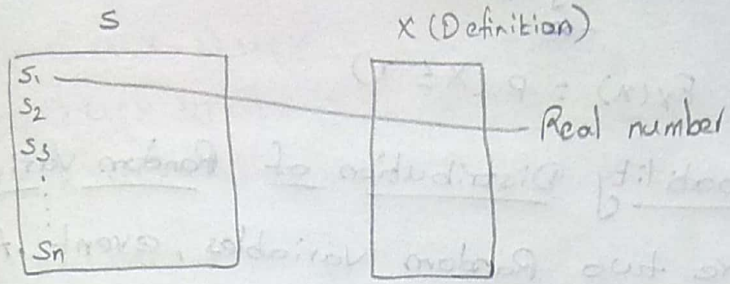
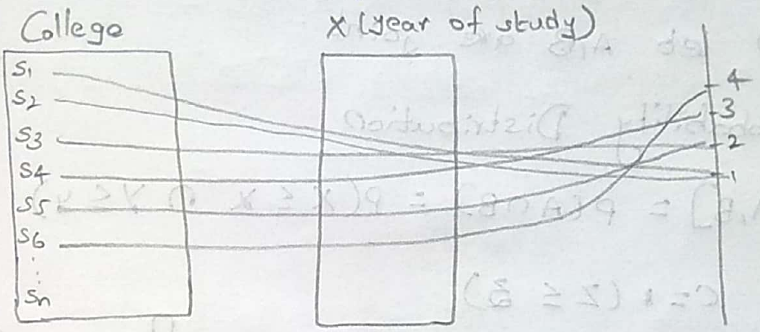
UNIT-3

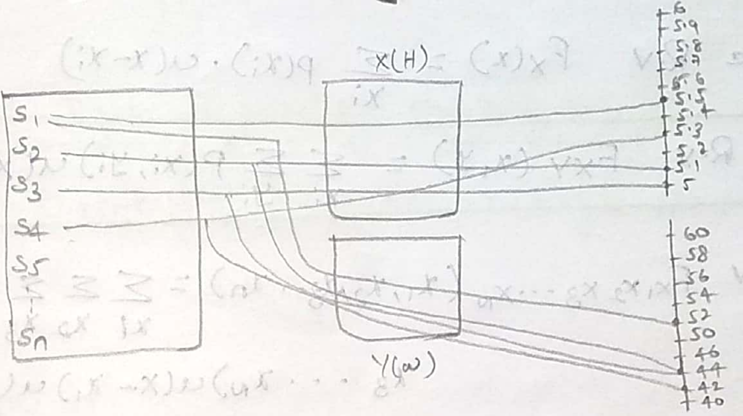
MULTIPLE RANDOM VARIABLES



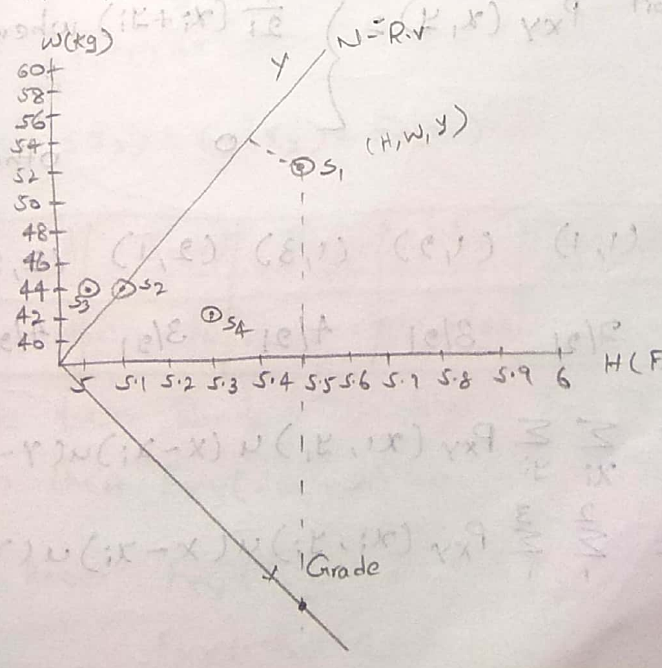
Random variable → Sample space → Real "$P(X \leq x) = F_X(x)$"



$X = \{1, 1, 2, 3, 2, 4 \ldots \ldots\}$



	s_1	s_2	s_3	s_4	s_5	s_6	s_7 . . . . . s_n
X(H)	5.5	5.1	5	5.3			
Y(ω)	52	44	44	42	; .	.	.



Probability Distribution of Random Variable

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$F_X(x) = P(X \le x)$$

Joint probability Distribution of Random Variable

X, Y are two Random Variables, event A is represented as $\{X \le x\}$ and event B is represented a $\{Y \le y)$ let A, B are joint.

Joint probability Distribution

$$P[A,B] = P(A \cap B) = P(X \le x \cap Y \le y) \quad F_{XY}(x,y)$$

$$C = (Z \le z)$$

$$P(A,B,C) = P(X \le x \cap Y \le y \cap Z \le z)$$

Single R.V $F_X(x) = \sum_{x_i} p(x_i) \cdot u(x - x_i)$

$$\boxed{\text{Two R.V} \quad F_{XY}(x,y) = \sum_{x_i} \sum_{y_i} P(x_i, y_i)\, u(X - x_i)\, u(Y - y_i)}$$

N R.V $F_{X_1 X_2 X_3 \cdots X_N}(x_1, x_2, x_3 \cdots x_n) = \sum_{x_1} \sum_{x_2} \sum_{x_3} \cdots \sum_{x_N} P(x_1, x_2,$

$x_3 \cdots x_N)\, u(X - x_1)\, u(x - x_2) \cdots u(X - x_N$

1.) Find Joint probability Distribution of given probability mass function $P_{XY}(x,y) = \begin{cases} \frac{1}{21}(x_i + y_i) & \text{where } x_i = 1,2, \\ & y_i = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$

(X, Y)	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)
$P_{XY}(x,y)$	2/21	3/21	4/21	3/21	4/21	5/21

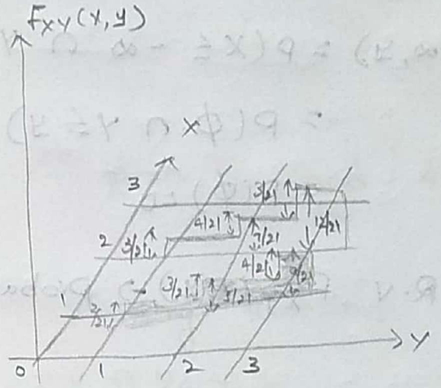
$$F_{XY}(x,y) = \sum_{x_i} \sum_{y_i} P_{XY}(x_1, y_1)\, u(X - x_i)\, u(Y - y_i)$$

$$= \sum_{1}^{2} \sum_{1}^{3} P_{XY}(x_i, y_i)\, u(X - x_i)\, u(Y - y_i)$$

$$= \sum_{i=1}^{2} \left[P_{XY}(x_i, 1)\, u(X - x_i)\, u(y-1) + P_{XY}(x_i, 2)\, u(X - x_i)\, u(y-2) + P_{XY}(x_i, 3) \cdot u(X - x_i) \cdot u(y-3) \right]$$

$$= P_{XY}(1,1)\, u(X-1)\, u(y-1) + P_{XY}(1,2)\, u(X-1)\, u(y-3) + P_{XY}(2,1)\, u(X-2)\, u(y-1) + P_{XY}(2,2)\, u(X-2)\, u(y-2) + P_{XY}(2,3)\, u(X-2)\, u(y-3)$$

$$= \frac{2}{21} u(X-1)\, u(y-1) + \frac{3}{21} u(X-1)\, u(y-2) + \frac{4}{21} u(x-1)(y-3) + \frac{3}{21} u(X-2)\, u(y-1) + \frac{4}{21} u(X-2)\, u(y-2) + \frac{5}{21} u(x-2)\, u(y-3)$$



Properties of Joint Probability Distribution :-

Probability Distribution properties of single Rv :-

1) $F_X(-\infty) = 0$

2) $F_X(\infty) = 1$

3) $0 \leq F_X(x) \leq 1$

4) $F_X(x)$ is nondecreasing.

5) $F_X(x)$ is non negative

6) $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$

7) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

Properties of Joint Probability Distribution :-

X, Y are two Random Variables.

1) X, Y are two Random variables, $F_{XY}(x,y)$ is distribution function then $F_{XY}(-\infty, -\infty) = 0$.

Proof:- We know $F_{XY}(x, y) = P(X \leq x \cap Y \leq y)$

$F_{XY}(-\infty, -\infty) = ?$

$F_{xy}(-\infty,-\infty) = P(X \le -\infty \cap Y \le -\infty)$

$= P(\phi \cap \phi)$

$= P(\phi) = 0$

2) Let X, Y are two random variables $\rightarrow F_{xy}(x,y)$,

$F_{xy}(-\infty, y) = 0$.

Proof:- We know

$F_{XY}(x,y) = P(X \le x \cap Y \le y)$

$F_{XY}(-\infty, y) = P(X \le -\infty \cap Y \le y)$

$= P(\phi \cap Y \le y)$

$= P(\phi) = 0$

3.) X, Y are two R.V $F_{XY}(x,y) \rightarrow$ probability distribution

$F_{XY}(x,-\infty) = 0$.

Proof:- We know $F_{XY}(x,y) = P(X \le x \cap Y \le y)$

$F_{XY}(x,-\infty) = P(X \le x \cap Y \le -\infty)$

$= P(X \le x \cap \phi)$

$= P(\phi) = 0$.

It is like x 2^{nd} property of single R.v

4.) Let X, Y are two R.V, $F_{xy}(x,y)$ is the probability Distribution then $F_{xy}(\infty,\infty) = 1$

Proof:- $F_{xy}(x,y) = P(X \le x \cap Y \le y)$

$F_{XY}(\infty,\infty) = P(X \le \infty \cap Y \le \infty)$

$= P(S \cap S) = P(S) = 1$

5) Let X, Y are two R.V, $F_{xy}(x,y)$ is the probability distribution then $F_{xy}(\infty, y) = F_Y(y)$.

This is also called as "Marginal Distribution".

Proof:- $F_{xy}(x, y) = P(X \le x \cap Y \le y)$

$F_{XY}(\infty, y) = P(X \le \infty \cap Y \le y)$

$= P(S \cap Y \le y)$

$= P(Y \le y) = F_Y(y)$ [Marginal distribution of y]

Find marginal distribution of R.V y then we have to take $x = \infty$, $y = y$

6.) X, Y are two R.V, $F_{XY}(x, y)$ is the probability distribution $F_{XY}(x, \infty) = F_X(x)$. [Marginal distribution of X]

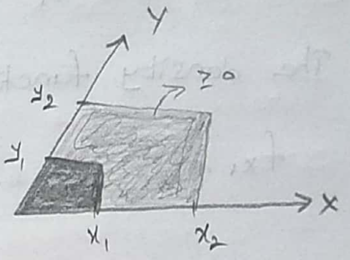
Proof:- $F_{XY}(x, y) = P(X \leq x \cap Y \leq y)$

$$F_{XY}(x, \infty) = P(X \leq x \cap Y \leq \infty)$$
$$= P(X \leq x \cap S)$$
$$= P(X \leq x) = F_X(x).$$

7.) $0 \leq F_{XY}(x, y) \leq 1$

Based on the 1st property and 4th property

8.) $F_{XY}(x, y)$ is nondecreasing function



$F_{XY}(x_2, y_2) = P(X \leq x_2 \cap Y \leq y_2)$

$F_{XY}(x_1, y_1) = P(X \leq x_1 \cap Y \leq y_1)$

$P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2)$

$$P(X \leq x_1 \cap Y \leq y_1) + \underbrace{P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2)}_{\geq 0} = P(X \leq x_2 \cap Y \leq y_2)$$

$$P(X \leq x_1 \cap Y \leq y_1) = P(X \leq x_2 \cap Y \leq y_2)$$

$$P(X \leq x_1 \cap Y \leq y_1) \leq P(X \leq x_2 \cap Y \leq y_2)$$

small

9.) $P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_1)$

Proof:- We know,

$$F_{XY}(x_2, y_2) = P(X \leq x_2 \cap Y \leq y_2)$$
$$= P(X \leq x_1 \cap Y \leq y_1) + P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2)$$

$$F_{XY}(x_2, y_2) = F_{XY}(x_1, y_1) + P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2)$$

$$P(x_1 < X \leq x_2 \cap y_1 < Y \leq y_2) = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_1)$$

Joint probability density function :-

density function $f_X(x) = \frac{d}{dx} F_X(x)$, $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$

Joint density function :- Let X, Y are two Random variables

$F_{XY}(x,y)$ is the Joint Distribution function,

$f_{XY}(x,y)$ is the Joint density function.

$$f_{XY}(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x,y)$$

Joint Distribution in density function :-

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dy\,dx.$$

Let we have $x_1, x_2, x_3, x_4 \cdots x_N$ Random variables

The density function is represented as

$$f_{x_1,x_2,X_3,X_4\ldots X_N}(x_1,x_2,x_3,\ldots x_N) = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_3}\cdots\frac{\partial}{\partial x_n} F_{x_1x_2x_3\ldots x_N}(x_1,x_2,x_3\ldots x_n)$$

$$F_{x_1,x_2,x_3\ldots x_N}(x_1,x_2,x_3,\ldots x_N) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_N} f_{x_1,x_2,x_3\ldots x_N}(x_1,x_2,x_3\ldots x_N)\,dx_1dx_2dx_3\ldots d$$

Properties of Joint Probability Density function:-

Let X, Y are two Random Variables and $f_{XY}(x,y)$ is the ~~x~~ joint density function

① $f_{XY}(x,y) \geq 0$

② $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

③ $F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x,y)\,dx\,dy$

④ $P(x_1 \leq X \leq x_2,\ y_1 \leq Y \leq y_2) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dy\,dx.$

① $f_X(x) \geq 0$

② $P(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\,dx$

③ $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

④ $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

⑤ *** Marginal density function

$$F_{XY}(x, \infty) = F_X(x)$$

$$F_{XY}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x, y)\,dy\cdot dx.$$

$$F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dy\cdot dx.$$

$$\frac{\partial}{\partial x} F_X(x) = \frac{\partial}{\partial x} F_{XY}(x, \infty) = \frac{\partial}{\partial x}\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dy\,dx$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy \rightarrow$$ Marginal density function of X.

Marginal density function of Y $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx.$

$$F_{XY}(\infty, y) = F_Y(y)$$

$$F_{XY}(x, y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x, y)\,dy\,dx.$$

$$F_Y(y) = F_{XY}(\infty, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{XY}(x, y)\,dy\cdot dx$$

$$\frac{\partial}{\partial y} F_Y(y) = \frac{\partial}{\partial y} F_{XY}(\infty, y) = \frac{\partial}{\partial y}\int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{XY}(x, y)\,dy\,dx.$$

$$F_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx$$

Statistically Independent R.V (S.I) :-

X, Y are two R.V Joint density function $f_{XY}(x, y)$. if ~~occurran~~ the probability of one variable is not depend on another they said to be Independent R.V.

$$\boxed{f_{XY}(x, y) = f_X(x)\cdot f_Y(y)}$$

Conditional Joint Distribution and density function:-

$$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

let X, Y are two R.V , $A = \{X \le x\}$

$B = \{Y \le y\}$

Conditional joint Distribution function

$$F_{X/Y}(x/y) = \frac{P(X \le x \cap Y \le y)}{P(Y \le y)} = \frac{F_{XY}(x, y)}{F_Y(y)}$$

$$\left(F_{Y/X}(y/x) = \frac{P(X \leq x \cap Y \leq y)}{P(X \leq x)} = \frac{F_{XY}(x,y)}{F_X(x)} \right.$$

conditional Joint Density function

$$\left\{ \begin{array}{l} f_{X/Y}(x/y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} \\ f_{Y/X}(y/x) = \dfrac{f_{XY}(x,y)}{f_X(x)} \end{array} \right.$$

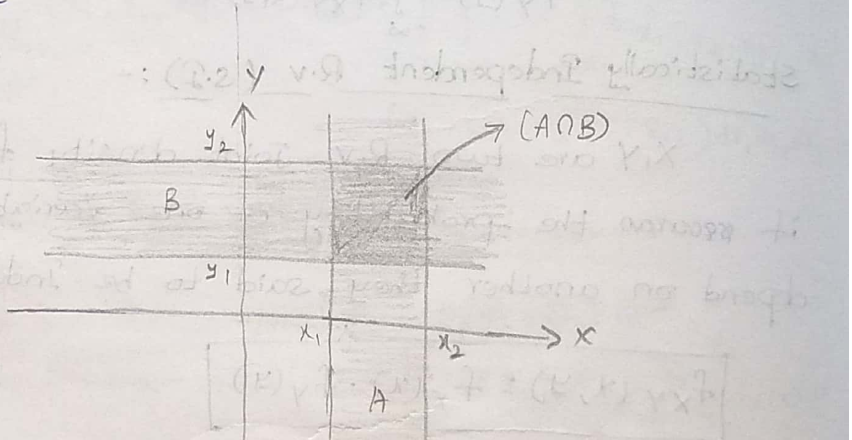
if X, Y are statistically Independent

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{f_X(x) \cdot \cancel{f_Y(y)}}{\cancel{f_Y(y)}} = f_X(x)$$

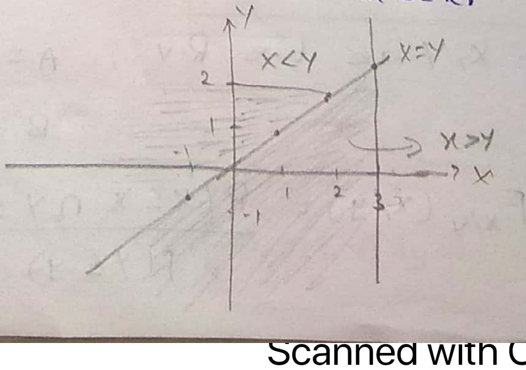
$$f_{Y/X}(y/x) = f_Y(y).$$

Sum of two Random Variables :-

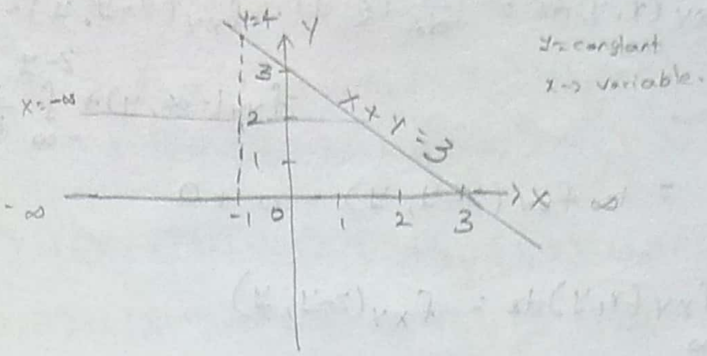
Two R.V A, B are generated by passing the sample space S through two Random variables X, Y and event A is represented as $\{x_1 \leq X \leq x_2\}$ and event B is represented as $B = \{y_1 \leq Y \leq y_2\}$. Find the Joint area on ~~X~~ Y plane.



Let two R.V X, Y are linear one with another map X = Y, X < Y, X > Y



Let X, Y are two R.V X + Y = 3 place the representation on X-Y plane.



** Sum of Two Random Variables

Statement:- Let X, Y are two R.V, having Joint probability density function $f_{XY}(x, y)$, if X, Y are added generates R.V $Z = X + Y$, $f_Z(z) = f_X(z) * f_Y(z)$ [∵ X, Y are S.I]

Leibnitz's rule:-

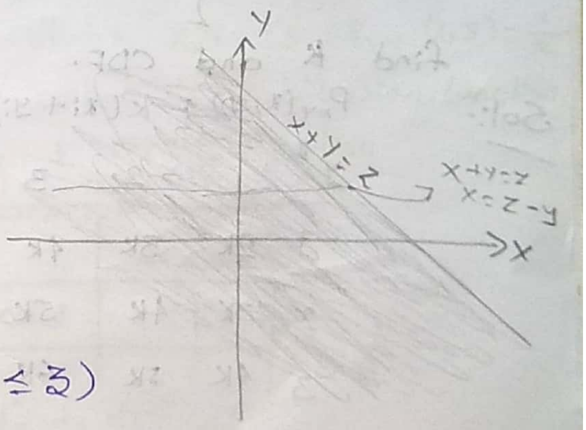
$$H(x) = \int_{a(x)}^{b(x)} g(x, y)\,dy.$$

Let diff H(x) w.r.t 'x'.

$$\frac{d}{dx} H(x) = \frac{d}{dx} b(x) \cdot g(x, b(x)) - \frac{d}{dx} a(x) \cdot g(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} g(x, y)\,dy.$$

$X, Y \rightarrow f_{XY}(x, y)$

$Z = X + Y \rightarrow f_Z(z) = ?$



probability distribution function of Z

$$F_Z(z) = P(Z \leq z) = P(X + Y \leq z)$$

$$F_Z(z) = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{z-y} f_{XY}(x, y)\,dx\,dy$$

$$F_Z(z) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{x=-\infty}^{z-y} f_{XY}(x, y)\,dx\right]}_{\text{Leibnitz's rule}} dy$$

perform diff w.r.t z to get density value $f_Z(z)$

$$\frac{d}{dz} F_Z(z) = \int_{y=-\infty}^{\infty} \underbrace{\frac{d}{dz}\left(\int_{x=-\infty}^{z-y} f_{XY}(x, y)\,dx\right]}\,dy$$

$$\frac{d}{dz}\int_{x=-\infty}^{z-y} f_{xy}(x,y)\,dx = \frac{d}{dz}(z-y)\cdot f_{xy}(z-y,y) - \frac{d}{dz}(\infty)\, f_{xy}(-\infty,y) + \int_{-\infty}^{z-y}\frac{\partial}{\partial x} f_{xy}(x,y).dz$$

$$= 1\cdot f_{xy}(z-y,y) - 0 + 0$$

$$\frac{d}{dz}\int_{x=-\infty}^{z-y} f_{xy}(x,y)\,dx = f_{xy}(z-y,y)$$

$$\boxed{f_Z(z) = \int_{y=-\infty}^{\infty} f_{xy}(z-y,y)\,dy.}$$

Let x, y are S.I $f_{xy}(x,y) = f_x(x)\cdot f_y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_x(z-y)\, f_y(y).dy.$$

$$\boxed{f_Z(z) = f_x(z) * f_y(z)}$$

Problems :-

1) Joint probability mass function is given by

$P_{xy}(x,y) = \begin{cases} K(x_i+y_i) \end{cases}$ for $x_i = 1,2,3$, $y_i = 1,2,3$

find K and CDF.

Sol:- $P_{xy}(x,y) = K(x_i+y_i)$

y \ x	1	2	3
1	2K	3K	4K
2	3K	4K	5K
3	4K	5K	6K

K = ? Total probability = 1

$$2K + 3K + 4K + 3K + 4K + 5K + 4K + 5K + 6K = 1$$

$$36K = 1 \Rightarrow \boxed{K = \frac{1}{36}}$$

$$F_{xy}(x,y) = \sum_{x_i}\sum_{y_i} P_{xy}(x_i,y_i)\,u(x-x_i)\,u(y-y_i)$$

$$= \sum_{x_i=1}^{3}\sum_{y_i=1}^{3} P_{xy}(x_i,y_i)\,u(x-x_i)\,u(y-y_i)$$

$$= \sum_{x_i=1}^{3} \Big[P_{XY}(x_i,1)\, u(X-x_i)\, u(Y-1) + P_{XY}(x_i,2)\, u(X-x_i)\, u(Y-2) + P_{XY}(x_i,3)\, u(X-x_i)\, u(Y-3) \Big]$$

$$= P_{XY}(1,1)\, u(X-1)\, u(Y-1) + P_{XY}(1,2)\, u(X-1)\, u(Y-2)$$
$$+ P_{XY}(1,3)\, u(X-1)\, u(Y-3) + P_{XY}(2,1)\, u(X-2)\, u(Y-1)$$
$$+ P_{XY}(2,2)\, u(X-2)\, u(Y-2) + P_{XY}(2,3)\, u(X-2)\, u(Y-3)$$
$$+ P_{XY}(3,1)\, u(X-3)\, u(Y-1) + P_{XY}(3,2)\, u(X-3)\, u(Y-2)$$
$$+ P_{XY}(3,3)\, u(Y-3)$$

$$= \frac{2}{36} u(X-1)\, u(Y-1) + \frac{3}{36} u(X-1)\, u(Y-2)$$
$$+ \frac{4}{36} u(X-1)\, u(Y-3) + \frac{3}{36} u(X-2)\, (Y-1)$$
$$+ \frac{4}{36} u(X-2)\, u(Y-2) + \frac{5}{36} u(X-2)\, u(Y-3)$$
$$+ \frac{4}{36} u(X-3)\, u(Y-1) + \frac{5}{36} u(X-3)\, u(Y-2)$$
$$+ \frac{6}{36} u(X-3)\, u(Y-3).$$

2) The Joint PDF of random variable X, Y is $f_{XY}(x,y) = \frac{1}{2}$ $\forall\ 0 \le X \le 1,\ 0 \le Y \le 2$ find probability distribution function find $F_{XY}(0.5, 1)$, find marginal density function $f_X(x)$, $f_Y(y)$ find X, Y are S.I (or) not.

Sol:- $f_{XY}(x,y) = \frac{1}{2}\ \ \forall\ 0 \le x \le 1,\ 0 \le Y \le 2$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dy\, dx.$$

$$= \int_0^x \left[\int_0^y \frac{1}{2}\, dy \right] dx$$

$$= \int_0^x \left[\frac{1}{2} \cdot y \right]_0^y dx.$$

$$= \frac{1}{2} \int_0^x [y-0]\, dx. = \frac{1}{2} \int_0^x y \cdot dx$$

$$= \frac{1}{2} y \int_0^x dx \Rightarrow \frac{1}{2} y [x]_0^x$$

$$= \frac{1}{2} y [x-0]$$

$$= \frac{xy}{2}$$

$$F_{XY}(x, y) = \frac{xy}{2}$$

(ii) $$F_{XY}(0.5, 1) = \int_{-\infty}^{0.5} \int_{-\infty}^{1} f_{XY}(x, y)\, dy\, dx$$

$$= \int_{-\infty}^{0.5} \int_{-\infty}^{1} \frac{1}{2}\, dy\, dx.$$

$$= \int_0^{0.5} \int_0^{1} \frac{1}{2}\, dy\, dx.$$

$$= \frac{1}{2} \int_0^{0.5} \int_0^{1} dy\, dx.$$

$$= \frac{1}{2} \int_0^{0.5} [y]_0^1\, dx \Rightarrow \frac{1}{2} \int_0^{0.5} (1)\, dx.$$

$$= \frac{1}{2} [x]_0^{0.5} \Rightarrow \frac{1}{2}(0.5) = \frac{1}{4}.$$

$$F_{XY}(0.5, 1) = \frac{1}{4}$$

3) Marginal density function

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^2 \frac{1}{2}\, dy = \frac{1}{2}[y]_0^2.$$

$$= \frac{1}{2}[2-0] = 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_0^1 \frac{1}{2}\, dx = \frac{1}{2}[x]_0^1$$

$$= \frac{1}{2}$$

4) If X, Y are two R.V statistically independent if $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$.

$$f_{XY}(x, y) = \frac{1}{2}$$

$$f_X(x) = 1$$

$$f_Y(y) = 1/2$$

$f_{xy}(x,y) = f_x(x) \cdot f_y(y)$

Both X, Y are S.I

3.) Two R.V X, Y have a Joint density function

$\overset{f_{xy}(x,y)}{f(x,y)} = \frac{5}{16}x^2y$ for $0 < y < x < 2$

① Find marginal density function of X, Y

② Find X, Y are S.I (or) not.

Sol:-

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy. = \int_0^x \frac{5}{16}x^2y\,dy$$

$$= \frac{5}{16}x^2\int_0^x y\,dy. = \frac{5}{16}x^2\left[\frac{y^2}{2}\right]_0^x$$

$$= \frac{5}{16}\cdot\frac{1}{2}x^2\cdot x^2$$

$$f_x(x) = \frac{5}{32}x^4$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_y^2 \frac{5}{16}x^2y\,dx.$$

$$= \frac{5}{16}\cdot y\int_y^2 x^2\,dx = \frac{5}{16}y\left[\frac{x^3}{3}\right]_y^2$$

$$= \frac{5}{48}y\left[8 - y^3\right] = \frac{5}{\cancel{48}_{6}}\cdot y\cdot\cancel{8} - \frac{5}{48}y\cdot y^3$$

$$= \frac{5}{6}y - \frac{5}{48}y^4.$$

$f_{xy}(x,y) = \frac{5}{16}x^2y$

$f_x(x) = \frac{5}{32}x^4$

$f_y(y) = \frac{5}{48}(8y - y^4)$

$f_x(x)\cdot f_y(y) = \frac{5}{32}x^4\cdot\frac{5}{48}(8y - y^4)$

$\neq f_{XY}(X,Y)$

Both X, Y are not statistically independent.

***

4.) The Joint Distribution of X, Y is given by $f_{xy}(x,y) = 4xy\,e^{-(x^2+y^2)}\,\underbrace{u(x)}_{x\geq 0}\cdot\underbrace{u(y)}_{y\geq 0}$ show X, Y are S.I.

Find marginal density functions

Sol:-

$$f_{xy}(x,y) = 4xy\,e^{-(x^2+y^2)} \quad u(x).u(y)$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_{0}^{\infty} 4xy\,e^{-(x^2+y^2)}\,dy.$$

$$= 4x\,e^{-x^2}\int_0^{\infty} y.e^{-y^2}\,dy.$$

Let $y^2 = \tau$

$2y\,dy = d\tau$

$dy = \frac{d\tau}{2y}$

$y=0 \Rightarrow \tau=0$

$y=\infty \Rightarrow \tau=\infty$

$$= 4x\,e^{-x^2}\int_0^{\infty} \cancel{y}.e^{-\tau}.\frac{d\tau}{2\cancel{y}}$$

$$= 4x\,e^{-x^2}\int_0^{\infty} e^{-\tau}.\frac{1}{2}.d\tau$$

$$= 2x\,e^{-x^2}\left[\frac{e^{-\tau}}{-1}\right]_0^{\infty} = 2x\,e^{-x^2}\left[\frac{0-1}{-1}\right]$$

$$= 2x\,e^{-x^2}.$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_0^{\infty} 4xy\,e^{-(x^2+y^2)}.\,dx.$$

$$= 4y\,e^{-y^2}\int_0^{\infty} x.e^{-x^2}\,dx.$$

Let $x^2 = p$

$2x\,dx = dp$

$dx = \frac{dp}{2x}$

$x=0 \Rightarrow p=0$

$x=\infty \Rightarrow p=\infty$

$$= 4y\,e^{-y^2}\int_0^{\infty} \cancel{x}.e^{-p}.\frac{dp}{2\cancel{x}}$$

$$= 2y\,e^{-y^2}\left[\frac{e^{-p}}{-1}\right]_0^{\infty}$$

$$= 2y\,e^{-y^2}\left[\frac{0-1}{-1}\right]$$

$$f_y(y) = 2y\,e^{-y^2}$$

$$f_{xy}(x,y) = 4xy\,e^{-(x^2+y^2)}$$

$$f_x(x).f_y(y) = 2.x.e^{-x^2}.2y\,e^{-y^2} = 4xy\,e^{-x^2}.e^{-y^2}$$

$$= 4xy\,e^{-(x^2+y^2)}$$

5.) Joint Density function of R.v X, Y is $f(x,y) = k\,e^{-(x+y)}$ in the range of $0 \le x \le \infty$, $0 \le y \le \infty$ and $f(x,y)=0$ otherwise

(a) Find value k.

(b) Find probability density of x by considering y is independent.

(c) Find $P(0 \le x \le 2,\ 2 \le y \le 3)$

(d) Prove R.V X and Y are S.I (or) not.

Sol:- Given $f(x,y) = Ke^{-(x+y)}$

(a) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = 1$$

$$\int_0^{\infty}\int_0^{\infty} K.e^{-(x+y)}\,dy\,dx = 1.$$

$$\int_0^{\infty}\int_0^{\infty} K.e^{-x}.e^{-y}\,dy\,dx = 1$$

$$K\int_0^{\infty} . e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} . dx = 1.$$

$$K\int_0^{\infty} e^{-x}\left[\frac{0-1}{-1}\right]dx = 1.$$

$$K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1;$$

$$K\left[\frac{0-1}{-1}\right] = 1$$

$$\boxed{K = 1}$$

(b) $f_X(x)$ considering y is independent.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^{\infty} 1.e^{-x}.e^{-y}\,dy.$$

$$= e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty} = e^{-x}\left[\frac{0-1}{-1}\right]$$

$$f_X(x) = e^{-x}$$

③ $P(0 \le X \le 2,\ 2 \le Y \le 3)$

$$\int_0^2\int_2^3 f_{XY}(x,y)\,dx\,dy.$$

$$\int_0^2\int_2^3 e^{-x}.e^{-y}\,dy\,dx.$$

$$\cancel{\int_0^2 e^{-x}\left[\frac{e^{-y}}{-1}\right.} \quad \int_0^2 e^{-x}\,dx \int_2^3 e^{-y}.\,dy$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^2 \cdot \left[\frac{e^{-y}}{-1}\right]_2^3$$

$$= \frac{e^{-2}-1}{-1} \cdot \frac{e^{-3}-e^{-2}}{-1}$$

$$= (1-e^{-2})(e^{-2}-e^{-3})$$

X, Y are statistically independent

Operations on Multiple R.V

Moments :-

1) Moment about mean

2) Moment about origin.

Expected value of Multiple R.V :-

$$E(x) = \int_{-\infty}^{\infty} x f_x(x) dx \quad \rightarrow \text{single R.V}$$

Let x, y are two R.V , $E(xy) = \int_{-\infty}^{\infty} xy \, f_{xy}(x,y) \, dx \, dy.$

N^{th} moment $E[x^n] = m_n \rightarrow$ for single R.V

Multiple

$(n+k)^{th}$ moment $= E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y) \, dx \, dy.$

0^{th} moment about origin of two r.v • X, Y

n = 0

K = 0 $\quad E[x^0 y^0] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^0 y^0 f_{xy}(x,y) dx \, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y) dy \, dx$

$E[1.1] = E[1] = 1 \qquad = 1$

1^{st} moment about origin of R.V X, Y

n = 1

k = 0

$(n+k)^{th}$ moment $= (1+0)^{th}$ moment $= 1^{st}$ moment

$$m_{10} = E[x^1 y^0] = E[x^1] = \mu_x$$

$n=0$
$k=1$ $(0+1)^{th}$ momenth $= E[x^0 y^1] = E[y] = \mu_y = m_{01}$

2nd moment about origin of m.R.v :-

$n=2, K=0$ $E(x^2 y^0] = m_{20} = E[x^2] =$ 2nd moment about origin of R.v x.

$n=0, K=2$ $E[x^0 y^2] = m_{02} = E[y^2] =$ 2nd moment about origin of R.V Y.

$n=1, K=1$ $E[x \cdot y] = m_{11} = E[xy] \rightarrow$ correlation of two R.v's
relation between 2 signals

Relaton :-

$x = \{1, 2, 3, 4\}$ $Y \subset X$
$Y = \{1, 2\}$

0th moment	1st moment	2nd moment
$n=K=0$	$n=1, K=0$	$n=2; K=0$
	$n=0, K=1$	$n=0; K=2$
		$n=1, K=1$

Properties of correlation :-

1) X, Y are two S.I R.v's $E[x, y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dy\, dx$.

If X, Y are S.I $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$.

$$E[x, y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\, dy\, dx$$

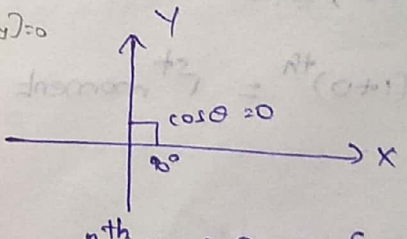
$$= \int_{-\infty}^{\infty} x f_X(x)\, dx \cdot \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$\boxed{E[x \cdot y] = E(x) \cdot E(y)}$$

X, Y are two RV orthogonal to each other $E[xy]=0$

2) X, Y are orthogonal

$R_{XY} = E[XY] = 0$



Moment about mean :-

n^{th} x → R.v $E[(x-\mu_x)^n]$

X, Y are two R.v's

$(n+k)^{th}$ moment about mean

$\mu_n \mu_k = E[(x-\mu_x)^n (y-\mu_y)^k]$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)^n (y-\mu_y)^k f_{xy}(x,y)\,dx\,dy$$

0th moment about mean :-

$n=0$, $k=0$

$$\mu_{0k_0} = \mu_{00} = E\left[(x-\mu_x)^0 (y-\mu_y)^0\right] = E[1\cdot1] = E[1] = 1.$$

Moment about mean :-

Single R.V = nth moment of single RV over mean

$$E\left[(x-\mu_x)^n\right]$$

Multiple R.V :- X, Y are two R.V , μ_x is the mean of the x, μ_y is the mean of the y

n,k $\quad E\left[(x-\mu_x)^n (Y-\mu_y)^k\right] = \mu_{nk}$

0th moment of M.R.V about mean :-

$n=0$, $k=0$

$(n+k)^{th}$ momenth $= (0+0)^{th} = 0^{th}$ moment about mean

$$E\left[(x-\mu_x)^0 (Y-\mu_y)^0\right] = E[1] = 1$$

1st moment about mean of M.R.V :-

$n=0$, $k=1$

$(n+k)^{th}$ moment $= (0+1)^{th} = 1^{st}$ moment

$$= E\left[(x-\mu_x)^0 (Y-\mu_y)^1\right] = E\left[Y-\mu_y\right]$$

$$= E(Y) - E(\mu_y)$$

$$= \mu_y - \mu_y = 0.$$

$n=1$, $k=0$.

$$(1+0)^{th} = 1^{st} \text{ moment} = E\left[(x-\mu_x)^1 (Y-\mu_y)^0\right]$$

$$= E\left[(x-\mu_x)^1\right]$$

$$= E(x) - E(\mu_x)$$

$$= \mu_x - \mu_x = 0$$

2^{nd} moment about mean of R.V x,Y :-

$(n+k)^{th}$ moment =

$n=2$, $k=0$ $\quad (2+0)^{th} = 2^{nd}$ moment :- $E\left[(X-\mu_x)^2 (Y-\mu_y)^0\right] = E\left[(x-\mu_x)^2\right]$

$= \sigma_x^2$ (Variance of x)

$n=0$, $k=2$ $\quad (0+2)^{th} = 2^{nd}$ moment $E\left[(x-\mu_x)^0 (Y-\mu_y)^2\right] = E\left[(Y-\mu_y)^2\right]$

$= \sigma_y^2$ (variance of y)

$n=1$, $k=1$ $\quad (1+1)^{th} = 2^{nd}$ moment $E\left[(x-\mu_x)^1 (Y-\mu_y)^1\right]$ = covariance

$x \Longleftrightarrow y$

Note:-

Covariance (x,Y) :- 2^{nd} moment of R.V x,Y over ~~rea~~ their respective mean with $n=1$, $k=1$

$$C_{XY} = E\left[(x-\mu_x)(Y-\mu_y)\right] = \sigma_{XY}$$

Properties of co-variance :-

1) $\operatorname{cov}(x,Y) = \underset{\text{corelation}}{R_{xy}} - \overset{\text{mean}(x)}{\bar{x}}\,\overset{\text{mean}(y)}{\bar{Y}}$

$= E\left[(x-\mu_x)(Y-\mu_y)\right]$

Proof:- $\operatorname{cov}(X,Y) = E\left[XY - X\mu_y - Y\mu_x + \mu_x\mu_y\right]$

$= E[XY] - E[X\mu_y] - \cancel{Y}\, E[\mu_x \cdot y] + E[\mu_x\mu_y]$

$= E[XY] - \mu_y E(x) - \mu_x E(y) + \mu_x \cdot \mu_y.$

$= E[XY] - \mu_y\mu_x - \cancel{\mu_x\mu_y} + \cancel{\mu_x\mu_y}$

$= E[XY] - \mu_x\mu_y$

$= E[XY] - \bar{x}\bar{Y}$

$\operatorname{cov}(x,Y) = R_{XY} - \bar{x}\bar{y}$

2.) x, Y are s.I R.v, $\operatorname{cov}(X,Y) = 0$

As per $\operatorname{cov}(X,Y) = R_{XY} - \bar{X}\bar{Y}$

We know x,Y are s.I

$f_{XY}(x,y) = f_X(x)\cdot f_Y(y)$

$$R_{XY} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx \int_{-\infty}^{\infty} y\, f_Y(y)\,dy.$$

$$= E(X)\cdot E(Y) \Rightarrow \mu_X \cdot \mu_Y \Rightarrow \bar{X}\cdot\bar{Y}$$

$$\text{cov}(X,Y) = E(XY) - \bar{X}\cdot\bar{Y} = \bar{X}\cdot\bar{Y} - \bar{X}\cdot\bar{Y} = 0.$$

③ "Scalar multiplication" $\text{cov}(aX, bY) = a\cdot b\cdot \text{cov}(X, Y)$

Proof:- ~~$\text{cov}(ax, b$~~ $\text{cov}(X,Y) = E((X-\mu_X)(Y-\mu_Y))$

$$= E\left[(X - E(X))(Y - E(Y))\right]$$

$$\text{cov}(aX, bY) = E\left[(aX - E(aX))(bY - E(bY))\right]$$

$$= E(aX\cdot bY) - E(aX)\cdot E(bY)$$

$$= E(abXY) - aE(X)\cdot bE(Y)$$

$$= ab\,E(XY) - ab\,E(X)E(Y)$$

$$= ab\left[E(XY) - E(X)\cdot E(Y)\right]$$

$$= ab\left[R_{XY} - \mu_X\cdot\mu_Y\right]$$

$$\text{Cov}(aX,bY) = ab\,\text{cov}(X,Y).$$

④ $\text{cov}(X+a,\ Y+b) = \text{cov}(X, Y)$

$$\text{cov}(X,Y) = E(XY) - E(X)\cdot E(Y)$$

$$= E\left[(X+a)(Y+b)\right] - E(X+a)\,E(Y+b)$$

$$= E\left[XY + Xb + aY + ab\right] - \left\{\left[E(X) + E(a)\right]\left[E(Y) + E(b)\right]\right\}$$

$$= E[XY] + E[Xb] + E[aY] + E[ab] - \{E(X)\cdot E(Y) + E(X)\cdot E(b) + E(a)\cdot E(Y) + E(a)E(b)\}$$

$$= E(XY) + b\,\cancel{E(X)} + a\,\cancel{E(Y)} + a\cancel{b} - E(X)\cdot E(Y) - b\cancel{E(X)} - a\cancel{E(Y)} - \cancel{E(a)}E(b)$$

$$= E(XY) - E(X)\cdot E(Y).$$

$$= E(XY) - \bar{X}\cdot\bar{Y} = \text{cov}(X,Y)$$

UNIT-3

MULTIPLE RANDOM VARIABLES

Properties of Co-variance:-

⑤ $Var(X+Y) = var(X) + var(Y) + 2C_{XY}$

Proof:- $Var(z) = m_2 - (m_1)^2$

$var(z) = E(z^2) - [E(z)]^2$

$Var(X+Y)$ Let $X+Y = z$

$$var(z) = E(z^2) - [E(z)]^2$$
$$= E[(X+Y)^2] - [E(X+Y)]^2$$
$$= E[X^2+Y^2+2XY] - [E(X)+E(Y)]^2$$
$$= E[X^2] + E[Y^2] + E[2XY] - \{[E(X)]^2 + [E(Y)]^2 + 2E[X]E[Y]\}$$
$$= E[X^2] - [E(X)]^2 + E(Y2) - [E(Y)]^2 + 2E[XY] - 2E[X]E[Y]$$
$$= var(X) + var(Y) + 2[R_{XY} - \bar{X}\bar{Y}]$$
$$= var(X) + var(Y) + 2\,cov(XY)$$

$Var(X+Y) = var(X) + Var(Y) + 2C_{XY}$

⑥ $Var(X-Y) = var(X) + var(Y) - 2C_{XY}$

Proof:- $Var(z) = m_2 - (m_1)^2$

$var(z) = E(z^2) - [E(z)]^2$

$var(X-Y)$ Let $X-Y = z$

$$var(z) = E(z^2) - [E(z)]^2$$
$$= E[(X-Y)^2] - [E(X-Y)]^2$$
$$= E[X^2+Y^2-2XY] - [E(X) - E(Y)]^2$$
$$= E(X^2) + E[Y^2] + E[-2XY] - \{[E(X)]^2 + [E(Y)]^2 - 2E[X]E[Y]\}$$
$$= E[X^2] - [E(X)]^2 + E[Y^2] - [E(Y)]^2 - 2E[XY] + 2E[X]E[Y]$$
$$= Var(X) + var(Y) - 2[E[XY] - E[X]E[Y]]$$
$$= var(X) + var(Y) - 2[R_{XY} - \bar{X}\bar{Y}]$$
$$= var(X) + var(Y) - 2\,cov(XY)$$

$var(X+Y) = var(X) + var(Y) - 2C_{XY}$

***

⑦ $\sigma_{xy} \leq \sigma_x \cdot \sigma_y$

$\text{cov}(X,Y) \leq \text{S.D}(X) \cdot \text{SD}(Y)$

Proof:- Assumption:- $[(Y-\bar{Y}) - K(X-\bar{X})]^2 \geq 0$

Apply Exp on both sides

$$E\{[(Y-\bar{Y}) - K(X-\bar{X})]^2\} \geq E(0)$$

$$E[(Y-\bar{Y})^2 + K^2(X-\bar{X})^2 - 2K(Y-\bar{Y})(X-\bar{X})] \geq 0$$

$$E[(Y-\bar{Y})^2] + E[K^2(X-\bar{X})^2] - E[2K(X-\bar{X})(Y-\bar{Y})] \geq 0$$

$$\sigma_y^2 + K^2\sigma_x^2 - 2K\,E[(X-\bar{X})(Y-\bar{Y})] \geq 0$$

$$\sigma_y^2 + K^2\sigma_x^2 - 2K\sigma_{xy} \geq 0$$

Divide on both sides using σ_x^2

$$\frac{\sigma_y^2 + K^2\sigma_x^2 - 2K\sigma_{xy}}{\sigma_x^2} \geq \frac{0}{\sigma_x^2}$$

$$\frac{\sigma_y^2}{\sigma_x^2} + K^2 - 2K\frac{\sigma_{xy}}{\sigma_x^2} \geq 0$$

Add & sub with $\left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2$

$$K^2 - 2K\frac{\sigma_{xy}}{\sigma_x^2} + \frac{\sigma_y^2}{\sigma_x^2} + \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 - \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 \geq 0$$

$$K^2 - 2K\frac{\sigma_{xy}}{\sigma_x^2} + \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 + \frac{\sigma_y^2}{\sigma_x^2} - \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 \geq 0$$

$$\left(K - \frac{\sigma_{xy}}{\sigma_x}\right)^2 + \frac{\sigma_y^2}{\sigma_x^2} - \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 \geq 0$$

$$\frac{\sigma_y^2}{\sigma_x^2} - \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2 \geq 0$$

$$\frac{\sigma_y^2}{\sigma_x^2} \geq \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^2$$

$$\left(\frac{\sigma_y}{\sigma_x}\right)^{\cancel{2}} \geq \left(\frac{\sigma_{xy}}{\sigma_x^2}\right)^{\cancel{2}}$$

$$\frac{\sigma_y}{\sigma_x} \geq \frac{\sigma_{xy}}{\sigma_x^2}$$

$$\frac{\sigma_y \sigma_x^2}{\sigma_x} \geq \sigma_{xy}$$

$$\sigma_x \sigma_y \geq \sigma_{xy}$$

$$\boxed{\sigma_{xy} \leq \sigma_x \cdot \sigma_y}$$

Characteristic function of Normal density function :-

Normal density function :- $N(0,1) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \forall \ -\infty \text{ to } \infty$

$$\phi_X(\omega) = E\left[e^{j\omega x}\right] = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\, dx.$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega x} \cdot e^{-x^2/2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2} + j\omega x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2} - j\omega x\right)} \cdot dx.$$

Add & sub $\left(\frac{j\omega}{\sqrt{2}}\right)^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{x^2}{2} - x \cdot j\omega + \left(\frac{j\omega}{\sqrt{2}}\right)^2 - \left(\frac{j\omega}{\sqrt{2}}\right)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{x^2}{2} - 2 \cdot \frac{x \cdot j\omega}{2} + \left(\frac{j\omega}{\sqrt{2}}\right)^2 - \left(\frac{j\omega}{\sqrt{2}}\right)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\frac{x^2}{2} - 2 \cdot \frac{x}{\sqrt{2}} \cdot \frac{j\omega}{\sqrt{2}} + \left(\frac{j\omega}{\sqrt{2}}\right)^2 - \left(\frac{j\omega}{\sqrt{2}}\right)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{2}} - \frac{j\omega}{\sqrt{2}}\right)^2 - \left(\frac{j\omega}{\sqrt{2}}\right)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left[-\left(\frac{x}{\sqrt{2}} - \frac{j\omega}{\sqrt{2}}\right)^2 + \left(\frac{j\omega}{\sqrt{2}}\right)^2\right]} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{2}} - \frac{j\omega}{\sqrt{2}}\right)^2} \cdot e^{\left(\frac{j\omega}{\sqrt{2}}\right)^2} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{(j\omega)^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(x-j\omega)^2}{2}} dx$$

let $x - j\omega = t$

$dx = dt$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{2}} \int_{-\infty}^{\infty} e^{-t^2/2} \, dt$$

$$= e^{-\omega^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt.$$

$$= e^{-\omega^2/2} \int_{-\infty}^{\infty} f_T(t) \, dt.$$

$$= e^{-\omega^2/2} \cdot 1 = e^{-\omega^2/2}$$

Center limit theorem :-

Statement:- Let N identically independent R.v as $x_1, x_2, x_3, \dots x_N$, have mean $\bar{x}_1, \bar{x}_2, \bar{x}_3 \dots \bar{x}_N$, Variance $\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_3} \dots \sigma_{x_N}$ by adding all the R.v we get $Y = x_1 + x_2 + x_3 + \dots + x_N$, $\bar{Y} = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \dots + \bar{x}_n$ $\sigma_Y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 + \dots + \sigma_{x_N}^2$. A normalized standard deviation represented as $W = \frac{Y - \bar{Y}}{\sigma_Y}$ its characteristic function is $e^{-\omega^2/2}$.

Proof:- $Y = x_1 + x_2 + x_3 + \dots + x_N = \sum_{n=1}^{N} x_n \rightarrow$ ①

$\bar{Y} = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \dots + \bar{x}_N = \sum_{n=1}^{N} \bar{x}_n \rightarrow$ ②

$\sigma_Y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2 + \sigma_{x_3}^2 + \dots + \sigma_{x_N}^2 = \sum_{n=1}^{N} \sigma_{x_n}^2 \rightarrow$ ③

$\boxed{W = \frac{Y - \bar{Y}}{\sigma_Y}} \rightarrow$ ④

sub ①, ②, ③ in ④

$$W = \frac{\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \bar{x}_n}{\sqrt{\sigma_Y^2}}$$

$$= \frac{\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} \bar{x}_n}{\sqrt{\sum_{n=1}^{N} \sigma_{x_n}^2}} \text{ — ③}$$

As per the statement all the R.v are i.i.d [S.I]

$x_1 = x_2 = x_3 \dots = x_N = X$

$\bar{x}_1 = \bar{x}_2 = \bar{x}_3 \dots = \bar{x}_N = \bar{X}$

$$\sigma_{x_1}^2 = \sigma_{x_2}^2 = \sigma_{x_3}^2 = \cdots = \sigma_{x_N}^2 = \sigma_x^2$$

⑤ $\Rightarrow$

$$W = \frac{\sum_N x - \sum_N \bar{x}}{\sqrt{\sum_N \sigma_x^2}} \Rightarrow \frac{\sum_N (x-\bar{x})}{\sqrt{\sum_{N} \sigma_x^2}}$$

$$= \frac{\sum_N (x-\bar{x})}{\sqrt{N \sigma_x^2}} \Rightarrow \frac{\sum_N (x-\bar{x})}{\sqrt{N}\ \sigma_x}$$

$$W = \frac{\sum_N (x-\bar{x})}{\sqrt{N}\ \sigma_x} \quad \text{---} ⑥$$

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General C.F

$$\phi_x(\omega) = E\left[e^{j\omega x}\right]$$

C.F of W $\phi_W(\omega) = E\left[e^{j\omega[W]}\right]$ —— ⑦

Sub ⑥ in ⑦

$$\phi_W(\omega) = E\left[e^{j\omega\left[\frac{\sum_N (x-\bar{x})}{\sqrt{N}\ \sigma_x}\right]}\right] = E\left[e^{j\omega\left[\frac{\sum_N (x-\bar{x})}{\sqrt{N}\ \sigma_x}\right]}\right]$$

$$= E\left[e^{\frac{\sum_N j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}\right]$$

$e^{\sum_3 x} = e^{x+x+x} = e^x \cdot e^x \cdot e^x$

$$= E\left[\prod_N \left[e^{\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}\right]\right]$$

$\prod_3 [e^x] = [e^x]^3$

$$= E\left\{\left\{e^{\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}\right\}^N\right\}$$

all the R.V are iid

x,y are S.I

$E[X Y] = E[X] \cdot E[Y]$

$$\phi_W(\omega) = \left\{E\left[e^{j\omega(x-\bar{x})/\sqrt{N}\ \sigma_x}\right]\right\}^N$$

$E\left[[e^x]^3\right]$

$E[e^x] \cdot E[e^x] \cdot E[e^x$

apply Natural log on both sides on ⑦

$$\ln[\phi_W(\omega)] = \ln\left\{E\left\{e^{\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}\right\}\right\}^N$$

$\ln[a^n] = n\ln(a)$

$$\ln[\phi_W(\omega)] = N\ln\left\{E\left[e^{\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}\right]\right\}$$

$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots$

$$= N\ln\left\{E\left[1 + \frac{\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}}{1!} + \frac{\left(\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}\right)^2}{2!} + \frac{\left(\frac{j\omega(x-\bar{x})}{\sqrt{N}\ \sigma_x}\right)^3}{3!} + \cdots\right]\right\}$$

$$= N\ln\left\{E\left[1+\frac{j\omega(x-\bar{x})}{\sqrt{N}\sigma_x}+\frac{1}{2!}\frac{(j\omega)^2(x-\bar{x})^2}{(\sqrt{N}\sigma_x)^2}+\frac{1}{3!}\frac{(j\omega)^3(x-\bar{x})^3}{(\sqrt{N}\sigma_x)^3}+\cdots\right]\right\}$$

$$= N\ln\left[E(1)+E\left[\frac{j\omega(x-\bar{x})}{\sqrt{N}\,\sigma_x}\right]+E\left[\frac{1}{2!}\frac{(j\omega)^2(x-\bar{x})^2}{(\sqrt{N}\,\sigma_x)^2}\right]+E\left[\frac{1}{3!}\frac{(j\omega)^3(x-\bar{x})^3}{(\sqrt{N}\,\sigma_x)^3}\right]+\cdots\right]$$

$$= N\ln\left[1+\frac{j\omega}{\sqrt{N}\sigma_x}\underbrace{E(x-\bar{x})}_{0}+\frac{1}{2}\frac{(j\omega)^2}{(\sqrt{N}\sigma_x)^2}\underbrace{E(x-\bar{x})^2}_{\sigma_x^2}+\underbrace{\frac{1}{6}\frac{(j\omega)^3}{(\sqrt{N}\sigma_x)^3}E(x-\bar{x})^3+\cdots}_{\frac{-j\omega^3}{\sigma_x^3\cdot N\sqrt{N}}E(x-\bar{x})^3\quad\frac{R(N)}{N}}\right]$$

$$= \frac{(j\omega)^3}{N\sqrt{N}\sigma_x^3}E(x-\bar{x})^3+\frac{(j\omega)^4}{N\sqrt{N}\sigma_x^4}E(x-\bar{x})^4$$

all the remaining terms

$$= N\ln\left[1+\frac{1}{2}\frac{(j\omega)^2}{(\sqrt{N}\sigma_x)^2}\cdot\sigma_x^2+\frac{1}{N}R(N)\right]$$

$$= N\ln\left[1-\frac{1}{2}\cdot\frac{\omega^2}{N\cancel{\sigma_x^2}}\cdot\cancel{\sigma_x^2}+\frac{R(N)}{N}\right]$$

$$= N\ln\left[1-\underbrace{\frac{\omega^2}{2N}+\frac{R(N)}{N}}_{z}\right]$$

$$= N\ln[1-z] \rightarrow ⑧ \qquad \text{let } z=\frac{\omega^2}{2N}+\frac{R(N)}{N}$$

$$\ln(1-z) = -\left[z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]$$

$$⑧ \Rightarrow \ln[\phi_W(\omega)] = N\ln[1-z]$$

$$= -N\left[z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right]$$

$$= -N\left[\frac{\omega^2}{2N}+\frac{R(N)}{N}+\frac{\left[\frac{\omega^2}{2N}+\frac{R(N)}{N}\right]^2}{2!}+\frac{\left[\frac{\omega^2}{2N}+\frac{R(N)}{N}\right]^3}{3!}+\cdots\right]$$

$$= -N\left[\frac{\omega^2}{2N} + \frac{R(N)}{N} + \frac{1}{2}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^2 + \frac{1}{3!}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^3 + \cdots\right]$$

$$\ln[\phi_W(\omega)] = -\frac{\omega^2}{2} - R(N) - \frac{N}{2}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^2 - \frac{N}{3!}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^3 - \cdots$$

~~$N \to R.V$ $x_1\ x_2\ x_3 \cdots x_N \to \infty$~~

Let the number of R.V are ∞

$$N\to\infty \quad \ln[\phi_W(\omega)] = \underset{N\to\infty}{Lt} -\frac{\omega^2}{2} - R(N) - \frac{N}{2}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^2 - \frac{N}{3!}\left[\frac{\omega^2}{2N} + \frac{R(N)}{N}\right]^3 - \cdots$$

$$\ln[\phi_W(\omega)]_{N\to\infty} = -\frac{\omega^2}{2}$$

$$\boxed{\phi_W(\omega)/_{N\to\infty} = e^{-\omega^2/2}} = \text{C.F (Normal density function)}$$

Normalized S.D

$Y = X_1 + X_2 + X_3 + \cdots \infty$ (i.i.d)

$\bar{Y} = \bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \cdots$

$\sigma_Y^2 = \sigma_{X_1}^2 + \sigma_{X_2}^2 + \sigma_{X_3}^2 + \cdots$

N^{th} order Gaussian density function :-

$X_1, X_2 \cdots X_N$ are N no. of RV, they have respective mean as $\bar{X}_1, \bar{X}_2 \cdots \bar{X}_N$, covariance C_X, Gaussian Density function is represented as

$$f_{X_1, X_2 X_3 \cdots X_N}(x_1, x_2, x_3 \cdots x_N) = \frac{1}{[2\pi]^{N/2}\,[C_X]^{1/2}} \cdot e^{-\frac{1}{2}\left[(x-\bar{x})^T [C_X]^{-1} (x-\bar{x})\right]}$$

$C_X \to$ covariance Matrix

$$\begin{bmatrix} C_{X_1X_1} & C_{X_1X_2} & C_{X_1X_3} & \cdots & C_{X_1X_N} \\ C_{X_2X_1} & C_{X_2X_2} & C_{X_2X_3} & \cdots & C_{X_2X_N} \\ C_{X_3X_1} & C_{X_3X_2} & C_{X_3X_3} & \cdots & C_{X_3X_N} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{X_NX_1} & C_{X_NX_2} & C_{X_NX_3} & \cdots & C_{X_NX_N} \end{bmatrix}_{(N\times N)}$$

$(x - \bar{x}) \to$ column matrix $\begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \\ \vdots \\ x_N - \bar{x}_N \end{bmatrix}_{N\times 1}$

$(x-\bar{x})^T \rightarrow$ Row matrix $\left[x_1 - \bar{x}_1 \quad x_2 - \bar{x}_2 \quad x_3 - \bar{x}_3 \cdots \quad x_N - \bar{x}_N \right]_{1\times N}$

Gaussian Density function

$$\frac{1}{\sqrt{2\pi \sigma_x^2}} \, e^{\frac{-(x-\bar{x})^2}{2\sigma_x^2}}$$

N^{th} order Gaussian

$$\frac{1}{[2\pi]^{N/2} [C_x]^{1/2}} \cdot e^{\frac{-(x-\bar{x})^T (x-\bar{x})}{2[C_x]}}$$

$$\frac{1}{[2\pi]^{N/2} [C_x]^{1/2}} \, e^{-1/2 \left[(x-\bar{x})^T (C_x)^{-1} (x-\bar{x}) \right]}$$

1^{st} order Gaussian from n^{th} order Gaussian :-

$N=1$

$$f_{x_1 x_2 \cdots x_N}(x_1 x_2 \cdots x_N) = \frac{1}{[2\pi]^{N/2} [C_x]^{1/2}} \, e^{-1/2 \left[(x-\bar{x})^T [C_x]^{-1} (x-\bar{x}) \right]}$$

on sub $N=1$

$$f_{x_1}(x_1) = \frac{1}{[2\pi]^{1/2} [C_x]^{1/2}} \cdot e^{-1/2 \left[(x_1 - \bar{x}_1)^T [C_x]^{-1} (x_1 - \bar{x}_1) \right]}$$

$C_x \rightarrow N \times N$ $\quad C_x = \left[C_{x_1 x_1} \right] = E\left[(x_1 - \bar{x}_1)(x_1 - \bar{x}_1) \right]$

$$= E\left[(x_1 - \bar{x}_1)^2 \right] = \sigma_{x_1}^2$$

$\Rightarrow (x_1 - \bar{x}_1)^T [C_x]^{-1} (x_1 - \bar{x}_1) = (x_1 - \bar{x}_1)^T = (x_1 - \bar{x}_1)$

$$(x_1 - \bar{x}_1) = (x_1 - \bar{x}_1)$$

$$\Rightarrow \frac{(x_1 - \bar{x}_1)^2}{[C_x]} = \frac{(x - \bar{x}_1)^2}{\sigma_{x_1}^2}$$

$$f_{x_1}(x_1) = \frac{1}{[2\pi]^{1/2} (\sigma_{x_1})^{1/2}} \cdot e^{-\frac{(x-\bar{x}_1)^2}{2\sigma_{x_1}^2}}$$

$\Rightarrow$ Covariance coefficient :- $(\rho) = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{C_{xy}}{\sigma_x \sigma_y}$

2nd order Gaussian from n^{th} order Gaussian :-

$N=2$

$$f_{x_1 x_2}(x_1, x_2) = \frac{1}{[2\pi]^{2/2} [C_x]^{1/2}} \cdot e^{-1/2 \left[(x-\bar{x}) (C_x)^{-1} (x-\bar{x})^T \right]}$$

Covariance matrix $[C_x]_{2\times2} = \begin{bmatrix} C_{x_1x_1} & C_{x_1x_2} \\ C_{x_2x_1} & C_{x_2x_2} \end{bmatrix}$

$$C_{x_1x_1} = E\left[(x_1-\bar{x}_1)^2\right] = \sigma_{\bar{x}_1}^2$$

$$C_{x_1x_2} = E\left[(x_1-\bar{x}_1)(x_2-\bar{x}_2)\right] = C_{x_1x_2} = \sigma_{\bar{x}_1x_2}$$

$$C_{x_2x_1} = E\left[(x_2-\bar{x}_2)(x_1-\bar{x}_1)\right] = C_{x_2x_1} = \sigma_{x_2x_1}$$

$$C_{x_2x_2} = E\left[(x_2-\bar{x}_2)^2\right] = \sigma_{\bar{x}_2}^2$$

$$[C_x] = \begin{bmatrix} \sigma_{\bar{x}_1}^2 & \sigma_{\bar{x}_1x_2} \\ \sigma_{\bar{x}_2x_1} & \sigma_{\bar{x}_2}^2 \end{bmatrix}$$

$$[C_x]^{-1} = \frac{1}{|C_x|}\begin{bmatrix} \sigma_{\bar{x}_2}^2 & -\sigma_{\bar{x}_1x_2} \\ -\sigma_{\bar{x}_2x_1} & \sigma_{\bar{x}_1}^2 \end{bmatrix}$$

$$= \frac{1}{\sigma_{\bar{x}_1}^2\sigma_{\bar{x}_2}^2 - \sigma_{\bar{x}_1x_2}\sigma_{\bar{x}_2x_1}}\begin{bmatrix} \sigma_{\bar{x}_2}^2 & -\sigma_{\bar{x}_1}\sigma_{\bar{x}_2} \\ -\sigma_{\bar{x}_2x_1} & \sigma_{\bar{x}_1}^2 \end{bmatrix}$$

$$= \frac{1}{\sigma_{\bar{x}_1}^2\cdot\sigma_{\bar{x}_2}^2\left[1 - \frac{\sigma_{\bar{x}_1x_2}\sigma_{\bar{x}_2x_1}}{\sigma_{\bar{x}_1}^2\sigma_{\bar{x}_2}^2}\right]}\begin{bmatrix} \sigma_{\bar{x}_2}^2 & -\sigma_{\bar{x}_1x_2} \\ -\sigma_{\bar{x}_2x_1} & \sigma_{\bar{x}_1}^2 \end{bmatrix}$$

$$* \quad \rho = \frac{\sigma_{xy}}{\sigma_x\cdot\sigma_y} \qquad = \frac{1}{\sigma_{\bar{x}_1}^2\sigma_{\bar{x}_2}^2\left[1-\rho^2\right]}\begin{bmatrix} \sigma_{\bar{x}_2}^2 & -\sigma_{\bar{x}_1x_2} \\ -\sigma_{\bar{x}_2x_1} & \sigma_{\bar{x}_1}^2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{\sigma_{\bar{x}_1}^2(1-\rho^2)} & -\dfrac{\sigma_{\bar{x}_1x_2}}{\sigma_{\bar{x}_1}^2\sigma_{\bar{x}_2}^2(1-\rho^2)} \\ -\dfrac{\sigma_{\bar{x}_2x_1}}{\sigma_{\bar{x}_1}^2\sigma_{\bar{x}_2}^2(1-\rho^2)} & \dfrac{1}{\sigma_{\bar{x}_2}^2(1-\rho^2)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{\sigma_{\bar{x}_1}^2(1-\rho^2)} & \dfrac{-\rho}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}(1-\rho^2)} \\ \dfrac{-\rho}{\sigma_{\bar{x}_1}\sigma_{\bar{x}_2}(1-\rho^2)} & \dfrac{1}{\sigma_{\bar{x}_2}^2(1-\rho^2)} \end{bmatrix}$$

Problems

1) Gaussian R.v has first and 2nd order moment $m_{10} = -1.1$, $m_{20} = -1.16$, $m_{01} = 1.5$, $m_{02} = 2.89$, $R_{xy} = -1.72$. Find C_{xy} and $\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$.

Sol:- $C_{xy} = ?$

$\rho =$ covariance coefficient

Given data:-

$$m_{10} = E[x^1 y^0] = E[x] = \mu_x = \bar{X} = -1.1$$

$$m_{20} = E[x^2 y^0] = E[x^2] = -1.16$$

$$m_{01} = E[x^0 y^1] = E[y] = \mu_y = \bar{Y} = 1.5$$

$$m_{02} = E[x^0 y^2] = E[y^2] = 2.89$$

$$C_{xy} = E[(X-\bar{X})(Y-\bar{Y})] = E[XY] - \bar{X}\bar{Y}$$

$$= R_{xy} - \bar{X}\cdot\bar{Y}$$

$$= -1.72 - (-1.1 \times 1.5)$$

$$= -1.72 + 1.65$$

$$= -0.07$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{E[x^2] - \{E(x)\}^2} \Rightarrow \sqrt{E[x^2] - [\bar{x}]^2}$$

$$= \sqrt{-1.16 - [-1.1]^2} \Rightarrow \sqrt{-1.16 - 1.21}$$

$$= \sqrt{-2.37}$$

$$\sigma_y = \sqrt{\sigma_y^2} = \sqrt{E(y^2) - [E(y)]^2} = \sqrt{2.89 - (1.5)^2}$$

$$= \sqrt{2.89 - 2.25} = \sqrt{0.64}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$

$$= \frac{-0.07}{\sqrt{-2.37}\cdot\sqrt{0.64}} \quad \cancel{-0.07}$$

2.) Prove $var(X+Y) = var(X) + var(Y)$ if X, Y are S.I.

Sol:- Proof:-

Let $X+Y = a$

$$var(a) = E(a^2) - [E(a)]^2$$

$$= E[(X+Y)^2] - \{E[(X+Y)]\}^2$$

$$= E[X^2+Y^2+2XY] - [E(X)+E(Y)]^2$$

$$= E[X^2]+E[Y^2]+2E[XY] - \{[E(X)]^2+[E(Y)]^2+2E(X)\cdot E(Y)\}$$

$$= E(X^2)+E(Y^2)+2E(XY) - [E(X)]^2 - [E(Y)]^2 - 2E(X)\cdot E(Y)$$

$$= \underbrace{E(X^2)-[E(X)]^2}_{} + \underbrace{E(Y^2)-[E(Y)]^2}_{} + 2E(XY) - 2E(X)\cdot E(Y)$$

$$= Var(X) + Var(Y) + 2E(XY) - 2E(X)\cdot E(Y).$$

If X, Y are S.I then $E[XY] = E(X)\cdot E(Y)$.

$$= Var(X) + Var(Y) + 2\cancel{E(X)\cdot E(Y)} - 2\cancel{E(X)\cdot E(Y)}$$

$$= Var(X) + Var(Y).$$

3.) Let X, Y be R.V having Joint density function $f_{XY}(x,y) = x+y$ for all $0 \le x \le 1$, $0 \le y \le 1$. Find $var(X)$, $var(Y)$, σ_{XY}, ρ.

Sol:- $f_{XY}(x,y) = x+y \ \forall \ 0 \le x \le 1$

$0 \le y \le 1$

$$var(x) = E(x-\mu_x)^2 = E[x^2] - (E(x))^2$$

$$E(x) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$f_X(x)$ from $f_{XY}(x,y)$ using Marginal density function

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^1 (x+y)\,dy = \int_0^1 x\,dy + \int_0^1 y\,dy$$

$$= x[y]_0^1 + \left[\frac{y^2}{2}\right]_0^1 \Rightarrow x[1] + \frac{1}{2}$$

$$= x + \frac{1}{2} \ \forall \ 0 \text{ to } 1.$$

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \int_0^1 x\cdot\left(x+\frac{1}{2}\right)dx = \int_0^1 x^2\,dx + \int_0^1 \frac{x}{2}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 \Rightarrow \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$$

$$E[x^2] = \int_0^1 x^2\left(x+\tfrac{1}{2}\right)dx = \int_0^1 x^3\,dx + \int_0^1 \frac{x^2}{2}\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1 + \frac{1}{2}\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{4}+\frac{1}{6} = \frac{10}{24} = \frac{5}{12}.$$

$$\mathrm{var}(x) = E(x^2) - [E(x)]^2 = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{5}{12} - \frac{49}{144}$$

$$= \frac{11}{144}.$$

$$\mathrm{var}(y) = E(y-\mu_y)^2 = E(y^2) - [E(y)]^2$$

$$E(y) = \int_{-\infty}^{\infty} y.f_y(y)\,dy$$

$f_y(y)$ from $f_{xy}(x,y)$ using marginal density function.

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_0^1 (x+y)\,dx = \int_0^1 x\,dx + \int_0^1 y\,dx.$$

$$= \left[\frac{x^2}{2}\right]_0^1 + y[x]_0^1 = \frac{1}{2} + y \quad \forall\ 0 \le y \le 1$$

$$E(y) = \int_{-\infty}^{\infty} y.f_y(y)\,dy = \int_0^1 y\left(\tfrac{1}{2}+y\right)dy$$

$$= \int_0^1 \frac{y}{2}\,dy + \int_0^1 y^2\,dy \Rightarrow \left[\frac{y^2}{4}\right]_0^1 + \left[\frac{y^3}{3}\right]_0^1$$

$$= \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$$

$$E(y^2) = \int_{\cancel{-\infty}\ 0}^{\cancel{\infty}\ 1} y^2\left(\tfrac{1}{2}+y\right)dy = \int_0^1 \frac{y^2}{2}\,dy + \int_0^1 y^3\,dy$$

$$= \left[\frac{y^3}{6}\right]_0^1 + \left[\frac{y^4}{4}\right]_0^1 = \frac{1}{6}+\frac{1}{4} = \frac{10}{24} = \frac{5}{12}$$

$$\mathrm{var}(y) = E(y^2) - [E(y)]^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$$

$$\sigma_{xy} = E[xy] - \bar{x}.\bar{y}$$

$$E[xy] = \int_0^1\int_0^1 xy.f_{xy}(x,y)\,dy\,dx$$

$$= \int_0^1\int_0^1 xy[x+y]\,dy\,dx$$

$$= \int_0^1\int_0^1 (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2.\frac{y^2}{2} + x.\frac{y^3}{3}\right]_0^1 dx.$$

$$= \int_0^1 \left(\frac{x^2}{2} + \frac{x}{3}\right) dx = \left[\frac{x^3}{6} + \frac{x^2}{6}\right]_0^1 \Rightarrow \frac{1}{6} + \frac{1}{6} = \frac{2}{6}$$

$$= \frac{1}{3}$$

$$\sigma_{xy} = E[xy] - \bar{x}\cdot\bar{y} = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$$

$$= -\frac{1}{144}$$

$$\rho = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y} = \frac{-\frac{1}{144}}{\sqrt{\sigma_x^2}\cdot\sqrt{\sigma_y^2}} = \frac{-\frac{1}{144}}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}}$$

$$= \frac{\frac{-1}{\cancel{144}}}{\frac{11}{\cancel{144}}} = \frac{-1}{11}.$$

4.) Two random variables x and Y have the joint density

$$f_{xy}(x,y) = \begin{cases} \frac{xy}{9} & 0 < x < 2,\ 0 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$ show that X and Y are uncorrelated and also statistically independent.

Sol:- $f_{xy}(x,y) = f_x(x)\cdot f_y(y) \rightarrow$ X, Y are S.I

"Marginal density"

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_0^3 \frac{xy}{9}\,dy = \frac{x}{9}\left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{x}{\cancel{9}}\left[\frac{\cancel{9}}{2}\right] = \frac{x}{2}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_0^2 \frac{xy}{9}\,dx = \frac{y}{9}\left[\frac{x^2}{2}\right]_0^2 = \frac{y}{9}\left[\frac{4}{2}\right]$$

$$f_y(y) = \frac{2y}{9}.$$

$$f_x(x)\cdot f_y(y) = \frac{x}{\cancel{2}}\cdot\frac{\cancel{2}y}{9} = \frac{xy}{9} = f_{xy}(x,y) \rightarrow X, Y \text{ are S.I}$$

5.) Find the marginal densities of the joint density

$$f_{xy}(x,y) = \begin{cases} b(x+y)^2 & -2 < x < 2 \text{ and } -3 < y < 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Sol:- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{-3}^{3} b(x+y)^2 \cdot dy = b\left[\int_{-3}^{3}(x^2+y^2+2xy)\,dy\right]$

$$= b\left[\int_{-3}^{3} x^2 dy + \int_{-3}^{3} y^2 dy + \int_{-3}^{3} 2xy \cdot dy\right]$$

$$= b\left[x^2[1]_{-3}^{3} + \left[\frac{y^3}{3}\right]_{-3}^{3} + \cancel{2}x\left[\frac{y^2}{\cancel{2}}\right]_{-3}^{3}\right]$$

$$= b\left[x^2(6) + \frac{1}{3}[27+27] + x[9-9]\right]$$

$$= b[6x^2+18]$$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-2}^{2} b(x^2+y^2+2xy)\,dx.$

$$= b\left[\frac{x^3}{3}\right]_{-2}^{2} + y^2[x]_{-2}^{2} + \cancel{2}y\left[\frac{x^2}{\cancel{2}}\right]_{-2}^{2}$$

$$= b\left[\frac{1}{3}[8+8] + y^2[4]\right]$$

$f_Y(y) = b\left[\frac{16}{3} + 4y^2\right]$

$f_X(x)\cdot f_Y(y) = b[6x^2+18]\cdot b\left[\frac{16}{3}+4y^2\right]$

$x^2 \cdot y^2$
↗
$(x+y)^2$

X, Y are not s.I

6) For two random variables X and Y

$$f_{X,Y}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.5\,\delta(x-1)\,\delta(y-3).$$

Find the correlation coefficient of X and Y.

Sol:- $R_{XY} = \sum_x \sum_y x \cdot y \cdot P_{xy}(x,y).$

y \ x	-1	0	1
0	0.15	0.1	
1			0.2
2		0.1	
3			0.5
-2			0.4

$\therefore \delta(x+1)$
$x+1=0$
$x=-1$

$R_{XY} = \sum_{x=-1}^{1} \sum_{y=-2}^{3} x \cdot y \cdot P_{xy}(x,y)$

$$\sum_{x=-1}^{1}\Big[x(-2)\cdot P_{XY}(x,-2) + x(-1)\,P_{XY}(x,-1) + x(0)\cdot P_{XY}(x,0) + x(1)\cdot P_{XY}(x,1) + x(2)\cdot P_{XY}(x,2) + x(3)\cdot P_{XY}(x,3)\Big]$$

$$= (-1)(-2)\,P_{XY}(-1,-2) + \cancel{0(-2)\,P_{XY}(0,-2)} + (1)(-2)\,P_{XY}(1,-2) + \cancel{(-1)(0)\,P_{XY}(-1,0)} + \cancel{0(0)\cdot P_{XY}(0,0)} + (1)(0)\cdot \cancel{P_{XY}(1,0)}$$

$$+ (-1)(1) \cdot P_{xy}(-1,1) + \cancel{(0)(1) P_{xy}}(0,1) + (1)(1) P_{xy}(1,1) +$$

$$(-1)(2) \cdot P_{xy}(-1,2) + \cancel{(0)(2) P_{xy}}(0,2) + (1)(2) P_{xy}(\overset{1}{0},2) +$$

$$(-1)(3) P_{xy}(-1,3) + \cancel{(0)(3) P_{xy}}(0,3) + (1)(3) P_{xy}(1,3).$$

$$= 2P_{xy}(-1,-2) - 2P_{xy}(1,-2) - 1P_{xy}(-1,1) + P_{xy}(1,1) - 2P_{xy}(-1,$$

$$+ 2P_{xy}(1,2) - 3P_{xy}(-1,3) + 3P_{xy}(1,3).$$

$$= \cancel{2(0)} - 2(0.4) - 1\cancel{(0)} + 0.2 - 2\cancel{(0)} + 2\cancel{(0)} - 3\cancel{(0)} + 3(0.5)$$

$$= -0.8 + 0.2 + 1.5$$

$$= 0.9$$