} \cancel{1\, dy}}{f_Y(y) \int_{y-\Delta y}^{y+\Delta y} \cancel{1\, dy}}$$

$$F_{X/Y}(x/Y=y) = \frac{\int_{-\infty}^{x} f_{X,Y}(x,y)\, dx}{f_Y(y)}$$

Apply differentiation to the above equation

$$\frac{d}{dx} F_{X/Y}(x/Y=y) = \frac{\frac{d}{dx}\left\{\int_{-\infty}^{x} f_{X,Y}(x,y)\, d\cancel{x}\right\}}{\left\{f_Y(y)\right\}}$$

$$f_{X/Y}(x/Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

lll^{ly} $$f_{Y/X}(y/x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_{X,Y}(x,y) = f_{X/Y}(x/y)\, f_Y(y)$$

and $f_{X,Y}(x,y) = f_{Y/X}(y/x)\, f_X(x)$.

* Interval Conditioning:

The conditional distribution function of random variable 'X', given that 'Y' is in a specified interval $y_1 < Y \le y_2$ is defined as

$$\boxed{F_{X/Y}(x/y) = \frac{\int_{y_1}^{y_2} \int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy}{\int_{y_1}^{y_2} f_Y(y)\, dy}}$$

Explanation

CDF of $F_{X/Y}(x/y) = \dfrac{\displaystyle\int_{y_1}^{y_2}\int_{-\infty}^{x}\left[\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)\right] dx\, dy}{\displaystyle\int_{y_1}^{y_2} f_Y(y)\, dy}$ $\quad \because f_{X,Y}(x,y) = \dfrac{\partial^2}{\partial x \partial y}[F_{X,Y}(x,y)$

$$= \frac{\displaystyle\int_{y_1}^{y_2} \frac{d}{dy}\int_{-\infty}^{\infty}\left(\left[\frac{\cancel{d}}{\cancel{dx}} F_{X,Y}(x,y)\right]\cancel{dx}\right) dy}{\displaystyle\int_{y_1}^{y_2} \frac{d}{\cancel{dy}}\left[F_Y(y)\right]\cancel{dy}}$$

$$= \frac{\displaystyle\int_{y_1}^{y_2} \frac{d}{\cancel{dy}}\left(F_{X,Y}(x,y)\Big|_{-\infty}^{x}\right]\cancel{dy}}{\displaystyle\int_{y_1}^{y_2} 1\, d\left[F_Y(y)\right]}$$

$$= \frac{\displaystyle\int_{y_1}^{y_2} d\, F_{X,Y}(x,y) - F_{X,Y}(-\infty,y)}{F_Y(y_2) - F_Y(y_1)}$$

$$= \frac{\displaystyle\int_{y_1}^{y_2} 1\cdot d\left[F_{X,Y}(x,y)\right]}{F_Y(y_2) - F_Y(y_1)}$$

$$= \frac{F_{X,Y}(x,y)\Big|_{y_1}^{y_2}}{F_Y(y_2) - F_Y(y_1)}$$

$$\boxed{\therefore F_{X/Y}(x/y) = \frac{F_{X,Y}(x, y_2) - F_{X,Y}(x, y_1)}{F_Y(y_2) - F_Y(y_1)}}$$

Apply $\frac{d}{dx}$ we get

$$PDF = \frac{d}{dx} F_{X/Y}(x/y) = \frac{d}{dx}\left[\frac{\int_{y_1}^{y_2}\int_{-\infty}^{x} f_{X,Y}(x,y)\,dx\,dy}{\int_{y_1}^{y_2} f_Y(y)\,dy}\right]$$

$$\boxed{f_{X/Y}(x/y) = \frac{\int_{y_1}^{y_2} f_{X,Y}(x,y)\,dy}{\int_{y_1}^{y_2} f_Y(y)\,dy}}$$

~~Sum of two~~

V-Imp

* Statistical Independent Random Variables:

Let the event $A = \{X \leq x\}$ of random variable X and $B = \{Y \leq y\}$ of random variable 'Y'. The two random variables X and Y are said to be statistically independent, if and only if $\boxed{P(X \leq x, Y \leq y) = P(X \leq x)\, P(Y \leq y)}$

The probability distribution is

$$\boxed{F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)}$$

The probability density is

$$\boxed{f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)}$$

Proof: We know if the events A and B are independent then, $P(A \cap B) = P(A) \cdot P(B)$.

$$P(X \leq x, Y \leq y) = P(X \leq x \cap Y \leq y)$$

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y) \rightarrow ①$$

The joint CDF of X, Y $= F_{X,Y}(x,y) = P(X \leq x, Y \leq y)$

$$= P(X \leq x) \cdot P(Y \leq y)$$

$$F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y) \rightarrow ②$$

$\because F_X(x) = P(X \leq x)$

The joint CDF X,Y is differentiated to the above equation

$$\frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y}\left[F_X(x) \cdot F_Y(y)\right]$$

$$= \frac{d}{dx}\left[F_X(x)\right] \cdot \frac{d}{dy}\left[F_Y(y)\right]$$

We know PDF of X & Y = $f_{X,Y}(x,y)$

$\because f_X(x) = \frac{d}{dx}\left[F_X(x)\right]$ $\qquad = \frac{\partial^2}{\partial x \partial y}\left[F_{X,Y}(x,y)\right]$

$f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right]$ $\qquad = \frac{d}{dx} F_X(x) \cdot \frac{d\left[F_Y(y)\right]}{dy}$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y) \longrightarrow ③$$

The conditional CDF of X and Y is given as:

$$F_{X/Y}(x/y) = \frac{F_{X,Y}(x,y)}{F_Y(y)}$$

$$= \frac{F_X(x)\,\cancel{F_Y(y)}}{\cancel{F_Y(y)}}$$

$$\boxed{\therefore F_{X/Y}(x/y) = F_X(x)}$$

III^{ly}

$$F_{Y/X}(y/x) = \frac{F_{X,Y}(x,y)}{F_X(x)}$$

$$= \frac{\cancel{F_X(x)}\,F_Y(y)}{\cancel{F_X(x)}}$$

$$\boxed{F_{Y/X}(y/x) = F_Y(y)}$$

The condition PDF fcs is given as

$$f_{X/Y}(x/y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{X/Y}(x/y) = \frac{f_X(x)\,f_Y(y)}{f_Y(y)}$$

$$\boxed{f_{X/Y}(x/y) = f_X(x)}$$

III^{ly}

$$\boxed{f_{Y/X}(y/x) = f_Y(y)}$$

* Sum of two statistical independent random variables:

Let the sum of two independent random variables X and Y is $W = X + Y$, then the probability density function of sum of two statistically independent random variables is equivalent to the convolution of their individual density functions i.e,

$$\boxed{f_W(w) = f_X(x) * f_Y(y)}$$

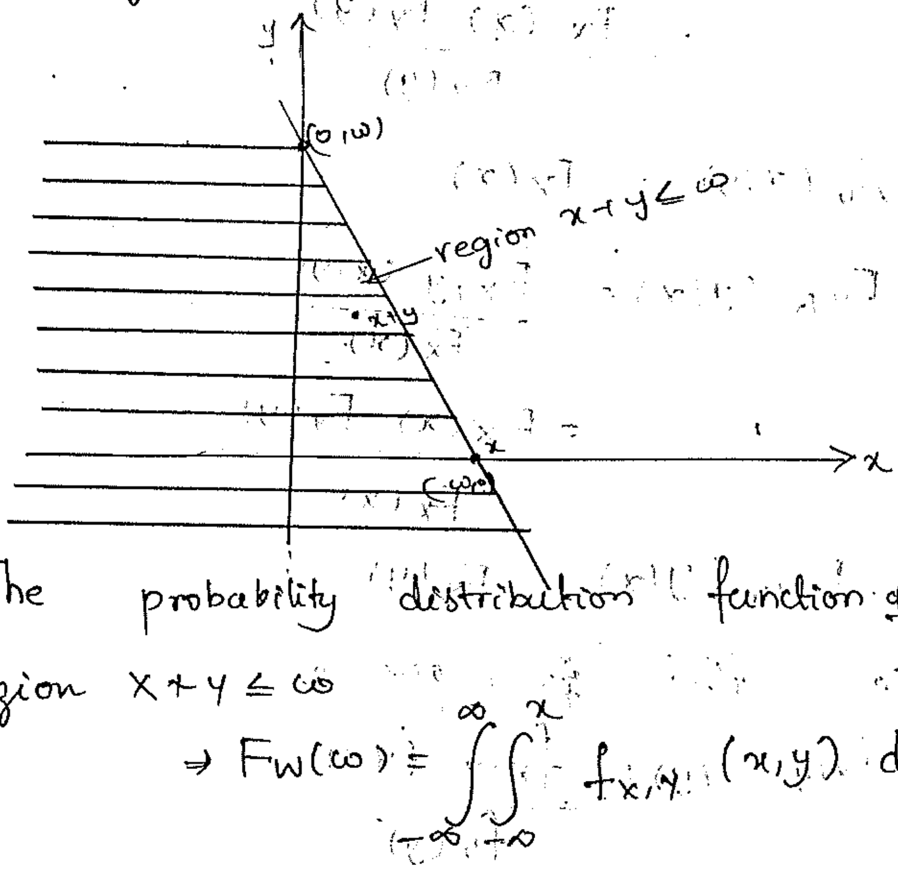
Proof: Let the sum of two random variables

$$W = X + Y$$

The probability distribution function of random variable

$$W = F_W(w) = P(W \leq w) = P(X + Y < w)$$

The region $X + Y \leq w$ is as shown in figure



The probability distribution function of w in the region $X + Y \leq w$

$$\Rightarrow F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy$$

$$x + y = w \Rightarrow x = w - y$$

$$\Rightarrow F_W(w) = \int_{-\infty}^{\infty}\int_{-\infty}^{w-y} f_{X,Y}(x,y)\, dx\, dy$$

Given X and Y are independent $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$

$$\boxed{\Rightarrow F_W(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\omega-y} f_X(x) f_Y(y)\, dx\, dy}$$

The PDF of ω = $f_W(\omega) = \frac{d}{d\omega}[F_W(\omega)]$

$$f_W(\omega) = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\omega-y} f_X(x) f_Y(y)\, dx\, dy\right]$$

By using Leibnitz's rule, we get

$$f_W(\omega) = \int_{-\infty}^{\infty} f_Y(y) \frac{d}{d\omega}\left[\int_{-\infty}^{\omega-y} f_X(x)\, dx\right] dy$$

$\because \frac{dF_X(x)}{dx} = f_X(x)$

$$= \int_{-\infty}^{\infty} f_Y(y) \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} \frac{d F_X(x)}{\cancel{dx}} \cancel{dx}\right] dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \frac{d}{d\omega}\left[F_X(x)\Big|_{\infty}^{\omega-y}\right] dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \frac{d}{d\omega}\left[F_X(\omega-y) - F_X(-\infty)\right] dy$$

$$= \int_{-\infty}^{\infty} f_Y(y) \frac{d}{d\omega}\left[F_X(\omega-y)\right] dy \qquad \boxed{\because F_X(-\infty)=0}$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, f_X(\omega-y)\, dy \qquad \because f_X(x) = \frac{d}{dx}[F_X(x)] \text{ where } x = \omega - y$$

$f_W(\omega) = f_Y(y) * f_X(x)$ $\because$ From convolution theorem, $y(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$

$f_W(\omega) = f_X(x) * f_Y(y)$

Hence, statement is proved.

$$\boxed{f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(y) \cdot f_Y(\omega-y)\, dy}$$

*Sum of Several or Multiple N Random Variables :-

let the sum of 'N' number of independent random variables $Y = X_1 + X_2 + X_3 + \cdots + X_N$, then the density function of sum of 'N' indenpendent random variables = convolution of their individual functions

i.e, $\boxed{f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * \cdots * f_{X_N}(x_N)}$

<u>Proof</u> let $Y = X_1 + X_2 + X_3 + \cdots + X_N$

let us consider three random variables is

$$Y_2 = X_1 + X_2 + X_3$$

Consider $Y_1 = X_2 + X_3$

then $Y_2 = X_1 + Y_1$

$\because w = x + y$

$f_W(w) = f_X(x) * f_Y(y)$

$$f_{Y_2}(y_2) = f_{X_1}(x_1) * f_{Y_1}(y_1) \rightarrow ①$$

$$\Rightarrow Y_1 = X_2 + X_3 \rightarrow ②$$

$$f_{Y_1}(y_1) = f_{X_2}(x_2) * f_{X_3}(x_3)$$

Substitute eq ② in eq ①

$$f_{Y_2}(y_2) = f_{X_1}(x_1) * (f_{X_2}(x_2) * f_{X_3}(x_3))$$

$$f_{Y_2}(y_2) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3)$$

Now one more random variable is added i.e 4 R.V's.

Consider $Y_3 = X_1 + X_2 + X_3 + X_4$

$$Y_3 = Y_2 + X_4$$

$$f_{Y_3}(y_3) = f_{Y_2}(y_2) * f_{X_4}(x_4)$$

$$= (f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3)) * f_{X_4}(x_4)$$

$$\therefore f_{Y_3}(y_3) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) * f_{X_4}(x_4)$$

In general for N - random variables,

$$Y = X_1 + X_2 + X_3 + \cdots\cdots + X_N$$

$$\therefore f_Y(y) = f_{X_1}(x_1) * f_{X_2}(x_2) * f_{X_3}(x_3) + \cdots\cdots + f_{X_N}(x_N)$$

* Problems:

Q. The joint density function of random variables X and Y

$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & ; x \geq 0 \ \& \ y \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Verify is it a valid density function.

(ii) Find joint distribution function $F_{X,Y}(x,y) = ?$

(iii) Find marginal distribution functions $F_X(x) = ?$ & $F_Y(y) = ?$

(iv) Find marginal density functions $f_X(x) = ?$ & $f_Y(y) = ?$

(v) Check whether X and Y are independent or not.

(vi) Find $P(X \geq 1, Y > 3)$

(vii) Find $P(X > 1)$

(viii) Find $P(Y > 3)$

(ix) Find $P(X > 1 / Y > 3)$

(x) Find $P(Y > 3 / X \geq 1)$

(xi) $P(X \geq 1, Y \geq 3)$

(xii) $P(X \geq 1)$

(xiii) $P(Y < 3)$

(xiv) $P(X < 1 / Y < 3)$

(xv) $P(Y < 3 / X < 1)$

We know area under J. density function is unity

i.e

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1$$

$$\Rightarrow 0 + \int_{0}^{\infty} \int_{0}^{\infty} e^{-(x+y)}\, dx\, dy = \int_{0}^{\infty} \int_{0}^{\infty} e^{-x} \cdot e^{-y}\, dx\, dy$$

$$\Rightarrow \int_{0}^{\infty} e^{-y} \int_{0}^{\infty} e^{-x}\, dx\, dy$$

$$\Rightarrow \int_{0}^{\infty} e^{-y} \; -e^{-x}\Big|_{0}^{\infty} dy$$

$$= \int_0^\infty e^{-y} \; -e^{-x}\Big|_0^\infty \, dy$$

$$= -\int_0^\infty e^{-y} \; [e^{-\infty} - e^{-0}] \, dy$$

$$= -\int_0^\infty e^{-y} \; (-1) \, dy$$

$$= \int_0^\infty e^{-y} \, dy$$

$$= -e^{-y} \Big|_0^\infty$$

$$= -[e^{-\infty} - e^{-0}]$$

$$= -[0 - 1]$$

$$= -(-1)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

$\therefore$ The function is a valid density function.

(ii) $$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{0} (0)\,dx\,dy + \int_0^x\int_0^y e^{-(x+y)}\,dx\,dy$$

$$= \int_0^x\int_0^y e^{-x} \cdot e^{-y} \, dx\,dy$$

$$= \int_0^x e^{-x} \int_0^y e^{-y}\,dy\,dx$$

$$= \int_0^x e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^y dx$$

$$= \int_0^x \frac{e^{-x}[e^{-y} - e^{-0}]}{-1} \, dx$$

$$= \int_0^x \frac{e^{-x} e^{-y} - 1}{-1} \, dx$$

$$= \int_0^\infty e^{-x}(1 - e^{-y}) \, dx$$

$$= (1-e^{-y}) \int_0^x e^{-x} \, dx$$

$$= (1-e^{-y}) \left. \frac{e^{-x}}{-1} \right|_0^x$$

$$= (1-e^{-y}) \left(\frac{e^{-x} - 1}{-1} \right)$$

$$= (1-e^{-y}) (1-e^{-x})$$

$$\boxed{\therefore F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y})}$$

(iii) Marginal distribution function of $x = F_X(x) = F_{X,Y}(x,\infty)$

$$= \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy$$

We know from the above solution

$$F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y})$$

$$F_{X,Y}(x,\infty) = (1-e^{-x})(1-e^{-\infty})$$

$$\boxed{\therefore F_{X,Y}(x,\infty) = 1-e^{-x}} \; ; \; x \geq 0$$

Marginal distribution function of $Y = F_Y(y) = F_{X,Y}(\infty, y)$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{y} f_{X,Y}(x,y) \, dx \, dy$$

From the above solution

$$F_{X,Y}(\infty, y) = (1-e^{-\infty})(1-e^{-y})$$

$$\boxed{F_{X,Y}(\infty, y) = 1-e^{-y}} \; ; \; y \geq 0$$

(iv) Marginal density function of 'x' $= f_{x,y}(x,y) = f_x(x) = ?$

$$= \frac{d}{dx}[F_x(x)]$$

$$= \int_{-\infty}^{\infty} f_{x,y}(x,y)\, dy$$

$$= \frac{d}{dx}\left[1 - e^{-x}\right]$$

$$= 0 - \frac{(-1)e^{-x}}{1}$$

$$\boxed{\therefore f_x(x) = e^{-x}} \quad ; x \geq 0$$

Marginal density function of $Y = f_y(y)$

$$= \int_0^{\infty} e^{-(x+y)}\, dx + \int_{-\infty}^{0} (0)\, dx$$

$$= \int_0^{\infty} e^{-x} e^{-y}\, dx$$

$$= e^{-y} \int_0^{\infty} e^{-x}\, dx$$

$$= e^{-y}\left[\frac{e^{-\infty} - e^{-0}}{-1}\right]$$

$$= e^{-y}\left[\frac{0-1}{-1}\right]$$

$$\boxed{\therefore f_y(y) = e^{-y}} \quad ; y \geq 0$$

(v) We know condition for statistical independence of two random variables, i.e.

$$f_{x,y}(x,y) = f_x(x) * f_y(y)$$

$$= e^{-x} \cdot e^{-y} \quad ; x \geq 0, y \geq 0$$

$$f_x(x) * f_y(y) = f_{x,y}(x,y) = e^{-(x+y)} \quad ; x \geq 0, y \geq 0$$

Hence X, Y are independent variables statistically

(xi) $\cancel{P(x>1, y>3)}$ $P(x<1, y<3)$

$$P(X \le x, Y \le y) = P(X < x, Y < y) = F_{X,Y}(x,y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(x,y)\, dx\, dy$$

$$P(X < x, Y < y) = F_{X,Y}(x,y) = (1-e^{-x})(1-e^{-y}) \; ; \; x \ge 0, \; y \ge 0$$

$$\Rightarrow \boxed{P(x<1, y<3) = F_{X,Y}(1,3) = (1-e^{-1})(1-e^{-3})}$$

(xii) $P(x<1) \Rightarrow P(X \le x) = P(X < x) = F_X(x)$

$$P(X < x) = F_{X,Y}(x, \infty)$$

$$= F_X(x)$$

$$P(X < x) = (1 - e^{-x}) \; ; \; x \ge 0$$

$$\boxed{P(X<1) = 1 - e^{-1} \; ; \; x \ge 0}$$

(xiii) $P(Y<3) \Rightarrow P(Y \le y) = P(Y < y) = F_Y(y)$

$$P(Y < y) = F_{X,Y}(\infty, y) = F_Y(y) = 1 - e^{-y} \; ; \; y \ge 0$$

$$\boxed{P(Y<3) = 1 - e^{-3}}$$

(xiv) $P(x<1 / y<3) \Rightarrow P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$= \frac{P(X<1 \cap Y<3)}{P(Y<3)}$$

$$= \frac{F_{X,Y}(x,y)}{F_Y(y)}$$

$$P(X<x / Y<y) = \frac{F_{X,Y}(x,y)}{F_Y(y)}$$

$$P(X<1 / Y<3) = \frac{F_{X,Y}(1,3)}{F_Y(3)} = \frac{(1-e^{-1})(1-e^{-3})}{(1-e^{-3})}$$

$$\boxed{P(X<1 / Y<3) = 1 - e^{-1}}$$

(XV) $P(Y<3/X<1)$

$$P(Y<y/X<x) = \frac{F_{X,Y}(1,3)}{F_X(1)}$$

$$= \frac{(1-e^{-1})(1-e^{-3})}{(1-e^{-1})}$$

$$\boxed{P(Y<3/X<1) = 1-e^{-3}}$$

(Vi) $P(X>1, Y>3)$

$$P(X \geq x, Y \geq y) = P(X>x, Y>y) = \int_x^\infty \int_y^\infty f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_x^\infty \int_y^\infty e^{-(x+y)}\,dx\,dy$$

$$= \int_x^\infty \int_y^\infty e^{-x} \cdot e^{-y}\,dx\,dy$$

$$= \int_x^\infty e^{-x} \left[\int_y^\infty e^{-y}\,dy\right] dx$$

$$= \int_x^\infty e^{-x} \left[\frac{e^{-y}}{-1}\right]_y^\infty dx$$

$$= \int_x^\infty e^{-x} \left[\frac{e^{-\infty} - e^{-y}}{-1}\right] dx$$

$$= \int_x^\infty e^{-x} e^{-y}\,dx$$

$$= e^{-y} \int_x^\infty e^{-x}\,dx$$

$$P(X>x, Y>y) = e^{-x} e^{-y} \quad ; \quad x \geq 0, y \geq 0$$

$$\boxed{\therefore P(X>1, Y>3) = e^{-1} e^{-3}}$$

(X

(vii) $P(x>1)$

$$P(X \geq x) = P(X>x) = \int f_{X,Y}(x,y)\, dy\, x$$

$$= \int_{x}^{\infty} f_X(x)\, dx$$

$$= \int_{x}^{\infty} 1 \times e^{-x}\, dx$$

$$= \int_{x}^{\infty} e^{-x}\, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_{x}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-x}\right]$$

$$= -\left[0 - e^{-x}\right]$$

$$P(x>x) = e^{-x}$$

$$\boxed{P(x>1) = e^{-1}}$$

(or) $P(X \leq x) + P(X>x) = 1$

$F_X(x) + P(X>x) = 1$

$P(X>x) = 1 - F_X(x)$

$= 1 - (1 - e^{-x})$

$P(X>x) = e^{-x}$

(viii) $P(Y>1)$.

$$P(Y \leq y) + P(Y>y) = 1$$

$$F_Y(y) + P(Y>y) = 1$$

$$= 1 - F_Y(y)$$

$$= 1 - (1 - e^{-y})$$

$$P(Y>y) = e^{-y}$$

$$\boxed{\therefore P(Y>3) = e^{-3}}$$

(ix) $P(X>1/Y>3)$

$$\Rightarrow \frac{P(X>1, Y>3)}{P(Y>3)} = \frac{e^{-1} \cdot e^{-3}}{e^{-3}}$$

$$\Rightarrow \boxed{P(X>1/Y>3) = e^{-1}}$$

(x) $P(Y>3/X>1) = \dfrac{e^{-1} e^{-3}}{e^{-1}} = e^{-3}$

$$\boxed{P(Y>3/X>1) = e^{-3}}$$

→ And also find $f_{X/Y}(x/y)$ and $f_{Y/X}(y/x)$

We know $f_{X/Y} = \dfrac{f_{X,Y}(x,y)}{f_Y(y)} = \dfrac{e^{-x+y}}{e^{-y}} = e^{-x} = P(X>x)$

$$f_{Y/X} = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-x+y}}{e^{-x}} = e^{-y} = P(Y>y).$$

→ And also find $F_{X/Y}(x/y)$ and $F_{Y/X}(y/x)$

$$F_{X/Y}(x/y) = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{(1-e^{-x})(1-e^{-y})}{1-e^{-y}} = 1-e^{-x}$$

$$F_{Y/X}(y/x) = \frac{F_{X,Y}(x,y)}{F_X(x)} = \frac{(1-e^{-x})(1-e^{-y})}{1-e^{-x}} = 1-e^{-y}$$

3. The joint ~~distribution~~ density function of X and Y is

$$f(x,y) = \begin{cases} kxy & ; \ 0<x<y<1 \\ 0 & ; \ \text{otherwise} \end{cases}$$ Find

(i) constant k

(ii) Marginal density functions of X and Y.

~~(iii)~~

Sol: (i) We know area under density function unity i.e.,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1\int_0^y kxy\,dx\,dy = 1 \qquad \left\{\begin{matrix} 0<y<1 \\ \because 0<x<y<1 \\ 0<x<y \end{matrix}\right\}$$

$$\Rightarrow \int_0^1 ky\left(\int_0^y x\,dx\right)dy = 1$$

$$\Rightarrow \int_0^1 ky\left[\frac{x^2}{2}\right]_0^y dy = 1$$

$$\Rightarrow k\int_0^1 \frac{y^3}{2}\,dy = 1$$

$$\Rightarrow k \cdot \frac{y^4}{2 \cdot 4}\Big|_0^1 = 1$$

$$\Rightarrow \frac{k}{8}\left[1-0\right]$$

$$\Rightarrow \frac{k}{8} = 1$$

$$\boxed{\Rightarrow k = 8}$$

(2) Marginal density of $X = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$

$$= \int_x^1 kxy\, dy \qquad \because 0 < x < y < 1$$

$$= \int_x^1 8xy\, dy$$

$$= 8x \int_x^1 y\, dy$$

$$= 8x\left[\frac{y^2}{2}\right]_x^1$$

$$= 8x\left[\frac{1}{2} - \frac{x^2}{2}\right]$$

$$\boxed{\therefore f(x) = f_X(x) = 4x(1-x^2)}$$

Marginal density of $Y = [f_Y(y)] = \int_{-\infty}^{\infty} f(x,y)\, dx$

$$= \int_0^y 8xy\, dx \qquad \because 0 < x < y < 1, \quad 0 < x < y$$

$$= 8y \int_0^y x\, dy$$

$$= 8y\left[\frac{x^2}{2}\right]_0^y$$

$$= 8y\left[\frac{y^2}{2}\right]$$

$$= \frac{8y^3}{2}$$

$$\boxed{f(y) = f_Y(y) = 4y^3}$$

* PTSP Assignment *

1. For the given Joint density function

$$f(x,y) = \begin{cases} c(2x+y) & ; 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find the value of c $\int_0^2 \int_0^1 c(2x+y)\, dx\, dy = 1$

(ii) Joint distribution function $F(x,y) = \int_0^y \int_0^x f(x,y)\, dx\, dy$

(iii) Marginal distribution function $\int_0^2 \int_0^x dx\,dy$; $\int_0^y \int_0^1 dx\,dy$

(iv) Joint density functions. $f(x) = \int_0^2 c(2x+y)dy$; $f(y) = \int_0^1 c(2x+y)\,dx$

(V) Conditional distribution $F(x/y) = \frac{F(x,y)}{F(y)}$; $F(y/x) = \frac{F(x,y)}{F(x)}$;

(vi) Examine X and Y are indepent or not. $f(x,y) \neq f(x) * f(y)$ $f(x/y) = \frac{f(x,y)}{f(x)}$; $f(y/x) = \frac{f(x,y)}{f(y)}$

2. Joint density function is $f(x,y) = \begin{cases} b\, e^{-(x+y)} & ; 0 < x < a,\ 0 < y < \infty \\ 0 & ; \text{otherwise} \end{cases}$

(i) Find the constant 'b'.

(ii) Joint distribution functions.

(iii) Marginal density functions

3. $f(x,y) = \begin{cases} 5/16\, x^2 y & ; 0 < y < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$

(i) Is it valid PDF ?

(ii) Marginal density functions

4. $f(x,y) = \begin{cases} b\,(x+y)^2 & ; -2 < x < 2;\ -3 < y < 3 \\ 0 & ; \text{elsewhere} \end{cases}$

(i) Find constant 'b'

(ii) Marginal density functions of X and Y

5. A joint sample space for two random variables X and Y forms $(x y)$ has four elements $(1,1),(2,2),(3,3)$ & $(4,4)$. Probabilities of these events are 0.1, 0.35, 0.05 and 0.5 respectively

(i) Find the probability of event $\{X \le 2.5, Y \le 6\}$

$P(X=1, Y=1) + P(X=2, Y=2) = 0.1 + 0.35 = 0.45$

(ii) $P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = 0.1 + 0.35 + 0.05 = 0.5$

6. Given $f(x,y) = \frac{3}{16} x^2 y$; $0 < x < 2$, $0 < y < 2$

(i) Is it valid PDF?

(ii) Find marginal density functions of X and Y.

(iii) Examine X and Y independent or not.

(iv) Conditional density functions of X and Y

7. $f(x,y) = a(2x + y^2)$, $0 \le x \le 2$, $2 \le y \le 4$

(i) Find constant 'a'

(ii) Find $P(X \le 1, Y > 3)$ $\int_3^4 \int_0^1 a(2x+y^2)\, dx\, dy$

8. $f(x,y) = xy \exp\left[-\left(\frac{x^2+y^2}{2}\right)\right]$ $x > 0$, $y > 0$

$f_X(x) = x e^{-x^2/2}$; $x > 0$

(i) Examine X and Y are independent or not $f_Y(y) = y e^{-y^2/2}$; $y > 0$

(ii) $P(X \le 1, Y \le 1) = \int_0^1 \int_0^1 xy\, e^{-\left(\frac{x^2+y^2}{2}\right)} dx\, dy = \left(1 - \frac{1}{\sqrt{e}}\right)^2$

9. $f(x,y) = \frac{1}{4} e^{-|x| - |y|}$ $|x| = \begin{cases} x ; & x \ge 0 \\ -x ; & x < 0 \end{cases}$ $f_X(x) = \frac{1}{2} e^{-|x|}$

(i) Examine X and Y are independent are not. $f_Y(y) = \frac{1}{2} e^{-|y|}$

(ii) $P(-1 < x < 2, 0 < y < 2) = \frac{1}{4}\left[(1-e^{-1}) - (e^{-2}-1)\right](1-e^{-2})$

10. $f(x,y) = a x^2 y$; $0 < x < y < 1$

(i) Find constant $a = 10$

(ii) Marginal density functions $f_X(x) = 5x^2(1-x^2)$, $f_Y(y) = \frac{10}{3} y^4$.

* Central Limit Theorem:

Statement: Let us consider 'N' no of independent continous random varcables $X_n, n = 1, 2, 3, \ldots N$ having equal distributions and densities. Let $Y = \sum_{n=1}^{N} X_n = X_1 + X_2 + \cdots + X_N$

Now let us define the normalized random variable 'Z'

$$\boxed{Z = \frac{Y - \overline{Y}}{\sigma_Y}}$$ $\because \overline{Y} = E[Y] = \sum_{n=1}^{N} \overline{X_n}$

$$Var(Y) = Var(X_1) + Var(X_2) + \cdots + Var(X_N)$$

$$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \cdots + \sigma_{X_N}^2$$

$$\sigma_{X_1}^2 = \sigma_{X_2}^2 = \cdots = \sigma_{X_N}^2 = \sigma_X^2$$

$$\sigma_Y^2 = \sigma_X^2 + \sigma_X^2 + \cdots + N \text{ times}$$

$$\sigma_Y^2 = N\sigma_X^2$$

$$\boxed{\sigma_Y = \sqrt{N}\,\sigma_X}$$

$$\therefore Z = \frac{\sum_{n=1}^{N} X_n - \left(\sum_{n=1}^{N} \overline{X_n}\right)}{\sqrt{N}\,\sigma_X}$$

$$\boxed{Z = \frac{\sum_{n=1}^{N} \left[X_N - \overline{X_n}\right]}{\sqrt{N}\,\sigma_X}}$$

Hence, Z is a guassian random variable

* World Statement:

" The central limit theorem states that the density of 'N' number of independent, equally distributed random variables approaches the guassian density function as the limit $N \to \infty$".

* Characteristic function of Gaussian R.V: $\lim_{N\to\infty} \phi_z(\omega) = e^{-\omega^2/2}$

Proof: The characteristic function of random variable $Z = \phi_z(\omega)$

$$= E\left[e^{j\omega z}\right]$$

We know $Z = \sum_{n=1}^{N} \frac{(X_n - \bar{X}_n)}{\sqrt{N}\,\sigma_X}$

$$\phi_z(\omega) = E\left[e^{j\omega\left[\sum_{n=1}^{N} \frac{(X_n - \bar{X}_n)}{\sqrt{N}\,\sigma_X}\right]}\right]$$

$$= E\left[e^{j\omega\left[\frac{(X_1 - \bar{X}_1)}{\sqrt{N}\,\sigma_X} + \frac{(X_2 - \bar{X}_2)}{\sqrt{N}\,\sigma_X} + \cdots + \frac{[X_N - \bar{X}_N]}{\sqrt{N}\,\sigma_X}\right]}\right]$$

$$= E\left[e^{j\omega \frac{X_1 - \bar{X}_1}{\sqrt{N}\,\sigma_X}} \cdot e^{\frac{j\omega(X_2 - \bar{X}_2)}{\sqrt{N}\,\sigma_X}} \cdots e^{j\omega\frac{[X_N - \bar{X}_N]}{\sqrt{N}\,\sigma_X}}\right]$$

$$= E\left(e^{j\omega\frac{[X_1 - \bar{X}_1]}{\sqrt{N}\,\sigma_X}}\right) E\, e^{j\omega\frac{[X_2 - \bar{X}_2]}{\sqrt{N}\,\sigma_X}} \cdots E\left[e^{j\omega\frac{[X_N - \bar{X}_N]}{\sqrt{N}\,\sigma_X}}\right]$$

($\because$ If X and Y are independent then $E[XY] = E[X]E[Y]$)

Consider $E\left[e^{\frac{j\omega(X_1 - \bar{X}_1)}{\sqrt{N}\,\sigma_X}}\right]$ $\qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$$= E\left[1 + \frac{j\omega(X_1 - \bar{X}_1)}{\sqrt{N}\,\sigma_X} + \frac{\left(\frac{j\omega(X_1 - \bar{X}_1)}{\sqrt{N}\,\sigma_X}\right)^2}{2!} + \cdots\right]$$

$$= E(1)\ \frac{j\omega E[X_1 - \bar{X}_1]}{\sqrt{N}\,\sigma_X} + \frac{1}{2}\left[\frac{j\omega}{\sqrt{N}\,\sigma_X}\right]^2 E[X_1 - \bar{X}_1]^2 + E\left[\frac{R_N}{N}\right]$$

$\because E(1) = 1$

$$= 1 + \frac{j\omega}{\sqrt{N}\,\sigma_X} \times 0 + \frac{1}{2}\,\frac{-\omega^2}{N\cancel{\sigma_X^2}}\left[\cancel{\sigma_X^2}\right] + E\left[\frac{R_N}{N}\right]$$

$E[X - \bar{X}_1] = E[X_1] - \bar{X}_1 E(1) = \bar{X}_1 - \bar{X}_1 = 0$

$E[[X_1 - X_2]^2] = \sigma_X^2$

$$= 1 + E\left[\frac{R_N}{N}\right] - \frac{\omega^2}{2N}$$

$$= 1 - \frac{\omega^2}{2N} + \frac{E[R_N]}{N} \qquad \because E\left[\frac{R_N}{N}\right] = \frac{1}{N}E[R_N]$$

$X_1, X_2, X_3, \ldots X_N$ are equally distributed random variables then

$$E\left[e^{j\omega\frac{[X_1-\bar{X}_1]}{\sqrt{N}\sigma_X}}\right] = E\left[e^{j\omega\frac{[X_2-\bar{X}_2]}{\sqrt{N}\sigma_X}}\right] = \cdots = E\, e^{j\omega\frac{[X_N-\bar{X}_N]}{\sqrt{N}\sigma_X}}$$

$$= 1 - \frac{\omega^2}{2N} + E\left[\frac{R_N}{N}\right]$$

$$= 1 - \left[\frac{\omega^2}{2N} - E\left[\frac{R_N}{N}\right]\right]$$

$$\therefore \phi_Z(\omega) = \left(1 - \left(\frac{\omega^2}{2N} - E\left[\frac{R_N}{N}\right]\right)\right) \cdot 1 - \left[\frac{\omega^2}{2N} - E\left[\frac{R_N}{N}\right]\right] \cdots N \text{ times}$$

$$\phi_Z(\omega) = \left[1 - \left[\frac{\omega^2}{2N} - E\frac{(R_N)}{N}\right]\right]^N$$

Applying natural logarith on both sides we have

$$\ln[\phi_Z(\omega)] = \ln\left[1 - \left(\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right)\right]^N$$

$$\ln(\phi_Z(\omega)) = N \ln\left[\left[1 - \frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right]\right]$$

$$\ln(1-x) = -\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$= N\left[-\left[\left[\frac{\omega^2}{2N} - \frac{E(R_N)}{N}\right] + \frac{\left[\frac{\omega^2}{2N} - E\left(\frac{R_N}{N}\right)\right]^2}{2!} + \cdots\right]\right]$$

$$\boxed{\begin{aligned} \ln(\phi_Z(\omega)) &= -\frac{\omega^2}{2} + E[R_N] + \frac{\left[\frac{\omega^2}{2} - E[R_N]\right]^2}{2N} + \cdots \\ \Rightarrow \phi_Z(\omega) &= \mathrm{Exp}\left(-\frac{\omega^2}{2} + E[R_N] + \frac{\left[\frac{\omega^2}{2} - E[R_N]\right]^2}{2N} + \cdots\right) \end{aligned}}$$

Apply $\lim_{N\to\infty}$, we have

$$\underset{N\to\infty}{Lt}\ \phi_Z(\omega) = \underset{N\to\infty}{Lt}\left[\text{Exp}\left(\frac{-\omega^2}{2} + E[R_N] + \frac{\frac{\omega^2}{2} - E[R_N]^2}{2N} + \cdots\right)\right]$$

$$= \text{Exp}\left[\frac{-\omega^2}{2} + 0 + 0 + 0 + \cdots\right]$$

$$\Rightarrow \underset{N\to\infty}{lt}\ \phi_Z(\omega) = e^{-\omega^2/2}$$

Hence proved.

* Operation On Single Random Variable :

→ Expected or Mean or Average value of R.V. X :

The expected value of random variable 'x' is defined as $= E[X] = \bar{X} = m = \mu = m_x = \mu_x = a_x$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

For discrete,

$$E(X) = \bar{X} = \sum_{i=1}^{N} x_i \, P(x_i)$$

→ Expected Value of function of random variable :

The expected value of function $g(x)$ of R.V. 'X' is defined as

$$= E[g(x)] = \overline{g(x)} = m = \mu = m_x = \mu_x = a_x = \int_{-\infty}^{\infty} g(x) f_X(x)\, d$$

For discrete

$$E[g(x)] = \overline{g(x)} = \sum_{i=1}^{N} g(x_i)\, P(x_i)$$

Properties:

1. $E[1] = 1$
2. $E[ax + b] = a\,E[X] + b$
3. $E[a_1 g_1(x) + a_2 g_2(x)] = a_1 E[g_1(x)] + a_2 E[g_2(x)]$
4. $E[kx] = K\,E[X]$

Moments:

There are two types (i) Moments about origin

(ii) Moments about mean or Central moments.

(i) Momem about Origine.

The expected value of function $g(x) = x^n$ of R.V. 'X' with PDF $f_X(x)$ is called as 'nth order moment about origin.

n^{th} order moment about origin of 'X' = m_n

$$= E[X^n]$$

$$= \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

Here n represents the order of the moments.

For zero order, $n=0$; $m_0 = E[X^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\, dx$

$$m_0 = E[1] = \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

For first order, $n=1$; $m_1 = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$

$$m_1 = E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

(1^{st} order moment - Expected or average value of X)

For second order, $n=2$; $m_2 = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$

(2^{nd} order moment - Mean square value of X)

(ii) Central Moments or Moments about Mean:

The expected value of function $g(x) = (x-\bar{X})^n$ of R.V. 'X' with PDF $f_X(x)$ is called as 'nth order moment about mean (or) central moment.

n^{th} order about mean of 'X' $= \mu_n = E[(X-\bar{X})^n] = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\,dx$

$n=0 \Rightarrow \mu_0 = E((X-\bar{X})^0) = \int_{-\infty}^{\infty} (x-\bar{x})^0 f_X(x)\,dx = 1$

$n=1 \Rightarrow \mu_1 = E[(X-\bar{X})] = E(X) - \bar{X}E(1) = \bar{X} - \bar{X} = 0$

$\mu_1 = 0 \rightarrow$ 1st order central moment

$n=2 \Rightarrow \mu_2 = E[(X-\bar{X})^2] = E[X^2] - E[\bar{X}]^2 \rightarrow$ 2nd order central moment

*Variance:

The second order central moment or second order about mean of random variable 'X' is known as variance of X.

$$Var(X) = \sigma_X^2 = \mu_2 = E[(X-\bar{X})^2]$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\,dx$$

For discrete, $\sigma_X^2 = \sum_{i=1}^{N} (x_i - \bar{X})^2 P(x_i)$

*Properties:

1. $Var(kX) = k^2 Var(X)$
2. $Var(k) = 0$
3. $Var(aX+b) = a^2 Var(X)$

* Skew :

The third order central moment is called as skew of random variable 'X'.

Skew of 'X' $= \mu_3 = E[(X-\bar{X})^3] = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_X(x)\,dx$

$$Skew = \mu_3 = \sum_{i=1}^{N} (x_i - \bar{X})^3 P(x_i)$$

* Coefficient of Skewness:

The normalised thirdorder central moment or the ratio of the skew to the cube of the standard deviation is called as coeff. of skewness.

$$\text{Coeff. of skewness} = \frac{\text{skew}}{(S.D)^3} = \frac{\mu_3}{\sigma_X^3} = \frac{E[(X-\overline{X})^3]}{\sigma_X^3}$$

$$= \frac{E[(X-\overline{X})^3]}{E[(X-\overline{X})^2]^{3/2}}$$

$\because \sigma_X^2 = E[(X-\overline{X})^2] = var(X)$

$\sigma_X = [E(X-\overline{X})^2]^{1/2}$

* Standard Deviation:

Standard deviation of X is defined as the squareroot of var (X)

$$\sigma_X = S.D. \text{ of } 'X' = \sqrt{Var(X)} = \sqrt{E[(X-\overline{X})^2]}$$

→ Moments can be calculated by using 2 function.

(a) Characteristic function ($\phi_X(\omega)$):

The characteristic function of randomvariable X is defined as expected value of $e^{j\omega X}$

$$= \phi_X(\omega) = E[e^{j\omega X}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

The PDF of 'X' $= f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\phi_X(\omega)]\, e^{-j\omega x}\, d\omega$

$$f_X(x) \xleftrightarrow{F.T} \phi_X(\omega)$$

The PDF and characteristic function both are fourier transform with sign reversal of R.V. X

(b) Moment Generating Function ($M_X(t)$):

MGF of 'X' $= M_X(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} f_X(x) e^{tx}\,dx$

Problem:

1. The probability of R.V. 'X' is as show in table. Find (i) $E[X]$ (ii) $E[2x+3]$ (iii) $E[X^2]$ (iv) $E[(3X+1)^2]$.
(v) $E[-5X^2+2X-1]$.

X	-2	-1	0	1	2	3
P	1/10	2/10	2/10	1/10	2/10	2/10

Sol: (i) $E[X] = \mu = \sum_{i=1}^{N} x_i P(x_i) = \sum_{i=1}^{6} x_i P(x_i)$

$= x_1 P(x_1) + x_2 P(x_2) + \cdots + x_6 P(x_6)$

$= -2 \times \frac{1}{10} + -1 \times \frac{2}{10} + 0 + 1 \times \frac{1}{10} + 2 \times \frac{2}{10} + 3 \times \frac{2}{10}$

$E[X] = \frac{7}{10} = 0.7$

(ii) $E[X^2] = \sum_{i=1}^{6} x_i^2 P(x_i)$

$= x_1^2 P(x_1) + x_2^2 + \cdots + x_6^2 P(x_6)$

$= \frac{4}{10} + \frac{2}{10} + 0 + \frac{1}{10} + \frac{4 \times 2}{10} + \frac{9 \times 2}{10}$

$= 32/10 = 3.2$

(iii) $E[2x+3] = 2E[X] + 3$

$= 2 \cdot \frac{7}{10} + 3$

$= \frac{7+15}{5}$

$E[2x+3] = \frac{23}{5} = 4.6$

(v) $E[-5X^2+2x-1]$

$= -5E[X^2] + 2E[X] - 1$

$= -5 \times 3.2 + 2 \times 0.7 - 1$

$= -15.6$

(iii) $E[(3X+1)^2] = E[9X^2+1+6X] = 9E[X^2]+1+6\cdot E[X]$

$= 34.$

2. Prove that sum of density of two gaussian random variables is a gaussian density function.

OR

let X and Y are two gaussian random variables. Find density of random variable $Z = X+Y$.

OR

let X and Y are two random variables with zero mean and unit variance. Then find density of random variable such that $Z = X+Y$.

Sol: Given X and Y are two gaussian R.V so the

PDF of gaussian r.v of $X = f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; $x \geq 0$

" " " " $Y = f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$; $y \geq 0$

Here random variable with zero mean and unit variance means that normalized gaussian density function.

We know the density of R.V. $Z = X+Y$

$$f_Z(z) = f_X(x) * f_Y(y)$$

The PDF of $Z = f_Z(z) = \int_{-\infty}^{\infty} f_X(y) f_Y(z-y) \, dy$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\right) \left(\frac{1}{\sqrt{2\pi}} e^{-(z-y)^2/2}\right) dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \cdot e^{-\frac{(z^2+y^2-2zy)}{2}} \, dy$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{-\frac{y^2}{2}+zy-\frac{y^2}{2}}\,dy$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{zy-y^2}\,dy$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-(y^2-zy)}\,dy$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-(y^2-zy)}\,dy$$

$$\left\{ \begin{aligned} y^2 - zy &= (y)^2 - 2y\left(\tfrac{z}{2}\right) + \left(\tfrac{z}{2}\right)^2 - \left(\tfrac{z}{2}\right)^2 \\ y^2 - zy &= \left(y - \tfrac{z}{2}\right)^2 - \left(\tfrac{z}{2}\right)^2 \end{aligned} \right.$$

$$= \frac{e^{-z^2/2}}{2\pi}\int_{-\infty}^{\infty} e^{-\left[\left(y-\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^2\right]}\,dy$$

$$= \frac{e^{-z^2/2+\frac{z^2}{4}}}{2\pi}\int_{-\infty}^{\infty} e^{-\left(y-\frac{z}{2}\right)^2}\,dy$$

Put $y - \frac{z}{2} = t$

$dy = dt$

$y \to -\infty \Rightarrow t = -\infty$

$y \to \infty \Rightarrow t = \infty$

$$= \frac{e^{-z^2/4}}{2\pi}\int_{-\infty}^{\infty} e^{-t^2}\,dt \quad (\text{even function})$$

$$= \frac{e^{-z^2/4}}{2\pi}\, 2\int_{0}^{\infty} e^{-t^2}\,dt \qquad \left\{ \because \int_0^{\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} \right\}$$

$$= \frac{e^{-z^2/4}}{2\pi} \cdot 2 \cdot \frac{\sqrt{\pi}}{2}$$

$$\boxed{f_Z(z) = \frac{e^{-z^2/4}}{2\sqrt{\pi}}}$$

Hence the density of R.V. 'Z' is a gaussian density function.

* Operation on Multiple R.V's:

→ Expected Value of Function of R.V's:

M.P

The expected value of function $g(X,Y)$ of random variable X and Y with joint PDF $f_{X,Y}(x,y)$ is defined as

$$\boxed{E[g(X,Y)] = \bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy}$$

For discrete random variables,

$$E[g(x_n, y_m)] = \bar{g} = \sum_{\langle n \rangle}\sum_{\langle m \rangle} g(x_n, y_m)\, P(x_n, y_m)$$

For 'N' random variables,

$$E[g(X_1, X_2, \ldots X_N)] = \bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2, \ldots x_N) f_{X_1,X_2,\ldots X_N}(x_1, x_2, \ldots x_N)\, dx_1\, dx_2 \ldots dx_N$$

Q.T

* Properties of Expectation or Mean or Average value of R.V's:

1. $E[C\cancel{X}X] = C\, E[X]$

Proof: $E[CX] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cx\, f_{X,Y}(x,y)\, dx\, dy$

The above one is from $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$

Let $g(X,Y) = CX$

$$= C \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx\, dy$$

$$= C\; E[X] \qquad \because E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx\, dy$$

2. $E[aX+b] = aE[X]+b$

Proof: Wkt, $E[g(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$

Put $g(x,y) = ax+b$

$$E[ax+b] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+b)\, f_{X,Y}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx\,dy + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= aE[X] + b * 1$$

$$= aE[X] + b$$

3. $E[aX+bY] = aE[X] + b[E(Y)]$

Proof - Put $g(x,y) = ax+by$

$$E[ax+by] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by)\, f_{X,Y}(x,y)\,dx\,dy$$

$$= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dx\,dy + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= aE[X] + bE[Y]$$

lly $a=b=1$ then

$$E[X+Y] = E(X) + E[Y]$$

$$E[X-Y] = E[X] - E[Y]$$

4. $E[a_1 g_1(x,y) + a_2 g_2(x,y)] = a_1 E[g_1(x,y)] + a_2 E[g_2(x,y)]$

Proof: Put $g(x,y) = a_1 g_1(x,y) + a_2 g_2(x,y)$

$$E[a_1 g_1(x,y) + a_2 g_2(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_1 g_1(x,y)\, f_{X,Y}\,dx\,dy + a_2 g_2(x,y)$$

$$= a_1 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x,y)\, f_{X,Y}(x,y)\, dx\, dy + a_2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= a_1 E[g_1(X,Y)] + a_2 E[g_2(X,Y)].$$

* Joint Moments:

(i) Moments about origin

(ii) Joint Central moments or Moments about mean

(i) Joint Moments about Origin:

M.P The expected value of function $g(X,Y) = X^n Y^k$ of two R.V's X and Y with joint PDF $f_{X,Y}(x,y)$ is ~~defined~~ called as $(n+k)$ order joint moment about origin.

$$E[g(X,Y)] = \bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Here $g(x,y) = x^n y^k$

$m_{nk} = (n+k)$ order joint moment about origin $= E[X^n Y^k]$.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\, dx\, dy$$

Here n and k are +ve integers ; n+k is order of the joint moments.

* For zero order joint moments:

$$n = k = 0$$

$$m_{00} = E[X^0 Y^0] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^0 y^0 f_{X,Y}(x,y)\, dx\, dy$$

$$m_{00} = E[1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy$$

$$\therefore m_{00} = E[1] = 1$$

y

* For 1st order joint moments:

$\Rightarrow n=1, k=0 \;,\; n+k=1$

$$m_{10} = E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\, dx\, dy$$

→ Mean or average value of 'X'

$\Rightarrow n=0, k=1, n+k=1$

$$m_{01} = E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)\, dx\, dy$$

→ Mean or average value of 'Y'

* For 2nd order joint Moments:

$n=2, k=0 \;;\; n+k=2$

$$m_{20} = E[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\, dx\, dy$$

→ Mean square value of 'X'

$n=0, k=2 \;;\; n+k=2$

$$m_{02} = E[Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{X,Y}(x,y)\, dx\, dy$$

→ Mean square value of 'Y'

$n=1, k=1 \;;\; n+k=2$

$$m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

→ Correlation of 'X' 'Y'.

* For 'N' number of R.V's

$$E[X_1^{n_1} X_2^{n_2} \cdots X_N^{n_N}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x_1^{n_1} x_2^{n_2} \cdots x_N^{n_N} f_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N)\, dx_1\, dx_2 \cdots dx_N$$

Here $n_1, n_2, \dots, n_N$ are integers

$n_1 + n_2 + \dots + n_N$ → order of joint moment

about origin.

* Correlation of r.v's X & Y:

The second order joint moment about origin is called correlation b/w r.v's X and Y, i.e.,

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\,dy$$

(ii) Joint Central Moments:

M.P The joint central moment (expected value) of function $g(X,Y)$ is given as

$$g(X,Y) = (X-\bar{X})^n \cdot (Y-\bar{Y})^k$$ of random variable X, Y with joint PDF $= f_{X,Y}(x,y)$ is called $(n+k)$ order joint central moments i.e.,

$$(n+k) \text{ order JCM} = \mu_{nk} = E[(X-\bar{X})^n (Y-\bar{Y})^k]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{X,Y}(x,y)\, dx\, dy$$

Here n, k are integers and $(n+k)$ represent order of JCM.

→ Zero order JCM:

$$\mu_{00} = E[1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

$$\mu_{00} = 1$$

→ First order JCM:

$$\mu_{10} = E[(X-\bar{X})^1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^n f_{X,Y}(x,y)\, dx\, dy$$

$$E(X-\bar{X}) = E[X] - \bar{X}\,E(1)$$

$$= \bar{X} - \bar{X}$$

$$\because E(X) = \bar{X}$$

$$\therefore \mu_{10} = 0$$

$$\mu_{01} = E[(Y-\bar{Y})^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\bar{y})^k f_{X,Y}(x,y)\, dx\, dy$$

$$= \bar{Y} - \bar{Y} = 0$$

∴ First order JCM's are absolutely zero.

→ Second order JCM:

$$\mu_{20} = E[(X-\bar{X})^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})^2 f_{X,Y}(x,y)\,dx\,dy \rightarrow var(X)$$

$$\mu_{02} = E[(Y-\bar{Y})^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y-\bar{Y})^2 f_{X,Y}(x,y)\,dx\,dy \rightarrow var(Y)$$

$$\mu_{11} = E[(X-\bar{X})(Y-\bar{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})(y-\bar{Y}) f_{X,Y}(x,y)\,dx\,dy \rightarrow \text{covariance of } X \,\&\, Y$$

→ For 'N' no. of random variables:

$$E[(X_1-\bar{X}_1)^{n_1}(X_2-\bar{X}_2)^{n_2}\cdots(X_N-\bar{X}_N)^{n_N}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (x_1-\bar{X}_1)^{n_1}(x_2-\bar{X}_2)^{n_2}\cdots(x_N-\bar{X}_N)^{n_N} f_{X_1,X_2\ldots X_N}(x_1,x_2,\ldots x_N)\,dx_1\,dx_2\cdots dx_N$$

Here $n_1, n_2 \ldots$ are the +ve integers and $n_1+n_2+\cdots+n_N$ order of the joint central moments

* Properties of Correlation:

1. If X and Y are two statistically independent random variables, then two are said to be unco-rrelated random variables.

$$R_{XY} = E[XY] = E[X]\,E[Y]$$

Proof: Correlation b/w X and Y $= R[XY] = E[XY]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

We know that if X and Y are statistically independent,

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$$

$$R_{X,Y} = E[XY]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E[X]\, E[Y] \qquad \left\{ \begin{array}{l} \because E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \\ E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\, dy \end{array} \right.$$

Hence proved.

2. If X and Y are two orthogonal R.V's, then correlation b/w two R.V's X and Y is zero.

$$R_{XY} = E[XY] = 0$$

<u>Proof</u>: Correlation b/w X and Y if they are orthogonal

$$= R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y\, f_{XY}(x,y)\, dx\, dy$$

If X and Y are orthogonal means that their joint probability occurence is zero i.e.,

$$f_{X,Y}(x,y) = 0$$

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,(0)\, dx\, dy$$

$$R_{XY} = E[XY] = 0$$

Hence proved.

P: Let X and Y are two R.V's such that $Y = -4X + 20$. The mean and variance of X is 4 and 2, respectively. Find the correlation and comment on the result.

Sol: Given, $Y = -4X + 20$

Mean of 'X' $= E[X] = \bar{X} = m = 4$

Variance of 'X' $= Var[X] = \sigma_X^2 = 2$

Correlation b/w X and Y $= R_{XY} = E[XY]$

$= E[X[-4X+20]]$ $\quad \because E[aX+bY] = aE[X] + bE[Y]$

$= E[-4X^2 + 20X]$

$= -4\,E[X^2] + 20\,E[X]$

From previous results,

$Var(X) = \sigma_X^2 = E[(X-\bar{X})^2] = E[X^2] - E[(X)]^2$

$2 = E[X^2] - (4)^2$

$E(X^2) = 18$

$\therefore \quad R_{XY} = -4(18) + 20(4)$

$= -72 + 80$

$\therefore R_{XY} = 8$

Condition for uncorrelated or independent R.V's:

$R_{XY} = E[XY] = E[X]\,E[Y]$

$E[X] = 4$

$E[Y] = E[-4X + 20] = -4\,E[X] + 20\,E[1]$

$E[Y] = -4(4) + 20 = 4$

$E[X]\,E[Y] = 4 \times 4 = 16$

$R_{XY} = 8$

$R_{XY} = E[XY] \neq E[X]\,E[Y]$

Hence X and Y are not independent ~~and~~ or ~~are not~~ uncorrelated R.V's. and $E[XY] \neq 0$, therefore, X and Y are not orthogonal R.V's.

→ Comment:

Therefore, X and Y are neither independent nor orthogonal R.V's.

* Co-Variance:

The second order joint central moment is called Co-variance of R.V's X and Y.

Covariance of X & Y $= \underline{Cov(X,Y) = C_{XY}} = \sigma_{XY} = \mu_{11} = E[(X-\bar{X})(Y-\bar{Y})]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{X})(y-\bar{Y})\, f_{X,Y}(x,y)\, dx\, dy$$

* Correlation Coefficient (ρ):

The normalised second order joint central moment is called correlation coefficient (ρ).

Correlation coefficient $= \rho = \dfrac{\mu_{11}}{\sqrt{\mu_{02}\,\mu_{20}}}$

$$= \frac{E[(X-\bar{X})(Y-\bar{Y})]}{\sqrt{\sigma_X^2 \cdot \sigma_Y^2}}$$

$$= \frac{C_{XY}}{\sqrt{\sigma_X^2\, \sigma_Y^2}}$$

$$\rho = \frac{C_{XY}}{\sigma_X\, \sigma_Y}$$

or

$$\boxed{\rho = \frac{E[(X-\bar{X})(Y-\bar{Y})]}{\sqrt{\underbrace{E[(X-\bar{X})^2]}_{\mu_{20}}\; \underbrace{E[(Y-\bar{Y})^2]}_{\mu_{02}}}}}$$

*Note:

1. The range of correlation coefficient is $-1 \leq \rho \leq 1$.
2. If X and Y are independent statistically then
$$\rho = 0$$
3. If the correlation b/w X and Y is perfect then
$$\rho = \pm 1$$
4. If $X = Y$ then $\rho = 1$

S.T

* Properties of Co-Variance:

1. If X and Y are two R.V's, then the co-variance

$$C_{XY} = R_{XY} - E[X]\,E[Y] = E[XY] - E[X]E[Y]$$

Proof:

Covariance of X and Y = Cov(X, Y)

$\because E[aX+bY] = aE[X] + bE[Y]$

$E[\bar{X}\bar{Y}] = \bar{X}\,\bar{Y}\,E(1) = \bar{X}\,\bar{Y}\cdot 1$

$$C_{XY} = E\left[(X-\bar{X})(Y-\bar{Y})\right]$$

$$= E\left[XY - X\bar{Y} - \bar{X}Y + \bar{X}\,\bar{Y}\right]$$

$$= E[XY] - E[X\bar{Y}] - E[\bar{X}Y] + E[\bar{X}\,\bar{Y}]$$

$$= E[XY] - \bar{Y}\,E[X] - \bar{X}\,E[Y] + \bar{X}\,\bar{Y}\,E[1]$$

$$= E[XY] - \bar{Y}\,\bar{X} - \cancel{\bar{X}\,\bar{Y}} + \cancel{\bar{X}\,\bar{Y}}$$

$$= E[XY] - \bar{X}\,\bar{Y}$$

We know $R_{XY} = E[XY]$

$$\therefore C_{XY} = R_{XY} - \bar{X}\,\bar{Y}$$

$$= E[XY] - \bar{X}\,\bar{Y}$$

Here $\bar{X} = E[X]$, $\bar{Y} = E[Y]$

$$\therefore C_{XY} = R_{XY} - E[X]E[Y] = E[XY] - E[X]E[Y]$$

2. If X and Y are statistically independent, then the Covariance is zero, i.e. two are uncorrelated R.V's.

$$C_{XY} = 0$$

Proof: Covariance of X & Y $= R_{XY} - E[X]E[Y]$

$$= E[XY] - E[X]E[Y]$$

We know, if X and Y are independent

$$R_{XY} = E[XY] = E[X]E[Y]$$

$$\therefore C_{XY} = E[X]\cancel{E[Y]} - E\cancel{[X]}E[Y] = 0$$

3. $Cov(X+a, Y+b) = Cov(X, Y) = C_{XY}$

Proof: The Covariance of X & Y $= C_{XY} = E[(X-\bar{X})(Y-\bar{Y})]$

$$Cov(X+a, Y+b) = E\left[\left((X+a) - (\overline{X+a})\right)\left((Y+b) - (\overline{Y+b})\right)\right]$$

$$\overline{X+a} = E[X+a] = E[X] + a\,E[1] = E[X] + a$$

$$\therefore \overline{X+a} = \bar{X} + a$$

$$\overline{Y+b} = E[Y+b] = E[Y] + b\,E[1] = E[Y] + b$$

$$\therefore \overline{Y+b} = \bar{Y} + b$$

$$= E\left[\left((X+a) - (\bar{X}+a)\right)\left((Y+b) - (\bar{Y}+b)\right)\right]$$

$$= E\left[(X + \cancel{a} - \bar{X} - \cancel{a})(Y + \cancel{b} - \bar{Y} - \cancel{b})\right]$$

$$= E\left[(X - \bar{X})(Y - \bar{Y})\right]$$

$$= Cov(X, Y)$$

$$= C_{XY}$$

Hence proved

4. $Cov(aX, bY) = ab\, Cov(X, Y)$ or

$C_{aX\,bY} = ab\, C_{XY}$

Proof: Covariance of X & Y = $C_{XY} = E[(X-\overline{X})(Y-\overline{Y})]$

$$Cov(aX, bY) = E[(aX-\overline{aX})(bY-\overline{bY})]$$

$$\overline{aX} = E[aX] = a\,E[X] = a\overline{X}$$

$$\overline{bY} = E[bY] = b\,E[Y] = b\overline{Y}$$

$$Cov(aX, bY) = E[(aX-a\overline{X})(bY-b\overline{Y})]$$

$$= E[ab\,(X-\overline{X})(Y-\overline{Y})]$$

$$= ab\,E[(X-\overline{X})(Y-\overline{Y})] \qquad \because E[kX] = kE[X]$$

$$= ab\,Cov(X, Y)$$

$$= ab\,C_{XY}.$$

5. $Cov(X+Y, Z) = Cov(X, Z) + Cov(Y, Z)$

Proof: Covariance of X & Y = $C_{XY} = E[(X-\overline{X})(Y-\overline{Y})]$

$$C_{(X+Y,Z)} = E[((X+Y)-(\overline{X+Y}))(Z-\overline{Z})]$$

$$\overline{X+Y} = E[X+Y] = E[X] + E[Y] = \overline{X}+\overline{Y}$$

$$\overline{Z} = E[Z] = \overline{Z}$$

$$\therefore C_{(X+Y,Z)} = E[((X+Y)-(\overline{X}+\overline{Y}))(Z-\overline{Z})]$$

$$= E[((X-\overline{X})+(Y-\overline{Y}))(Z-\overline{Z})]$$

$$= E[(X-\overline{X})(Z-\overline{Z}) + (Y-\overline{Y})(Z-\overline{Z})]$$

$$= E[(X-\overline{X})(Z-\overline{Z})] + E[(Y-\overline{Y})(Z-\overline{Z})]$$

$$= Cov(X, Z) + Cov(Y, Z)$$

6. Express Theorems:

(i) $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

or

V.imp

$$\sigma^2_{X+Y} = \sigma_X^2 + \sigma_Y^2 + 2\,C_{XY}$$

Proof: Variance of 'X' = $Var(X) = \sigma_X^2 = E[(X-\bar{X})^2]$

$$= E[X^2] - [E[X]]^2$$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$Var(X+Y) = E[(X+Y)^2] - [E(X+Y)]^2$$

$$= E[X^2+Y^2+2XY] - [E(X)+E(Y)]^2$$

$$= E(X^2) + E[Y^2] + 2\,E[XY] - [E(X)]^2 + [E(Y)]^2 + 2\,E[X]\,E[Y]$$

$$= E[X^2] - [E(X)]^2 + E[Y^2] - [E(Y)]^2 + 2\,[E(XY) - E[X]E[Y]]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

$$= Var(X) + Var(Y) + 2C_{XY}$$

$$\therefore Var(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2C_{XY}$$

Hence proved

(ii) $Var(X-Y) = Var(X) + Var(Y) - 2\,Cov(X,Y)$

$$\sigma^2_{X-Y} = \sigma_X^2 + \sigma_Y^2 - 2C_{XY}$$

Proof: Variance of 'X' = $Var(X) = \sigma_X^2 = E[(X-\bar{X})^2]$

$$Var(X) = E[X^2] - [E(X)]^2$$

$$Var(X-Y) = [E(X-Y)^2] - [E(X-Y)]^2$$

$$= E[X^2+Y^2-2XY] - [E(X)-E(Y)]^2$$

$$= E[X^2] + E[Y^2] - 2E[XY] - \left([E(X)]^2 + E[Y]^2 - 2E[X]E[Y]\right)$$

$$= E[X^2] - [E(X)]^2 + E[Y^2] [E(Y)]^2 - 2[E(XY) - E[X]E(Y)]$$

$$= Var(X) + Var(Y) - 2\,Cov(X,Y)$$

$$Var(X-Y) = \sigma_X^2 + \sigma_Y^2 - 2C_{XY}$$

Hence proved.

(iii) $Var(aX+bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)$

$$\sigma^2_{aX+bY} = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,C_{XY}$$

Sol: Proof: $Var[X] = E[X^2] - (E[X])^2$

$$Var(aX+bY) = E[(aX+bY)^2] - [E(aX+bY)]^2$$

$$= E[a^2X^2 + b^2Y^2 + 2abXY] - [[E(aX)]^2 + [E(bY)]^2 + 2E(aX)E(bY)]$$

$$= E[a^2X^2] - [E(aX)]^2 + E[b^2Y^2] - [E[(bY)]]^2 + 2ab[E(XY) - E[X]E[Y]]$$

$$= a^2[E[X^2] - E[X]^2] + b[E(Y^2) - [E(Y)]^2] + 2ab[E(XY) - E(X)E(Y)]$$

$$= a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X,Y)$$

$$= a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\,C_{X,Y}$$

(iv) $Var(ax - by) = a^2 Var(X) + b^2 Var(Y) - 2ab\,Cov(X,Y)$

$$\sigma^2_{(ax-by)} = a^2 \sigma_x^2 + b^2 \sigma_y^2 - 2ab\,C_{xy}$$

Proof: $Var(X) = E[X^2] - [E[X]]^2$

$$Var(ax - by) = E[(ax-by)^2] - [E(ax-by)]^2$$

$$= E[a^2x^2 + b^2y^2 - 2abxy] - [[E(ax)]^2 + [E(by)]^2 - 2ab\,E[X][Y]]$$

$$= a^2[E[X^2] - [E(X)]^2] + b^2[E[Y^2] - [E(Y)]^2] - 2ab[E(XY) - E(X)(Y)]$$

$$= a^2 Var(X) + b^2 Var(Y) - 2ab\,Cov(X,Y)$$

$$= a^2 \sigma_x^2 + b^2 \sigma_y^2 - 2ab\,C_{xy}$$

Problems:

2. The joint PDF of random variables X and Y is

Given $f_{X,Y}(x,y) = \begin{cases} \frac{1}{100}; & 0 \le x \le 1,\ 0 \le y \le 2. \\ 0; & \text{elsewhere} \end{cases}$

(i) Find $E[X]$ (ii) $E(Y)$ (iii) $E(XY)$ (iv) $E[X^2]$ (v) $E[Y^2]$

(vii) $E[XY^2]$ (vi) $E[X^2Y]$ (viii) $E[X+Y]$ (ix) $E[X-Y]$

Sol: (i) $E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dx\,dy$

$$= \int_0^2 \int_0^1 x \left(\frac{1}{100}\right) dx\,dy$$

$$= \frac{1}{100} \int_0^2 1\,dy \int_0^1 x\,dx$$

$$= \frac{1}{100} [y]_0^2 \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{100}\left[2-0\right]\left[\frac{1^2}{2}-\frac{0^2}{2}\right]$$

$$= \frac{1}{100} \times \cancel{2} \times \frac{1}{\cancel{2}}$$

$$\boxed{\therefore E[X] = \frac{1}{100}}$$

(ii) $E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \, f_{X,Y}(x,y) \, dx \, dy$

$$= \int_0^2\int_0^1 y \left(\frac{1}{100}\right) dx \, dy$$

$$= \int_0^2 y \, dy \int_0^1 \frac{1}{100} \, dx$$

$$= \frac{1}{100}\left[\frac{y^2}{2}\right]_0^2 \left[x\right]_0^1$$

$$= \frac{1}{200}\left[2^2\right] \cdot 1$$

$$= \frac{4}{200}$$

$$\boxed{\therefore E[Y] = \frac{1}{50}}$$

(iii) $E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{X,Y}(x,y) \, dx \, dy$

$$= \int_0^2\int_0^1 xy \left(\frac{1}{100}\right) dx \, dy$$

$$= \frac{1}{100}\int_0^2 y \, dy \int_0^1 x \, dx$$

$$= \frac{1}{100}\left[\frac{y^2}{2}\right]_0^2 \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{100}\left[\frac{4}{2}\right]\left[\frac{1}{2}\right]$$

$$= \frac{4}{400}$$

$$\boxed{\therefore E[XY] = \frac{1}{100}}$$

(iv) $E[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 \, f_{X,Y}(x,y) \, dx \, dy$

$$= \int_0^2 \int_0^1 x^2 \cdot \frac{1}{100} \, dx \, dy$$

$$= \frac{1}{100} \int_0^2 y \cdot dy \cdot \int_0^1 x^2 \, dx$$

$$= \frac{1}{100} \left[y\right]_0^2 \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{100} \times 2 \times \frac{1}{3}$$

$$\boxed{\therefore E[X^2] = \frac{2}{300}}$$

(v) $E[Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 \, f_{X,Y}(x,y) \, dx \, dy$

$$= \int_0^2 \int_0^1 y^2 \cdot \frac{1}{100} \, dx \, dy$$

$$= \frac{1}{100} \int_0^2 y^2 \, dy \int_0^1 x^0 \, dx$$

$$= \frac{1}{100} \left[\frac{y^3}{3}\right]_0^2 \left[x\right]_0^1$$

$$= \frac{1}{100} \times \frac{8}{3} \times 1$$

$$\boxed{\therefore E[Y^2] = \frac{8}{300}}$$

(vi) $E[X^2 Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 y \, f_{X,Y}(x,y) \, dx \, dy$

$$= \int_0^2 \int_0^1 x^2 y \left(\frac{1}{100}\right) dx \, dy$$

$$= \frac{1}{100}\int_0^2 y\,dy \int_0^1 x^2\,dx$$

$$= \frac{1}{100}\left[\frac{y^2}{2}\right]_0^2 \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{600} \times 4 \times 1$$

$$\boxed{\therefore E[X^2 Y] = \frac{2}{300}}$$

(vii) $E[Y^2 X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy^2 f_{XY}(x,y)\,dx\,dy$

$$= \int_0^2\int_0^1 xy^2 \left(\frac{1}{100}\right) dx\,dy$$

$$= \frac{1}{100}\int_0^2 y^2\,dy \int_0^1 x\,dx$$

$$= \frac{1}{100}\left[\frac{y^3}{3}\right]_0^2 \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{600} \times 8 \times 1$$

$$E(XY^2) = \frac{8}{600}$$

$$\boxed{E[X^1 Y^2] = \frac{4}{300}}$$

(viii) $E[X+Y] = E[X] + E[Y] = \frac{1}{100} + \frac{1}{50} = \frac{3}{100}$

(ix) $E[X-Y] = E[X] - E[Y] = \frac{1}{100} - \frac{1}{50} = \frac{-1}{100}$

3 I) Show that mean value of weighted sum of random variables is equal to the weighted sum of mean value of r.v's M.P V.imp

II) Show that var. of weighted sum of R.V's be equal to the weighted sum of variance of uncorrelated R.V's. M.P

I. $E[a_1X_1 + a_2X_2 + \cdots + a_NX_N] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_N E[X_N]$

Proof: let us consider X_n, $n = 1, 2, 3 \cdots N$ (i.e $X_1, X_2, \cdots X_N$)

Here $a_1, a_2, a_3 \cdots a_N$ are constants

$$L.H.S = E[a_1X_1 + a_2X_2 + \cdots + a_NX_N] \quad \because a_1X_1 + a_2X_2 + \cdots = \sum_{n=1}^{N} a_nX_n$$

$$= E\left[\sum_{n=1}^{N} a_nX_n\right]$$

$$= \sum_{n=1}^{N} a_n E[X_n] = \sum_{n=1}^{N} a_n \bar{X}_n$$

$$= a_1 E[X_1] + a_2 E[X_2] + a_3 E[X_3] + \cdots + a_N E[X_N]$$

$$= R.H.S$$

Hence property is proved

II. $Var[a_1X_1 + a_2X_2 + \cdots + a_NX_N] = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + \cdots + a_N^2\sigma_{X_N}^2$

Proof: let us consider 'N' number of R.V's

$X_n, n = 1, 2, 3 \cdots N$

Here $a_1, a_2, a_3 \cdots a_N$ are constants.

$$Var[a_1X_1 + a_2X_2 + \cdots + a_NX_N] = a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cdots + a_N^2 Var(X_N)$$

Consider $L.H.S = Var(a_1X_1 + a_2X_2 + \cdots + a_NX_N)$

$$\because a_1X_1 + a_2X_2 + \cdots + a_NX_N = \sum_{n=1}^{N} a_nX_n$$

$$= Var\left(\sum_{n=1}^{N} a_nX_n\right)$$

$$= E\left[\left(\sum_{n=1}^{N} a_N X_N - \overline{\sum_{n=1}^{N} a_n X_N}\right)^2\right] \quad \left\{ \because Var(X) = E\left[(X-\overline{X})^2\right] \right.$$

$$\overline{\sum_{n=1}^{N} a_n x_n} = E\left[\sum_{n=1}^{N} a_n x_n\right] = \sum_{n=1}^{N} a_n \overline{X_n}$$

$$= E\left[\left(\sum_{n=1}^{N} a_n X_m - \sum_{n=1}^{N} a_n \overline{X_n}\right)^2\right]$$

$$= E\left[\left(\sum_{n=1}^{N} a_n (X_n - \overline{X_n})\right)^2\right]$$

$$= E\left[\left(\sum_{n=1}^{N} a_n (X_n - \overline{X_n})\right)\left(\sum_{n=1}^{N} a_n (X_n - \overline{X_n})\right)\right]$$

Put $n = m$ in second term

$$= E\left[\left(\sum_{n=1}^{N} a_n (X_n - \overline{X_n})\right)\left(\sum_{m=1}^{N} a_m (X_m - \overline{X_m})\right)\right]$$

$$= E\left[\sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m (X_n - \overline{X_n})(X_m - \overline{X_m})\right]$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m \, E\left[(X_n - \overline{X_n})(X_m - \overline{X_m})\right]$$

$$\left\{ \begin{array}{l} \because C_{XY} = E\left[(X-\overline{X})(Y-\overline{Y})\right] \\ \text{III}^{ly}\ C_{X_n X_m} = E\left[(X_n - \overline{X_n})(X_m - \overline{X_m})\right] \end{array} \right\}$$

$$L.H.S = \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m C_{X_n X_m}$$

Here $C_{X_n X_m}$ is the covariance of X_n and X_m.

This means that variance of weighted sum of r.v's equal to weighted sum of their covariances.

For uncorrelated r.v's $C_{X_n X_m} = \begin{cases} 0 & ; n \neq m \\ \sigma_{X_m}^2 & ; n = m \end{cases}$

For $n=m$; $C_{X_n X_m} = E[(X_n-\overline{X}_n)(X_m-\overline{X}_m)]$

$$C_{X_n X_n} = E[(X_n-\overline{X}_n)(X_n-\overline{X}_n)]$$

$$= E[(X_n-\overline{X}_n)^2]$$

$$= Var(X_n)$$

$$= \sigma_{X_n}^2$$

Now $L.H.S = \sum_{n=1}^{N}\sum_{n=1}^{N} a_n a_n C_{X_n X_n}$

$$= \sum_{n=1}^{N} a_n^2 \sigma_{X_n}^2$$

$$= a_1^2 \sigma_{X_1}^2 + a_2^2 \sigma_{X_2}^2 + \cdots + a_N^2 \sigma_{X_n}^2$$

$[a_1^2 Var(X_1) + a_2^2 Var(X_2) + \cdots + a_N^2 Var(X_N)]$

$= R.H.S$

Hence proved

4. The Joint PDF of R.V's X and Y is

$$f_{X,Y}(x,y) = \begin{cases} x+y & ; 0 \le x \le 1 ; 0 \le y \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find mean values of X & Y

Sol: Given

(ii) Find mean square values of X & Y

(iii) Variances of X and Y

(iv) Correlation b/w X & Y

(v) Covariance of X & Y

(vi) Correlation Coefficient of X and Y

(vii) First order joint moments and second order about origin

(viii) First and second order joint central moments (ix)(a) Examine X & Y are independent or not.

(x)(b) Uncorrelated or not

(xi)(c) Orthogonal or not

Sol: Given $f_{X,Y}(x,y) = \begin{cases} x+y \ ; & 0 \le x \le 1;\ 0 \le y \le 1 \\ 0 \ ; & \text{otherwise} \end{cases}$

(i) Mean value $E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^1 x(x+y)\, dx\, dy$

$= \int_0^1 \int_0^1 x^2 + xy \ dx\, dy$

$= \int_0^1 \frac{x^3}{3}\Big|_0^1 + y \frac{x^2}{2}\Big|_0^1 \ dy$

$= \int_0^1 \left(\frac{1}{3} + \frac{y}{2}\right) dy$

$= \left(\frac{y}{3} + \frac{y^2}{4}\right)_0^1$

$= \frac{1}{3} + \frac{1}{4}$

$E[X] = \frac{7}{12} = \bar{X} = m$

Mean value of Y $= E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$

$= \int_0^1 \int_0^1 y(x+y)\ dx\, dy$

$= \int_0^1 \int_0^1 xy + y^2 \ dx\, dy$

$$= \int_0^1 \frac{x^2}{2}\Big|_0^1 \cdot y + x\Big|_0^1 \cdot y^2 \; dy$$

$$= \int_0^1 \frac{1}{2} \cdot y + y^2 \; dy$$

$$= \frac{y^2}{4}\Big|_0^1 + \frac{y^3}{3}\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{3}$$

$$E[Y] = \frac{7}{12} = \overline{Y} = m$$

(ii) $E[X^2]$ = Mean square of X

$$E[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x^2(x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 (x^3 + x^2 y)\, dx\, dy$$

$$= \int_0^1 \frac{x^4}{4}\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 y \; dy$$

$$= \int_0^1 \frac{1}{4} + \frac{1}{3} y \; dy$$

$$= \frac{y}{4}\Big|_0^1 + \frac{y^2}{6}\Big|_0^1$$

$$= \frac{1}{4} + \frac{1}{6}$$

$$= \frac{5}{12}$$

Mean square of $Y = E[Y^2]$.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 y^2 (x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy^2 + y^3\,dx\,dy$$

$$= \int_0^1 \frac{x^2 y^2}{2}\Big|_0^1 + xy^3\Big|_0^1\,dy$$

$$= \int_0^1 \frac{y^2}{2} + y^3\,dy$$

$$= \frac{y^3}{6}\Big|_0^1 + \frac{y^4}{4}\Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{4}$$

$$E[Y^2] = \frac{5}{12}$$

(iii) Variance of 'X' $= \sigma_X^2 = E[(X-\bar{X})^2]$

$$= E[X^2] - [E(X)]^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$= \frac{5}{12} - \frac{49}{144}$$

$$\sigma_X^2 = \frac{11}{144}$$

Variance of 'Y' $\sigma_Y^2 = E[(Y-\bar{Y})^2]$

$$= E[Y^2] - [E(Y)]^2$$

$$= \frac{5}{12} - \left(\frac{7}{12}\right)^2$$

$$\sigma_Y^2 = \frac{11}{144}$$

(iv) Correlation of X & Y

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x^2y + xy^2\, dx\, dy$$

$$= \int_0^1 \frac{x^3}{3}\Big|_0^1 y + \frac{x^2}{2}\Big|_0^1 y^2\, dy$$

$$= \int_0^1 \frac{y}{3} + \frac{y^2}{2}\, dy$$

$$= \frac{y^2}{6}\Big|_0^1 + \frac{y^3}{6}\Big|_0^1$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$\therefore R_{XY} = E[XY] = \frac{1}{3}$$

(v) Covariance of X & Y:

$$Cov(X,Y) = \sigma_{XY} = E[(X-\bar{X})(Y-\bar{Y})]$$

$$= R_{XY} - E[X][Y]$$

$$= E[XY] - E[X]E[Y]$$

$$= \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right)$$

$$= \frac{1}{3} - \frac{49}{144}$$

$$C_{XY} = \frac{-1}{144}$$

(vi) Correlation Coefficient

$$\rho = \frac{C_{xy}}{\sigma_x \sigma_y}$$

$$Var(x) = \sigma_x^2 = \frac{11}{144} \Rightarrow \sigma_x = \sqrt{\frac{11}{144}}$$

$$Var(y) = \sigma_y^2 = \frac{11}{144} \Rightarrow \sigma_y = \sqrt{\frac{11}{144}}$$

$$C_{xy} = \frac{-1}{144}$$

$$\rho = \frac{\frac{-1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}$$

$$= \frac{-1}{144} \times \frac{144}{11}$$

$$= \frac{-1}{11}$$

$$\boxed{\therefore \rho = -0.09}$$

(vii) First and second order joint moments about origin:

Zeroth order: $= m_{00} = E[1] = 1$

First order: $= m_{10} = E[X] = \frac{7}{12}$ Mean value of X

" $= m_{01} = E[Y] = \frac{7}{12}$ Mean value of Y

Second order: $= m_{20} = E[X^2] = \frac{5}{12}$ Meansquare of x

" $= m_{02} = E[Y^2] = \frac{5}{12}$ Meansquare of y

" $= m_{11} = E[XY] = \frac{1}{3}$ Correlation of X&Y

(viii) Joint Central Moments (μ)

Zero order: $= \mu_{00} = E[1] = 1$

First order: $= \mu_{10} = E[X - \bar{X}] = 0$

" " $= \mu_{01} = E[Y - \bar{Y}] = 0$

Second order: Variance of X $\mu_{20} = E[(X - \bar{X})^2] = \sigma_X^2 = \frac{11}{144}$

Variance of Y $\mu_{02} = E[(Y - \bar{Y})^2] = \sigma_Y^2 = \frac{11}{144}$

Covariance of X & Y $\mu_{11} = E[(X - \bar{X})(Y - \bar{Y})] = C_{XY} = \frac{-1}{144}$

(ix)(a)(b) Examine X, Y are independent or not and uncorrelated or not

Condition for statistically independent variables is

$$\Rightarrow E[XY] = E[X]E[Y]$$

$$E[XY] = \frac{1}{3}$$

$$E[X] = \frac{7}{12} \quad ; \quad E[Y] = \frac{7}{12}$$

$$E[X]E[Y] = \frac{7}{12} \cdot \frac{7}{12} = \frac{49}{144}$$

$$E[X][Y] \neq E[XY]$$

Hence the r.v's X and Y are not independent and are not

~~(b) X, Y are~~ uncorrelated ~~or not~~ random variables.

(c) X, Y are orthogonal, or not.

Condition for orthogonality is $E[XY] = 0$

But $E[XY] \neq 0$

Since $E[XY] = \frac{1}{3}$ so these are not

orthogonal random variables.

Comment:
From this we conclude that the two random variables X and Y are neither independent nor orthogonal random variable

5. Two R.V's X and Y have mean values $\bar{X}=1$, $\bar{Y}=1$ and variances $\sigma_X^2 = 4$, $\sigma_Y^2 = 2$ and a correlation Coefficients $\rho_{XY} = 0.1$. Determine two new R.V's $V = -X-Y$ & $W = 2X+Y$.

(i) Find mean values of V & W
(ii) Mean square values of V & W
(iii) Variances of V & W.
(iv) Correlation b/w V and W
(v) Covariance of V and W
(vi) Correlation coefficent of V & W
(vii) Examine V and W are independent or not, uncorrelated or not & orthogonal or not.

Sol: Given $E[X]=\bar{X}=1$, $E[Y]=\bar{Y}=1$ ⇒ Mean values of X & Y

$\sigma_X^2 = 4$, $\sigma_Y^2 = 2$ ⇒ Variances of X & Y

$\rho_{XY} = 0.1$ ⇒ Correlation of coefficient

(i) Mean value of V ($\bar{V}$)

$$\bar{V} = E[V] = E[-X-Y]$$

$\because E[aX+bY] = a\,E[X]+b\,E[Y]$

$$= (-1)\,E[X] + (-1)\times E[Y]$$

$$= -1\times 1 + -1\times 1$$

$$= -1 - 1$$

$$\therefore \bar{V} = -2$$

Mean value of W ($\overline{W}$)

$$\overline{W} = E[W] = E[2X+Y]$$

$$= 2\,E[X] + 1\cdot E[Y]$$

$$= 2\cdot 1 + 1$$

$$\therefore \overline{W} = 3$$

(ii) Mean square value of V:

$$\overline{V^2} = E[V^2] = E[(-X-Y)^2]$$

$$= E[X^2 + Y^2 + 2XY]$$

$$= E[X^2] + E[Y^2] + 2E[XY]$$

Given $Var(X) = \sigma_X^2 = E[X^2] - [E(X)]^2$

$$E[X^2] = \sigma_X^2 + [E(X)]^2$$

$$= 4 + 1$$

$$E[X^2] = 5$$

$$Var(Y) = \sigma_Y^2 = E[Y^2] - [E(Y)]^2$$

$$E[Y^2] = \sigma_Y^2 + [E(Y)]^2$$

$$= 2 + 1$$

$$E[Y^2] = 3$$

From Given data $\sigma_X^2 = 4 \Rightarrow \sigma_X = 2$

$\sigma_Y^2 = 2 \Rightarrow \sigma_Y = \sqrt{2}$

$$\rho_{XY} = \frac{C_{XY}}{\sigma_X \sigma_Y}$$

$$C_{XY} = \rho_{XY}\,\sigma_X\,\sigma_Y$$

$$= 0.1 \times 2 \times \sqrt{2}$$

$$C_{XY} = 0.2828$$

$$Covariance\ (X,Y) = C_{XY} = E[XY] - E[X]E[Y]$$

$$E[XY] = C_{XY} + E[X]E[Y]$$

$$= 0.2828 + 1\times 1$$

$$= 0.2828 + 1$$

$$E[XY] = 1.2828$$

$$\therefore \overline{V^2} = 5 + 3 + 2(1.2828)$$

$$\overline{V^2} = E[V^2] = 10.5656$$

Mean square value of W:

$$\overline{W^2} = E[W^2] = E[(2X+Y)^2]$$

$$= E[4X^2 + Y^2 + 4XY]$$

$$= 4E[X^2] + E[Y^2] + 4E[XY]$$

$$= 4\times 5 + 3 + 4\times 1.2828$$

$$E[W^2] = 28.1312$$

(iii) Variance of V:

$$Var(V) = \sigma_{V^2} = E[(V-\overline{V})^2]$$

$$= E[V^2] - [E(V)]^2$$

$$= 10.5656 - (-2)^2$$

$$\therefore \sigma_{V^2} = 6.5656$$

Variance of W:

$$Var(W) = \sigma_{W^2} = E[(W-\overline{W})^2]$$

$$= E[W^2] - [E(W)]^2$$

$$= 28.1312 - (3)^2$$

$$\therefore \sigma_{W^2} = 19.1312$$

(iv) Correlation b/w V and W:

$R_{VW} = E[VW] = E[(-X-Y)(2X+Y)]$

$= E[-2X^2 - XY - 2XY - Y^2]$

$= E[-2X^2 - 3XY - Y^2]$

$= -2E[X^2] - 3E[XY] - E[Y^2]$

$= -2(5) - 3(1.2828) - 3$

$\therefore R_{VW} = -16.8484$

(v) Covariance of V and W:

$Cov(V,W) = R_{VW} = E[VW] - E[V]E[W]$

$= -16.8484 - (-2)\cdot(3)$

$\therefore C_{VW} = -10.8484$

(vi) Correlation coefficient of V and W:

$$\rho_{VW} = \frac{C_{VW}}{\sigma_V \sigma_W}$$

$$= \frac{-10.8484}{\sigma_V \sigma_W}$$

$\sigma_V^2 = 6.5656 \Rightarrow \sigma_V = 2.5623$

$\sigma_W^2 = 19.1312 \Rightarrow \sigma_W = 4.3739$

$$= \frac{-10.8484}{(2.5623)(4.3739)}$$

$\therefore \rho_{VW} = -0.96798$

(vii) Examine V, W are independent, uncorrelated, and orthogonal or not:

$$E[VW] = -16.8484$$

$$E[V] = -2 \quad ; \quad E[W] = 3$$

$$\therefore E[VW] \neq E[V]E[W]$$

Hence these V and W are not independent and are not uncorrelated.

$$E[VW] = -16.8484 \neq 0$$

Hence this is not orthogonal.

6. A random variable with PDF anothe random variables $X = Z$ and $Y = Z^2$ then show that X and Y are uncorrelated R.V's. $f_Z(z) = \begin{cases} 1/2 & ; -1 \leq z \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$

Sol: We know condition for uncorrelated random variables X and Y is $E[XY] = E[X]E[Y]$ or $C_{XY} = 0$

$$E[X] = \int_{-\infty}^{\infty} x \, f_Z(z) \, dz$$

Given $X = Z$

$$E[X] = \int_{-\infty}^{\infty} z \, f_Z(z) \, dz$$

$$= \int_{-1}^{1} z \frac{1}{2} \, dz$$

$$= \frac{1}{2} \left[\frac{z^2}{2} \right]_{-1}^{1}$$

$$= \frac{1}{4} [1 - 1]$$

$$E[X] = 0$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Z(z)\, dz$$

Given $y = z^2$

$$= \int_{-\infty}^{\infty} z^2 f_Z(z)\, dz$$

$$= \int_{-1}^{1} \frac{z^2}{2}\, dz$$

$$= \frac{1}{2}\left[\frac{z^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2\times 3}\left[1^3 - (-1)^3\right]$$

$$= \frac{1}{2\times 3}[2]$$

$$E[Y] = \frac{1}{3}$$

$$E[XY] = \int_{-\infty}^{\infty} xy\, f_Z(z)\, dz$$

$$= \int_{-1}^{1} z \cdot z^2 f_Z(z)\, dz$$

$$= \frac{1}{2}\left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(1 - 1\right)$$

$$= 0$$

$\therefore\ E[XY] = 0$; $E(X)\,E[Y] = 0 \times \frac{1}{3} = 0$

$\therefore\ E[XY] = E[X]\,E[Y]$

$C_{XY} = E[XY] - E[X]\,E[Y] = 0$

Hence, X & Y are uncorrelated random variables.

7. If X and Y be independent with marginal PDF
$f_X(x) = 3e^{-3x}; x \geq 0$ & $f_Y(y) = 3e^{-3y}; y \geq 0$. Find
(a) $E[X^2+Y^2]$ (b) $E[XY]$

Sol: Given X & Y are independents. Their marginal PDF's are $f_X(x) = 3e^{-3x}; x \geq 0$

$$f_Y(y) = 3e^{-3y}; y \geq 0$$

(a) $E[X^2+Y^2] = E[X^2] + E[Y^2]$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^{\infty} x^2\, 3e^{-3x}\, dx$$

$$= 3\left[x^2 \frac{e^{-3x}}{-3} - \left(2x \frac{e^{-3x}}{(-3)(-3)}\right) + \frac{2e^{-3x}}{(-3)(-3)(-3)}\right]_0^{\infty}$$

$$= 3\left[-3x^2 e^{-3x} - \frac{2}{9}xe^{-3x} - \frac{2}{27}e^{-3x}\right]_0^{\infty}$$

$$= 3\left[0 - \left[-0 - 0 - \frac{2}{27}e^0\right]\right]$$

$$= \frac{6}{27}$$

$$E[X^2] = \frac{2}{9}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy$$

$$= \int_0^{\infty} y^2\, 3e^{-3y}\, dy$$

$$= 3\left[y^2 \frac{e^{-3y}}{-3} - 2x \frac{e^{-3y}}{(-3)(-3)} + \frac{2e^{-3y}}{(-3)(-3)(-3)}\right]_0^{\infty}$$

$$E[Y^2] = 3\left[0 - \left(-0 - 0 - \frac{2e^0}{-27}\right)\right]$$

$$= \frac{6}{27}$$

$$\therefore E[Y^2] = \frac{2}{9}$$

So $E[X^2+Y^2] = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$.

$$\boxed{E[X^2+Y^2] = \frac{4}{9}}$$

(b) $E[XY]$

If X and Y are independent $E[XY] = E[X]\,E[Y]$

$$E[X] = \int_0^\infty x\, 3e^{-3x}\, dx$$

$$= 3\left[x\frac{e^{-3x}}{(-3)} - 1\cdot\frac{e^{-3x}}{(-3)(-3)}\right]_0^\infty$$

$$= 3\left[0 - \left[0 + \frac{e^0}{9}\right]\right]$$

$$\therefore E[X] = \frac{1}{3}$$

$$E[Y] = \int_0^\infty y \cdot 3e^{-3y}\, dy$$

$$= 3\left[y\frac{e^{-3y}}{(-3)} - \frac{1\cdot e^{-3y}}{(-3)(-3)}\right]_0^\infty$$

$$= 3\left[0 - \left(0 + \frac{e^0}{9}\right)\right]$$

$$= \frac{1}{3}$$

$$\therefore E[XY] = \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}$$

$$f_{X,Y}(x,y) = f_X(x)\ f_Y(y)$$

$$= 3e^{-3x} \cdot 3e^{-3y}$$

$$\therefore f_{X,Y}(x,y) = 9\, e^{-3x} e^{-3y} \quad ; \quad x \geq 0,\ y \geq 0$$

(OR)

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\ f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{\infty} xy\ 9\, e^{-3x} e^{-3y}\, dx\, dy$$

$$= 9\left[\int_0^{\infty} x\, e^{-3x}\, dx \int_0^{\infty} y\, e^{-3y}\, dy\right]$$

$$= 9\left[x\frac{e^{-3x}}{-3} - \frac{e^{-3x}}{(-3)(-3)}\right]\left[\frac{y\, e^{-3y}}{(-3)} - \frac{e^{-3y}}{(-3)(-3)}\right]$$

03/9/14 Assignment - 2

1. $\overline{X}=0$, $\overline{Y}=-1$, $\overline{X^2}=2$, $\overline{Y^2}=4$, $R_{XY}=-2$, $W=2X+Y$, $U=X-3Y$. Find $\overline{W}$, $\overline{U}$, $\overline{W^2}$, $\overline{U^2}$, R_{UW}, σ_{X^2}, σ_{Y^2}, C_{WU}, ρ_{W+U}, σ_{W^2}, σ_{U^2}

2. Two random variables X and Y have means $\overline{X}=1$, $\overline{Y}=2$, Variances $\sigma_{X^2}=4$, $\sigma_{Y^2}=1$ and $\rho_{XY}=0.4$. New random variables defined by $V=-x+2y$, $W=x+3y$. Find all possibilities.

3. Assign-2 M.P X is a random variable with mean 4 and variance 3, another random variable Y is related to X, $Y=2X+7$. Determine $E[X^2]$, $E[Y]$, $E[Y^2]$, $Var(Y)$, R_{XY}, C_{XY}, ρ_{XY} and also examine all possibilities. $E[Y]=E[2X+7]$

4. If X and Y are independent $f_X(x)=2e^{-2x}$; $x\geq 0$, $f_Y(y)=2e^{-2y}$; $y\geq 0$. Find $E[X+Y]$, $E[X^2+Y^2]$, $E[XY]$, $E[X^2Y^2]$

5. Let $f(x,y)=\begin{cases} x(y+1.5) & ; 0<x<1 \ \& \ 0<y<1 \\ 0 & ; \text{elsewhere} \end{cases}$. Find all the joint moments $m_{nk}=E[X^nY^k]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)\,dx\,dy$

$$=\int_0^1\int_0^1 x^n y^k \cdot x(y+1.5)\,dx\,dy$$

$$=\int_0^1\int_0^1 x^{n+1}(y^{k+1}+1.5y^k)\,dx\,dy$$

$$=\int_0^1 x^{n+1}\,dx \int_0^1 (y^{k+1}+1.5y^k)\,dy$$

$$=\frac{x^{(n+1)+1}}{(n+1)+1}\Big|_0^1 \times \left[\frac{y^{k+1+1}}{(k+1)+1}\Big|_0^1 + 1.5\frac{y^{k+1}}{k+1}\Big|_0^1\right]$$

$$=\frac{1}{n+2}\left[\frac{1}{k+2}+\frac{1.5}{k+1}\right]$$

$$=\frac{1}{n+2}\left[\frac{(k+1)+(1.5)(k+2)}{(k+1)(k+2)}\right]$$

$$=\frac{1}{n+2}\left[\frac{2.5k+4}{(k+1)(k+2)}\right].$$

Discrete type:

Q7. The joint PDF of random variables X and Y is

$$f(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.05\,\delta(x-1)\,\delta(y-3)$$

Find (i) mean value of X and Y, (ii) Mean square value of X and Y (iii) Variances of X and Y

(iv) Correlation b/w. X and Y

(v) Covariance of X and Y

(vi) Correlation Coeff of X and Y.

Sol:

XY	(-1, 0) $x_1 y_1$	(0, 0) $x_2 y_2$	(0, 2) x_3, y_3	(1, -2) $x_4 y_4$	(1, 1) $x_5 y_5$	(1, 3) $x_6 y_6$
$P(x_n, y_n)$	0.15	0.1	0.1	0.4	0.2	0.05

Mean value of $X = \bar{X} = E[X] = \sum_{\langle n \rangle} x_n P(x_n)$

$$= \sum_{n=1}^{6} x_n P(x_n)$$

$$= x_1 P(x_1) + x_2 P(x_2) + \cdots + x_6 P(x_6)$$

$$= -1 \times 0.15 + 0.1 \times 0 + 0.1 \times 0 + 1 \times 0.4 + 1 \times 0.2 + 1 \times 0.05$$

$$= 0.5$$

Mean value of $Y = \bar{Y} = E[Y] = \sum_{\langle n \rangle} y_n P(y_n)$

$$= \sum_{n=1}^{6} y_n (P(y_n))$$

$$= 0 + 0 + 2 \times 0.1 + -2 \times 0.4 + 1 \times 0.2 + 3 \times 0.05$$

$$= -0.25$$

(ii) Mean square value of $X = E[X^2] = \sum_{\langle n \rangle} x_n^2 P(x_n)$

$$= \sum_{n=1}^{6} x_n^2 P(x_n)$$

$$= x_1^2 P(x_1) + x_2^2 P(x_2) + \cdots + x_6^2 P(x_6)$$

$$= (-1)^2 \times 0.15 + 0^2 + 0^2 + (1)^2 \times 0.4 + (1)^2 \times 0.2 + (1)^2 \times 0.05$$

$E[X^2] = 0.8$

Mean square value of $Y = E[Y^2] = \sum_{\langle n \rangle} y^2 P(y_n)$

$= (0^2) \times 0.15 + (0)^2 \times 0.1 + (2)^2 \times 0.1 + (-2)^2 \times 0.4$

$+ (1)^2 \times 0.2 + (3)^2 \times 0.05$

$= 2.65$

(iii) Variance of $X = \sigma_X^2 = E[X^2] - [E[(X)]]^2$

$= 0.8 - (0.5)^2$

$= 0.55$

Variance of $Y = \sigma_Y^2 = E[Y^2] - [E(Y)]^2$

$= 2.65 - (-0.25)^2$

$= 2.5875$

(iv) Correlation b/w X and Y

$= R_{XY} = E[X \cdot Y] = \sum_{n=1}^{6} \sum_{k=1}^{6} x_n y_k \, [P(x_n, y_k)]$

$= x_1 y_1 P(x_1, y_1) + \ldots + x_6 y_6 P(x_6, y_6)$

$= -1 \times 0 \times 0.15 + 0 \times 0 \times 0.1 + 0 \times 2 \times 0.1 + 1 \times -2 \times 0.4$

$+ 1 \times 1 \times 0.2 + 1 \times 3 \times 0.05$

$= -0.45$

(v) Covariance of X and Y

$C_{XY} = E[XY] - E[X]E[Y]$

$= -0.45 - (0.5 \times (-0.25))$

$= -0.325$

(vi) Correlation coefficient of X & Y

$\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y} = \dfrac{-0.325}{\sqrt{0.55}\sqrt{2.5875}}$

$\therefore \rho_{XY} = -0.272$

* Joint Characteristics functions:

The joint characteristic function of random variables X and Y is defined simply as Expectation of the function $g(X, Y) = e^{j\omega_1 X} e^{j\omega_2 Y}$

Mathematically $= \phi_{XY}(\omega_1, \omega_2)$

$$\Rightarrow E[e^{j\omega_1 x} e^{j\omega_2 y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x} e^{j\omega_2 y} f_{X,Y}(x,y)\, dx\, dy$$

$$\phi_{XY}(\omega_1, \omega_2) = E[e^{j\omega_1 x + j\omega_2 y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y) e^{j\omega_1 x + j\omega_2 y}\, dx\, dy$$

The characteristics of function 'X' $= \phi_X(\omega) = E[e^{j\omega x}]$

Single random variable $= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$

The PDF of X

$$= f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

The joint PDF of X and Y $= f_{XY}(x,y) \Rightarrow$

$$\left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{XY}(\omega_1, \omega_2)\, e^{-j\omega_1 x - j\omega_2 y}\, d\omega_1\, d\omega_2$$

For discrete case

$$\phi_{XY}(\omega_1, \omega_2) = E[e^{j\omega_1 x + j\omega_2 y}] = \sum_{n}\sum_{k} e^{j\omega_1 x_n} e^{j\omega_2 y_k} P(x_n, y_k)$$

For 'n' random variables:-

$$\phi_{X_1, X_2, \ldots X_n}(\omega_1, \omega_2 \ldots \omega_n) = E[e^{j\omega_1 x_1} + e^{j\omega_2 x_2} + \ldots + e^{j\omega_N x_N}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 X_2 \ldots X_N}(x_1, x_2 \ldots x_N)\, e^{j\omega_1 x_1 + j\omega_2 x_2 + \ldots + j\omega_N x_N}\, dx_1\, dx_2 \ldots dx_N.$$

* Properties of Joint Characteristic function:

1. The marginal characteristic function can be obtained from joint characteristic function. i.e.

$\phi_X(\omega_1) = \phi_{XY}(\omega_1, 0)$ and $\phi_Y(\omega_2) = \phi_{XY}(0, \omega_2)$

and also $\phi_{XY}(0,0) = 1$

Proof: Joint characteristic function of X and Y

$= \phi_{XY}(\omega_1, \omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}]$

let $\omega_2 = 0 \Rightarrow \phi_{XY}(\omega_1, 0) = E[e^{j\omega_1 X} e^{j(0)Y}]$

$= E[e^{j\omega_1 X}]$

The characteristic function of X

$= \phi_X(\omega) = E[e^{j\omega X}]$

$= \phi_X(\omega_1) = E[e^{j\omega_1 X}]$

The marginal char. function of Y

let $\omega_1 = 0 \Rightarrow \phi_{XY}(0, \omega_2) = E[e^{j(0)X} e^{j\omega_2 Y}]$

$= E[e^{j\omega_2 Y}]$

$= \phi_Y(\omega_2)$

$\therefore$ The characteristic function of 'Y'

$= \phi_Y(\omega) = E[e^{j\omega Y}]$

$= \phi_Y(\omega_2) = E[e^{j\omega_2 Y}]$

$\phi(\omega_1, \omega_2) = E[e^{j\omega X} e^{j\omega Y}] = E[1]$

$\phi(0,0) = 1.$

2. If X and Y are statistically independent, then the joint characteristic function is equal to the product of their individual characteristics functions

Proof: If X and Y are independent, then

$$\phi_{XY}(\omega_1, \omega_2) = \phi_X(\omega_1)\, \phi_Y(\omega_2)$$

The joint characteristic function of X and Y

$$= \phi_{XY}(\omega_1, \omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x} e^{j\omega_2 y} f_{XY}(x,y)\, dx\, dy$$

We know if X and Y are independent, then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$\phi_{XY}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x} e^{j\omega_2 y} f_X(x) f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} f_X(x)\, e^{j\omega_1 x}\, dx\right)\left(\int_{-\infty}^{\infty} f_Y(y)\, e^{j\omega_2 y}\, dy\right)$$

$$= \phi_X(\omega_1)\, \phi_Y(\omega_2)$$

Hence proved.

3. If X and Y are independent, then

$$\phi_{X+Y}(\omega) = \phi_X(\omega)\, \phi_Y(\omega)$$

Proof: If X and Y are independent, then

$$= \phi_{XY}(\omega_1, \omega_2) = E[e^{j\omega_1 X} e^{j\omega_2 Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x} e^{j\omega_2 y} f_{XY}(x,y)\, dx\, dy$$

We know that if X and Y are independent, then

$$f_{XY}(x,y) = f_X(x)\, f_Y(y)$$

$$\phi_{X+Y}(\omega) = E[e^{j\omega(X+Y)}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega(x+y)} f_X(x)\, f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx\right)\left(\int_{-\infty}^{\infty} f_Y(y)\, e^{j\omega y}\, dy\right)$$

$$= \phi_X(\omega)\, \phi_Y(\omega)$$

4. If x and y are two random variables then the joint moments can be derived from the joint characteristic function as

$$m_{nk} = (-j)^{n+k} \frac{\partial^{n+k} [\phi_{xy}(\omega_1, \omega_2)]}{\partial \omega_1^n \, \partial \omega_2^k} \Bigg|_{\omega_1=\omega_2=0}$$

Proof: let joint characteristics of x and y

$$= \phi_{xy}(\omega_1, \omega_2) = E[e^{j\omega_1 x} e^{j\omega_2 y}] \quad \left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)$$

$$\phi_{xy}(\omega_1 \omega_2) = E\left[1 + j\omega_1 x + \frac{(j\omega_1 x)^2}{2!} + \frac{(j\omega_1 x)^3}{3!} + \cdots\right]$$

$$+ \left(1 + j\omega_2 y + \frac{(j\omega_2 y)^2}{2!} + \frac{(j\omega_2 y)^3}{3!} + \cdots\right)$$

$$= E\left[1 + j\omega_1 x + \frac{j^2 \omega_1^2 x^2}{2!} + \frac{j^3 \omega_1^3 x^3}{3!} + j\omega_2 y + j^2 \omega_1 \omega_2 x y + \frac{j^2 \omega_2^2 y^2}{2!} + \frac{j^3 \omega_2^3 y^3}{3!} + \cdots\right]$$

$$= 1 + j\omega_1 E[x] + j\omega_2 E[y] + \frac{j^2 \omega_1^2}{2!} E[x^2] + \frac{j^2 \omega_2^2}{2!} E[y^2] + \cdots \quad \to ①$$

$$= \sum_{\langle n \rangle} \sum_{\langle k \rangle} (j)^{n+k} \frac{\omega_1^n \omega_2^k}{n! \, k!} E[x^n y^k]$$

$$= \phi_{xy}(\omega_1 \omega_2)$$

From the above equation ①

$$\frac{\partial}{\partial \omega_1} [\phi_{xy}(\omega_1, \omega_2)] \Big|_{\omega_1=\omega_2=0} = jE[x] = j m_{10}$$

$\because E[\,] = m_{10}$

$E[y] = m_{01}$

$$\frac{\partial}{\partial \omega_2} [\phi_{xy}(\omega_1, \omega_2)] \Big|_{\omega_1=\omega_2=0} = jE[y] = j m_{01}$$

$$\frac{\partial^2}{\partial \omega_1^2}\left[\phi_{XY}(\omega_1, \omega_2)\right]\Big|_{\omega_1=\omega_2=0} = j^2 E[X^2] = j^2 m_{20}$$

$$\frac{\partial^2}{\partial \omega_2}\left[\phi_{XY}(\omega_1 \omega_2)\right]\Big|_{\omega_1=\omega_2=0} = (j)^2 E[Y^2] = j^2 m_{02}$$

$$\frac{\partial}{\partial \omega_1 \partial \omega_2}\left[\phi_{XY}(\omega_1, \omega_2)\right]\Big|_{\omega_1=\omega_2=0} = j^2 E[XY] = j^2 m_{11} \text{ etc}$$

$$\vdots$$

$$\frac{j^{n+k}}{\partial \omega_1^n \partial \omega_2^n} \phi_{XY}(\omega_1 \omega_2)\Big|_{\omega_1=\omega_2=0} = (j)^{n+k} E[X^n Y^k] = j^{n+k} m_{nk}$$

$$m_{nk} = (-j)^{n+k} \frac{\partial^{n+k}\left[\phi_{XY}(\omega_1, \omega_2)\right]}{\partial \omega_1^n \partial \omega_2^k}\Big|_{\omega_1=\omega_2=0}$$

Hence proved.

* Moment Generating Joint Function : (MGF)

The joint moment generating function of random variables X and Y is defined simply as the function

$$g(x, y) = e^{t_1 x} e^{t_2 y}$$

The MGF of X and Y = $M_{XY}(t_1, t_2) = E[e^{t_1 X} e^{t_2 Y}]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} e^{t_2 y} f_{X,Y}(x, y)\, dx\, dy$$

Single MGF $= M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$

$$= E[e^{t_1 X + t_2 Y}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\, e^{t_1 x + t_2 y}\, dx\, dy$$

For discrete random variables :-

$M_{XY}(t_1,t_2) = E[e^{t_1X} e^{t_2Y}] = \sum_{\langle n\rangle}\sum_{\langle k\rangle} e^{t_1 x_n} e^{t_2 y_k} p(x_n, y_k)$

S.T

* Properties of MGF:

1. The marginal moment generating function can be obtained by Joint MGF i.e.,

$$M_X(t_1) = M_{XY}(t_1, 0) \text{ and } M_X(t_2) = M_{XY}(0, t_2)$$

and also $M_{XY}(0,0) = 1$

<u>Proof:</u> Joint moment generating function of X and Y

$\equiv M_{XY}(t_1 t_2) = E[e^{t_1X} e^{t_2Y}]$

let $t_2 = 0 \Rightarrow M_{XY}(t_1, 0) = E[e^{t_1X} e^{(0)Y}]$

$\therefore M_X(t_1) = E[e^{t_1X}]$

$= E[e^{t_1X}]$

$= M_X(t_1) \rightarrow$ Marginal MGF of X

let $t_1 = 0 \Rightarrow M_{XY}(0, t_2) = E[e^{0X} e^{t_2Y}]$

$= E[e^{t_2Y}]$

$= M_Y(t_2) \rightarrow$ Marginal MGF of Y

let $t_1 = 0 = t_2 = E[e^{0X} e^{0Y}]$

$= E[1]$

$= 1 \rightarrow$ Marginal MGF of X & Y

Hence proved.

2. If X and Y are independent,

MP

$$M_{XY}(t_1, t_2) = M_X(t_1) \cdot M_Y(t_2)$$

3. If X and Y are independent

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

4. $$\left.\frac{\partial^{n+k} M_{XY}(t_1, t_2)}{\partial t_1^n \, \partial t_2^k}\right|_{t_1 = t_2 = 0}$$

2. ~~Proof~~: The joint moment generating function of X and Y

$$= M_{XY}(t_1, t_2) = E\left[e^{t_1 X} \cdot e^{t_2 Y}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} e^{t_2 y} f_{X,Y}(x,y)\, dx\, dy$$

We know X and Y are independent then

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$M_{XY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} \cdot e^{t_2 y} \cdot f_X(x) \, f_Y(y) \, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} f_X(x) \cdot e^{t_1 x} \, dx\right) \cdot \left(\int_{-\infty}^{\infty} f_Y(y) \, e^{t_2 y} \, dy\right)$$

$$= M_X(t_1) \cdot M_Y(t_2)$$

$$= M_X(t_1) \cdot M_Y(t_2)$$

Hence proved.

3. Proof: The joint moment generating function of X & Y

$= M_{XY}(t_1, t_2) = E[e^{t_1 X} \cdot e^{t_2 Y}]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} \cdot e^{t_2 y} f_{X,Y}(x,y)\, dx\, dy$$

We know if X and Y are independent then

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$M_{X+Y}(t) = E[e^{t_1 X + t_2 Y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} \cdot e^{t_2 y} \cdot f_X(x) \cdot f_Y(y)\, dx\, dy$$

$$= \left(\int_{-\infty}^{\infty} e^{t_1 x} \cdot f_X(x)\, dx\right) \left(\int_{-\infty}^{\infty} f_Y(y) \cdot e^{t_2 y}\, dy\right)$$

$= M_X(t_1) \cdot M_Y(t_2)$ Here $t_1 = t_2 = t$

$= M_X(t) \cdot M_Y(t)$

Hence proved.

4. Proof: If two variables are X and Y then the joint moments can be derived from the joint moment generating function.

i.e $$m_{nk} = \left.\frac{\partial^{n+k}[M_{XY}(t_1, t_2)]}{\partial t_1^n \cdot \partial t_2^k}\right|_{t_1 = t_2 = 0}$$

The joint moment generating function of X & Y

$= M_{XY}(t_1, t_2) = E[e^{t_1 X} \cdot e^{t_2 Y}]$

$$\left(\because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

$$M_{XY}(t_1 t_2) = E\left[\left(1 + t_1 X + \frac{(t_1 X)^2}{2!} + \frac{(t_1 X)^3}{3!} \dots\right)\left(1 + t_2 Y + \frac{(t_2 Y)^2}{2!} + \frac{(t_2 Y)^3}{3!} \dots\right)\right]$$

$$= E\left[1 + t_1 X + \frac{t_1^2 X^2}{2!} + \frac{t_1^3 X^3}{3!} + t_2 Y + t_1 t_2 XY + \frac{t_1^2 t_2 X^2 Y}{2!} + \frac{t_1^3 t_2 X^3 Y}{3!} + \dots\right]$$

$$= E[1] + t_1 E[X] + \frac{t_1^2}{2!} E[X^2] + \frac{t_1^3}{3!} E[X^3] + t_2 E[Y] + t_1 t_2 E[XY] + \frac{t_1^2 t_2}{2!} E[X^2 Y] + \frac{t_1^2 t_2}{3!} E[X^3 Y] + \dots \rightarrow (1)$$

$$M_{XY}(t_1, t_2) = \sum_{\langle n \rangle} \sum_{\langle k \rangle} \frac{t_1^n t_2^k}{n!\, k!} E[X^n Y^k]$$

from the above equation (1)

* Jointly Gaussian Random Variables:

(i) Two gaussian random variables:

M.P. If two random variables, X and Y are said to be jointly gaussian, then the jointo density function is given as $\left[f(x,y) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sigma_X \sigma_Y} \right] \times$

$$f_{XY}(x,y) = \frac{1}{(2\pi)^{1/2} \sigma_X \sigma_Y \sqrt{1-\rho}} Exp\left[\frac{-1}{2(1-(\rho)^2)} \left[\frac{(x-\bar{x})^2}{\sigma_{X^2}} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_X \sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_{Y^2}} \right] \right]$$

This is also called a bivariate guassian density function

The mean value of 'X' $= \bar{X} = E[X]$

The mean value of 'Y' $= \bar{Y} = E[Y]$

Variance of 'X' $= Var(X) = \sigma_X^2$

Variance of 'Y' $= Var(Y) = \sigma_Y^2$

Correlation coefficient of X & Y $= \rho_{XY} = \rho$

(ii) 'N' - random variables:

Consider 'N' random variables X_n, $n = 1, 2, \ldots N$, they are said to be jointly gaussian if their joint PDF is given by

$$f_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots x_N) = \frac{1}{(2\pi)^{N/2} |[C_X]|^{1/2}} Exp\left[\frac{-(x-\bar{x})^t [C_X]^{-1} [x-\bar{x}]}{2} \right]$$

where covariance matrix of 'N' random variables is

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}_{N \times N}$$

$$[x-\overline{X}] = \begin{bmatrix} x_1 - \overline{X_1} \\ x_2 - \overline{X_2} \\ \vdots \\ x_N - \overline{X_N} \end{bmatrix}_{1 \times N}$$

$[x-\overline{X}]^t$ = transpose of $[x-\overline{X}]$

$|[C_X]|$ = Determinant of $[C_X]$

$[C_X]^{-1}$ = Inverse of $[C_X]$

The elements of covariance matrix of C_X are given by

$$C_{ij} = E[(X_i - \overline{X_i})(X_j - \overline{X_j})] = C_{X_i X_j}$$

$$C_{ij} = \begin{cases} \sigma_{X_i}^2 & ; i = j \\ C_{ij} & ; i \neq j \end{cases}$$

* Properties of Gaussian R.V's :

1. Gaussian R.V's are completely defined by their mean, variances and covariances
2. If gaussian R.V's are uncorrelated, then they are independent
3. All marginal density functions derived from N-variate gaussian density function are gaussian.
4. All conditional PDF are also gaussian.
5. The linear transformations of gaussian R.V's are gaussian.

* Transformations of Multiple Random Variables:

M.P. let N random variables X_n, $n = 1, 2, 3, \ldots N$ be continuous or discrete. Now define another set of random variables Y_n, $n = 1, 2, 3, \ldots N$ by the transformation of X_n.

$$\boxed{Y_n = T_n(x_1, x_2, \ldots x_N) \; ; \; n = 1, 2, \ldots N}$$

where transformation T_n can be linear, non-linear, continuous etc.

$$\boxed{X_n = T_m^{-1}(y_1, y_2, \ldots y_N) \; ; \; m = 1, 2, \ldots N}$$

where T_m^{-1} is inverse continuous function.

If R_x & R_y are the closed regions of X and Y, respectively, then $\int\int_{R_x} \cdots \int f_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots x_N)\, dx_1 dx_2 \ldots dx_N$

$$= \int\int_{R_y} \cdots \int f_{Y_1, Y_2, \ldots, Y_N}(y_1, y_2, y_3 \ldots y_N)\, dy_1 dy_2 \ldots dy_N \rightarrow (1)$$

By applying transformations on random variables X_n, we get.

$$= \int\int_{R_x} \cdots \int f_{X_1, X_2 \ldots X_N}(x_1, x_2 \ldots, x_N)\, dx_1 dx_2 \ldots dx_N$$

$$= \int\int_{R_y} \cdots \int f_{Y_1, Y_2, \ldots Y_N}(x_1 = T_1^{-1}, x_2 = T_2^{-1}, \ldots, x_N = T_N^{-1})\, |J|\, dy_1 dy_2 \ldots dy_N \rightarrow (2)$$

where $|J|$ is the magnitude of Jacobian (J) of the transformations.

The Jacobian is the determinant of a matrix of derivatives and defined by

$$J = \begin{vmatrix} \frac{\partial T_1^{-1}}{\partial y_1} & \frac{\partial T_1^{-1}}{\partial y_2} & \cdots & \frac{\partial T_1^{-1}}{\partial y_N} \\ \frac{\partial T_2^{-1}}{\partial y_1} & \frac{\partial T_2^{-1}}{\partial y_2} & \cdots & \frac{\partial T_2^{-1}}{\partial y_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial T_N^{-1}}{\partial y_1} & \frac{\partial T_N^{-1}}{\partial y_2} & \cdots & \frac{\partial T_N^{-1}}{\partial y_N} \end{vmatrix}$$

Equating ① and ②, we get

$$\iint_{R_y} \cdots \int f_{Y_1, Y_2, \ldots Y_N}(y_1, y_2, y_3, \ldots, y_N)\, dy_1\, dy_2 \cdots dy_N$$

$$= \iint_{R_y} \cdots \int f_{X_1, X_2, \ldots X_N}(x_1 = T_1^{-1}, x_2 = T_2^{-1}, \ldots, x_N = T_N^{-1})\, |J|\, dy_1\, dy_2 \ldots dy_N$$

$$\boxed{\therefore f_{Y_1, Y_2, \ldots Y_N}(y_1, y_2, \ldots y_N) = f_{X_1, X_2, \ldots X_N}(x_1 = T_1^{-1}, x_2 = T_2^{-1}, \ldots, x_N = T_N^{-1})\, |J|}$$

Note 1:

For single r.v's transformation b/w X and Y, i.e., $N=1$ so new random variable $Y = TX$ and $X = T^{-1}Y$ then

$$f_Y(y) = f_X(x) \left|\frac{\partial T^{-1}}{\partial y}\right| \Rightarrow \boxed{f_Y(y) = f_X(x) \left|\frac{\partial x}{\partial y}\right|}$$

Note 2:

Two random variables transformation b/w (X_1, X_2) and (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = f_{X_1, X_2}(x_1, x_2) \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

* Linear Transformations of Gaussian Random Variables:

Consider N gaussian random variables Y_n, $n = 1, 2, \ldots, N$ having linear transformation with the set of X_n, $n = 1, 2, \ldots N$

The linear transformation can be written as.

$$Y_1 = a_{11} X_1 + a_{12} X_2 + \cdots + a_{1N} X_N$$

$$Y_2 = a_{21} X_1 + a_{22} X_2 + \cdots + a_{2N} X_N$$

$$\vdots$$

$$Y_{N} = a_{N1} X_1 + a_{N2} X_2 + \cdots + a_{NN} X_N$$

where the elements a_{ij} $i \& j = 1, 2, \ldots, N$ are real numbers.

Therefore the transformation in matrix form is

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

The transformation matrix $= [T] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$

$$\therefore [Y] = [T][X]$$

If the transformation is not singular, then

$$\therefore [X] = [T]^{-1}[Y]$$

also $[X - \bar{X}] = [T]^{-1}[Y - \bar{Y}]$

$$[Y - \bar{Y}] = [T][X - \bar{X}]$$

let the elements of $[T]^{-1}$ be b_{ij} i.e.,

$$[T]^{-1} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1N} \\ b_{21} & b_{22} & \cdots & b_{2N} \\ \vdots & & & \\ b_{N1} & b_{N2} & \cdots & b_{NN} \end{bmatrix}$$

$$[X] = [T]^{-1} Y$$

$$X_i = b_{i1} Y_1 + b_{i2} Y_2 + \cdots + b_{iN} Y_N$$

$$X_i - \overline{X_i} = b_{i1}(Y_1 - \overline{Y_1}) + b_{i2}(Y_2 - \overline{Y_2}) + \cdots + b_{iN}(Y_N - \overline{Y_N})$$

$$\frac{\partial x_i}{\partial y_i} = \frac{\partial T_i^{-1}}{\partial Y_i} = b_{ij}$$

$\therefore |J| =$ The determinant of the matrix $[T]^{-1}$

$$|J| = |[T]^{-1}| = \frac{1}{|[T]|}$$

$$C_{x_i x_j} = E\left[(X_i - \overline{X_i})(X_j - \overline{X_j})\right]$$

$$C_{x_i x_j} = E\Big[b_{i1}(Y_1 - \overline{Y_1}) + b_{i2}(Y_2 - \overline{Y_2}) + \cdots + b_{iN}(Y_N - \overline{Y_N}) \big(b_{j1}(Y_1 - \overline{Y_1}) + b_{j2}(Y_2 - \overline{Y_2}) + \cdots + b_{jN}(Y_N - \overline{Y_N})\big)\Big]$$

$$C_{x_i x_j} = \sum_{k=1}^{N} \sum_{m=1}^{N} b_{ik} b_{jm} C_{Y_k Y_m}$$

Here $C_{x_i x_j}$ is the ij^{th} element of $[C_x]$

$C_{Y_k Y_m}$ is the km^{th} element of $[C_Y]$

b_{ik} is the ik^{th} element of $[T]^{-1}$

$$\therefore [C_X] = [T]^{-1} [C_Y] [[T]^t]^{-1}$$

$$[C_Y] = [T][C_X][T]^t$$

$$[C_X]^{-1} = [T]^t \, [C_Y]^{-1} [T]$$

$$|[C_X]^{-1}| = |[T]^t [C_Y]^{-1} [T]|$$

$$= |[C_Y]^{-1}| \; |[T]|^2$$

The n-variant gaussian density function is

$$f_{X_1, X_2, \ldots X_N}(x_1, x_2, \ldots x_N) = \frac{1}{(2\pi)^{N/2} |[C_X]|^{1/2}} \, Exp\left[\frac{-(x-\bar{X})^t \, [C_X]^{-1} (x-\bar{X})}{2}\right]$$

The transformation of Y is

$$f_{Y_1, Y_2, \ldots Y_N}(y_1, y_2, \ldots, y_N) = f_{X_1, X_2, \ldots X_N}(x_1 = T_1^{-1}, x_2 = T_2^{-1}, \ldots x_N = T_N^{-1}) |J|$$

$$= \frac{1}{(2\pi)^{N/2} |[C_X]|^{1/2}} \, Exp\left[\frac{-(x-\bar{X})^t [T]^t [C_Y]^{-1} [T] (x-\bar{X})}{2}\right] |[T]^{-1}|$$

$$= \frac{|[C_X]^{-1}|^{1/2} |[T]^{-1}|}{(2\pi)^{N/2}} \, Exp\left[\frac{-(y-\bar{Y})^t [C_Y]^{-1} (y-\bar{Y})}{2}\right]$$

$$= \frac{|[T]^{-2}|^{1/2} \, |[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}} \, Exp\left[\frac{-(y-\bar{Y}) [C_Y]^{-1} (y-\bar{Y})}{2}\right]$$

$$= \frac{\left| |[T]^{-1}| \, [C_X^{-1}] \right|^{1/2}}{(2\pi)^{N/2}} \, Exp\left[\frac{-(y-\bar{Y}) [C_Y]^{-1} (y-\bar{Y})}{2}\right]$$

$$= \frac{|[C_Y]^{-1}|^{1/2}}{(2\pi)^{N/2}} \, Exp\left[\frac{-(y-\bar{Y}) [C_Y]^{-1} (y-\bar{Y})}{2}\right]$$

1. The joint characteristic function of R.V's X and Y is

v.imp $\phi_{XY}(\omega_1, \omega_2) = k \exp(-2\omega_1^2 - 8\omega_2^2)$

(i) Show that the mean values of X and Y are zero.

(ii) or X and Y uncorrelated ?

Sol: We know joint moments from joint characteristic function, i.e,
$$m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k}[\phi_{XY}(\omega_1, \omega_2)]}{\partial\omega_1^n \, \partial\omega_2^k} \right|_{\omega_1=\omega_2=0}$$

Given that $\phi_{XY}(\omega_1, \omega_2) = k \exp(-2\omega_1^2 - 8\omega_2^2)$

(i) The mean value of 'X' $= E[X] = m_{10} = -j \left. \frac{\partial[\phi_{XY}(\omega_1, \omega_2)]}{\partial\omega_1} \right|_{\omega_1=\omega_2=0}$

$$= -j \frac{\partial}{\partial\omega_1}\left[k \exp(-2\omega_1^2 - 8\omega_2^2)\right]\Big|_{\omega_1=\omega_2=0}$$

$$= -jk \exp(-2\omega_1^2 - 8\omega_2^2) \times (-4\omega_1 - 0)\Big|_{\omega_1=\omega_2=0}$$

$$= -jk \exp(-2(0)^2 - 8(0)^2) \times -4(0)$$

$\Rightarrow E[X] = 0$

The mean value of 'Y' $= E[Y] = m_{01} = -j \frac{\partial}{\partial\omega_2}[\phi_{XY}(\omega_1, \omega_2)]\Big|_{\omega_1=\omega_2=0}$

$$= -j \frac{\partial}{\partial\omega_2} k \exp(-2\omega_1^2 - 8\omega_2^2)\Big|_{\omega_1=\omega_2=0}$$

$$= -jk \exp(-2\omega_1^2 - 8\omega_2^2) \times (0 - 16\omega_2)\Big|_{\omega_1=\omega_2=0}$$

$$= -jk \exp(-2(0)^2 - 8(0)^2) \times -16(0)$$

$\Rightarrow E[Y] = 0$

Hence proved

(ii) We know the condition for uncorrelated random variables i.e., $R_{XY} = E[XY] = E[X]\,E[Y]$ (or)

$$C_{XY} = 0$$

The correlation b/w X and Y, $R_{XY} = E[XY]$

$$m_{11} = (-j)^2 \left.\frac{\partial^2 (\phi_{XY}(\omega_1, \omega_2))}{\partial \omega_1 \, \partial \omega_2}\right|_{\omega_1 = \omega_2 = 0}$$

$$= (-j^2) \frac{\partial}{\partial \omega_1}\left[\frac{\partial}{\partial \omega^2}\left(k \exp(-2\omega_1^2 - 8\omega_2^2)\right)\right]\Bigg|_{\omega_1=\omega_2=0}$$

$$= (-j)^2 \frac{\partial}{\partial \omega_1} k \exp(-2\omega_1^2 - 8\omega_2^2)(-16\omega_2)\Big|_{\omega_1=\omega_2=0}$$

$$= -1 \cdot k \times -16\omega_2 \frac{\partial}{\partial \omega_1} \exp(-2\omega_1^2 - 8\omega_2^2)\Big|_{\omega_1=\omega_2=0}$$

$$= k \times 16\omega_2 \exp(-2\omega_1^2 - 8\omega_2^2)(-4\omega_1)\Big|_{\omega_1=\omega_2=0}$$

$$= k \times 16(0) \exp(-2(0) - 8(0))(-4(0))$$

$\therefore E[XY] = 0$; $E[X]\,E[Y] = 0 \times 0 = 0$

$$E[XY] = E[X]\,E[Y]$$

Hence, $C_{XY} = E[XY] - E[X]\,E[Y] = 0 - 0$

$$C_{XY} = 0$$

Hence, the r.v's X and Y are uncorrelated.

2. Gaussian random variables X_1 & X_2 whose $\overline{X_1} = 2$, $\sigma_{X_1}^2 = 9$, $\overline{X_2} = -1$, $\sigma_{X_2}^2 = 4$ and $C_{X_1X_2} = -3$ are transformed to new random variables Y_1 & Y_2 such that $Y_1 = -X_1 + X_2$ & $Y_2 = -2X_1 - 3X_2$. Determine $\overline{X_1^2}$, $\overline{X_2^2}$, $R_{X_1X_2}$, $\rho_{X_1X_2}$, $\overline{Y_1}$, $\overline{Y_2}$, $\sigma_{Y_1}^2$, $\sigma_{Y_2}^2$, $R_{Y_1Y_2}$ & $\rho_{Y_1Y_2}$.

2.Sol: Given X_1, X_2 are gaussian R.V's

$\overline{X_1} = 2, \quad \sigma_X^2 = 9$

$\overline{X_2} = -1 \quad \sigma_Y^2 = 4 \quad C_{X_1X_2} = -3$

We know that $Var(X) = \sigma_X^2 = E[X^2] - [E(X)]^2$

$$\sigma_X^2 = \overline{X^2} - [\overline{X}]^2$$

$$\sigma_{X_1}^2 = \overline{X_1^2} - [\overline{X_1}]^2$$

$$\overline{X_1^2} = \sigma_{X_1}^2 + [\overline{X_1}]^2$$

$$= 9 + (2)^2$$

$$\boxed{\Rightarrow \overline{X_1^2} = 13}$$

$$\sigma_{X_2}^2 = \overline{X_2^2} - [\overline{X_2}]^2$$

$$\overline{X_2^2} = \sigma_{X_2}^2 + [\overline{X_2}]^2$$

$$= 4 + (-1)^2$$

$$\boxed{\Rightarrow \overline{X_2^2} = 5}$$

The correlation b/w X_1 and X_2 is

$$R_{X_1X_2} = E[X_1 X_2] = E[X_1]\,E[X_2] = ?$$

$$C_{X_1X_2} = R_{X_1X_2} - \overline{X_1}\,\overline{X_2}$$

$$C_{X_1X_2} = E[X_1X_2] - E[X_1]\,E[X_2]$$

$$\Rightarrow R_{X_1X_2} = [C_{X_1X_2}] + \overline{X_1}\,[\overline{X_2}]$$

$$= -3 + (2)(-1)$$

$$\Rightarrow \boxed{R_{X_1X_2} = -5}$$

$$\rho_{X_1X_2} = \frac{C_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}}$$

$$= \frac{-3}{\sqrt{9}\sqrt{4}}$$

$$= \frac{-3}{3 \times 2}$$

$$= \frac{-1}{2}$$

$$\Rightarrow \boxed{\rho_{X_1X_2} = -0.5}$$

Transformed variables are $Y_1 = -X_1 + X_2$; $Y_2 = -2X_1 - 3X_2$

$$\overline{Y_1} = E[Y_1] = E[-X_1 + X_2]$$

$$= -1\, E[X_1] + E[X_2]$$

$$= -1\,\overline{X_1} + \overline{X_2}$$

$$= -1(2) + (-1)$$

$$\boxed{\overline{Y_1} = -3}$$

$$\overline{Y_2} = E[Y_2] = E[-2X_1 - 3X_2]$$

$$= -2(2) - 3(-1)$$

$$\boxed{\overline{Y_2} = -1}$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$Y = [T]\, X$$

$\therefore$ The transformation matrix $= [T] = \begin{bmatrix} -1 & +1 \\ -2 & -3 \end{bmatrix}$

Covariance matrix of $Y = [C_Y] = [T]\,[C_X]\,[T]^t$

Covariance matrix of $X = [C_X] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} \\ C_{X_2X_1} & C_{X_2X_2} \end{bmatrix}$

$$[C_x] = \begin{bmatrix} C_{x_1x_1} & C_{x_1x_2} \\ C_{x_2x_1} & C_{x_2x_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{x_1}^2 & C_{x_1x_2} \\ C_{x_2x_1} & \sigma_{x_2}^2 \end{bmatrix} \quad \left\{ \begin{aligned} C_{x_1x_1} &= E[(x_1-\bar{x}_1)(x_1-\bar{x}_1)] \\ &= E[(x_1-\bar{x}_1)^2] \\ &= \sigma_{x_1}^2 \end{aligned} \right.$$

$$[C_x] = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \quad \left\{ \begin{aligned} C_{x_1x_2} &= E[(x_1-\bar{x}_1)(x_2-\bar{x}_2)] \\ C_{x_2x_1} &= E[(x_2-\bar{x}_2)(x_1-\bar{x}_1)] \end{aligned} \right.$$

Now $[C_y] = [T]\,[C_x]\,[T]^t$

$$= \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 9\times -1 + -3\times 1 & 9\times -2 + -3\times -3 \\ -3\times -1 + 4\times 1 & -3\times -2 + 4\times -3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -12 & -9 \\ 7 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 12+7 & +9-6 \\ 24-21 & +18+18 \end{bmatrix}$$

$$\therefore [C_y] = \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix}$$

(OR)

$$[C_y] = \begin{bmatrix} C_{y_1y_1} & C_{y_1y_2} \\ C_{y_2y_1} & C_{y_2y_2} \end{bmatrix} = \begin{bmatrix} \sigma_{y_1}^2 & C_{y_1y_2} \\ C_{y_2y_1} & \sigma_{y_2}^2 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 3 \\ 3 & 36 \end{bmatrix}$$

Variance of Y_1 = $Var(Y_1)$ = $\boxed{\sigma_{Y_1}^2 = .19}$

Variance of Y_2 = $Var(Y_2)$ = $\boxed{\sigma_{Y_2}^2 = 36}$

Covariance of Y_1 & Y_2 = $C_{Y_1Y_2}$ = $C_{Y_2Y_1} = 3$

The correlation b/w Y_1 & Y_2 = $R_{Y_1Y_2} = E[Y_1 Y_2]$

$$C_{Y_1Y_2} = R_{Y_1Y_2} - \overline{Y_1}\,\overline{Y_2}$$

$$R_{Y_1Y_2} = 3 + (-3)(-1)$$

$$= 3+3$$

$$\boxed{R_{Y_1Y_2} = 6}$$

$$\rho_{Y_1Y_2} = \frac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{3}{\sqrt{19}\sqrt{36}}$$

$$\boxed{\rho_{Y_1Y_2} = 0.1147}$$

Examind Y_1 and Y_2 are independent or not

$$E[Y_1 Y_2] = E[Y_1]\,E[Y_2]$$

$$E[Y_1 Y_2] \neq (-3)(-1) \neq E[Y_1]\,E[Y_2]$$

$\therefore$ Y_1, Y_2 are not independent and not uncorrelated

$$E[Y_1 Y_2] \neq 0$$

Y_1, Y_2 are not orthogonal.

3. The covariance matrix of X is $[C_X] = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.1 & 1 & 0.1 \\ 0.2 & 0.1 & 1 \end{bmatrix}$ and the transformation matrix is

$$[T] = \begin{bmatrix} 1 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$ Find covariance of new r.v's Y_1, Y_2 & Y_3 i.e. C_Y.

Sol: We know $[C_Y] = [T][C_X][T]^t$

$$= \begin{bmatrix} 1 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.1 & 1 & 0.1 \\ 0.2 & 0.1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & -2 \\ -2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5.4 & \cancel{-1.3}\ -7.1 & \cancel{-4}\ -2.8 \\ -1.3 & \cancel{8.6}\ 8.2 & \cancel{-5}\ -5.0 \\ \cancel{4} & \cancel{-5}\ 0.2 & 12 \end{bmatrix}$$

4. Two gaussian r.v's X_1 and X_2 are defined by the mean and covariance $[\bar{X}] = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $[C_X] = \begin{bmatrix} 5 & -2/\sqrt{5} \\ -2/\sqrt{5} & 4 \end{bmatrix}$

$[T] = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ → Two new r.v's transformed by this transformation matrix. Find $\bar{Y}$, $[C_Y]$, $\sigma_{Y_1}^2$, $\sigma_{Y_2}^2$, $C_{Y_1Y_2}$, $\rho_{Y_1Y_2}$.

Sol: We know $[Y] = [T][X]$

$$[\bar{Y}] = [T][\bar{X}]$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0.5 \\ 1-1 \end{bmatrix}$$

$$\therefore [\bar{Y}] = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

$$[C_Y] = [T][C_X][T]^t$$

$$= \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -2/\sqrt{5} \\ -\frac{2}{\sqrt{5}} & 4 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

$$[C_Y] = \begin{bmatrix} 5.1 & 3.382 \\ 3.382 & 4.3556 \end{bmatrix} \simeq \begin{bmatrix} \sigma_{Y_1}^2 & C_{Y_1Y_2} \\ C_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

$$Var(Y_1) = \sigma_{Y_1}^2 = 5.1$$

$$Var(Y_2) = \sigma_{Y_2}^2 = 4.3536$$

$$C_{Y_1Y_2} = C_{Y_2Y_1} = 3.382$$

$$\therefore \rho_{Y_1Y_2} = \frac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \frac{3.382}{\sqrt{5.1}\sqrt{4.3536}}$$

$$\boxed{\rho_{Y_1Y_2} = 0.717}$$

From the given $[C_X]$; $\sigma_{X_1}^2 = 5$

$\sigma_{X_2}^2 = 4$

$C_{X_1X_2} = C_{X_2X_1} = -\frac{2}{\sqrt{5}}$

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = [\bar{X}]$$

$$[\bar{Y}] = \begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

5. Let two r.v's Y_1, Y_2 be linear transformation of X_1 and X_2 given by $Y_1 = X_1 + X_2$, $Y_2 = 2X_1 + 3X_2$ if $f_{X_1X_2}(x_1, x_2)$ is a joint PDF then find joint PDF of Y_1 & Y_2.

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$[Y] = [T][X]$$

$$= [T] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= f_{y_1 y_2}(y_1 y_2) = f_{x_1 x_2}(x_1 = T_1^{-1}, x_2 = T_2^{-1}) \, |J|$$

$$|J| = \left| [T]^{-1} \right| = \left| \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \right|^{-1}$$

$$= \frac{1}{3-2} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$[T]^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$$

$$|J| = \begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix} = 3 - 2 = 1$$

$$[T]^{-1}[Y] = \frac{3y_1 - y_2}{-2y_1 + y_2}$$

$$= f_{x_1 x_2}(3y_1 - y_2, \; -2y_1 + y_2) \times 1$$

$$= f_{x_1 x_2}(3y_1 - y_2, \; -2y_1 + y_2)$$

6. Gaussian r.v's X_1 and X_2 whose $\overline{X_1} = 2$, $\overline{X_2} = -1$, $\sigma_{x_1}^2 = 9$, $\sigma_{x_2}^2 = 4$, $C_{x_1 x_2} = -3$, $y_1 = -x_1 + y_2$, $y_2 = -2x_1 - 3x_2$

Find $\overline{x_1^2}$, $\overline{x_2^2}$, $R_{x_1 x_2}$, $\rho_{x_1 x_2}$, $\overline{y_1^2}$, $\overline{y_2^2}$, $R_{y_1 y_2}$, $C_{y_1 y_2}$, $\sigma_{y_1}^2$, $\sigma_{y_2}^2$, $\rho_{y_1 y_2}$

Hint: $y_1 = (+x_1) + (-2x_1 - 3x_2) = -3x_1 - 3x_2$

9. In a control system a random voltage x is known $\overline{x} = m_1 = -2V$ and the second moment $\overline{x^2} = m_2 = 9V^2$, if the voltage x is amplified by an amplifier that gives output $y = -1.5x + 2$. Find σ_x^2, $\overline{y}$, $\overline{y^2}$, σ_y^2, R_{xy}, C_{xy}, $\rho_{x,y}$.

Sol:

8. Two random variables X & Y have density functions

Second c.w

$$f_{XY}(x,y) = \begin{cases} 2/43\,(x+0.5y)^2 & ; 0<x<2,\ 0<y<3 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) Find all the first and second order moments. about origin and about mean

(ii) Find covariance $C_{XY} = E[XY] - E[X]E[Y]$

(iii) or X and Y uncorrelated

9. Determine the variance of $y = -6x + 22$. Determine

Sol: Given $\overline{y} = -6\overline{x} + 22 \Rightarrow y^2 = (-6x+22)^2$

$$y^2 = 36x^2 + 144 - 264x$$

$$\overline{y^2} = 36\overline{x^2} + 144 - 264\overline{x}$$

$$\text{Var}(y) = E[y^2] - [E[y]]^2$$

$$= \overline{y^2} - [\overline{y}]^2$$

$$= 36\overline{x^2} + 144 - 264\overline{x} - [-6x+22]^2$$

$$= \cancel{36\overline{x^2}} + \cancel{144} - \cancel{264\overline{x}} - \cancel{36x^2} - 144 + 264x = 0$$

10. Let x and y be two independent variables, then prove that $\text{Var}(xy) = \text{Var}(x)\,\text{Var}(y)$ if $E[x]\,E[y] = 0$

Sol: We know $\text{Var}(y) = E(y^2) - [E(y)]^2$

$$\text{Var}(x) = E[x^2] - [E(x)]^2$$

$$\text{Var}(xy) = E[(xy)^2] - [E(xy)]^2$$

If x and y are independent $E(xy) = E(x)E(y)$

$$E(xy)^2 = E[x^2y^2] = E(x^2)\,E(y^2)$$

$$\text{Var}(xy) = E[x^2]E[y^2] - [E(x)\,E(y)]^2$$

But given $E[x] = E[y] = 0$

$|||^{ly}$ $\text{Var}(x)\,\text{Var}(y) = 0$ is also proved

Hence the solution is proved.

A random variable has PDF $f_Z(z) = a e^{-a(z-b)} u(z-b)$

M.P. Show that the characteristic function Z is $\phi_Z(\omega) = \frac{a}{a-j\omega} e^{-jab}$

has probability function $P(x) = \frac{1}{3^{2x}}$; $x = 1, 2, \ldots N$

Sol: The characteristic function $= \phi_Z(\omega)$

$$= E[e^{j\omega z}] = \int_{-\infty}^{\infty} e^{j\omega z} f_Z(z)\, dz$$

$$\phi_Z(\omega) = \int_{-\infty}^{\infty} a\, e^{-a(z-b)} u(z-b)\, e^{j\omega z}\, dz$$

$$= a \int_{-\infty}^{\infty} e^{-az} \cdot e^{ab}\, e^{j\omega z}\, u(z-b)\, dz$$

$$u(z-b) = \begin{cases} 1 ; & z \geq b \\ 0 ; & z < b \end{cases}$$

$$= a\, e^{ab} \int_{-\infty}^{\infty} e^{-(a-j\omega)z}\, u(z-b)\, dz$$

$$= a e^{ab} \left[\int_{-\infty}^{b} e^{-(a-j\omega)z} (0)\, dz + \int_{b}^{\infty} e^{-(a+j\omega)z} (1)\, dz \right]$$

$$= a e^{ab} \left[0 + \int_{b}^{\infty} e^{-(a-j\omega)z}\, dz \right]$$

$$= a e^{ab} \left[\frac{e^{-z(a-j\omega)}}{-(a-j\omega)} \right]_b^{\infty}$$

$$= \frac{a e^{ab}}{\cancel{-(a-j\omega)}} \left[0 - e^{-\cancel{a}(a-j\omega)} \right]$$

$$= a e^{ab} \left[\frac{0 - e^{-(a-j\omega) \times b}}{-(a-j\omega)} \right]$$

$$= a \cancel{e^{ab}} \times \frac{\cancel{e^{-ab}}\, e^{j\omega b}}{a-j\omega}$$

$$= \frac{a\, e^{j\omega b}}{a-j\omega}$$

12. The joint PDF of X and Y is $f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{3}} e^{-2/3(x^2-xy+y^2)}$

Previous. Find the marginal PDF of X and Y.

Sol: Given $f_{X,Y}(x,y) = \frac{1}{\pi\sqrt{3}} e^{-2/3(x^2-xy+y^2)}$

Marginal PDF of 'x' $= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{\pi\sqrt{3}} e^{-2/3(x^2-xy+y^2)}\,dy$$

$$= \frac{1}{\pi\sqrt{3}} e^{-2/3x^2} \int_{-\infty}^{\infty} e^{+2/3xy} \cdot e^{-2/3y^2}\,dy$$

$$= \frac{e^{-2/3x^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3(y^2-xy)}\,dy$$

$$= \frac{e^{-2/3x^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3}\left(\left(y-\frac{x}{2}\right)^2 - \frac{x^2}{4}\right)dy$$

Here $y^2 - xy = y^2 - 2xy\left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^2$

$$= \left(y-\frac{x}{2}\right)^2 - \frac{x^2}{4}$$

$$= \frac{e^{-2/3x^2} \cdot e^{\frac{2x^2}{12}}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3\left(y-\frac{x}{2}\right)^2}\,dy$$

$$= \frac{e^{-2/3x^2} \cdot e^{x^2/6}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3\left(y-\frac{x}{2}\right)^2}\,dy$$

Put $y - \frac{x}{2} = t \Rightarrow dy = dt$

$y \to -\infty \Rightarrow t \to -\infty$

$y \Rightarrow \infty \Rightarrow t \to \infty$

$$= \frac{e^{-\frac{2}{3}x^2 + \frac{x^2}{6}}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{2}{3}(t)^2} dt$$

$$= \frac{e^{\left(-\frac{2}{3}x^2 + \frac{x^2}{6}\right)}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\frac{2}{3}(t)^2} dt$$

$$= \frac{e^{-\frac{x^2}{2}}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{2}}{\sqrt{3}}t\right)^2} dt$$

Put $\frac{\sqrt{2}}{\sqrt{3}} t = \tau \Rightarrow \frac{\sqrt{2}}{\sqrt{3}} dt = d\tau$

$$dt = \frac{\sqrt{2}}{\sqrt{3}} d\tau$$

$$= \frac{e^{-\frac{x^2}{2}}}{\pi\cancel{\sqrt{3}}} \int_{-\infty}^{\infty} e^{-(\tau)^2} \frac{\cancel{\sqrt{3}}}{\sqrt{2}} \, d\tau$$

$$= \frac{e^{\frac{-x^2}{2}}}{\sqrt{2}\,\pi} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau$$

$$= \frac{e^{-x^2/2}}{\sqrt{2}\,\pi} \cdot 2\int_{0}^{\infty} e^{-\tau^2} d\tau \quad (\because \text{ even function})$$

$$= \frac{e^{-\frac{x^2}{2}}}{\sqrt{\pi}\cancel{\pi}\sqrt{2}} \times \cancel{2} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$\boxed{\therefore f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}}$$

Marginal PDF of $y = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = f_Y(y)$

$$f_Y(y) = \frac{1}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3\,(x^2 - xy + y^2)} dx$$

$$= \frac{e^{-2/3\, y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3\,(x^2 - xy)} dx$$

$$= \frac{e^{-2/3 y^2}}{\sqrt{3}\,\pi} \int_{-\infty}^{\infty} e^{-2/3 (x^2 - xy)}\, dx$$

$$\text{Let } x^2 - xy = x^2 - xy \cdot 2 \times \frac{y}{2} + \left(\frac{y}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$x^2 - xy = \left(x - \frac{y}{2}\right)^2 - \frac{y^2}{4}$$

$$= \frac{e^{-2/3 y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3 \left(x - \frac{y}{2}\right)^2 - \cancel{\frac{y^2}{4}}}\, dx$$

$$= \frac{e^{-2/3 y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3 \left(x - \frac{y}{2}\right)^2}\, dx$$

$$\text{Put } x - \frac{y}{2} = t \Rightarrow dx = dt.$$

$$x \to -\infty \Rightarrow t \to -\infty$$
$$x \to \infty \Rightarrow t \to \infty$$

$$= \frac{e^{-2/3 y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-2/3 (t)^2}\, dt$$

$$= \frac{e^{-2/3 y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-\left(\frac{\sqrt{2}}{\sqrt{3}} t\right)^2}\, dt$$

$$\text{Put } \frac{\sqrt{2}}{\sqrt{3}} t = r$$

$$\frac{\sqrt{2}}{\sqrt{3}}\, dt = dr$$

$$dt = \frac{\sqrt{3}}{\sqrt{2}}\, dr$$

$$= \frac{e^{-2/3 y^2}}{\pi\sqrt{3}} \int_{-\infty}^{\infty} e^{-r^2} \frac{\sqrt{3}}{\sqrt{2}}\, dr$$

$$= \frac{e^{-2/3 y^2}}{\pi \cancel{\sqrt{3}}} \cdot \frac{\cancel{\sqrt{3}}}{\sqrt{2}} \cdot \cancel{2} \cdot \frac{\sqrt{\pi}}{\cancel{2}}$$

$$\boxed{f_Y(y) = \frac{e^{-2/3 y^2}}{\sqrt{2\pi}}}$$

6 Sol: Given $\overline{X_1} = 2, \overline{X_2} = -1$

M.P

$\sigma_{X_1}^2 = 9 \; ; \; \sigma_{X_2}^2 = 4$

$C_{X_1X_2} = -3, \quad y_1 = -x_1 + y_2 \; ; \; y_2 = -2x_1 - 3x_2$

$y_1 = -x_1 - 2x_1 - 3x_2 = -3x_1 - 3x_2$

Variance of $X_1 = \sigma_X^2 = E[X^2] - [E(X)]^2$

$\sigma_{X_1}^2 = \overline{X_1^2} - (\overline{X_1})^2$

$\overline{X_1^2} = \sigma_{X_1}^2 + (\overline{X_1})^2$

$= 9 + 4$

$\therefore \overline{X_1^2} = 13$

Variance of $X_2 = \sigma_{X_2}^2 = E[X_2^2] - [E(X_2)]^2$

$\sigma_{X_2}^2 = \overline{X_2^2} - (\overline{X_2})^2$

$\therefore \overline{X_2^2} = \sigma_{X_2}^2 + (\overline{X_2})^2$

$= 4 + 1 = 5$

$R_{X_1X_2} = E[X_1 X_2] = E[X_1]\, E[X_2] = ?$

$C_{X_1X_2} = R_{X_1X_2} - \overline{X_1}\,\overline{X_2}$

$R_{X_1X_2} = C_{X_1X_2} + \overline{X_1}\,\overline{X_2}$

$= -3 + (2)(-1)$

$R_{X_1X_2} = -5$

$\rho_{X_1X_2} = \dfrac{C_{X_1X_2}}{\sigma_{X_1}\sigma_{X_2}}$

$= \dfrac{-3}{\sqrt{9}\sqrt{4}}$

$\rho_{X_1X_2} = \dfrac{-1}{2} = -0.5$

$\overline{Y_1} = E[y_1] = E[-3x_1 - 3x_2]$

$= -3\,E[X_1] - 3\,E[X_2]$

$= -3(2) - 3(-1)$

$= -6 + 3$

$\overline{Y_1} = -3$

$\overline{Y_2} = -2E[\overline{X_1}] - 3\,E[\overline{X_2}]$

$= -2(2) - 3(-1)$

$= -4 + 3$

$\therefore \overline{Y_2} = -1$

$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ → Transformation matrix

Covariance of matrix $Y = [C_Y] = [T][C_X][T]^t$

Covariance matrix of $X = [C_X] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} \\ C_{X_2X_1} & C_{X_2X_2} \end{bmatrix}$

$= \begin{bmatrix} \sigma_{X_1}^2 & C_{X_1X_2} \\ C_{X_1X_2} & \sigma_{X_2}^2 \end{bmatrix}$

$[C_X] = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}$

Now $[C_Y] = \begin{bmatrix} -3 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -3 & -3 \end{bmatrix}$

$[C_Y] = \begin{bmatrix} 63 & 45 \\ 45 & 36 \end{bmatrix} = \begin{bmatrix} \sigma_{Y_1}^2 & C_{Y_1Y_2} \\ C_{Y_1Y_2} & \sigma_{Y_2}^2 \end{bmatrix}$

$\therefore$ Variance of $Y_1 = \boxed{\sigma_{Y_1}^2 = 63}$

Variance of $Y_2 = \boxed{\sigma_{Y_2}^2 = 36}$

Covariance of $Y_1, Y_2 = \boxed{C_{Y_1Y_2} = 45}$

Correlation b/w Y_1 & $Y_2 = R_{Y_1Y_2} - E[Y_1]\,E[Y_2]$

$C_{Y_1Y_2} = R_{Y_1Y_2} - E[Y_1]\,E[Y_2]$

$R_{Y_1Y_2} = 45 - (-3)(-1)$

$\boxed{\therefore R_{Y_1Y_2} = 42}$

$\rho_{Y_1Y_2} = \dfrac{C_{Y_1Y_2}}{\sigma_{Y_1}\sigma_{Y_2}} = \dfrac{45}{\sqrt{63}\sqrt{36}} = 0.9449$

$\boxed{\therefore \rho_{Y_1Y_2} = 0.9449}$

7 - Sol: Given $x = m_1 = -2V$

$$\overline{x^2} = m_2 = 9V^2$$

$$y = -1.5x + 2$$

Variance of x $\sigma_x^2 = E(x^2) - [E(x)]^2$

$$= 9 - (-2)^2$$

$$= 9 - 4$$

$$\therefore \sigma_x^2 = 5$$

$$\overline{y} = E[y] = E[-1.5(x) + 2]$$

$$= -1.5\,E[x] + 2$$

$$= -1.5 \times (-2) + 2$$

$$\therefore \overline{y} = 5$$

$$y^2 = (-1.5x + 2)^2 = 2.25x^2 + 4 + 6x$$

$$\overline{y^2} = 2.25E(x^2) + 4 + 6E[x]$$

$$= 2.25(9) + 4 + 6(-2)$$

$$\overline{y^2} = 3\cancel{2}6.25$$

$$\sigma_y^2 = E[y^2] - [E[y]]^2$$

$$= 3\cancel{2}6.25 - (5)^2$$

$$\sigma_y^2 = 11.25$$

$$R_{xy} = \cancel{E[xy]}\ E(x)\,E(y)$$

$$= \overline{x}\ \overline{y}$$

$$= (-2)(5)$$

$$R_{xy} = -10$$

$$C_{xy} = R_{xy} - E(x)\,E(y)$$

$$= -10 - (-10)$$

$$C_{xy} = 0$$

$$\rho_{xy} = \frac{C_{xy}}{\sigma_y \sigma_x} = 0$$

13/9/19.