UNIT - II

UNIT-III

3. Operation on single random variable

(3.1)

Introduction :-

The random variable was introduced in chapter-2 as a means of providing a systemetic defination of events defind as a sample. space specifically if formed a mathamatical model for describing cherecterstics of some real physical word random phenomenon.

Mathamatical expectation :-

→ The average value of or mean value of a density function is known as mathametical expectation and it is denoted by $E[x]$ (or) m (or) μ. (or) $\bar{x}$.

Mathamatical expectation of random variable (or) Expected value of "x" (or) Mean value of "x" :-

Expected value of random variable is denoted by $E[x]$ (or) $\bar{x}$ (or) μ.

If "x" is a continuous random variable with density function $f_x(x)$ then the expected value of the random variable is

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_x(x)\, dx.$$

provided that the R.H.S series is absoluttey convergent

$$E[x] = \left| \int_{-\infty}^{\infty} x\, f_x(x)\, dx \right| < \infty$$

If "x" is a discrete random variable with assigned values $x_1, x_2, \ldots x_n$ having probabilities $p(x_1), p(x_2) \ldots p(x_n)$, respectively. then the density function of ...

$$f_x(x) = \sum_{i=1}^{N} P(x=x_i)\, \delta(x-x_i)$$

The expected value of a discrete random variable is defind as

$$E[x] = \sum_{\text{all } x} x\, f_x(x) \quad (\text{or}) \quad = \sum_{i=1}^{N} x_i f_x(x_i).$$

"By using expected value of random variable "x" we will find out the centerd value of density function."

Let us consider all assigned values of random variable "x" having equal probabilities

i.e. $p(x_1) = p(x_2) = \cdots\cdots = p(x_N) = \frac{1}{N}.$

$$E[x] = x_1 p(x_1) + x_2 p(x_2) + \cdots + x_n p(x_n)$$

$$= \frac{1}{N} p(x_1) + \frac{1}{N} x_2 + \cdots + \frac{1}{N} x_n$$

$$E[x] = \frac{1}{N} (x_1 + x_2 + \cdots + x_n)$$

Hence the probabilities of all assigned values are equal then the expected value is equal to "Arithemetic mean (or) Average mean"

Expected value of a function of a random variable :-

Let us consider a random variable "x" and $g(x)$ is a function of random variable "x."

If "x" is continuous random variable then the expected value of a function of random variable is

$$\bar{g} = E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_x(x)\, dx.$$

If "x" is a discrete random variable then the expected value of a function is defind as (3.2)

$$\bar{g} = E[g(x)] = \sum_{\text{all } x} g(x) f_x(x) = \sum_{i=1}^{N} g(x_i) f_x(x_i)$$

Theorem's on expectation's :-

Let us consider random variable "x" with density function $f_x(x)$ is

(i) E[constant] = constant

ie $E[K] = K$

Proof :- From the defination of expectation

$$E[x] = \sum_{i=1}^{N} x_i f_x(x_i)$$

$$E[K] = \sum_{i=1}^{N} K f_x(x_i) \qquad \left[\because \sum_{i=1}^{N} f_x(x_i) = 1\right]$$

$$= K \sum_{i=1}^{N} f_x(x_i)$$

$$= K(1)$$

$$\boxed{E[K] = K}$$

(ii) $E(kx) = K E(x)$

Proof :- From the defination of expectation

$$E(x) = \sum_{i=1}^{N} x_i f_x(x_i)$$

$$E(kx) = \sum_{i=1}^{N} k x_i f_x(x_i)$$

$$= K \sum_{i=1}^{N} x_i f_x(x_i)$$

$$= K . E[x]$$

$$\boxed{\therefore E[kx] = K E[x]}$$

(iii) $E[ax+b] = a E[x] + b$

Proof :- From the defination of expectation

$$E[x] = \sum_{i=1}^{N} x_i f_x(x_i)$$

$$E[ax+b] = \sum_{i=1}^{N} (ax_i+b) f_x(x_i)$$

$$= \sum_{i=1}^{N} a x_i f_x(x_i) + \sum_{i=1}^{N} b\, f_x(x_i)$$

$$= a \sum_{i=1}^{N} x_i f_x(x_i) + b \sum_{i=1}^{N} f_x(x_i)$$

$$\boxed{E[ax+b] = a E[x] + b}$$

(iv) Additional Theorem on expectation :-

Let us consider two random variables "X & Y" with the joint density function $f_{xy}(x,y)$ then expectation of

$$E[X+Y] = E[X] + E[Y]$$

Proof :- Let us consider two random variables "X and y" with joint function $f_{xy}(x,y)$

From the defination of expectation

$$E[X] = \sum_{i=1}^{N} x_i f_x(x_i)$$

$$E[Y] = \sum_{j=1}^{N} y_j f_y(y_j)$$

$$E(X+Y) = \sum_{i=1}^{N} \sum_{j=1}^{N} [x_i + y_j] \cdot f_{xy}(x_i, y_j)$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} x_i f_{xy}(x_i, y_j) + \sum_{i=1}^{N} \sum_{j=1}^{N} y_j f_{xy}(x_i, y_j)$$

We know that the marginal density functions of "X & Y" are

$$f_x(x_i) = \sum_{j=1}^{N} f_{xy}(x_i, y_j) \longrightarrow (1)$$

$$f_y(y_j) = \sum_{i=1}^{N} f_{xy}(x_i, y_j) \longrightarrow ②$$

$$\therefore E[X+Y] = \sum_{i=1}^{N} x_i \sum_{j=1}^{N} f_{xy}(x_i, y_j) + \sum_{j=1}^{N} y_j \sum_{i=1}^{N} f_{xy}(x_i, y_j)$$

$$= \sum_{i=1}^{N} x_i f_x(x_i) + \sum_{j=1}^{N} y_j f_y(y_j)$$

$$\boxed{E(X+Y) = E[X] + E[Y]}$$

(v) **Multiplication theorem of expectation :-**

If X & Y are independent random variables then

$$E[XY] = E[X] \cdot E[Y]$$

Proof :- From the defination of expectation

$$E[X] = \sum_{i=1}^{N} x_i f_x(x_i)$$

$$E[Y] = \sum_{j=1}^{N} y_j f_y(y_j)$$

$$E(XY) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j f_{xy}(x_i, y_j)$$

If 'X & Y' are independent random variables then

$$f_{xy}(x_i, y_j) = f_x(x_i) f_y(y_j)$$

$$E(XY) = \sum_{i=1}^{N} \sum_{j=1}^{N} x_i y_j f_x(x_i) f_y(y_j)$$

$$= \sum_{i=1}^{N} x_i f_x(x_i) * \sum_{j=1}^{N} y_j f_y(y_j)$$

$$\boxed{E(XY) = E[X] \cdot E[Y]}$$

(vi) **Theorem-6 :-** If $X \geq 0$, then $E(X) \geq 0$.

Proof :- From the defination of expectation

$$E[X] = \sum_{i=n}^{N} x_i f_x(x_i)$$

Here, as per problem. $x_i \geq 0$.

From the property of $f_x(x)$.

$$f_x(x_i) \geq 0$$

$$x_i f_x(x_i) \geq 0.$$

$$\sum_{i=1}^{N} x_i f_x(x_i) \geq 0.$$

$$E[X] \geq 0.$$

Theorem-7 :- If $X \geq Y$, then $E(X) \geq E(Y)$

Proof :- Given $X \geq Y$.

but $X - Y \geq Y - Y$

$$X - Y \geq 0.$$

Taking expectation on both sides

$$E(X-Y) \geq 0$$

$$E(X) - E(Y) \geq 0$$

$$E(X) \geq E(Y)$$

Moments of a random variable :-

The n^{th} moments of a random variable can be divided into two types

(i) Moments about origin

(ii) Moments about mean (or) central moments :-

Moments about origin (m_n or μ_n') :-

They are denoted by m_n or μ_n' and is defined as

$$m_n = \mu_n^1 = E(x^n)$$

(3.4)

If "x" is a continuous random variable then

$$m_n = E(x^n) = \int_{-\infty}^{\infty} x^n f_x(x)\,dx.$$

If "x" is a discrete random variable then

$$m_n = \mu_n^1 = E(x^n) = \sum_{i=1}^{N} x_i^n f_x(x_i)$$

Case(i):- If $n=1$, then $m_1 = E[x^1] = E(x)$

i.e. the first moment about origin is the mean value of a random variable "x".

Case(ii):- If $n=0$, then $m_0 = E(x^0) = E(1) = 1$

i.e the zeroth moment about origin is the area under the PDF

Case(iii):- If $n=2$, then $m_2 = E(x^2)$

the second moment about origin is the mean squard value of a random variable "x" and this is also equal to the "total average power."

Moments about mean or central moments:-

The n^{th} moment about mean (or) n^{th} central moment is denoted by μ_n and is defind as $\mu_n = E[(x-\bar{x})^n]$

$$\mu_n = E[(x-\bar{x})^n] \text{ (or) } \mu_n = E[(x-\mu)^n].$$

Here $\bar{x}, \mu$ are mean of random variable "x". If x is

$$\mu_n = E(x-\bar{x})^n = \int_{-\infty}^{\infty} (x-\bar{x})^n f_x(x)\,dx.$$

If "x" is discrete random variable then

$$\mu_n = E(x-\bar{x})^n = \sum_{i=1}^{N} (x_i-\bar{x}_i)^n f_x(x_i)$$

Case(i):- If $n=0$, then

$$\mu_0 = E[(x-\bar{x})^0] = E(1) = 1.$$

i.e the zeroth centerd moment about mean is the area under the PDF.

Case(ii):- If $n=1$, then

$$\mu_1 = E(x-\bar{x})^1 = E(x) - \bar{x} = \bar{x}-\bar{x} = 0$$

ie the first centerd moment about mean is equal to the "zero".

Case(iii):- If $n=2$, then

$$\mu_2 = E[(x-\bar{x})^2] = \sigma_x^2$$

ie. the second moment about mean is the varience of random variable "x"

Varience of random variable "x":-

The varience of random variable "x" is denoted by $Var(x)$ or σ_x^2. and is defined as second central moment

ie. $Var(x) = \sigma_x^2 = E[(x-\bar{x})^2]$

If "x" is continuous random variable then the varience of "x" is $Var(x) = \sigma_x^2 = E[(x-\bar{x})^2] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\,dx.$

If "x" is discrete random variable then

$$var(x) = E[(x-\bar{x})^2] = \sum_{all\,x} (x-\bar{x})^2 f_x(x)$$ (3.5)

The varience of is to measure the dispersion (varience) about it's mean value.

If the assigned values are nearer to the mean value then the "varience is small."

If the assigned values are away to the mean value then the "varience is large."

It is dimensionless (no units) quantity because of the reason for measuring the deviation we will define the "standard deviation".

The standard deviation is denoted by "σ_x" and is defind as square root of varience

ie. $\boxed{\sigma_x = \sqrt{var(x)}}$

It is having the units same that as random variable units

Theorems on varience :-

(i) Theorem-1 :- $var(x) = E[x^2] - [E(x)]^2$ (or) $var(x) = E[x^2] - \bar{x}^2$

Proof :- From the defination of the varience.

$$Var(x) = E[(x-\bar{x})^2]$$
$$= E[x^2 + (\bar{x})^2 - 2x\bar{x}]$$
$$= E[x^2] + E[\bar{x}^2] - 2\bar{x}\bar{x}$$
$$= E[x^2] + E[\bar{x}]^2 - 2[E(x)]^2$$
$$\boxed{Var(x) = E[x^2] - [E(x)]^2 \text{ (or) } E[x^2] - \bar{x}^2}$$

(ii) Theorem-2 :- $Var(cx) = c^2 Var(x)$

Proof :- From the defination of varience

$$Var(x) = E((x-\bar{x})^2) \to ①$$
$$Var(cx) = E((cx - c\bar{x})^2)$$
$$= E[c^2(x-\bar{x})^2]$$
$$= c^2[E(x-\bar{x})^2]$$
$$= c^2 \cdot Var(x)$$
$$\boxed{\therefore Var(cx) = c^2 Var(x)}$$

(iii) Theorem-3 :- If "x & y" are independent random variable then

$$Var(x+y) = Var(x) + Var(y). \text{ and}$$
$$Var(x-y) = Var(x) - Var(y).$$

Proof :- From the defination of varience.

$$\left.\begin{aligned} Var(x) &= E[(x-\bar{x})^2] \\ Var(y) &= E[(y-\bar{y})^2] \end{aligned}\right\} \cdots ①$$

Here $\bar{x} = E[x]$ and $\bar{y} = E(y)$.

$$Var(x+y) = E[[(x+y) - E(x+y)]^2]$$
$$= E[(x+y) - (E(x) + E(y))]^2$$
$$= E(x+y - \bar{x} - \bar{y})^2$$
$$= E((x-\bar{x}) + (y-\bar{y}))^2$$
$$= E[(x-\bar{x})^2 + (y-\bar{y})^2 + 2(x-\bar{x})(y-\bar{y})]$$
$$= E((x-\bar{x})^2) + E(y-\bar{y})^2 + 2\,E(x-\bar{x})\,E(y-\bar{y})$$

$= Var(x) + Var(y) + 2E(x-\bar{x}) \cdot E(y-\bar{y})$

$= Var(x) + Var(y) + 2[E(x) - E(\bar{x})](E(y) - E(\bar{y}))$

$= Var(x) + Var(y) + 2(\bar{x}-\bar{x})(\bar{y}-\bar{y})$

$= Var(x) + Var(y) + 2(0)$

$\therefore \boxed{Var(x+y) = Var(x) + Var(y)}$

(or)

$Var(x-y) = Var(x+(-y))$

$= Var(x) + Var(-y)$

$= Var(x) + (-1)^2 Var(y)$

$= Var(x) + Var(y)$

$\therefore \boxed{Var(x-y) = Var(x) + Var(y)}$

(iv) **Theorem-4:-** $Var(ax+b) = a^2 Var(x)$

proof:- From the defination of varience

$Var(x) = E[(x-\bar{x})^2]$

$= E((ax+b) - E(ax+b))^2$

$= E((ax+b) - (aE(x)+b))^2$

$= E(ax+b-aE(x)-b)^2$

$= E(ax-aE(x))^2$

$= a^2[E(x)^2 + E(E(x))^2 - 2E(x)E(x)]$

$= a^2[E(x^2) + (E(x))^2 - 2(E(x))^2]$

$= a^2[E(x^2) - (E(x))^2]$

$\boxed{Var(ax+b) = a^2 Var(x)}$

3.6

(v) **Theorem-5:-** $Var(\text{constant}) = 0$ (or) $Var(k) = 0$

proof:- we know that $Var(ax+b) = a^2 Var(x)$

let $a = 0$

$Var(ax+b) = 0^2 Var(x)$

$Var(b) = 0$

$\boxed{Var(K) = 0}$

Skew and coefficient of skewness:-

Skew is discribing the asymmetry of the density function.

The skew is defined as the third central moment about of mean i.e. $\boxed{\mu_3 = E[(x-\bar{x})^3]}$

The measure of assymetry is known as coefficient of the skewness or skewness.

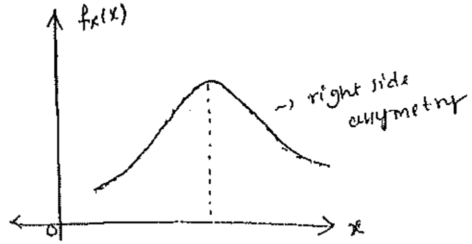
$$\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{E(x-\bar{x})^3}{\sigma^3}$$

($\because$ Here σ = standard deviation)

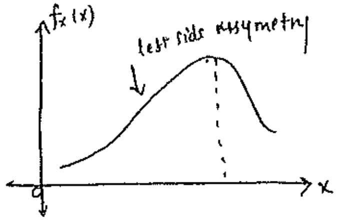
The skewness is dimentionless quantity.

The coefficent of skewness is either positive or negative.

If α_3 is positive, then the function is assymetry to right side.



If α_3 is negative, then the function is asymmetry to left side

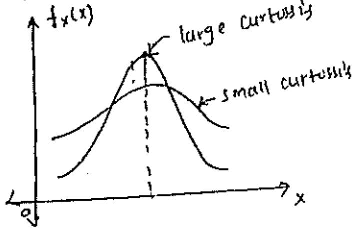


Curtosis or Kurtosis :-

It measures the degree of peakness (maximum) is called coefficient of curtosis and it is denoted by α_4

$$\alpha_4 = \frac{\mu_4}{\sigma^4} = \frac{\mu_4}{\sigma^4}$$

$$\alpha_4 = \frac{E\left((x-\bar{x})^4\right)}{\sigma^4}$$



Here μ_4 is called the 4^{th} central moment.

σ = standard deviation

Moment generating function :-

The moment generating function of random variable "X" is denoted by $M_X(t)$ and is defind as

$$M_X(t) = E\left(e^{tx}\right)$$

If "X" is continuous random variable, then

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

If "X" is discrete random variable, then

$$M_X(t) = E\left(e^{tx}\right) = \sum_{\text{all } x} e^{tx} f_X(x) = \sum_{i=1}^{n} e^{tx_i} f_X(x_i)$$

Properties :-

① If the moment generating function of random variable is $M_X(t)$ then $M_{cX}(t) = M_X(ct)$.

Proof :- From the defination of moment generating function

$$M_X(t) = E\left(e^{tx}\right)$$

$$M_{cX}(t) = E\left(e^{cxt}\right)$$

$$= E\left[e^{(tc)x}\right]$$

$$\boxed{M_{cX}(t) = M_X(ct).}$$

② The moment generating function of random variable is $M_X(t)$ then moment generating function of $y = ax+b$ is $M_X(t) = e^{bt} M_X(at)$.

Proof :- Given the moment generating function is

$$M_X(t) = E\left(e^{tx}\right)$$

$$y = ax+b$$

$$M_y(t) = E\left(e^{ty}\right)$$

$$= E\left(e^{t(ax+b)}\right)$$

$$= E\left(e^{tax+tb}\right)$$

$$= E\left(e^{tb}\, e^{tax}\right)$$

$$= e^{tb}\, E\left[e^{(ta)x}\right]$$

$$\boxed{M_y(t) = e^{tb} M_X(ta)}$$

③ If X and Y are independent random variable with moment generating function $M_X(t)$, $M_Y(t)$ then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

Proof :- The moment generating function of X and Y are

$$M_X(t) = E\left(e^{tx}\right); \quad M_Y(t) = E\left(e^{ty}\right) \longrightarrow ①$$

If x and y are independent random variables then (3.8)

$$M_{x+y}(t) = E\left[e^{t(x+y)}\right]$$

$$M_{x+y}(t) = E\left(e^{tx} \cdot e^{ty}\right)$$

W.K.T if x and y are independent random variables then

$$E(x \cdot y) = E(x) \cdot E(y).$$

$$M_{x+y}(t) = E\left(e^{tx} \cdot e^{ty}\right)$$

$$= E(e^{tx}) \cdot E(e^{ty})$$

$$= M_x(t) \cdot M_y(t)$$

$$\therefore \boxed{M_{x+y}(t) = M_x(t)\, M_y(t)}$$

(4) If the moment generating function of random variable "x" is $M_x(t)$ then the moment generating function of random variable "y" is $M_y(t)$. then $y = \frac{x+a}{b}$ $e^{a/b\, t} \cdot M_x(t/b)$

proof :- the moment generating function

$$M_x(t) = E\left(e^{tx}\right)$$

$$M_y(t) = E\left(e^{ty}\right)$$

$$= E\left[e^{t\left(\frac{x+a}{b}\right)}\right]$$

$$= E\left(e^{tx/b} \cdot e^{ta/b}\right)$$

$$= e^{ta/b}\, E\left(e^{t/b\, x}\right)$$

$$\boxed{M_y(t) = e^{ta/b} \cdot M_x(t/b)}$$

Hence proved

The following steps gives the procedure for obtaining the moment's about origin from moment generating function :-

proof :- we know that $M_x(t) = E(e^{tx})$.

$$M_x(t) = E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^n}{n!} + \cdots \infty\right]$$

$$= E\left[1 + \frac{tx}{1} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^n x^n}{n!} + \cdots \infty\right]$$

$$= 1 + E\left(\frac{t\,x}{1}\right) + E\left(\frac{t^2x^2}{2!}\right) + \cdots + E\left(\frac{t^n x^n}{n!}\right) + \cdots \rightarrow ①$$

differentiate with respect to "t" on both sides

$$\frac{\partial}{\partial t}(M_x(t))\Big|_{t=0} = 0 + E(x) + \frac{2t}{2!}E(x^2) + \cdots + \frac{n\,t^{n-1}}{n!}E(x^n) \rightarrow ②$$

$$\frac{\partial}{\partial t}(M_x(t))\Big|_{t=0} = E(x) + 0 + 0.$$

$$\therefore E(x) = \frac{\partial}{\partial t}(M_x(t))\Big|_{t=0}$$

this is the first movement about origin

Again differentiate with respect to "t" on both sides on eq ②

$$\frac{\partial^2}{\partial t^2}[M_x(t)] = 0 + 0 + \cdots + \frac{2}{2!}E(x^2) + \frac{6t}{3!}E(x^3) + \cdots$$

$$\frac{\partial^2}{\partial t^2}[M_x(t)]\Big|_{t=0} = E(x^2)$$

$$E(x^2) = \frac{\partial^2}{\partial t^2}[M_x(t)]\Big|_{t=0}.$$

Cherecterstic function's:-

the cherecterstic function of a random variable is denoted by $\phi_x(\omega)$. and is defind as

$$\phi_x(\omega) = E\left(e^{j\omega x}\right).$$

If "x" is continuous random variable then

$$\phi_x(\omega) = E\left(e^{j\omega x}\right) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$$

If "x" is discrete random variable. then

$$\phi_x(\omega) = E\left(e^{j\omega x}\right) = \sum_{\text{all } x} e^{j\omega x} f_x(x) = \sum_{i=1}^{n} e^{j\omega x_i} f_x(x_i)$$

Properties of cherecterstic function :-

(i) $|\phi_x(\omega)| \leq 1$

Proof :- From the defination of cherecterstic function

$$\phi_x(\omega) = E\left[e^{j\omega x}\right]$$

$$\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$$

$$= \left|\int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx\right|$$

$$\phi_x(\omega) \leq \int_{-\infty}^{\infty} |e^{j\omega x}|\,|f_x(x)|\,dx \qquad \left[\because |ab| \leq |a||b|\right]$$

$$\phi_x(\omega) \leq \int_{-\infty}^{\infty} |f_x(x)|\,dx \qquad \left[\because \int_{-\infty}^{\infty} f_x(x)\,dx = 1\right]$$

$$\phi_x(\omega) \leq \int_{-\infty}^{\infty} f_x(x)\,dx$$

$$\boxed{\phi_x(\omega) \leq 1}$$

(2) $\phi_x(0) = 1$

Proof :- From the defination of cherecterstic function

$$\phi_x(\omega) = E\left(e^{j\omega x}\right)$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx$$

$$\phi_x(0) = \int_{-\infty}^{\infty} e^{0} \cdot f_x(x)\,dx \qquad \left[\because \int_{-\infty}^{\infty} f_x(x)\,dx = 1\right]$$

$$= \int_{-\infty}^{\infty} 1 \cdot f_x(x)\,dx$$

$$\boxed{\phi_x(0) = 1}$$

(3) $\phi_x(\omega)$ and $\phi_x(-\omega)$ are complex conjugative function's

i.e $\phi_x(\bar{\omega}) = \phi_x(-\omega)$

Proof :- From the defination of cherecterstic function

$$\phi_x(\omega) = E\left(e^{j\omega x}\right) \longrightarrow (1)$$

$$\phi_x(\bar{\omega}) = E\left(e^{j\bar{\omega} x}\right)$$

$$\phi_x(\bar{\omega}) = E\left(e^{-j\omega x}\right) \longrightarrow (2)$$

from eq (1) $\quad \phi_x(-\omega) = E\left[e^{j(-\omega)x}\right]$

$$\phi_x(-\omega) = E\left(e^{-j\omega x}\right) \longrightarrow (3)$$

From (2) and (3)

$$\phi_x(\bar{\omega}) = \phi_x(-\omega)$$

(4) $\phi_x(\omega)$ is continuous function $\forall\ \omega$ $(-\infty < \omega < \infty)$

(5) $\phi_{cx}(\omega) = \phi_x(c\omega)$

Proof :- From the cherecterstic function

$$\phi_x(\omega) = E\left(e^{j\omega x}\right)$$

$$\phi_{xc}(\omega) = E\left(e^{j\omega xc}\right)$$

$= E\left[e^{j(c\omega)x}\right]$ (3.10)

$$\boxed{\phi_{cx}(\omega) = \phi_x(c\omega)}$$

(6) $\phi_{ax+b}(\omega) = e^{j\omega b}\phi_x(a\omega)$

Proof:- From the cherecterstic function

$$\phi_x(\omega) = E\left(e^{j\omega x}\right)$$

$$\phi_{ax+b}(\omega) = E\left[e^{j\omega(ax+b)}\right]$$

$$= E\left[e^{j\omega ax + j\omega b}\right]$$

$$= E\left(e^{j\omega ax}\, e^{j\omega b}\right)$$

$$= e^{j\omega b}\, E\, e^{j\omega(ax)}$$

$$\boxed{\phi_{ax+b}(\omega) = e^{j\omega b}\phi_{ax}(\omega)}$$

(7) If "x and y" are Individual random variables then the

$$\phi_{x+y}(\omega) = \phi_x(\omega)\cdot\phi_y(\omega)$$

Proof:- From the defination of cherecterstic function

$$\left.\begin{aligned}\phi_x(\omega) &= E\left(e^{j\omega x}\right)\\ \phi_y(\omega) &= E\left(e^{j\omega y}\right)\end{aligned}\right\}\longrightarrow (1)$$

$$\therefore\ \phi_{x+y}(\omega) = E\left[e^{j\omega(x+y)}\right]$$

$$= E\left[e^{j\omega x + j\omega y}\right]$$

$$= E\left[e^{j\omega x}\cdot e^{j\omega y}\right]$$

$$= E\left(e^{j\omega x}\right)\cdot E\left(e^{j\omega y}\right)$$

If "x & y" are Independent random variable. then

$$E(xy) = E(x)\cdot E(y)$$

$$\therefore\ \phi_{x+y}(\omega) = E\left[e^{j\omega x}\right]\ E\left[e^{j\omega y}\right]$$

$$\therefore\ \boxed{\phi_{x+y}(\omega) = \phi_x(\omega)\cdot\phi_y(\omega)}$$

Steps for obtaining the moments from Cherecterstic function:-

From the defination of cherecterstic function

$$\phi_x(\omega) = E\left(e^{j\omega x}\right)$$

$$\phi_x(\omega) = E\left[1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \cdots + \frac{(j\omega x)^n}{n!}\right]$$

$$= 1 + \frac{j\omega}{1!}E(x) + \frac{(j\omega)^2}{2!}E(x^2) + \cdots + \frac{(j\omega)^n}{n!}E(x^n) \rightarrow (1)$$

differentiate eq (1) w.r.to "ω" on both sides, we get

$$\frac{\partial}{\partial\omega}\left[\phi_x(\omega)\right] = 0 + \frac{j}{1!}E[x] + \frac{j^2(2\omega)}{2!}E[x^2] + \cdots + \frac{j^n n j\omega^{n-1}}{n!}E(x^n) + \longrightarrow (2)$$

$$\left.\frac{\partial}{\partial\omega}\left(\phi_x(\omega)\right)\right|_{\omega=0} = 0 + jE(x) + 0 + \cdots 0$$

$$\boxed{E(x) = \frac{1}{j}\left.\frac{\partial}{\partial\omega}\left(\phi_x(\omega)\right)\right|_{\omega=0}}$$

this expression gives the 1st moment about the origin or a mean value of "x".

Again differentiate w.r.t "ω" in eq (2), we get

$$\frac{\partial^2}{\partial\omega^2}\left[\phi_x(\omega)\right] = 0 + \frac{2j^2}{2}E(x^2) + \frac{j^3 6\omega}{3!}E(x^3) + \cdots + j^n\frac{n(n-1)\omega^{n-2}}{n!}E(x^n)$$

$$\left.\frac{\partial^2}{\partial\omega^2}\left[\phi_x(\omega)\right]\right|_{\omega=0} = 0 + 0 + j^2E(x^2) + 0 + \cdots 0$$

$$\left.\frac{\partial^2}{\partial\omega^2}\left[\phi_x(\omega)\right]\right|_{\omega=0} = j^2E(x^2)$$

(3.11)

$$\frac{d^r}{dw^r}(\phi_x(\omega))\Big|_{\omega=0} = j^r E(x^r)$$

$$E(x^r) = \frac{1}{j^r}\frac{d^r}{dw^r}\left(\phi_x(\omega)\right)\Big|_{\omega=0}$$

this expression gives the 2^{nd} moment about the origin. (or) mean squared value of "x".

similarly. the n^{th} moment about origin of "x" is given by-

$$E(x^n) = \frac{1}{j^n}\frac{d^n}{dw^n}\left(\phi_x(\omega)\right)\Big|_{\omega=0}$$

the steps shows the cherecterstic function having more adwantage than moment generating function:-

→ the cherecterstic function is absolutley convergent

ie. $|\phi_x(\omega)| \le 1$

→ If the cherecterstic function is known then the distribution function can be find. by using cherecterstic function.

→ If cherecterstic function is known then we can find the density function by using cherecterstic function

ie. $\phi_x(\omega) = E(e^{j\omega x}) = \int_{-\infty}^{\infty} f_x(x)\, e^{j\omega x}\, dx$

$$f_x(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_x(\omega)\cdot e^{-j\omega x}\, dx$$

ie the cherecterstic function and density functions are fourier transform pairs

$$F_X(x) \xleftrightarrow{F.T} \phi_X(\omega)$$

$$\phi_X(\omega) \xleftrightarrow{I.F.T} F_X(\omega)$$

Transformation of random variable:-

Transformation means to change one random variable to (x) new random variable (y)

ie $Y = T[x]$.

the block diagram of this transformation is

$$\underset{f_X(x)}{\xrightarrow{X}}\boxed{f(x)}\underset{f_Y(y)}{\xrightarrow{Y}}$$

[∵ only assigned values are changed but not the probabilities]

In general X - is continuous, discrete, and mixed. and T is linear, non-linear, segmented stair case etc

But we will consider only fallowing two cases:-

(i) 'X' - continuous & 'T' continuous

(ii) 'X' - Discrete & 'T' continuous

→ X - continuous & T continuous :-

this transformation can be divided into two types. they are.

→ Monotonic transformation

→ Non monotonic transformation

→ Monotonic transformation :-

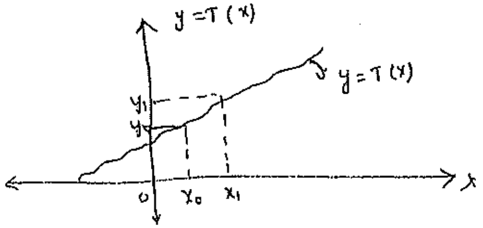
Monotonic transformation mean's one - one transformation and is divided into two types. they are

→ Monotonically increasing transformation

→ Monotonically decreasing transformation

Monotonically Increasing transformation :-

A transformation is said to be monotonically increasing if $T(x_1) < T(x_2)$ for all $x_1 < x_2$ and as shown in figure



From figure $y_0 = T(x_0)$
$y_1 = T(x_1)$

From the definition of transformation, $P(y = y_0) = P(x = x_0)$

$$\Rightarrow P(y \le y_0) = P(x \le x_0)$$

$$\Rightarrow \int_{-\infty}^{y_0} f_y(y)\,dy = \int_{-\infty}^{x_0} f_x(x)\,dx$$

taking differentiation on both sides, by using libentz rule

$$f_y(y_0) = f_x(x_0) \cdot \frac{\partial x_0}{\partial y_0}\Big|_{x_0 = T^{-1}[y_0]}$$

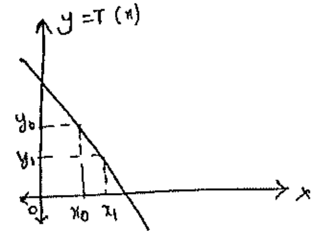
For all values of "x"

$$f_y(y) = f_x(x)\,\frac{\partial x}{\partial y}\Big|_{x = T^{-1}(y)}$$

Monotonically decreasing transformation :-

A transformation is said to be monotonically decreasing if $T(x_1) > T(x_2)$ for all $x_1 < x_2$ and shown in figure

From Figure $\cdot\ y_0 = T(x_0)$
$y_1 = T(y_1)$



$$P(y = y_0) = P(x = x_0)$$

$$P(y \le y_0) = P(x \ge x_0) = 1 - P(x < x_0)$$

$$\int_{-\infty}^{\infty} f_y(y)\,dy = 1 - \int_{-\infty}^{\infty} f_x(x)\,dx$$

differentiate by using Lienbnitz theorem.

$$f_y(y_0) = 0 - f_x(x_0)\,\frac{\partial x_0}{\partial y_0}\Big|_{x_0 = T^{-1}(y_0)}$$

$$\boxed{f_y(y) = -f_x(x)\,\frac{\partial x}{\partial y}\Big|_{x = T^{-1}(y)}}$$

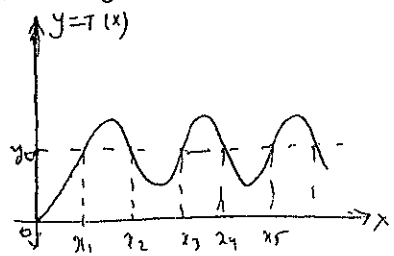
Finally for monotonic transformation

$$f_y(y) = f_x(x)\,\left|\frac{\partial x}{\partial y}\right|\Big|_{x = T^{-1}(y)}$$

→ NON-MONOTONIC TRANSFORMATION :-

A transformation is said to be not monotonic. then it is known as "non-monotonic transformation".

Non-monotonic transformation is many-one-transformation and is shown in figure



From figure.

$$P(y=y_0) = P(X=x_1) + P(X=x_2) + P(X=x_3) + P(X=x_4) \qquad (3.13)$$

$$P(y \le y_0) = P(x/_{x \le x_n})$$

$$\int_{-\infty}^{y_0} f_Y(y)\,dy = \int_{\{x/x \le x_n\}} f_X(x)\,dx$$

taking transformation on both sides by using Leibnitz's theorem.

$$f_Y(y) = \sum_{n=1}^{N} f_X(x_n) \cdot \left|\frac{dx_n}{dy}\right| \Bigg/_{x_n = T^{-1}(y_n)}$$

(ii) 'X' discrete and 'T' continuous :-

the density and distribution functions of "X" is

$$f_X(x) = \sum_{n=1}^{N} P(X=x_n)\,\delta(x-x_n)$$

$$F_Y(y) = \sum_{n=1}^{N} P(X=x_n)\,u(x-x_n)$$

Let us consider the transformation

$$y = T(x)$$

$$x = T^{-1}(y)$$

the density and distribution functions of "Y" is

$$f_Y(y) = \sum_{n=1}^{N} P(y=y_n)\cdot\delta(y-y_0)$$

$$F_Y(y) = \sum_{n=1}^{N} P(y=y_n)\,u(y-y_n)$$

here x_n is $T^{-1}[y_n]$

For monotonic transformation $P(y=y_n) = P(X=x_n)$

For non monotonic transformation $P(y=y_n) = P(X=x_1) + P(X=x_2) + \cdots + P(X=x$.

in the fallowing way.

(i) Ist moment about origin :-

$$\mu_1' = \frac{1}{j}\frac{d}{d\omega}(\phi_x(\omega))\Big|_{\omega=0}$$

$$E[e^{j\omega x}] = 1 + \frac{j\omega x}{1!} + \frac{(j\omega x)^2}{2!} + \cdots$$

$$\frac{d}{d\omega}E[e^{j\omega x}]\Big|_{\omega=0} = 0 + jE(x) + 2\frac{j^2\omega}{2!}E(x^2) + \cdots$$

$$= jE(x).$$

$$\boxed{E(x) = \frac{1}{j}\frac{d}{d\omega}\left[E(e^{j\omega x})\right]\Big|_{\omega=0}}$$

(ii) IInd Moment about origin :-

$$\mu_2' = \left(\frac{1}{j}\right)^2 \frac{d^2}{d\omega^2}(\phi_x(\omega))\Big|_{\omega=0}$$

nth moment about origin :-

$$\mu_n' = \left(\frac{1}{j}\right)^n \frac{d^n}{d\omega^n}(\phi_x(\omega))\Big|_{\omega=0}$$

* Cheby chev's inequality :-

→ A powerful inequality that is use. to determine an upper boundary of the distributing is "Chebchev's inequality"

If 'x' is a random variable with mean 'm' & varience σ_x^2. then for any positive value of 'K'. the Chebchev's inequality is given by probability of

$$\boxed{\begin{array}{l} P(|x-m| > K\sigma_x) \le \frac{1}{K^2} \quad \text{(or)} \\ \hline P|(x-m) < K\sigma_x) \le 1-\frac{1}{K^2} \end{array}}$$

proof :- If the mean of a random variable "x" is m and varience σ_x^2. with density function $f_x(x)$. then from the defination of varience.

$$\sigma_x^2 = E[(x-\mu)^2]$$

$$\sigma_x^2 = E[(x-m)^2] \qquad (\because \text{mean} = m)$$

$$= \int_{-\infty}^{\infty} (x-m)^2 f_x(x)\,dx$$

$$= \int_{-\infty}^{m-K\sigma_x} (x-m)^2 f_x(x)\,dx + \int_{m-K\sigma_x}^{m+K\sigma_x} (x-m)^2 f_x(x)\,dx + \int_{m+K\sigma_x}^{\infty} (x-m)^2 f_x(x)\,dx \longrightarrow (1)$$

From eq(1). right side of eq(1). the upper limit of "x" is

$m-K\sigma_x$ for part (1)

$$x \le m-K\sigma_x$$

$$x-m \le -K\sigma_x$$

$$-(x-m) \ge K\sigma_x$$

$$(m-x) \ge K\sigma_x \longrightarrow (2)$$

→ From the second part of eq (1). the lower limit of 'x' is $m-K\sigma_x$.

$$x \le m-K\sigma_x$$

$$x-m \le -K\sigma_x$$

$$\boxed{m-x \ge K\sigma_x}$$

→ From the second part of the eq (1). the upper limit of 'x' is

$$x \ge m+K\sigma_x$$

$$x-m \ge K\sigma_x \longrightarrow (3)$$

$$(2) \Rightarrow (m-x)^2 \ge (K\sigma_x)^2$$

$$(x-m)^2 \ge (K\sigma_x)^2$$

$$(3) \Rightarrow (x-m)^2 \ge (K\sigma_x)^2$$

$$\sigma_x^2 \ge \int_{-\infty}^{m-K\sigma_x} (x-m)^2 f_x(x)\,dx + \int_{m+K\sigma_x}^{\infty} (x-m)^2 f_x(x)\,dx$$

$$\sigma_x^2 \ge \int_{-\infty}^{m-K\sigma_x} K^2\sigma_x^2 f_x(x)\,dx + \int_{m+K\sigma_x}^{\infty} K^2\sigma_x^2 f_x(x)\,dx$$

$$\cancel{\sigma_x^2} \ge K^2\cancel{\sigma_x^2}\left[\int_{-\infty}^{m-K\sigma_x} f_x(x)\,dx + \int_{m+K\sigma_x}^{\infty} f_x(x)\,dx\right]$$

$$\frac{1}{K^2} \ge P(-\infty \le x \le m-K\sigma_x) + P(m+K\sigma_x \le x \le \infty)$$

$$\because P(-\infty \le x \le m-K\sigma_x) = P(x < m-K\sigma_x).$$

$$\frac{1}{K^2} \ge P(x \le m-K\sigma_x) + P(m+K\sigma_x \le x)$$

$$= \frac{1}{K^2} \ge P(|x-m| \ge K\sigma_x)$$

$$\therefore P(|x-m| \ge K\sigma_x) \le \frac{1}{K^2}$$

W.K.T $P((x-m) \ge K\sigma_x) + P((x-m) < K\sigma_x) = 1$

$$\therefore P((x-m) < K\sigma_x) = 1 - P(|x-m| \ge K\sigma_x)$$

$$= 1 - 1/K^2$$

$$\boxed{P((x-m) < K\sigma_x) = 1 - 1/K^2}$$

* Markov's inequality :-

→ This inequality gives the relation between probability of an event $x \ge 0$ and expected value of "x" i.e. for non negative values of random variable $X\ (x \ge 0)$. The markov's inequality is given by. $P(x \ge a) \ge \frac{E[x]}{a}$.

proof:- Let us consider r.v "x" is not negative i.e. $(x \ge 0)$. and continuous function $f_x(x)$.

$$E[x] = \int_{-\infty}^{\infty} f_x(x) \cdot x \cdot dx$$

$$= \int_0^{\infty} x \cdot f_x(x)\,dx$$

$$= \int_0^{a} x \cdot f_x(x)\,dx + \int_a^{\infty} x \cdot f_x(x)\,dx$$

$$E[X] \geq \int_a^{\infty} x \cdot f_X(x)\,dx$$

$$E(X) \geq \int_a^{\infty} a \cdot f_X(x)\,dx$$

[$\because$ For inequality we will sub. the lower limit value in x place]

$$\frac{E(X)}{a} \geq \int_a^{\infty} f_X(x)\,dx$$

$$\frac{E[X]}{a} \geq P(X \geq a)$$

$$P(X \geq a) \leq \frac{E(X)}{a}$$

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

** Find out mean, varience, moment generating function, chorecterstic function for binomial distribution :- (3.17)

NOTE :- $(p+q)^n = n_{c_0} p^0 q^n + n_{c_1} p^1 q^{n-1} + \dots + n_{c_n} p^n q^0$

$$\sum_{x=0}^{n} (n_{c_x})\, p^x q^{n-x} = 1$$

$$n_{c_x} = \frac{n}{x} \cdot \left(^{n-1}c_{x-1}\right) = \left(\frac{n}{x}\right)\left(\frac{n-1}{x-1}\right)\left(^{n-2}c_{x-2}\right) \dots$$

Proof: the binomial density function is

$$f_X(x) = n_{c_x} \cdot p^x q^{n-x}$$

mean of binomial distribution :-

$$\text{mean } (m_1) = \bar{x} = E(X)$$

$$= \sum_{\text{all } x} x \cdot f_X(x)$$

$$= \sum_{x=0}^{n} x \cdot (n_{c_x})\, p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \cdot \frac{n}{x} \cdot \left(^{n-1}c_{x-1}\right) p^x q^{n-x}$$

$$= \sum_{x=1}^{n} n \cdot \left(^{n-1}c_{x-1}\right) p^x p^1 p^{-1} q^{n-x} q^1 q^{-1}$$

$$= n \sum_{x=1}^{n} \left(^{n-1}c_{x-1}\right) p^{x-1} \cdot p \cdot q^{n-x-1+1}$$

$$= np \cdot \sum_{x=1}^{n} \left(^{n-1}c_{x-1}\right) \cdot p^{x-1} q^{(n-1)-(x-1)}$$

$$= np(1)$$

$$\boxed{\text{mean} = m_1 = \bar{x} = np}$$

mean square value of binomial distribution :-

$$m_2 = E(x^2)$$

$$= \sum x^2 (n_{c_x})\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x^2 \cdot (n_{c_x}) \cdot p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \left(x(x-1) + x\right) \cdot (n_{c_x})\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \cdot (n_{c_x})\, p^x q^{n-x} + \sum_{n=0}^{N} x\, (n_{c_x})\, p^x q^{n-x}$$

$$= \sum_{x=2}^{n} x(x-1) \cdot \frac{n}{x} \frac{(n-1)}{(x-1)} \left(^{n-2}c_{x-2}\right) p^{x-2} p^2 \cdot q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 + np$$

$$\boxed{E[x^2] = n(n-1)p^2 + np}$$

Varience of binomial distribution :-

$$Var(x) = \sigma_x^2 = E\left[(x-\bar{x})^2\right]$$

$$= E(x^2) - (E(x))^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$\boxed{Var(x) = npq}$$

moment generating function of binomial distribution :-

$$M_X(t) = E(e^{tx})$$

$$= \sum_{\text{all } x} e^{tx} f_X(x)$$

$$= \sum_{\text{all } x} e^{tx} (n_{c_x})\, p^x q^{n-x}$$

$$= \sum_{\text{all}} (n_{c_x})\, (e^t)^x p^x \cdot q^{n-x}$$

$$= \sum_{n=0}^{N} (n_{c_x}) (p e^t)^x q^{n-x}$$

(3.18)

$$\boxed{M_X(t) = (q + e^t p)^n}$$

Cherecterstic function of binomial distribution :-

$$\phi_X(\omega) = E(e^{j\omega x})$$

$$= \sum_{all\, x} e^{j\omega x} f_X(x)$$

$$= \sum_{all\, x} e^{j\omega x} (n_{c_x}) p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (n_{c_x}) (e^{j\omega})^x p^x q^{n-k}$$

$$= \sum_{x=0}^{n} (n_{c_x}) (p e^{j\omega})^x q^{n-k}$$

$$\boxed{\phi_X(\omega) = (q + e^{j\omega} p)^n}$$

** Find out mean, varience, moment generating function, cherecterstic function of poisson's distribution :-

NOTE!

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \infty$$

$$e^b = 1 + \frac{b}{1!} + \frac{b^2}{2!} + \dots \infty$$

$$\sum_{x=0}^{\infty} \frac{b^x}{x!} = 1 + \frac{b}{1!} + \frac{b^2}{2!} + \dots = e^b$$

Proof :- Mean or First moment about origin of a poissons distributions :-

the density function of poisson's distribution is

$$f_X(x) = \frac{e^{-b} b^x}{x!}$$

$$\text{mean} = m_1 = \bar{x} = E[x]$$

$$= \sum x . f_X(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-b} b^x}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} x \cdot \frac{b^x}{x!}$$

$$= e^{-b} \sum_{x=1}^{\infty} \cancel{x} \cdot \frac{b^{x-1} \cdot b}{\cancel{x} (x-1)!}$$

$$= e^{-b} b \cdot e^b$$

$$\boxed{m_1 = \bar{x} = E(x) = b}$$

mean square value or second moment about origin of poissons distributions :-

$$m_2 = E[x^2]$$

$$= \sum_{all\, x} x^2 f_X(x)$$

$$= \sum_{all} x^2 \cdot \frac{e^{-b} b^x}{x!}$$

$$= \sum_{all\, x} [x(x-1) + x] \frac{e^{-b} b^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x(x-1) e^{-b} b^x}{x!} + \sum_{all\, x} x \cdot \frac{e^{-b} b^x}{x!}$$

$$= e^{-b} b^2 \sum_{x=2}^{\infty} \frac{b^{x-2}}{(x-2)!} + b$$

$$= e^{-b} b^2 e^b + b$$

$$= b + b^2$$

$$\boxed{E(x^2) = b(b+1)}$$

Varience of poissons distribution :-

$$Var(x) = \sigma_X^2 = E[(x - \bar{x})^2]$$

$$= E(x^2) - [E(x)]^2$$

$= b^2+b-b^2$

$= b$

$\boxed{\sigma_x^2 = b}$

moment generating function of poisson's distribution:-

$$M_X(t) = E(e^{tx})$$

$$= \sum_{\text{all } x} e^{tx} \cdot f_X(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-b} b^x}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} \frac{(b \cdot e^t)^x}{x!}$$

$$= e^{-b} \left[1 + \frac{be^t}{1!} + \frac{be^t}{2!} + \cdots \infty \right]$$

$$= e^{-b} \, e^{e^t b}$$

$$= e^{-b} \cdot e^{e^t b}$$

$$\boxed{M_X(t) = e^{(e^t-1)b}}$$

Cherecterstic function of poisson distribution:-

$$\phi_X(\omega) = E\left[e^{j\omega x}\right]$$

$$= \sum_{\text{all } x} e^{j\omega x} f_X(x)$$

$$= \sum_{x=0}^{\infty} e^{j\omega x} \cdot \frac{e^{-b} b^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{(e^{j\omega} b)^x \cdot e^{-b}}{x!}$$

$$= e^{-b} \sum_{x=0}^{\infty} \frac{(e^{j\omega} b)^x}{x!}$$

$$= e^{-b} \left[1 + \frac{e^{j\omega} b}{1!} + \frac{(e^{j\omega} b)^2}{2!} + \cdots \right]$$

$$= e^{-b} \cdot e^{e^{j\omega} b}$$

$$= e^{(e^{j\omega}-1)b}$$

$$\boxed{\phi_X(\omega) = e^{(e^{j\omega}-1)b}}$$

*** Find out mean, variences, moment generating function, and cherecterstic function of uniform distribution function of interval "a to b".

Proof:- The uniform density of a random variable x in interval (a, b) is $f_X(x) = \frac{1}{b-a}$; $a \le x \le b$

$= 0$; elsewere

(i) Mean or First moment about origin:-

$$\text{Mean} = \overline{X} = m_1 = E[x]$$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{b^2-a^2}{2\,b-a} = \frac{(b+a)(b-a)}{2(b-a)}$$

$$= \frac{a+b}{2}$$

$$\boxed{\therefore E[x] = \frac{a+b}{2}}$$

(ii) Mean square value (or) second moment about origin:-

$$m_2 = E(x^2)$$

$$= \sum_{\text{all } x} x^2 \cdot f_X(x)$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{(\cancel{b-a})(b^2+ab+a^2)}{3(\cancel{b-a})}$$

$$\boxed{E[x^2] = \frac{a^2+ab+b^2}{3}}$$

Varience of uniform distribution :-

$$Var(x) = E((x-\bar{x}))^2$$

$$= E(x^2) - (E(x))^2$$

$$= \frac{b^2+ab+a^2}{3} - \left[\frac{(b+a)}{2}\right]^2$$

$$= \frac{b^2+ab+a^2}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$= \frac{4b^2+4ab+4a^2-3b^2-3a^2-6ab}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

$$\boxed{Var(x) = \frac{(b-a)^2}{12}}$$

Moment generation function of uniform distribution :-

$$M_X(t) = E\left(e^{tx}\right)$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

$$= \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{b-a}\left(\frac{e^{tb}}{t} - \frac{e^{ta}}{t}\right)$$

$$\boxed{M_X(t) = \left[\frac{e^{tb}-e^{ta}}{bt-at}\right]}$$

Cherecterstic function of uniform distribution :-

$$\phi_X(\omega) = E\left[e^{j\omega x}\right]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \int_a^b e^{j\omega x}\, {}^{1}/_{b-a}\,dx$$

$$= \frac{1}{b-a}\left(\frac{e^{j\omega x}}{j\omega}\right)_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{j\omega b}-e^{j\omega a}}{j\omega}\right]$$

$$\boxed{\phi_X(\omega) = \left[\frac{e^{j\omega b}-e^{j\omega a}}{j\omega(b-a)}\right]}$$

*** Show that the cherecterstic function $\phi_X(\omega)$ satisfies the

$$|\phi_X(\omega)| \le \phi_X(0) = 1$$

proof:- Let us consider the density function of a r.v "x" is $f_X(x)$. from the defination

$$\phi_X(\omega) = E\left(e^{j\omega x}\right)$$

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$ (3.2)

$$|\phi_X(\omega)| = \left|\int_{-\infty}^{\infty} e^{j\omega x}\cdot f_X(x)\,dx\right|$$

$$|\phi_X(\omega)| \leq \int_{-\infty}^{\infty} |e^{j\omega x}|\,|f_X(x)|\,dx$$

$$\leq \int_{-\infty}^{\infty} 1\cdot|f_X(x)|\,dx$$

$$\leq \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$\therefore \boxed{|\phi_X(\omega)| \leq 1} \rightarrow ①$$

$$\phi_X(\omega) = E[e^{j\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} (1)\, f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$\boxed{\phi_X(0) = 1} \rightarrow ②$$

From ① and ②

$$\boxed{|\phi_X(\omega)| \leq |\phi_X(0)| = 1}$$

*** Find mean, varience, skewness, or coefficient of skewness, moment generating function and cherecterstic function of the exponential distribution.

(4) proof:- The exponential signal distribution function of random variable "X" is given by $f_X(x) = \frac{1}{b}\cdot e^{-(x-a)/b}$; $x \geq a$

(i) Mean of the first moment about origin:-

$$\text{mean} = \bar{X} = E[X]$$

$$= \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx$$

$$= \int_{0}^{\infty} x\cdot \frac{1}{b}\cdot e^{-(x-a)/b}\,dx$$

$$= \frac{1}{b}\cdot e^{a/b}\cdot \int_{a}^{\infty} x\, e^{-x/b}\,dx$$

$$= \frac{1}{b} e^{a/b}\left[e^{-x/b}\left(\frac{x}{-1/b} - \frac{1}{(-1/b)^2}\right)\right]_a^{\infty}$$

$$= \frac{1}{b} e^{a/b}\left[0+0 - e^{-a/b}\cdot\frac{a}{-1/b} - \frac{1}{(1/b)^2}\right]$$

$$= \frac{1}{b}\cdot e^{a/b}\cdot\left(0 - e^{-a/b}(-ab-b^2)\right)$$

$$= \frac{-1}{b} e^{a/b}\, e^{-a/b}\cdot(-ab-b^2)$$

$$= -1/b\,(-ab-b^2)$$

$$\boxed{E(X) = a+b}$$

(ii) Mean square value or second moment about origin:-

$$\overline{X^2} = E(X^2)$$

$$= \int_{-\infty}^{\infty} x^2\cdot f_X(x)\,dx$$

$$= \int_{a}^{\infty} x^2\, \frac{1}{b}\, e^{-(x-a)/b}\,dx$$

$$= \frac{1}{b} e^{a/b}\int_{a}^{\infty} x^2\, e^{-x/b}\,dx$$

$$= \frac{1}{b} e^{a/b}\left[e^{-x/b}\left[\frac{x^2}{-1/b} - \frac{2x}{-(1/b)^2} + \frac{2}{-1/b^3}\right]\right]_a^{\infty}$$

$$= \frac{1}{b} e^{a/b}\left[0 - e^{-a/b}\left[\frac{a^2}{-1/b} - \frac{2a}{-1/b^2} + \frac{2}{-1/b^3}\right]\right]$$

(3.22)

$$= \frac{1}{b} e^{a/b}\, e^{-a/b} \cdot (-a^2 b - 2ab^2 - 2b^3)$$

$$= -\frac{1}{b} \cdot (a^2 b + 2ab^2 + 2b^3)$$

$$\boxed{\overline{x^2} = a^2 + 2ab + 2b^2}$$

(iii) <u>Third moment about origin:-</u>

$$\overline{x^3} = E(x^3)$$

$$= \int_{-\infty}^{\infty} x^3 f_x(x)\, dx$$

$$= \int_a^{\infty} x^3 \, \frac{1}{b}\, e^{-(x-a)/b}\, dx$$

$$= e^{a/b} \frac{1}{b} \int_a^{\infty} x^3 \cdot e^{-x/b}\, dx$$

$$= \frac{e^{a/b}}{b}\left[e^{-x/b}\left[\frac{x^3}{-1/b} - \frac{3x^2}{(-1/b)^2} + \frac{6x}{(-1/b)^3} + \frac{6}{(-1/b)^4}\right]\right]_a^{\infty}$$

$$= -\frac{e^{a/b}}{b}\left[- e^{-a/b} \, -a^3 b - a^2 b^2 3 - 6ab^3 - 6b^4\right]$$

$$\boxed{\overline{x^3} = a^3 + 3a^2 b + 6ab^2 + 6b^3}$$

(iv) <u>Varience:-</u>

$$Var(x) = \sigma_x^2 = E(x^2) - (E(x))^2$$

$$= a^2 + 2ab + 2b^2 - (a+b)^2$$

$$= b^2$$

$$\boxed{Var(x) = b^2}$$

(v) <u>Skewness (α_3) coefficient of skewness:-</u>

$\alpha_3 = \mu_3 /$

$$= E\left[\frac{(x-\bar{x})^3}{\sigma^3}\right]$$

$$= E\left(x^3 - 3x^2\bar{x} + 3x\bar{x}^2 - \bar{x}^3\right)$$

$$= E(x^3) - 3E(x^2)\cdot\bar{x} + 3E(x)\cdot\bar{x}^2 - E(\bar{x}^3)$$

$$= a^3 + 3a^2 b + 6ab^2 + 6b^3 - (3a^2 - 6ab - 6b^2) * a+b + 2(a^3+b^3+3ab^2+3b^2a)$$

$$= -a^3 + 3a^2b + 6ab^2 + 6b^3 - 3a^3 - 3a^2b - 3a^2b + 12ab^2 + 6b^3 + 3a^3 + 3a^2b + 6a^2b + 6ab^2 + 9ab^2 + 3b^3 - a^3 + 3a^2b + 3ab^2 + b^3$$

$$= 2b^3$$

$$\sigma = \sqrt{var(x)}$$

$$\sigma = \sqrt{b^2}$$

$$\sigma = b$$

$$\sigma^3 = b^3$$

$\therefore$ coefficient of skewness $\alpha_3 = \frac{\mu_3}{\sigma^3} = \frac{2b^3}{b^3} = 2$

(vi) <u>moment generating function:-</u>

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{b} e^{-(x-a)/b} dx$$

$$= \int_a^{\infty} e^{tx} \frac{1}{b} e^{-x/b} \cdot e^{a/b}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} e^{(t-1/b)x}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} e^{-(1/b-t)x}\, dx$$

$$= \frac{1}{b} e^{a/b} \cdot \left[\frac{e^{-x(1/b - t)}}{-(1/b - t)} \right]_a^{\infty}$$

(3-23)

$$= \frac{1}{b} e^{a/b} \left[\infty + \frac{e^{-a(1/b - t)}}{1/b - t} \right]$$

$$= \frac{1}{b} \frac{e^{at}}{(1/b - t)}$$

$$\boxed{M_X(t) = e^{at} \frac{1/b}{(1/b - t)}}$$

(viii) Charecteristic function:-

$$\phi_X(\omega) = E[e^{j\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \frac{1}{b} e^{-(x-a)/b} dx$$

$$= \frac{1}{b} e^{a/b} \int_a^{\infty} e^{-x(1/b - j\omega)} dx$$

$$= \frac{1}{b} \cdot e^{a/b} \cdot \left[\frac{e^{-x(1/b - j\omega)}}{-(1/b - j\omega)} \right]_a^{\infty}$$

$$= \frac{1}{b} e^{a/b} \left[\infty + \frac{e^{-a(1/b - j\omega)}}{(1/b - j\omega)} \right]$$

$$\boxed{\phi_X(\omega) = e^{j\omega a} \cdot \frac{1/b}{1/b - j\omega}}$$

** Find mean varience, moment generating function and charecteristic function of gaussian density function:

NOTE:- (i) $\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(x-a_x)^2/2\sigma_x^2} dx = 1$

(ii) $\frac{1}{\ } \int_{\ }^{\infty} e^{-x^2/2} dx = \ $

Proof:- The gaussian density function of a random variable X is

ie $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-a_x)^2/2\sigma_x^2}$; $-\infty < x < \infty$

Here a_x = mean of R.V "X"

σ_x^2 = varience of r.v "X"

(i) Mean of gaussian random variable "X" :-

$$\text{Mean of } X = \bar{X} = E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-a_x)^2/2\sigma_x^2} dx$$

let $\frac{x - a_x}{\sigma_x} = z$; $x = z\sigma_x + a_x$

$dx = \sigma_x dz$

$$E(X) = \int_{-\infty}^{\infty} (\sigma_x z + a_x) \frac{1}{\sqrt{2\pi}\,\cancel{\sigma_x}} e^{-z^2/2} \cancel{\sigma_x} dz$$

$$= \int_{-\infty}^{\infty} (\sigma_x z + a_x) \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \underbrace{\int_{-\infty}^{\infty} \sigma_x z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz}_{(A)} + \underbrace{\int_{-\infty}^{\infty} a_x \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz}_{(B)} \rightarrow (1)$$

(1) $\int_{-\infty}^{\infty} \sigma_x z \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz =$

$$= \frac{\sigma_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d(-e^{-z^2/2}) dz$$

$$= \frac{\sigma_x}{\ } \left(-e^{-z^2/2}\right)_{\ }^{\infty} = 0$$

(B) $\frac{a_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$

$= a_x (1)$

$= a_x$

sub (A) & (B) in (1)

$E(x) = 0 + a_x$

$$\boxed{E(x) = a_x}$$

(ii) <u>Mean square value of random variable 'x' :-</u>

$\overline{x^2} = E(x^2) = \int_{-\infty}^{\infty} x^2 f_x(x)\, dx$

$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi \sigma_x^2}} \cdot e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$

let $\frac{x-a_x}{\sigma_x} = z \Rightarrow x = \sigma_x z + a_x$

$dx = \sigma_x dz$

$E[x^2] = \int_{-\infty}^{\infty} (\sigma_x z + a_x)^2 \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-z^2/2} \sigma_x\, dz$

$= \int_{-\infty}^{\infty} (\sigma_x^2 z^2 + a_x^2 + 2\sigma_x z a_x) \frac{1}{\sqrt{2\pi}\,\sigma_x} \cdot e^{-z^2/2} \sigma_x\, dz$

$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \sigma_x^2 z^2 e^{-z^2/2} dz}_{(A)} + \underbrace{\int_{-\infty}^{\infty} a_x^2 e^{-z^2/2} dz}_{(B)} + \underbrace{\int_{-\infty}^{\infty} 2\sigma_x z a_x e^{-z^2/2} dz}_{(C)} \right]$

(A) $\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma_x^2 z^2 e^{-z^2/2} dz$

$= \frac{\sigma_x^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot (z e^{-z^2/2})\, dz$

(3.24)

$= \frac{\sigma_x^2}{\sqrt{2\pi}} \left[z(-e^{-z^2/2})_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (-e^{-z^2/2})\, dz \right]$

$= \frac{\sigma_x^2}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{-z^2/2} dz$

(A) $= \sigma_x^2$

(B) $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sigma_x a_x \cdot z\, e^{-z^2/2} \cdot dz$

$\frac{2\sigma_x a_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot e^{-z^2/2} dz$

$= \frac{2\sigma_x a_x}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \partial(-e^{-z^2/2})\, dz$

$= \frac{2\sigma_x a_x}{\sqrt{2\pi}} (0)$

$= 0$

(C) $\frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} a_x^2 e^{-z^2/2} dz$

$\frac{a_x^2}{\sqrt{2\pi}} (1)$

$= a_x^2$

$E(x^2) = \sigma_x^2 + 0 + a_x^2$

$$\boxed{E(x^2) = \sigma_x^2 + a_x^2}$$

<u>Varience of random varieble 'x' :-</u>

$\mathrm{var}(x) = E(x^2) - (E(x))^2$

$= \sigma_x^2 + a_x^2 - (a_x)^2$

$\therefore \boxed{Var(x) = \sigma_x^2}$

(3.25)

Moment generating function :-

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-\infty}^{\infty} e^{tx} f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-a_x)^2}{2\sigma_x^2}} dx$$

let $\frac{x-a_x}{\sigma_x} = z \Rightarrow x = \sigma_x z + a_x$

$dx = \sigma_x dz$

$$M_x(t) = \int_{-\infty}^{\infty} e^{t(\sigma_x z + a_x)} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} e^{t\sigma_x z - z^2/2} e^{a_x t} dz$$

$$= \frac{e^{a_x t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{2t\sigma_x z - z^2}{2}} dz$$

$$= \frac{e^{a_x t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2 - 2z\sigma_x t}{2}\right)} dz$$

$$= \frac{e^{a_x t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2 - 2z\sigma_x t}{2} + \frac{\sigma_x^2 t^2}{2} - \frac{\sigma_x^2 t^2}{2}\right)} dz$$

$$= \frac{e^{a_x t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z^2 - 2z\sigma_x t + \sigma_x^2 t^2}{2}\right)} \cdot e^{\frac{\sigma_x^2 t^2}{2}} dz$$

$$= \frac{e^{a_x t - \frac{\sigma_x^2 t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(z-\sigma_x t)^2} dz$$

$$M_x(t) = e^{a_x t - \frac{\sigma_x^2 t^2}{2}} \quad (1)$$

$$\boxed{M_x(t) = e^{a_x t - \sigma_x^2 t^2/2}}$$

Charecterstic function :-

$$\phi_x(\omega) = E[e^{j\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} \cdot \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\left(\frac{x-a_x}{2\sigma_x}\right)^2} dx$$

let $\frac{x-a_x}{\sigma_x} = z \Rightarrow x = \sigma_x z + a_x$

$dx = \sigma_x dz$

$$= \int_{-\infty}^{\infty} e^{j\omega(\sigma_x z + a_x)} \times \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-z^2/2} \cdot dz\, \sigma_x$$

$$= \frac{e^{j\omega a_x}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{2j\omega\sigma_x z - z^2}{2}} dz$$

$$= \frac{e^{j\omega a_x}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z^2 - 2z\sigma_x j\omega)}{2}} dz$$

$$= \frac{e^{j\omega a_x}}{\sqrt{2\pi}}$$

$$\boxed{M_x(t) = \frac{e^{j\omega a_x}}{\sqrt{2\pi}}}$$

UNIT-III.

"Operations on single random variables", problems (3.26)

(1) A random variable "x" has possible values $x_i = i^2$, $i = 1,2,3,4,5$, which occur with probabilities 0.4, 0.25, 0.15, 0.1 and 0.1 respectively. find

(i) probability density function (ii) Distribution function (iii) Mean value of "x".

sol: Given that $x_i = i^2$

$i = 1,2,3,4,5 \ldots$

The assigned values of random variable "x" are

$x_1 = 1^2 = 1$

$x_2 = 2^2 = 4$

$x_3 = 3^2 = 9$

$x_4 = 4^2 = 16$

$x_5 = 5^2 = 25$

$\therefore$ The probabilities of assigned values are.

$P(X = x_1) = 0.4$

$P(X = x_2) = 0.25$

$P(X = x_3) = 0.15$

$P(X = x_4) = 0.1$

$P(X = x_5) = 0.1$

The density function is

X = x	1	4	9	16	25
P(X = x)	0.4	0.25	0.15	0.1	0.1

Here the assigned values are finite. Hence the random variable "x" is ... density function is given by

$$f_X(x) = \sum_{i=1}^{N} P(X = x_i)\, \delta(x - x_i)$$

Here $N = 5$

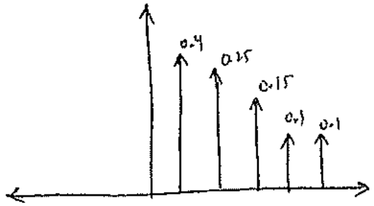
$$= \sum_{i=1}^{5} P(X = x_i)\, \delta(x - x_i)$$

$$= P(X = x_1)\,\delta(x - x_1) + P(X = x_2)\,\delta(x - x_2) + P(X = x_3)\,\delta(x - x_3) + P(X = x_4)\,\delta(x - x_4) + P(X = x_5)\,\delta(x - x_5).$$

$$= 0.4\,\delta(x-1) + 0.25\,\delta(x-4) + 0.15\,\delta(x-9) + 0.1\,\delta(x-16) + 0.1\,\delta(x-25).$$

$$\therefore f_X(x) = 0.4\,\delta(x-1) + 0.25\,\delta(x-4) + 0.15\,\delta(x-9) + 0.1\,\delta(x-16) + 0.1\,\delta(x-25).$$

The plot of density function is



The distribution function is given by

$$F_X(x) = \sum_{i=1}^{N} P(X = x_i)\, u(x - x_i).$$

Here $N = 5$

$$F_X(x) = \sum_{i=1}^{5} P(X = x_i)\, u(x - x_i).$$

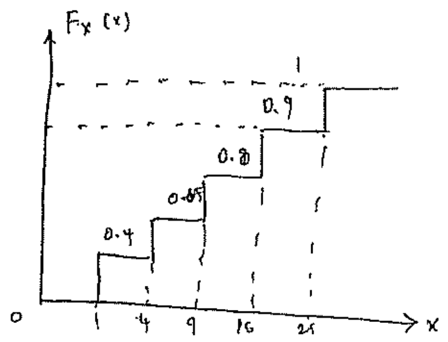
$$= P(X = x_1)\,u(x - x_1) + P(X = x_2)\,u(x - x_2) + P(X = x_3)\,u(x - x_3) + P(X = x_4)\,u(x - x_4) + P(X = x_5)\,u(x - x_5).$$

$$F_X(x) = 0.4\,u(x-1) + 0.25\,u(x-4) + 0.15\,u(x-9) + 0.1\,u(x-16) + 0.1\,u(x-25).$$

$$F_X(x) = 0.4\,u(x-1) + 0.25\,u(x-4) + 0.15\,u(x-9) + 0.1\,u(x-16) + 0.1\,u(x-25)$$

The plot of distribution function is

(3.29)



Mean of "x" is

$\bar{x} = \mu_1' = \mu = m_1 = m = E[x]$

$= \sum_{i=1}^{N} x_i \, p(x=x_i)$

$= \sum_{i=1}^{5} x_i \, p(x=x_i)$.

$= x_1 P(X=x_1) + x_2 P(X=x_2) + x_3 P(X=x_3) + x_4 P(X=x_4) + x_5 P(X=x_5)$

$= 1(0.4) + 4(0.25) + 9(0.15) + 16(0.1) + 25(0.1)$

$\boxed{\bar{x} = 6.85}$

(2) A random variable "X" has the following probability function

x	-2	-1	0	1	2	3
p(x)	0.1	k	0.2	2k	0.3	k

(i) find value of "k" (ii) Mean of 'x', (iii) variance of 'x'.

Sol: (i) We know that the sum of probabilitys = 1

$\sum_{all\,x} p(x) = 1$

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$\boxed{k = 0.1}$

x	-2	-1	0	1	2	3
$p(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

(ii) mean of "x".

$\bar{x} = \mu_1' = \mu = m_1 = m = E[x]$

$= \sum_{i=1}^{N} x_i \, p(x=x_i)$

$= \sum_{i=1}^{6} x_i \, p(x_i)$.

$= (-2)\cancel{0.1} + (-1)0.1 + 0 + 1\cancel{(0.2)} + 2(0.3) + 3(0.3)$

$= 0.8$

$\boxed{\bar{x} = \mu_1' = 0.8}$

(iii) The mean square value of "x" is. $\mu_2' = \sum_{all\,x} x^2 p(x)$

$\mu_2' = (4).0.1 + (0.1)(1) + 1(0.2) + 4(0.3) + 9(0.1)$

$\boxed{\mu_2' = 2.8}$

Varience of 'x' is $\sigma_x^2 = \mu_2' - (\mu_1')^2$

$= 2.8 - (0.8)^2$

$= 2.8 - 0.64$

$\boxed{\sigma_x^2 = 2.16}$

(3) let "X" be the random variable defind by the density function, is

$f_X(x) = \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right)$; $-4 \le x \le 4$

$= 0$; else were, find $E[3x]$, $E[x^2]$.

sol:

Given that $f_X(x) = \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right)$; $-4 \le x \le 4$ (3.29)

$= 0$; elsewere.

Here the random variable "x" is continuous random variable

for continuous random variable $E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$

(i) $E(3x) = \int_{-\infty}^{\infty} 3x \cdot f_X(x)\,dx.$

$$= 3\int_{-4}^{4} x\cos\left(\frac{\pi x}{8}\right)\frac{\pi}{16}\,dx$$

$$= \frac{3\pi}{16}\int_{-4}^{4} x\cdot\cos\left(\frac{\pi x}{8}\right)dx$$

let $g(x) = x\cos\left(\frac{\pi x}{8}\right).$

$$g(-x) = -x\cos\left(\frac{\pi(-x)}{8}\right).$$

$$= -x\cos\left(\frac{\pi x}{8}\right).$$

$$= -g(x)$$

$\therefore$ g(x) is odd function

for odd function, $\int_{-a}^{a} g(x)\,dx = 0$

$\because E(3x) = 3(0)$

$= 0$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\,dx$$

$$= \int_{-4}^{4} x^2 \frac{\pi}{16}\cos\left(\frac{\pi x}{8}\right)dx$$

$$= \frac{\pi}{16}\int_{-4}^{4} x^2\cos\left(\frac{\pi x}{8}\right)dx$$

$$= \frac{\pi}{16}\int_{-4}^{4} x^2\cos\left(\frac{\pi x}{8}\right)dx$$

$$= \frac{\pi}{16}\left[x^2\left(\frac{\sin(x\pi/8)}{\pi/8}\right)_{-4}^{4} - \int_{-4}^{4} 2x\cdot\frac{\sin\left(\frac{\pi x}{8}\right)}{\pi/8}\,dx\right]$$

$$= \frac{\pi}{16}\left[\left[\frac{16\times\sin(\pi/2)}{\pi/8} - 16\times\frac{\sin(-\pi/2)}{\pi/8}\right] - \frac{16}{\pi}\left[\int_{-4}^{4} x\cdot\sin\left(\frac{\pi x}{8}\right)dx\right]\right]$$

$$= \frac{\pi}{16}\cdot\left[16\cdot\frac{8}{\pi} + 16\times\frac{8}{\pi}\right] - \cancel{\frac{\pi}{16}}\times\cancel{\frac{16}{\pi}}\left[\int_{-4}^{4} x\sin\left(\frac{\pi x}{8}\right)dx\right]$$

$$= \cancel{\frac{\pi}{16}}\,\cancel{\frac{16}{\pi}}\left[(16)\right] - \int_{-4}^{4} x\sin\left(\frac{\pi x}{8}\right)dx$$

$$= 16 - \int_{-4}^{4} x\sin\left(\frac{\pi x}{8}\right)dx$$

$$= 16 - \left[x\left(\frac{-\cos(\pi x/8)}{\pi/8}\right)_{-4}^{4} - \int_{-4}^{4} 1\cdot\frac{-\cos\left(\frac{\pi x}{8}\right)}{\pi/8}\,dx\right]$$

$$= 16 - \left[-4\,\frac{\cos\pi/2}{\pi/8} - 4\,\frac{\cos\pi/2}{\pi/8}\right] + \frac{8}{\pi}\cdot\int_{-4}^{4}\cos\left(\frac{\pi x}{8}\right)dx$$

$$= 16 - \left[0 + \frac{8}{\pi}\int_{-4}^{4}\cos\left(\frac{\pi x}{8}\right)dx\right]$$

$$= 16 - \frac{8}{\pi}\left(\frac{\sin\left(\frac{\pi x}{8}\right)}{\pi/8}\right)_{-4}^{4}$$

$$= 16 - \frac{8}{\pi}\left[\frac{8}{\pi}(2)\right]$$

$$\boxed{E(x^2) = 16 - \frac{128}{\pi^2}}$$

(4) The density function of random variable of 'x' is $g(x) = 5e^{-x}$; $0 \le x \le$.
$= 0$; otherwise

find $E(x)$, $E(x-1)^2$; $E(3x-1)$

Sol: Given that $g(x) = 5e^{-x}$; $0 \le x \le \infty$

$= 0$; elsewere.

(3.29)

Here the random variable 'x' fallows continuous distribution function

$$E[x] = \int_{-\infty}^{\infty} x.g(x)\,dx$$

$$= \int_{0}^{\infty} x.\,5e^{-x}\,dx$$

$$= 5\int_{0}^{\infty} xe^{-x}\,dx.$$

$$= 5\left[x\left(\frac{e^{-x}}{-1}\right)_0^{\infty} + \int_0^{\infty}(1)\,e^{-x}\,dx\right]$$

$$= 5\left[[0] + \left(\frac{e^{-x}}{-1}\right)_0^{\infty}\right]$$

$$= 5\left[-e^{-\infty}+e^{0}\right]$$

$$\boxed{E(x) = 5}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2.g(x)\,dx$$

$$= \int_0^{\infty} x^2\,5e^{-x}\,dx$$

$$= 5\int_0^{\infty} x^2 e^{-x}\,dx$$

$$= 5\left[x^2\left(\frac{e^{-x}}{-1}\right)_0^{\infty} + \int_0^{\infty} 2x\,e^{-x}\,dx\right]$$

$$= 5\left[0 + 2\int_0^{\infty} x.e^{-x}\,dx\right]$$

$$= 5 + \cdot[0] + \left[2\int_0^{\infty} 5xe^{-x}\,dx\right]$$

$$= 2\times 5 = 10$$

$$\boxed{E(x^2) = 10}$$

$$E\left[(x-1)^2\right] = E\left[x^2+1-2x\right]$$

$$= E(x^2) + E(1) - 2E(x)$$

$$= 10 - 2(5) + 1$$

$$= \cancel{10} - \cancel{10} + 1$$

$$= 1$$

$$\boxed{\therefore\ E\left[(x-1)^2\right] = 1}$$

(or)

$$E\left[(x-1)^2\right] = \int_0^{\infty} (x-1)^2.\,5e^{-x}\,dx$$

$$= \int_0^{\infty} (x^2+1-2x)\ 5e^{-x}\,dx$$

$$= \int_0^{\infty} x^2.g(x)\,dx - 2\int_0^{\infty} x.g(x)\,dx + \int_0^{\infty} g(x)\,dx$$

$$= 10 - 2(5) + \int_0^{\infty} 5e^{-x}\,dx$$

$$= 10 - 10 + 5\left(\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$= 5\left(e^{-\infty} - e^{0}\right)$$

$$= 5$$

NOTE: The theorems on expectation and varience can be applicable when the density function is valid density function

$$E(3x-1) = \int_0^{\infty}(3x-1)\,5e^{-x}\,dx \Rightarrow 3.\int_0^{\infty} x\,g(x)\,dx - \int_0^{\infty} g(x)\,dx$$

$$= 3(5) - 5$$

$$= 15 - 5$$

$$= 10$$

$$\boxed{E(3x-1) = 10}$$

(5) For a random variable $y = \cos \pi x$. where x is a random variable follows uniform distribution over the interval $(-1/2, 1/2)$. find the mean and mean square value of 'y'. (2:10)

Sol: Given that the random variable y is $= \cos \pi x$

Here 'x' is a random variable which follows uniform distribution over the interval $(-1/2, 1/2)$

We know that, the random variable 'x' in follows uniform density function over the interval (a, b). Then the density function of

$$f_X(x) = \frac{1}{b-a} \; ; \; a \le x \le b$$

$$= 0 \; ; \text{ elsewere}$$

$$\therefore f_X(x) = \frac{1}{1/2 + (1/2)} = 1/1 = 1$$

$$\therefore f_X(x) = 1 \; ; \; -1/2 \le x \le 1/2$$

$$= 0 \; ; \text{ elsewere.}$$

mean of y is $= E[y]$

$$= E[\cos \pi x]$$

$$= \int_{-\infty}^{\infty} \cos(\pi x) \, f_X(x) \, dx$$

$$= \int_{-1/2}^{1/2} \cos(\pi x) \cdot (1) \, dx$$

$$= \left(\frac{\sin \pi x}{\pi}\right)_{-1/2}^{1/2} = \frac{\sin \pi/2}{\pi} + \frac{\sin \pi/2}{\pi}$$

$$\boxed{E[y] = \frac{2}{\pi}}$$

Mean square value of y is $= E[y^2]$

$$= E[\cos^2 \pi x]$$

$$= E\left[\frac{1 + \cos 2\pi x}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2} E[\cos(2\pi x)]$$

$$= \frac{1}{2} + \frac{1}{2} \int_{-\infty}^{\infty} \cos 2\pi x \, dx$$

$$= \frac{1}{2} + \frac{1}{2} \int_{-1/2}^{1/2} \cos 2\pi x \, dx$$

$$= \frac{1}{2} + \frac{1}{2} \left(\frac{\sin 2\pi x}{2\pi}\right)_{-1/2}^{1/2}$$

$$= \frac{1}{2} + \frac{1}{2} \left[\frac{\sin \pi}{2\pi} + \frac{\sin \pi}{2\pi}\right]$$

$$= \frac{1}{2} + \frac{1}{2}(0) = 1/2 = 0.5$$

$$\boxed{E[y^2] = 0.5}$$

(6) If the given function $f_X(x) = \frac{1}{2} \cos x \; ; \; -\frac{\pi}{2} \le x \le \pi/2$ is the density
$= 0 \; ;$ else were
function of a random variable 'x', then find the mean value of the functions. (i) $g(x) = 4x^2$ (ii) $g(x) = 4x^4$

Sol: Given that

$$f_X(x) = \frac{1}{2} \cos x \; ; \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$= 0 \; ; \text{ elsewere.}$$

Here, 'x' is a continuous random variable is

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$$

(i) $g(x) = 4x^2$

$\therefore$ Mean of $g(x) = E[g(x)] = E[4x^2]$

$$= \int_{-\infty}^{\infty} 4x^2 \cdot f_x(x)\,dx$$

$$= \int_{-\pi/2}^{\pi/2} 4x^2 \cdot \frac{1}{2} \cos x\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos x \cdot (x^2)\,dx$$

$$= 2\left[x^2 \cdot (\sin x)_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} 2x \sin x\,dx\right]$$

$$= 2\left[\frac{\pi^2}{4} + \frac{\pi^2}{4}\right] - 2\left(\int_{-\pi/2}^{\pi/2} x \sin x\,dx\right)$$

$$= 2\left(\frac{2\pi^2}{4}\right) - 2\left[x \cdot (-\cos x)_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos x\,dx\right]$$

$$= \pi^2/4 \,[/\pi$$

$$= 2\left[\frac{\pi^2}{2} - 2\,(\sin x)_{-\pi/2}^{\pi/2}\right]$$

$$= 2\left[\frac{\pi^2}{2} - 2.(1+1)\right]$$

$$\boxed{E(g(x)) = [\pi^2 - 8]}$$

(ii) $E[g(x)] = E[4x^4]$

$$= \int_{-\infty}^{\infty} 4x^4 \cdot f_x(x)\,dx$$

$$= \int_{-\pi/2}^{\pi/2} 4x^4 \frac{\cos x}{2}\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2} x^4 \cos x\,dx$$

(3.41)

$$= 2\left[x^4 (\sin x)_{-\pi/2}^{\pi/2} - \int_{-\pi/2} 4x^3 \sin x\,dx\right]$$

$$= 2\left[\left(\frac{\pi}{2}\right)^4 + \frac{\pi^4}{16} - 4\int_{-\pi/2}^{\pi/2} x^3 \sin x\,dx\right]$$

$$= 2\left[\frac{2\pi^4}{16} - 4\left[x^3 (-\cos x)_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} 3x^2 \cos x\,dx\right]\right]$$

$$= 2\left(\frac{\pi^4}{8} - 12\left[\int_{-\pi/2}^{\pi/2} x^2 \cos x\,dx\right]\right)$$

$$= 2\left[\frac{\pi^4}{8} - 12\left[x^2 (\sin x)_{-\pi/2}^{\pi/2} - \int_{-\pi/2}^{\pi/2} 2x \cdot \sin x\,dx\right]\right]$$

$$= 2\left[\frac{\pi^4}{8} - \left(12\left(\frac{\pi^2}{4} + \frac{\pi^2}{4}\right)\right) - 2\int_{-\pi/2}^{\pi/2} x \sin x\,dx\right]$$

$$= 2\left[\frac{\pi^4}{8} - 12\left(\frac{2\pi^2}{4}\right) - 2\left[x(-\cos x)_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \cos x\,dx\right]\right]$$

$$= 2\left[\frac{\pi^4}{8} - 12\left(\frac{\pi^2}{2} - 2\left[(0) + (\sin x)_{-\pi/2}^{\pi/2}\right]\right)\right]$$

$$= 2\left(\frac{\pi^4}{8} - 12\left(\frac{\pi^2}{2} - 0 + (4)\right)\right)$$

$$= 2\left[\frac{\pi^4}{8} - \frac{12\pi^2}{2} - 48\right]$$

$$\boxed{E[g(x)] = \frac{\pi^4}{4} - 12\pi^2 - 96}$$

(7) Find the expected value of the function $g(x) = x^2$, where x is a random variable defined by the density function $f_x(x) = a \cdot e^{-ax} u(x)$, where "$a$" is a constant

Sol: Given that $g(x) = x^2$.

Here "x" is a random variable

The density function of "x" is

$$f_X(x) = a \cdot e^{-ax} \cdot u(x)$$

Here $u(x) = 1$; $x \geq 0$

$= 0$; $x < 0$

$\therefore f_X(x) = a \cdot e^{-ax}$; $x \geq 0$

$= 0$; $x < 0$

$\therefore$ Mean of $g(x)$ is $= E[g(x)]$

$$= E[x^2]$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot a \cdot e^{-ax}\, dx$$

$$= a \int_{0}^{\infty} x^2 e^{-ax}\, dx$$

$$= a \cdot \left[x^2 \left(\frac{e^{-ax}}{-a}\right)_0^{\infty} - \frac{2x}{(-a^2)} + \frac{2}{(-a)^3} \right]_0^{\infty}$$

$$= a \left[\left(e^{-a\infty} \right) - \left[0 - \frac{2(0)}{a^2} + \frac{2}{-a^3} \right] \right]$$

$$= \frac{\cancel{a} \cdot 2}{a^{\cancel{3}}}$$

$$\boxed{E[g(x)] = \frac{2}{a^2}}$$

(8) In an experiment throwing a die. find the expected value of no. of points on the die.

Sol: Given. The experiment is throwing a die

(3.42)

Let us consider random variable 'x' that denotes no. of points on a die

$\therefore$ the assign values of "x" are 1, 2, 3, 4, 5, and 6.

$\therefore$ the probability of assign values are

$p(x=1) = 1/6$

$p(x=2) = 1/6$

$p(x=3) = 1/6$

$p(x=4) = 1/6$

$p(x=5) = 1/6$

$p(x=6) = 1/6$

$\therefore$ The PDF is.

$X = x$	1	2	3	4	5	6
$p(X=x)$	1/6	1/6	1/6	1/6	1/6	1/6

Mean of $x = \sum_{\text{all } x} x \cdot p(X=x)$.

$E[x] = 1 \cdot 1/6 + 2 \cdot 1/6 + 3 \cdot 1/6 + 4 \cdot 1/6 + 5 \cdot 1/6 + 6 \cdot 1/6$

$= 21/6$

$$\boxed{E[X] = 3.5}$$

(9) In an experiment two dice's are thrown simultaneously find the expected value of no. of points on them.

Sol: Given the experiment is two dies are thrown the sample space of experiment

$S = \{ (1,1) (1,2) (1,3) (1,4) (1,5) (1,6), (2,1) (2,2) (2,3) (2,4) (2,5) (2,6)$
$(3,1) (3,2) (3,3) (3,4) (3,5) (3,6), (4,1) (4,2) (4,3) (4,4) (4,5)$
$(4,6) (5,1) (5,2) (5,3) (5,4) (5,5) (5,6) (6,1) (6,2) (6,3)$
$(6,4) (6,5) (6,6) \}$

Let us consider random variable X and that denotes the no. of points on dies when two dies are thrown. (5.43)

The assign values of "x" are, $2, 3, 4, 5, 6, \dots 12$.

$\therefore$ The PDF is

$X=x$	2	3	4	5	6	7	8	9	10	11	12
$p(X=x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Mean of $X = E[X] = \sum_{\text{all } x} x \cdot p(X=x)$.

$= 2 \cdot 1/36 + 3 \cdot 2/36 + 4 \cdot 3/36 + 5 \cdot 4/36 + 6 \cdot 5/36 + 7 \cdot 6/36 + 8 \cdot 5/36 + 9 \cdot 4/36 + 10 \cdot 3/36$

$+ 11 \cdot 2/36 + 12 \cdot 1/36$

$= 7$.

$\boxed{\therefore \text{ Mean of } X = 7}$

(10) Define a function $g(\,)$ of random variable "x" by $g(x) = 1 \,;\, x \geq x_0$
$= 0 \,;\, x < x_0$
where "x_0" is a real number's show that $E[g(x)] = 1 - F_X(x_0)$

Sol: let us consider random variable 'x' with density function $f_X(x)$. from $-\infty$ to ∞

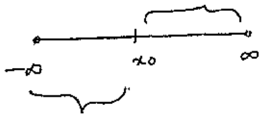
Given $g(x) = 1 \,;\, x \geq x_0$
$= 0 \,;\, x < x_0$

Mean of $g(x) = E[g(x)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,dx$

$= \int_{x_0}^{\infty} 1 \cdot f_X(x)\,dx$

$= \int_{x_0}^{\infty} f_X(x)\,dx$

We know that



$= \int_{-\infty}^{x_0} f_X(x)\,dx + \int_{x_0}^{\infty} f_X(x)\,dx = 1$

$= \int_{x_0}^{\infty} f_X(x)\,dx = 1 - \int_{-\infty}^{x_0} f_X(x)\,dx \rightarrow (1)$

from the defination of distribution function

$F_X(x) = \int_{-\infty}^{x_0} f_X(x)\,dx$

$\therefore$ eq (1) becomes

$\boxed{E[g(x)] = 1 - F_X(x_0)}$

(11) A random variable "x" has a density function is $f_X(x) = \frac{3}{32}(-x^2+8x-12)\,;\, 2 \leq x \leq 6$
$= 0 \,;$ elsewhere
find m_0, m_1, m_2, μ_2. $\mu_n' = m_n = E[x^n]$.

NOTE: The moment about origin are also denoted by m_n

Sol: Given the density function of a random variable is

$f_X(x) = \frac{3}{32}(-x^2+8x-12) \,;\, 2 \leq x \leq 6$

$= 0 \,;$ elsewhere.

$\mu_0' = E[x^0]$

$= E[1]$

$= \int_{-\infty}^{\infty} 1 \cdot f_X(x)\,dx$

$= \int_{2}^{6} \frac{3}{32}(-x^2+8x-12)\,dx$

$= \frac{3}{32}\left(\frac{-x^3}{3} + \frac{8x^2}{2} - 12x\right)_2^6$

$$= \frac{3}{32}\left[\left(\frac{-6^3}{3} + 8\cdot\frac{6^2}{2} - 12(6)\right) - \left(\frac{-2^3}{3} + 8\frac{2^2}{2} - 12(2)\right)\right]$$ (344)

$$= 1.$$

$$m_1 = E(X) = \int_{-\infty}^{\infty} x\cdot f_X(x)\,dx$$

$$= \int_2^6 x.\frac{3}{32}(-x^2+8x-12)\,dx$$

$$= \frac{3}{32}\int_2^6 (-x^3+8x^2-12x)\,dx$$

$$= \frac{3}{32}\left(\frac{-x^4}{4} + \frac{8x^3}{3} - \frac{12x^2}{2}\right)_2^6$$

$$= \frac{3}{32}\left[\left(\frac{-6^4}{4} + \frac{8\,6^3}{3} - 12\,6^2\right) - \left(-\frac{2^4}{4} + \frac{8\,2^3}{3} - 12\,2^2\right)\right]$$

$$=$$

$$\mu_2' = m_2 = E(x^2) = \int_{-\infty}^{\infty} x^2\cdot f_X(x)\,dx$$

$$= \int_2^6 x^2\cdot\frac{3}{32}(-x^2+8x-12)\,dx$$

$$= \frac{3}{32}\int_2^6 (-x^4+8x^3-12x^2)\,dx$$

$$= \frac{3}{32}\left(\frac{-x^5}{5} + \frac{8x^4}{4} - \frac{12x^3}{3}\right)_2^6$$

$$= \frac{3}{32}\left[\left(\frac{-6^5}{5} + 6^4\times 2 - 4\,6^3\right) - \left(\frac{-2^5}{5} + 2.2^4 - 12\,2^3\right)\right]$$

$$= 16.8$$

μ_2 = second moment about mean $= E\left[(X-\mu_1')^2\right]$

$$= \mu_2' - (\mu_1')^2$$

$$= m_2 - (\mu_1')^2 \qquad \boxed{\mu_2 = 0.8}$$

(12) A random variable "X" has $\bar{X} = -3$, $\overline{X^2} = 11$ and $\sigma_X^2 = 2$. for new random variable $Y = 2X-3$; find $\bar{Y}$, $\overline{Y^2}$ and σ_Y^2

Sol:

Given $\bar{X} = -3$; $\mu_1' = E[X]$

$\overline{X^2} = 11$; $\mu_2' = E[X^2]$.

We know that $\sigma_X^2 = \mu_2' - (\mu_1')^2 = 11 - 3^2 = 11-9$

$\sigma_X^2 = 2 \longrightarrow (1)$

As per problem $\sigma_X^2 = 2$.

$\therefore$ The given random variable "X" has valid density function

The new random variable $Y = 2X-3$

$$\bar{Y} = E[Y]$$
$$= E[2X-3]$$
$$= 2E[X] - E(3)$$
$$= 2(-3) - 3$$
$$= -9.$$

$$\overline{Y^2} = E[Y^2]$$
$$= E[(2X-3)^2]$$
$$= E[4x^2+9-12x]$$
$$= 4E[x^2] + E(9) - 12E[x]$$
$$= 4.(11) + 9 - 12(-3)$$
$$= 44+9+36$$
$$\boxed{\overline{Y^2} = 89}$$

$\therefore$ $\sigma_Y^2 = \mu_2' - (\mu_1')^2$

$$= E(Y^2) - (E(Y))^2$$
$$= 89 - 81$$
$$\boxed{\sigma_Y^2 = 8.}$$

(13) The exponential density function is given by $f_X(x) = \frac{1}{b} e^{-(x-a)/b}$; $x>a$

$= 0$; $x<a$

find out the varience, skew and coefficient of skewness ? (3.45)

Sol: Given that $f_X(x) = \frac{1}{b} e^{-(x-a)/b}$; $x>a$

$= 0$; $x<a$

We know that, $E[X] = \mu_1' = a+b$

$E[X^2] = \mu_2' = (a+b)^2 + b^2$

Varience $\sigma_X^2 = b^2$

Skew $= \mu_3 =$ third moment about mean

$$\mu_3 = E[(X-\mu_1')^3]$$

$$= E[x^3 - 3x^2\mu_1' + 3x\mu_1'^2 - (\mu_1')^3]$$

$$= \mu_3' - 3\mu_1' E[x^2] + 3\mu_1'^2 E[x] - (\mu_1')^3$$

$$= \mu_3' - 3\mu_1'\mu_2' + 3(\mu_1')^3 - (\mu_1')^3$$

$$= \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$$

$\mu_3' = E[x^3]$

$$= \int_{-\infty}^{\infty} x^3 \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^3 \cdot \frac{1}{b} e^{-(x-a)/b}\, dx$$

$$= \frac{e^{a/b}}{b} \int_a^{\infty} x^3 \cdot e^{-x/b}\, dx$$

$$= \frac{e^{a/b}}{b} \left[e^{-x/b} \left(\frac{x^3}{-1/b} - \frac{3x^2}{(-1/b)^2} + \frac{6x}{(-1/b)^3} - \frac{6}{(-1/b)^4} \right) \right]_a^{\infty}$$

$$= \frac{e^{a/b}}{b} \left[0 - e^{-a/b} \left(\frac{a^3}{-1/b} - \frac{3a^2}{(-1/b)^2} + \frac{6a}{(-1/b)^3} - \frac{6}{(-1/b)^4} \right) \right]$$

$$= \frac{e^{a/b}}{b} \left[-e^{-a/b} \left(-ba^3 - 3a^2b^2 - 6ab^3 - 6b^4 \right) \right]$$

$$= \frac{e^{a/b}}{b} \left[e^{-a/b} \left(ba^3 + 3a^2b^2 + 6ab^3 + 6b^4 \right) \right].$$

$\mu_3' = a^3 + 3a^2b + 6ab^2 + 6b^3$

$\therefore$ skew $= \mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$$\mu_3' = (a^3+3a^2b+6ab^2+6b^3) - 3(a+b)\left((a+b)^2+b^2\right) + 2(a+b)^3$$

$$= a^3+3a^2b+6ab^2+6b^3 - 3(a+b)\left[a^2+b^2+2ab+b^2\right] + 2(a^3+3a^2b+3ab^2+b^3)$$

$$= a^3+3a^2b+6ab^2+6b^3 - 3a^3+3ab^2+3a^2b+2b^3 + 2a^3+6a^2b+6ab^2+2b^3$$

$$= 2b^3$$

$\therefore$ coefficient of skewness $= \alpha_3 = \frac{\mu_3}{\sigma_x^3}$

$\sigma_x^2 = b^2$

$\sigma_x = b$

$\sigma_x^3 = b^3$

$\alpha_3 = \frac{2\cancel{b^3}}{\cancel{b^3}}$

$\boxed{\alpha_3 = 2}$

(14) The plot for the random variable "X". is given by $f_X(x) = 0.503\sqrt{x}$; $0<x<2$
$= 0$; else were.

find mean of "X". mean of the square of "X", variable. varience of "X".

(3.96)

sol:- Given that $f_X(x) = 0.503\sqrt{x}$; $0<x<2$

$= 0$; other wise.

mean of $x = M_1' = m_1 = E[x]$

$$= \int_{-\infty}^{\infty} x . f_X(x)\, dx$$

$$= \int_0^2 x . 0.503\sqrt{x}\, dx$$

$$= \int_0^2 0.503\, x^{3/2} dx$$

$$= 0.503 \cdot \frac{2}{5}\left(x^{5/2}\right)_0^2$$

$$\boxed{M_1' = 1.138}$$

$$M_2' = E[x^2]$$

$$= 0.503 \int_0^2 x^2 . x^{1/2} dx$$

$$= 0.503 \cdot \frac{2}{7} . \left(x^{7/2}\right)_0^2$$

$$\boxed{M_2' = 1.625}$$

Varience $= \sigma_X^2 = M_2' - (M_1')^2$

$$= 1.625 - (1.138)^2$$

$$\boxed{\sigma_X^2 = 0.325}$$

(15) Given random variable "X" and its density function. is

$f_X(x) = 1$; $0<x<1$ evaluate $\bar{X}$?
$= 0$; elsewere

sol:- Given that $f_X(x) = 1$; $0<x<1$

$= 0$; elsewere.

$$\bar{X} = E(x)$$

$$= \int_{-\infty}^{\infty} x . f_X(x)\, dx$$

$$= \int_0^1 x . (1)\, dx$$

$$= \left(\frac{x^2}{2}\right)_0^1$$

$$\boxed{\bar{X} = \frac{1}{2}}$$

(16) Find the expected value of the function. $g(x) = x^3$; where "x" is a random variable. defined by the density function $f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x} u(x)$

sol:- Given that $g(x) = x^3$

The density function of the random variable "X" is

$f_X(x) = \frac{1}{2} e^{-\frac{1}{2}x} u(x)$; $x \geq 0$

$= 0$; $x<0$.

The expected value of the function $g(x)$ is

$$= E[g(x)]$$

$$= E(x^3)$$

$$= \int_{-\infty}^{\infty} x^3 . f_X(x)\, dx$$

$$= \int_0^{\infty} x^3 . \frac{1}{2} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3 . e^{-x/2} dx$$

$$= \frac{1}{2}\left[e^{-\frac{1}{2}x} . \frac{x^3}{(-\frac{1}{2})^1} - \frac{3x^2}{(-\frac{1}{2})^2} + \frac{6x}{(-\frac{1}{2})^3} - \frac{6}{(-\frac{1}{2})^4}\right]_0^{\infty}$$

$= \frac{1}{2}\left(0 - \left(\frac{-6}{1/16}\right)\right)$ (2.47)

$= \frac{1}{2} \times 96$

$= 48$

$\boxed{E[g(x)] = 48}$

(17) "x" is a uniform random variable in the interval (x_1, x_2), find the expected value of "x".

Sol: Given 'x' is a uniform random variable of the density function is

$$f_X(x) = \frac{1}{b-a} \quad ; \; a \le x \le b$$

$$= 0 \; ; \text{ elsewere}$$

As per problem,

The density function is $f_X(x) = \frac{1}{x_2 - x_1} \; ; \; x_1 \le x \le x_2$

$$= 0 \; ; \text{ elsewere}$$

$$\text{Mean} = E(x) = \int_{-\infty}^{\infty} x . f_X(x)\, dx$$

$$= \int_{x_1}^{x_2} x . \left(\frac{1}{x_2 - x_1}\right) dx$$

$$= \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} x\, dx$$

$$= \left(\frac{1}{x_2 - x_1}\right)\left(\frac{x^2}{2}\right)_{x_1}^{x_2}$$

$$= \frac{(x_1^2 - x_2^2)}{2(x_2 - x_1)} = \frac{(x_2 + x_1)\cancel{(x_2 - x_1)}}{2.\cancel{(x_2 - x_1)}}$$

$$\boxed{\therefore \text{ Mean } M = \mu_1' = \frac{x_1 + x_2}{2}}$$

(18) Consider the random variable with exponential density $f_X(x) =$

$$f_X(x) = \frac{1}{b} e^{-(x-a)/b} \; ; \; x \ge 0 \quad \text{find it's charecteristic's function.}$$

$$= 0 \; ; \; x < a$$

and its first moment

Sol: Given the random variable "X" of exponential density function is

$$f_X(x) = \frac{1}{b} e^{-(x-a)/b} \; ; \; x \ge a$$

$$= 0 \; ; \; x < a$$

The first moment about origin = mean

$$E(X) = \int_{-\infty}^{\infty} x . f_X(x)\, dx$$

$$= \int_{a}^{\infty} x . \frac{1}{b} e^{-(x-a)/b}\, dx$$

$$= \frac{e^{a/b}}{b} . \int_{a}^{\infty} x . e^{-x/b}\, dx$$

$$= \frac{e^{a/b}}{b}\left[x\left(\frac{e^{-x/b}}{-1/b}\right)_a^{\infty} - \int_a^{\infty} (1) \frac{e^{-(x/b)}}{(-1/b)}\, dx \right]$$

$$= \frac{e^{a/b}}{b}\left[ab . e^{-a/b} + b^2 . e^{-a/b} \right]$$

$$= (a+b)$$

$$\boxed{\therefore \text{ mean } = a+b}$$

characteristic function $= E(e^{j\omega x})$ (3.48)

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \int_{a}^{\infty} e^{j\omega x}\ \tfrac{1}{b}\ e^{-(x-a)/b}\,dx$$

$$= \frac{1}{b}\ e^{a/b} \int_{a}^{\infty} e^{j\omega x}\cdot e^{-x/b}\,dx$$

$$= \frac{1}{b}\ e^{a/b} \int_{a}^{\infty} e^{-x\cdot(1/b - j\omega)}\,dx$$

$$= \frac{1}{b}\ e^{a/b} \left[\frac{e^{-x(1/b - j\omega)}}{-(1/b - j\omega)}\right]_{+a}^{\infty}$$

$$= \frac{1}{b}\ e^{a/b} \left[\frac{e^{-\infty(1/b - j\omega)}}{-(1/b - j\omega)} - \frac{e^{-a(1/b - j\omega)}}{-(1/b - j\omega)}\right]$$

$$= \frac{1}{b}\ e^{a/b} \left[\frac{e^{-a(1/b - j\omega)}}{(1/b - j\omega)}\right]$$

$$= \frac{1}{b} \times \frac{1}{(1/b - j\omega)} \left(e^{a/b - a/b + j\omega}\right)$$

$$= \frac{1}{b}\left(\frac{1}{1/b - j\omega}\right)\left(e^{j\omega}\right)$$

$$\boxed{\phi_X(\omega) = e^{j\omega}\left(\frac{1/b}{1/b - j\omega}\right)}$$

(19) Show that the mean value and varience of a random variable having the uniform density function $f_X(x) = \frac{1}{b-a}$; $a \le x \le b$. are

$\bar{X} = E(X) = \frac{a+b}{2}$; and $\sigma_X^2 = \frac{(b-a)^2}{12}$

Sol: Given the uniform density function of a random variable "x" is given by

$$f_X(x) = \frac{1}{b-a} \quad ; a \le x \le b$$
$$= 0 \quad ; \text{elsewere}$$

$$\mu_1' = \text{mean} = E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

$$= \int_a^b x\cdot\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\int_a^b x\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \left(\frac{1}{b-a}\right)\frac{1}{2}\left(b^2 - a^2\right)$$

$$= \frac{(b^2 - a^2)}{2(b-a)}$$

$$= \frac{(b+a)(b-a)}{2(b-a)}$$

$$\boxed{\mu_1' = \text{mean} = \frac{a+b}{2}}$$

$$\mu_2' = E[x^2] = \int_a^b x^2 \cdot f_X(x)\,dx$$

$$= \int_a^b x^2\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\left(\frac{x^3}{3}\right)_a^b = \frac{1}{3}\ \frac{1}{b-a}\left(b^3 - a^3\right)$$

$$= \frac{1}{3}\cdot\frac{(b-a)(b^2 + ab + a^2)}{(b-a)}$$

$$\boxed{\mu_2' = \frac{a^2 + ab + b^2}{3}}$$

Varience $\sigma_x^2 = \mu_2' - (\mu_1')^2$

(3.49)

$$\sigma_x^2 = \frac{a^2+ab+b^2}{3} - \frac{a^2+2ab+b^2}{4}$$

$$= \frac{4a^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$= \frac{(b-a)^2}{12}$$

$$\boxed{\sigma_x^2 = \frac{(b-a)^2}{12}}$$

(20) Prove that the moment generating function of sum of two independent random variable is the product of their moment generating functions. $M_{x+y}(t) = M_x(t) \cdot M_y(t)$.

Sol: From the definition of moment generating function

$$\phi_x(t) = E[e^{tx}]$$

$$M_{(x+y)}(t) = E\left[e^{t(x+y)}\right]$$

$$= E\left[e^{tx+ty}\right]$$

$$= E\left[e^{tx} \cdot e^{ty}\right]$$

$$= E\left(e^{tx}\right) \cdot E\left(e^{ty}\right)$$

$$= M_x(t) \cdot M_y(t)$$

$$\therefore \boxed{M_{x+y}(t) = M_x(t) \cdot M_y(t)}$$

(21) Show that any charecteristic function $\phi_x(\omega)$ satisfies the $|\phi_x(\omega)| \leq \phi_x(0) = 1$; (properties of 1st and 2nd of charecteristic function)

Sol: We know that $\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_x(x)\, dx$

$$|\phi_x(\omega)| = \left| \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_x(x)\, dx \right|$$

$$= \int_{-\infty}^{\infty} f_x(x)\, dx \qquad \left[\because |e^{j\omega x}| = 1\right] \quad \left[\because \int_{-\infty}^{\infty} f_x(x)\, dx = 1\right]$$

$$= 1.$$

$$|\phi_x(\omega)| \leq 1 \rightarrow (1)$$

$$|\phi_x(0)| = \int_{-\infty}^{\infty} e^{0} \cdot f_x(x)\, dx$$

$$= \int_{-\infty}^{\infty} f_x(x)\, dx$$

$$= 1$$

$$\phi_x(0) = 1 \rightarrow (2)$$

$\therefore$ from eq (1) and (2)

$$\boxed{|\phi_x(\omega)| \leq \phi_x(0) = 1}$$

(22) Find the moment generating function of random variable 'x' with $x = \frac{1}{2}$, with probability $\frac{1}{2}$, $x = -\frac{1}{2}$, with probability $\frac{1}{2}$ also find out first 4 moment's about origin from moment generating function?

Sol: Given that the random variable "x" has $x = \frac{1}{2}$ with probability $\frac{1}{2}$

i.e. $X = \frac{1}{2}$; $P(X = \frac{1}{2}) = \frac{1}{2}$

i.e $X = -\frac{1}{2}$; $P(X = -\frac{1}{2}) = \frac{1}{2}$

(3.50)

∴ The PDF is

$X = x$	$\frac{1}{2}$	$-\frac{1}{2}$
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

Here the random variable is "x" is discrete random variable

for discrete random variable, the moment generating function is given by

$$M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx} \cdot f_X(x)$$

$$= \sum_{\text{all } x} e^{tx} \cdot p(X=x).$$

$$= e^{t(\frac{1}{2})} \cdot p(X=\tfrac{1}{2}) + e^{t(-\frac{1}{2})} \cdot p(X=-\tfrac{1}{2})$$

$$= e^{t/2}\,\tfrac{1}{2} + e^{-t/2}\,\tfrac{1}{2}$$

$$M_X(t) = \frac{1}{2}\left(e^{t/2} + e^{-t/2}\right)$$

$$M_X(t) = \frac{1}{2}\left[1 + \frac{(t/2)}{1!} + \frac{(t/2)^2}{2!} + \cdots \infty + 1 + \frac{(-t/2)}{1!} + \frac{(-t/2)^2}{2!} + \cdots \infty\right]$$

$$= \frac{1}{2}\left[1 + \frac{t}{2 \cdot 1!} + \frac{t^2}{4\,2!} + \cdots + \infty + 1 - \frac{t}{2!} + \frac{t^2}{2!\,4} + \cdots + \infty\right]$$

$$= \frac{1}{2}\left[2 + \frac{t}{1!}\left(\frac{1}{2} - \frac{1}{2}\right) + \frac{t^2}{2!}\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{t^3}{3!}\left(\frac{1}{8} - \frac{1}{8}\right) + \cdots + \infty\right]$$

$$= \frac{1}{2}\left[2 + \frac{t^2}{2!}\left(\frac{2}{4}\right)\right] + \cdots$$

$$1 + \frac{t}{1!}(0) + \frac{t^2}{2!}\left(\frac{1}{4}\right) + \frac{t^3}{3!}(0) + \frac{t^4}{4!}\,\frac{1}{16} + \cdots \rightarrow (1)$$

We know that from the defination of moment generating function is

$$M_X(t) = E(e^{tX}) = 1 + \frac{t}{1!} \cdot E(X) + \frac{t^2}{2!} E(X^2) + \cdots + \cdots \infty \rightarrow (2)$$

from (1) and (2)

$$E(X) = 0 \,;\quad E(X^2) = \tfrac{1}{4} \,;\quad E(X^3) = 0 \,;\quad E(X^4) = \tfrac{1}{16}.$$

(or)

$$M_X(t) = \frac{1}{2}\left(e^{t/2} + e^{-t/2}\right).$$

$$= \frac{1}{2} e^{t/2} + \frac{1}{2} e^{-t/2}$$

∴ the I^{st} moment about origin $= \mu_1' = m_1 = \frac{d}{dt}\left(M_X(t)\right)\Big|_{t=0}$

$$= \frac{d}{dt}\left(\frac{1}{2} e^{t/2} + \frac{1}{2} e^{-t/2}\right)\Big|_{t=0}$$

$$= \frac{1}{2} \cdot e^{t/2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(e^{-t/2}\right)\left(-\frac{1}{2}\right)\Big|_{t=0}$$

$$= \frac{1}{4} - \frac{1}{4}$$

$$= 0.$$

∴ the 2^{nd} moment about origin $= \mu_2' = m_2 = \frac{d^2}{dt^2}\left(M_X(t)\right)\Big|_{t=0}$

$$\frac{d}{dt}\left[\frac{d}{dt}\left(M_X(t)\right)\right]_{t=0}$$

$$= \frac{d}{dt}\left(\frac{1}{4} e^{t/2} - \frac{1}{4} e^{-t/2}\right)\Big|_{t=0}$$

$$= \frac{1}{4} e^{t/2}\left(\frac{1}{2}\right) - \frac{1}{4} e^{-t/2}\left(-\frac{1}{2}\right)\Big|_{t=0}$$

$$= \frac{1}{8} + \frac{1}{8}$$

$$\boxed{\mu_2' = \frac{1}{4}}$$

(iii)rd moment about origin $= \mu_3' = m_3 = \frac{d^3}{dt^3}(M_x(t))\Big|_{t=0}$ (3-51)

$$= \frac{d}{dt}\left(\frac{d^2}{dt^2}\left(M_x(t)\right)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\left(\tfrac{1}{8}e^{t/2} + \tfrac{1}{8}e^{-t/2}\right)\Big|_{t=0}\right]$$

$$= \frac{1}{16}e^{t/2} - \frac{1}{16}e^{-t/2}\Big|_{t=0}$$

$$= \frac{1}{16} - \frac{1}{16}$$

$$= 0$$

(iv)th moment about origin $\mu_4' = m_4 = \frac{d^4}{dt^4}(M_x(t))\Big|_{t=0}$

$$\frac{d}{dt}\left(\frac{d^3}{dt}\left(M_x(t)\right)\right)\Big|_{t=0}$$

$$\frac{d}{dt}\left(\frac{1}{16}e^{t/2} - \tfrac{1}{16}e^{-t/2}\right)\Big|_{t=0}$$

$$\frac{1}{32}e^{t/2} + \frac{1}{32}e^{-t/2}\Big|_{t=0}$$

$$= \frac{1}{32} + \frac{1}{32}$$

$$= 1/16$$

(23) The moment generating function of a random variable 'x' is $2/_{2-t}$ find out it's mean and verience?

Sol: Given the moment generating function of random variable x is $\frac{2}{2-t}$

$\therefore \mu_1' = m_1 = \frac{\partial}{\partial t}(M_x(t))\Big|_{t=0}$

$$\frac{\partial}{\partial t}\left(\frac{2}{2-t}\right)_{t=0}$$

$$= \frac{(2-t)(0) - 2(-1)}{(2-t)^2}\Big|_{t=0}$$

$$= \frac{2}{(2-0)^2}$$

$$= \frac{2}{4}$$

$$= 1/2$$

Second moment about origin $= \mu_2' = m_2 = \frac{d^2}{dt^2}(M_x(t))\Big|_{t=0}$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}(M_x(t))\right)\Big|_{t=0}$$

$$\frac{\partial}{\partial t}\left(\frac{2}{(2-t)^2}\right)\Big|_{t=0}$$

$$\frac{(2-t)^2(0) - 2\cdot 2(2-t)(-1)}{(2-t)^4}\Big|_{t=0}$$

$$= \frac{4\cancel{(2-t)}}{(2-t)^{\cancel{4}}}\Big|_{t=0}$$

$$= \frac{4}{2^3}$$

$$= \frac{4}{8} = 1/2$$

Verience of 'x' $= \sigma_x^2 = \mu_2' - (\mu_1')^2$

$$= 1/2 - 1/4$$

$$= 1/4$$

(24) The moment generating function of random variable "x" having the density function $f_X(x) = e^{-x}$; $x \geq 0$ find moment generating function of
$= 0$; elsewere (3.52)

a variance?

Sol: Given $f_X(x) = e^{-x}$; $x \geq 0$
$= 0$; elsewere

$$M_X(t) = E[e^{tx}]$$
$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$
$$= \int_0^{\infty} e^{tx} e^{-x}\,dx.$$
$$= \int_0^{\infty} e^{x(t-1)}\,dx$$
$$= \left[\frac{e^{x(t-1)}}{(t-1)}\right]_0^{\infty}$$
$$= 0 + \left(\frac{-1}{t-1}\right)$$
$$= \frac{1}{1-t}$$

$$M_X(t) = \left(\frac{1}{1-t}\right)\Big|_{t=0}$$
$$\boxed{M_X(t) = 1}$$

$$\mu_1' = E(x) = \frac{\partial}{\partial t}(M_X(t))\Big|_{t=0}$$
$$= \frac{\partial}{\partial t}\left(\frac{1}{1-t}\right)\Big|_{t=0}$$
$$= \frac{-1}{(1-t)^2}\Big|_{t=0}$$
$$\boxed{\mu_1' = -1}$$

$$\mu_2' = \frac{\partial^2}{\partial t^2}(M_X(t))\Big|_{t=0}.$$
$$= \frac{-2}{(1-t)^2}\Big|_{t=0}$$
$$= -2(-1)$$
$$\boxed{\mu_2' = 2.}$$

$\because$ variance $\sigma_X^2 = \mu_2' - (\mu_1')^2$
$= 2-(1)^2$
$= 2-1$
$= 1$
$\boxed{\sigma_X^2 = 1}$

(25) If density function of a continuous random variable is $f_X(x) = \frac{1}{2}e^{-|x|}$, find moment generating function of "x". and It's mean and variance?

Sol: Given that $f_X(x) = \frac{1}{2}e^{-|x|}$

The moment generating function of random variable 'x" is

$$M_X(t) = E[e^{tx}]$$
$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$
$$= \int_{-\infty}^{\infty} e^{tx}\, \frac{1}{2} e^{-|x|}\,dx$$
$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx} e^{-(-x)}\,dx + \int_0^{\infty} e^{tx} e^{-x}\,dx\right]$$
$$= \frac{1}{2}\int_{-\infty}^{0} e^{tx+x}\,dx + \frac{1}{2}\int_0^{\infty} e^{tx-x}\,dx$$
$$= \frac{1}{2}\int_{-\infty}^{0} e^{(t+1)x}\,dx + \frac{1}{2}\int_0^{\infty} e^{(t-1)x}\,dx$$
$$= \frac{1}{2}\left(\frac{e^{(t+1)x}}{t+1}\right)_{-\infty}^{0} + \frac{1}{2}\left(\frac{e^{(t-1)x}}{(t-1)}\right)_0^{\infty}$$
$$= \frac{1}{2}\left[e^0 - \frac{1}{t+1}\right) + \frac{1}{2}\left[\frac{e^0}{t-1} - 0\right]$$
$$= \frac{-1}{2t+2} + \frac{1}{2t-2}$$
$$= \frac{-\cancel{2t}+2+\cancel{2t}+2}{(2t)^2-2^2} = \frac{4}{4t^2-4} = \frac{4}{(t^2-1)4} = \frac{1}{(t^2-1)}$$
$$\boxed{M_X(t) = \frac{1}{1-t^2}}$$

The mean of random variable "x" is $\mu_1' = E[x]$
$$= \int_{-\infty}^{\infty} x.f_X(x)\,dx \text{ (or) } \frac{d}{dt}(M_X(t))\Big|_{t=0}$$
$$\mu_1' = \frac{d}{dt}\left(\frac{1}{1-t^2}\right)\Big|_{t=0} = \frac{(1-t^2)(0) - 1(-2t)}{(1-t^2)^2}\Big|_{t=0}$$

$\boxed{\mu_1' = 0}$

(3.53)

$$\mu_2' = \frac{d^2}{dt^2}\left(\frac{1}{1-t^2}\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{2t}{(1-t^2)^2}\right]_{t=0}$$

$$= \frac{(1-t^2)^2\, 2 - 2t\; 2(1-t^2)(-2t)}{(1-t^2)^4}\Bigg|_{t=0}$$

$$= \frac{2\cdot(1-t^2)^2 + 8t^2(1-t^2)}{(1-t^2)^4} = \frac{(1-t^2)\left[2(1-t^2)+8t^2\right]}{(1-t^2)^4}$$

$$= \frac{2(1-t^2)+8t^2}{(1-t^2)^3}\Bigg|_{t=0}$$

$$= \frac{2}{1}$$

$\boxed{\mu_2' = 2}$

$\therefore$ variance $\sigma_x^2 = \mu_2' - (\mu_1')^2$

$= 2 - 0^2$

$= 2-0$

$= 2$

$\boxed{\sigma_x^2 = 2}$

(26) Show that the distribution function for which the characteristic function $e^{-|t|}$ has density function $f_X(x) = \frac{1}{\pi(1+x^2)}$; $-\infty < x < \infty$

Sol: Given the characteristic function $\phi_X(t) = e^{-|t|}$.

We know that, the density function of random variable "X" is inverse fourier transform of the characteristic function

$$\therefore f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(t)\, e^{-j\omega x}\, dt.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|t|}\, e^{-j\omega x}\, dt$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{t}\cdot e^{-j\omega x} dt + \int_{0}^{\infty} e^{-t}\, e^{-j\omega x} dt\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{t(1-jx)} dx + \int_{0}^{\infty} e^{-t(1+jx)} dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{e^{t(1-jx)}}{(1-jx)}\right)_{-\infty}^{0} + \left(\frac{e^{-t(1+jx)}}{-(1+jx)}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} + 0 + \frac{1}{1+jx}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} + \frac{1}{1+jx}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1+\cancel{jx}+1-\cancel{jx}}{1+x^2}\right)$$

$$= \frac{1}{\cancel{2}\pi}\left(\frac{\cancel{2}}{1+x^2}\right)$$

$$\boxed{f_X(x) = \frac{1}{\pi(1+x^2)} \; ; -\infty \le x \le \infty}$$

(27) Find the characteristic function of a random variable "X" with the density function $f_X(x) = x/2$; $0 \le x \le 2$

$= 0$; elsewhere.

Sol: Given the density function of a random variable "X" is

$$f_X(x) = x/2 \quad ; 0 \le x \le 2$$
$$= 0 \quad ; \text{elsewere}$$

(3.54)

The cherecterstic function of random variable "x" is

$$\phi_X(\omega) = E\left(e^{j\omega x}\right)$$
$$= \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\, dx$$
$$= \int_{-\infty}^{\infty} \frac{x}{2} e^{j\omega x}\, dx$$
$$= \frac{1}{2}\int_{0}^{2} x e^{j\omega x}\, dx$$
$$= \frac{1}{2}\left[x\left(\frac{e^{j\omega x}}{j\omega}\right)_0^2 - \int_0^2 \frac{e^{j\omega x}}{j\omega}\, dx\right]$$
$$= \frac{1}{2}\left[(2-0)\frac{e^{2j\omega}}{j\omega} - \frac{1}{j\omega}\left(\frac{e^{j\omega x}}{j\omega}\right)_0^2\right]$$
$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2} e^{2j\omega}}{j\omega} + \frac{1}{\omega^2}\left(e^{j2\omega}-1\right)\right]$$
$$\boxed{= \phi_X(\omega)' = \frac{e^{j2\omega}}{j\omega} + \frac{2}{\omega^2}\left(e^{j2\omega}-1\right)}$$

(28) A random variable "x" has a cherecterstic function $\phi_X(\omega) = 1-|\omega| ; |\omega| < 1$
$= 0 ; |\omega| > 1$
find the density function of a random variable "x".

Sol:- Given that $\phi_X(\omega) = 1-|\omega| ; |\omega| \le 1$
$= 0 ; |\omega| > 1$

$$\therefore \phi_X(\omega) = 1-|\omega| ; -1 \le |\omega| \le 1$$
$$= 0 ; \text{Otherwise.}$$

We know that the density function of a random variable "x" is inverse fourier transform of its cherecterstic function

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x} d\omega$$
$$= \frac{1}{2\pi}\int_{-1}^{1} (1-|\omega|) \cdot e^{-j\omega x} d\omega$$
$$= \frac{1}{2\pi}\left[\int_{-1}^{0} (1-(-\omega))\, e^{-j\omega x} d\omega + \int_0^1 (1-\omega)\, e^{-j\omega x} d\omega\right]$$
$$= \frac{1}{2\pi}\left[\int_{-1}^{0} (1+\omega)\, e^{-j\omega x} d\omega + \int_0^1 (1-\omega)\, e^{-j\omega x} d\omega\right]$$
$$= \frac{1}{2\pi}\left[(1+\omega)\left(\frac{e^{-j\omega x}}{-jx}\right)_{-1}^{0} - \int_{-1}^{0}\frac{e^{-j\omega x}}{-jx} d\omega\right] + \frac{1}{2\pi}\left[(1-\omega)\left(\frac{e^{-j\omega x}}{-jx}\right)_0^1 + \int_0^1 \frac{e^{-j\omega x}}{-jx} d\omega\right]$$
$$= \frac{1}{2\pi}\left[\left(\frac{1}{-jx}-0\right) - \frac{1}{(jx)^2}\left(e^{-j\omega x}\right)_{-1}^{0}\right] + \frac{1}{2\pi}\left[\left(0+\frac{1}{jx}\right) + \frac{1}{(jx)^2}\left(e^{-j\omega x}\right)_0^1\right]$$
$$= \frac{1}{2\pi}\left[\frac{-1}{jx} - \left(\frac{1}{(jx)^2} - \frac{e^{jx}}{(jx)^2}\right)\right] + \frac{1}{2\pi}\left[\frac{1}{jx} + \left(\frac{e^{-jx}}{(jx)^2} - \frac{1}{(jx)^2}\right)\right]$$
$$= \frac{1}{2\pi}\left[\cancel{\frac{-1}{jx}} - \cancel{\frac{1}{(jx)^2}} + \frac{e^{jx}}{(jx)^2} + \cancel{\frac{1}{jx}} + \frac{e^{-jx}}{(jx)^2} - \cancel{\frac{1}{(jx)^2}}\right]$$
$$= \frac{1}{2\pi}\left[\frac{e^{jx}}{x^2} + \frac{e^{-jx}}{x^2} + \frac{1}{x^2} + \frac{1}{x^2}\right]$$
$$= \frac{1}{2\pi}\left(\frac{1}{x^2}\right)\left(e^{jx} + e^{-jx} + 2\right)$$
$$= \frac{1}{\pi x^2}\left[1 + \left(\frac{e^{jx}+e^{-jx}}{2}\right)\right]$$
$$= \frac{1}{\pi x^2}\left[1+\cos x\right] = \frac{1+\cos x}{\pi x^2}$$

(29) for the rayleigh density function $f_X(x) = \frac{2}{b}(x-a)\, e^{-(x-a)^2/b}$; $x \geq a$

$= 0$; $x < a$.

Show that $E[X] = a + \sqrt{\frac{\pi b}{4}}$; $\sigma_X^2 = \frac{b(4-\pi)}{4}$. (255)

Sol: Given that $f_X(x) = \frac{2}{b}(x-a)\cdot e^{-(x-a)^2/b}$; $x \geq a$

$= 0$; $x \leq a$

let $(x-a) = t$ If $x=a \Rightarrow t = a-a \Rightarrow t=0$

$b/2 = \alpha^2$

$\therefore f_X(x) = \frac{t}{\alpha^2} e^{-t^2/2\alpha^2}$; $t \geq 0$

$= 0$; otherwise.

Mean of random variable 'x' is $= E[X]$

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{0}^{\infty} (t+a) \cdot \frac{t}{\alpha^2} e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty \left(\frac{t^2}{\alpha^2} + \frac{at}{\alpha^2}\right) e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty \frac{t^2}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt + \int_0^\infty \frac{at}{\alpha^2} e^{-t^2/2\alpha^2} dt \rightarrow (1)$$

let us consider, first term in eq (1).

$$\int_0^\infty \frac{t^2}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt = \int_0^\infty t\left(\frac{t}{\alpha^2} e^{-t^2/2\alpha^2}\right) dt$$

$$= \int_0^\infty t \cdot \left(-d\left(e^{-t^2/2\alpha^2}\right)\right) dt$$

$$= t \cdot \left(-e^{-t^2/2\alpha^2}\right)_0^\infty - \int_0^\infty 1 \cdot \left(-e^{-t^2/2\alpha^2}\right) dt$$

$$= 0\,(0-0) + \int_0^\infty e^{-t^2/2\alpha^2} dt.$$

$$\therefore \int_0^\infty \frac{t^2}{\alpha^2} e^{-t^2/2\alpha^2} dt = \int_0^\infty e^{-t^2/2\alpha^2} dt \rightarrow (2)$$

we know that the density function of a normal distribution function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot e^{-(x-m)^2/2\sigma_X^2}$$

let $m=0$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \cdot e^{-x^2/2\sigma_X^2}.$$

We know that gaussian density function is valid density function

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-x^2/2\sigma_X^2} dx = 1 \rightarrow (3)$$

eq (2) $\Rightarrow$ $\int_0^\infty \frac{t^2}{\alpha^2} e^{-t^2/2\alpha^2} dt = \int_0^\infty e^{-t^2/2\alpha^2} dt$

$$= \frac{1}{2} \cdot 2\left(\int_0^\infty e^{-t^2/2\alpha^2} dt\right)$$

$$= \frac{1}{2}\left[\int_{-\infty}^\infty e^{-t^2/2\alpha^2} dt\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\alpha^2}} \cdot \sqrt{2\pi\alpha^2} \cdot e^{-t^2/2\alpha^2} dt\right.$$

$$= \frac{\sqrt{2\pi\alpha^2}}{2} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\alpha^2}} \cdot e^{-t^2/2\alpha^2} dt$$

$$\therefore \int_0 \frac{t^2}{\alpha^2} \cdot e^{-t^2/2\alpha^2} \cdot dt = \int_0 \sqrt{\frac{\pi\alpha^2}{2}} \quad (1)$$

$$= \sqrt{\frac{\pi\alpha^2}{2}} \rightarrow (4)$$

(3.56)

let us consider (2)nd term in eq (1) is

$$\int_0^\infty \frac{at}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt = a \int_0^\infty d\left(-e^{-t^2/2\alpha^2}\right).$$

$$= a \cdot \left(-e^{-t^2/2\alpha^2}\right)_0^\infty$$

$$= a\left(-e^{-\infty} + e^0\right)$$

$$= a. \rightarrow (5)$$

Substitute eq (4) and (5) in (1)

$$\therefore E(x) = \sqrt{\frac{\pi\alpha^2}{2}} + a$$

but $\frac{b}{2} = \alpha^2$

$$\Rightarrow E(x) = \sqrt{\frac{\pi(b/2)}{2}} + a$$

$$\therefore E(x) = a + \sqrt{\frac{\pi b}{4}}$$

$$\therefore E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f_x(t)\, dt$$

$$= \int_0^\infty (t+a)^2 \cdot \frac{t}{\alpha^2} e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty \left(t^2 + a^2 + 2at\right) \frac{t}{\alpha^2} e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty \frac{t^3}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt + \int_0^\infty \frac{2at^2}{\alpha^2} e^{-t^2/2\alpha^2} dt + \int_0^\infty a^2 \frac{t}{\alpha^2} e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty \frac{t^3}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt + 2a \int_0^\infty \frac{t^2}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt + a^2 \int_0^\infty \frac{t}{\alpha^2} e^{-t^2/2\alpha^2} dt$$

$$E[x^2] = \int_0^\infty \frac{t^3}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt + 2a\sqrt{\frac{\pi\alpha^2}{2}} + a^2(1) \rightarrow (6)$$

$$\therefore = \int_0^\infty \frac{t^3}{\alpha^2} e^{-t^2/2\alpha^2} dt = \int_0^\infty t^2 \cdot \frac{t}{\alpha^2} \cdot e^{-t^2/2\alpha^2} dt$$

$$= \int_0^\infty t^2 \cdot d\left(-e^{-t^2/2\alpha^2}\right) dt$$

$$= \left[t^2 \left(-e^{-t^2/2\alpha^2}\right)_0^\infty + \int_0^\infty 2t \cdot e^{-t^2/2\alpha^2} dt\right]$$

$$= \left[0 + 2\int_0^\infty t \cdot e^{-t^2/2\alpha^2} dt\right]$$

$$= 2\int_0^\infty d\left[\left(-e^{-t^2/2\alpha^2}\right) \cdot \alpha^2\right]_0^\infty$$

$$= 2\left[-e^{-t^2/2\alpha^2} \cdot \alpha^2\right]_0^\infty$$

$$= 2\alpha^2$$

$$\therefore E(x^2) = 2\alpha^2 + \frac{2a\sqrt{\pi\alpha^2}}{\sqrt{2}} + a^2$$

$$= 2 \cdot b/2 + 2a\sqrt{\frac{\pi b}{4}} + a^2$$

$$= b + a\sqrt{\pi b} + a^2$$

$$\therefore E(x^2) = b + a\sqrt{\pi b} + a^2$$

Variance $\cdot \sigma_x^2 = \mu_2' - (\mu_1')^2 = b + a\sqrt{\pi b} + a^2 - \left(a + \sqrt{\frac{\pi b}{4}}\right)^2$

$$\sigma_x^2 = b + \cancel{a\sqrt{\pi b}} + \cancel{a^2} - \cancel{a^2} - \frac{\pi b}{4} - \cancel{2a\sqrt{\pi b}}$$

$$= b - \frac{\pi b}{4}$$
$$= \frac{4b - \pi b}{4}$$
$$= \frac{b(4-\pi)}{4}$$

(59)

(30) The cherecterstic function of a laplace density function is $\phi_x(\omega) = \frac{e^{jm\omega}}{1+(b\omega)^2}$. find mean and verience of r.v. x.

Sol: Given, The random variable 'x' fallows the laplace transform distribution with cherectestic function

$$\phi_x(\omega) = \left[\frac{e^{jm\omega}}{1+(b\omega)^2}\right]$$

$$\therefore \text{mean of } x = E[x] = \mu_1' = \left(\frac{1}{j}\right)\frac{\partial}{\partial\omega}(\phi_x(\omega))\Big|_{\omega=0}$$

$$= \left(\frac{1}{j}\right)\frac{\partial}{\partial\omega}\left(\frac{e^{j\omega m}}{1+(b\omega)^2}\right)\Big|_{\omega=0}$$

$$= \left(\frac{1}{j}\right)\left[\frac{(1+(b\omega)^2)\,e^{jm\omega}(jm) - e^{j\omega m}\,2b\omega}{(1+(b\omega)^2)^2}\right]\Big|_{\omega=0}$$

$$= \left(\frac{1}{j}\right)\left(\frac{jm}{1}\right)$$

$$\boxed{E(x) = m}$$

$$E(x^2) = \mu_2' = \left(\frac{1}{j}\right)^2 \cdot \frac{\partial^2}{\partial\omega^2}(\phi_x(\omega))\Big|_{\omega=0}$$

$$= (-1)\frac{\partial}{\partial\omega}\left[\frac{\partial}{\partial\omega}(\phi_x(\omega))\right]\Big|_{\omega=0}$$

$$= (-1)\frac{\partial}{\partial\omega}\left[\frac{(1+(b\omega)^2)\,jm\,e^{jm\omega} - e^{jm\omega}(2b\omega)}{(1+(b\omega)^2)}\right]\Big|_{\omega=0}$$

$$= m^2 + 2b^2.$$

$$\sigma_x^2 = E(x^2) - (E(x))^2$$

$$= \cancel{m^2} + 2b^2 - \cancel{m^2}$$

$$\boxed{\sigma_x^2 = 2b^2}$$

(31) The analog r.v. 'x' has a cherecteristic function $\phi_x(\omega) = \left(\frac{a}{a-j\omega}\right)^N$ for $a>0$, And $N=1,2,3,\ldots$ Show that $\bar{x} = N/a$, $\overline{x^2} = \frac{N(N+1)}{a^2}$; $\sigma_x^2 = \frac{N}{a^2}$

Sol: Given $\phi_x(\omega) = \left(\frac{a}{a-j\omega}\right)^N = \frac{a^N}{(a-j\omega)^N}$

$$\therefore \text{Mean of } x = E(x) = \mu_1' = \left(\frac{1}{j}\right)\cdot\frac{d}{d\omega}\,\phi_x(\omega)\Big|_{\omega=0}$$

$$= \left(\frac{1}{j}\right)\cdot\frac{\partial}{\partial\omega}\left(\frac{a^N}{(a-j\omega)^N}\right)\Big|_{\omega=0}$$

$$= \frac{a^N}{j}\,\frac{\partial}{\partial\omega}\left(\frac{1}{(a-j\omega)^N}\right)\Big|_{\omega=0}$$

$$= \frac{a^N}{j}\left[\frac{0 - N(a-j\omega)^{N-1}(-j)}{(a-j\omega)^{2N}}\right]\Big|_{\omega=0}$$

$$= \frac{a^N}{j}\left[\frac{jN(a-j\omega)^{N-1}}{(a-j\omega)^{2N}}\right]\Big|_{\omega=0}$$

$$= \frac{a^N}{j}\left(\frac{jN\,a^{N-1}}{a^{2N}}\right)$$

$$= \frac{a^N}{\cancel{j}}\,\cancel{j}N\,a^{N-1-2N}$$

$$= N a^{\cancel{N}+N-1-\cancel{2N}}$$

$$= \frac{N}{a}$$

(3.58)

$\therefore \quad \mu_1' = a^{-1} N$

$$\boxed{\mu_1' = m = \frac{N}{a}}$$

$$\overline{X^2} = E(X^2) = \left(\frac{1}{j}\right)^2 \frac{\partial^2}{\partial \omega^2} \cdot \left[\phi_x(\omega) \right] \Big|_{\omega=0}$$

$$= \left(\frac{1}{j}\right)^2 \cdot \frac{\partial}{\partial \omega}\left(\frac{\partial}{\partial \omega}\left(\frac{a^N jN \cdot (a-j\omega)^{N-1}}{(a-j\omega)^{2N}}\right)\right)\Big|_{\omega=0}$$

$$= \frac{1}{j^2} a^N jN \cdot \frac{\partial}{\partial \omega} \cdot \left(\frac{(a-j\omega)^{N-1}}{(a-j\omega)^{2N}}\right)\Big|_{\omega=0}$$

$$= \frac{a^N N}{j} \cdot \frac{\partial}{\partial \omega}\left[\frac{1}{(a-j\omega)^{N+1}}\right]\Big|_{\omega=0}$$

$$= \frac{a^N N}{j}\left[\frac{1 - (N+1)(a-j\omega)^N(-j)}{(a-j\omega)^{2N+2}}\right]\Big|_{\omega=0}$$

$$= \frac{a^N N}{\cancel{j}}\left(\frac{\cancel{j}(N+1)\, a^N}{a^{2N+2}}\right)$$

$$= \frac{a^N N \cdot (N+1)\, a^N}{a^{2N+2}}$$

$$= \frac{N(N+1)}{a^{\cancel{2N}+2\,\cancel{-2N}}}$$

$$\boxed{\overline{X^2} = \frac{N(N+1)}{a^2}}$$

$$\sigma_X^2 = E(X^2) - (E(X))^2 = \frac{N(N+1)}{a^2} - \frac{N^2}{a^2}$$

$$\boxed{\sigma_X^2 = N/a^2}$$

(32) A random variable has p.d.f $f_X(x) = \frac{1}{2^x}$; $x = 1,2,3,4 \ldots$ find the moment generating function. (3.09)

Sol: Given $f_X(x) = \frac{1}{2^x}$; $x = 1,2,3,4 \ldots$

here "x" is a discrete random variable.

$\therefore$ The moment generating function of "x" is

$$f_X(x) = E(e^{tx})$$
$$= \sum_{\text{all } x} e^{tx} f_X(x)$$
$$= \sum_{x=1}^{\infty} e^{tx} \frac{1}{2^x}$$
$$= \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$
$$= \left(\frac{e^t}{2}\right)^1 + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{3}\right)^3 + \ldots$$
$$= \left(\frac{e^t}{2}\right)\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots\right]$$
$$= \left(\frac{e^t}{2}\right)\left[1 - \frac{e^t}{2}\right]^{-1}$$
$$= \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1}$$
$$= \frac{e^t}{\cancel{2}} \frac{(2-e^t)^{-1}}{\cancel{2}}$$
$$= e^t (2-e^t)^{-1}$$
$$\boxed{M_X(t) = \frac{e^t}{2-e^t}}$$

(33) The probability density function of a random variable is given by $f_X(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x$, $x = 0,1,2,\ldots\infty$; find moment generating function and also find out 1st & 2nd moment?

Sol: Given $f_X(x) = \frac{2}{3}(1/3)^x$, $x = 0,1,2,\ldots\infty$

where "x" is discrete random variable.

$\therefore$ The moment generating function of "x" is

$$M_X(t) = E[e^{tx}]$$
$$= \sum_{\text{all } x} e^{tx} f_X(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} \frac{2}{3}(1/3)^x$$
$$= \frac{2}{3}\sum_{x=0}^{\infty} e^{tx}(1/3)^x$$
$$= \frac{2}{3}\sum_{x=0}^{\infty}\left(\frac{e^t}{3}\right)^x$$
$$= \frac{2}{3}\left[\left(\frac{e^t}{3}\right)^0 + \left(\frac{e^t}{3}\right)^1 + \left(\frac{e^t}{3}\right)^2 + \ldots\right]$$
$$= \frac{2}{3}\left[1 - \frac{e^t}{3}\right]^{-1}$$
$$= \frac{2}{3}\left[\frac{3-e^t}{3}\right]^{-1}$$
$$= \frac{2}{\cancel{3}} \frac{(3-e^t)^{-1}}{\cancel{3}}$$
$$= 2(3-e^t)^{-1}$$
$$\boxed{M_X(t) = 2(3-e^t)^{-1}}$$

First moment about origin :-

$$M_1 = E[x] = \frac{\partial}{\partial t}[M_X(t)]\Big|_{t=0}$$
$$= \frac{\partial}{\partial t}\left(\frac{2}{3-e^t}\right)\Big|_{t=0}$$
$$= \frac{3-e^t(0) - 2(0-e^t)}{(3-e^t)^2}\Big|_{t=0}$$
$$= \frac{2e^t}{(3-e^t)^2}\Big|_{t=0}$$
$$= \frac{2e^0}{(3-e^0)^2} = \frac{2}{4} = 1/4$$

Second moment about origin :-

$$M_2 = E[X^2] = \frac{\partial^2}{\partial t^2}[M_X(t)]\Big|_{t=0}$$

$$= \frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(M_X(t))\right]\Big|_{t=0}$$

$$= \frac{\partial}{\partial t}\left(\frac{2e^t}{(3-e^t)^2}\right)\Big|_{t=0}$$

$$= \frac{(3-e^t)^2\, 2e^t - 2e^t \cdot 2\,(3-e^t)(-e^t)}{(3-e^t)^4}\Big|_{t=0}$$

$$= \frac{(3-1)^2\, 2e^0 - 2e^0 \cdot 2\,(3-e^0)(-e^0)}{(3-e^0)^4}$$

$$= \frac{4 - 4\times 2\,(-1)}{2^4}$$

$$= \frac{16}{16} = 1$$

$$\boxed{M_2 = E(x^2) = 1}$$

(34) Find the charecteristic function of r.v. "X" having the density function

$$f_X(x) = \frac{1}{2a} \;;\; |x| < a$$
$$= 0 \;;\; \text{elsewere}$$

Sol: Given that

$$f_X(x) = \frac{1}{2a} \;;\; |x| < a$$
$$= 0 \;;\; \text{otherwise}$$

$$f_X(x) = \frac{1}{2a} \;;\; -a < x < a$$
$$= 0 \;;\; \text{elsewhere}$$

The charecteristic function of 'x' is

$$\phi_X(\omega) = E(e^{j\omega x})$$
$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$$

$$= \int_{-a}^{a} e^{j\omega x} \cdot \frac{1}{2a}\,dx$$

$$= \frac{1}{2a}\int_{-a}^{a} e^{j\omega x}\,dx$$

$$= \frac{1}{2a}\left(\frac{e^{j\omega x}}{\omega j}\right)_{-a}^{a}$$

$$= \frac{1}{2a\,j\omega}\left[e^{j\omega a} - e^{-j\omega a}\right]$$

$$= \frac{1}{a\omega}\left[\frac{e^{j\omega a} - e^{-j\omega a}}{2j}\right]$$

$$= \frac{1}{a\omega}\sin \omega a$$

$$\phi_X(\omega) = \frac{\sin(\omega a)}{\omega a}$$

(35) Find the charecteristic function of for $f_X(x) = e^{-|x|}$.

Sol: The density function of "x" is $f_X(x) = e^{-|x|}$

$$\phi_X(\omega) = E(e^{j\omega x})$$
$$= \int_{-\infty}^{\infty} e^{j\omega x} \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{0} e^{j\omega x} f(x)\,dx + \int_{0}^{\infty} e^{j\omega x} \cdot f_X(x)\,dx$$

$$= \int_{-\infty}^{0} e^{j\omega x} \cdot e^{-(-x)}\,dx + \int_{0}^{\infty} e^{j\omega x}\, e^{-(x)}\,dx$$

$$= \int_{-\infty}^{0} e^{(j\omega+1)x}\,dx + \int_{0}^{\infty} e^{(j\omega-1)x}\,dx$$

$$= \left(\frac{e^{x(j\omega+1)}}{j\omega+1}\right)_{-\infty}^{0} + \left(\frac{e^{(j\omega-1)x}}{j\omega-1}\right)_{0}^{\infty}$$

$$= \left[\frac{e^0}{j\omega+1} + \frac{e^{\infty} - e^0}{(j\omega-1)}\right] = \frac{1}{j\omega+1} - \frac{1}{j\omega-1}$$

$$\frac{1}{(1+jw)} + \frac{1}{(1-jw)}$$

$$= \frac{1-\cancel{jw}+1+\cancel{jw}}{1+w^2} = \frac{2}{1+w^2}$$

$$\boxed{\phi_X(w) = \frac{2}{1+w^2}}$$

(36) The charecterstic function of r.v is $f_X(x) = a e^{-bx}$; $x \geq 0$. find the charecterstic function and first two moments

Sol:- Given $f_X(x) = a e^{-bx}$; $x \geq 0$

$= 0$; $x < 0$

The cherecterstic function of 'x' is

$$\phi_X(x) = E(e^{jwx})$$

$$= \int_{-\infty}^{\infty} e^{jwx} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{jwx}\, a e^{-bx}\, dx$$

$$= a \int_0^{\infty} e^{(jw-b)x}\, dx$$

$$= a \int_0^{\infty} e^{-(b-jw)x}\, dx$$

$$= a\left[\frac{e^{-(b-jw)x}}{-(b-jw)}\right]_0^{\infty}$$

$$= a\left[0 + \frac{1}{(b-jw)}\right]$$

$$= \frac{a}{b-jw}$$

$$\therefore \boxed{\phi_X(w) = \frac{a}{b-jw}}$$

First moment about origin :-

the n^{th} moment about origin from cherecterstic functio

is $m_n = E(x^n) = \left(\frac{1}{j}\right)^n \frac{\partial^n}{\partial w^n}(\phi_X(w))\Big|_{w=0}$

$$m_1 = E(x) = \frac{1}{j} \cdot \frac{\partial}{\partial w}(\phi_X(w))\Big|_{w=0}$$

$$= \frac{1}{j} \frac{b-jw(0)-a(0-j)}{(b-jw)^2}\Bigg|_{w=0}$$

$$= \frac{1}{j} \cdot \frac{0+aj}{(b-jw)^2}\Bigg|_{w=0}$$

$$= \frac{a}{(b-jw)^2}\Bigg|_{w=0}$$

Second moment about origin :-

$$m_2 = E[x^2] = \left(\frac{1}{j}\right)^2 \frac{\partial^2}{\partial w^2}\left[\phi_X(w)\right]\Big|_{w=0}$$

$$= \frac{1}{j^2} \cdot \frac{\partial}{\partial w}\left[\frac{\partial}{\partial w}\left[\phi_X(w)\right]\right]\Bigg|_{w=0}$$

$$= \frac{1}{j^2} \frac{\partial}{\partial w}\left(\frac{a}{(b-jw)^2}\right)\Bigg|_{w=0}$$

$$= \frac{1}{j^2}\left[\frac{0+2abj(b-jw)}{(b-jw)^4}\right]\Bigg|_{w=0}$$

$$= \frac{2ab}{b^4} = \frac{2a}{b^3}$$

(37) Let us consider $f_X(x)$ is a density function of 'x' then find density function of $Y = aX+b$

Sol:- Given $Y = aX+b$

Here "x" is a random variable with density function. $f_X(x)$

Here the transformation is monotonic transformation

W.K.T For monotonic transformation, (3.62)

$$f_Y(y) = f_X(x) \left|\frac{\partial x}{\partial y}\right|$$

Here $x = T^{-1}(y) \rightarrow ①$

$y = ax+b$

$ax+b = y \quad = \quad ax+b=y$

$ax = y-b$

$x = \left(\frac{y-b}{a}\right)$

$x = T^{-1}(y) = \frac{\partial}{\partial y}\left(\frac{y-b}{a}\right) = 1/a$

$\frac{\partial x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y-b}{a}\right) = \frac{1}{|a|}$

substitute "x" value and $\left|\frac{\partial x}{\partial y}\right|$ in eq①

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{|a|}$$

This is the required density function of "y"

(38) Let "x" be a continuous random variable with probability density function $f_X(x) = \frac{x}{12}$; $1 < x < 12$ find the density function of $y = 2x-3$

$= 0$; elsewere

Sol:- given $Y = 2x-3$

Here "x" is a r.v with density function

$f_X(x) = \frac{x}{12}$; $1<x<5$

$= 0$; other wise

The limits of "Y" are if $x=1 \Rightarrow y = 2-3 = -1$

If $x=5 \Rightarrow y = 10-3 = 7$

∴ The limits of Y are $-1<y<7$

Here the transformation is monotonic transformation

For monotonic transformation

$$f_Y(y) = f_X(x)\left|\frac{\partial x}{\partial y}\right|$$

Here ∴ $x = T^{-1}(y) \rightarrow ①$

$y = 2x-3$

$y+3 = 2x$

$x = \frac{y+3}{2} = T^{-1}(y)$

$\frac{\partial x}{\partial y} = 1/2 = \frac{1}{|2|}$

∴ $f_X(x) = f_X\left(\frac{y+3}{2}\right) = \frac{\frac{y+3}{2}}{12} = \frac{y+3}{24}$

sub. x and $\left|\frac{\partial x}{\partial y}\right|$ in eq①

$f_Y(y) = \frac{y+3}{24} \times \frac{1}{|2|} = \frac{y+3}{48}$; $-1<x<7$

$= 0$; elsewere.

(39) given a r.v. having the density function $f_X(x) = 2x$; $0<x<1$ find the density function of $y = 8x^3$ $= 0$; otherwise

Sol:- given $y = 8x^3$

Here x is a r.v with density function $f_X(x) = 2x$; $0<x<1$

$= 0$; elsewere

The limits of "y" are if $x=0$ then $y=0$

If $x=1$ then $y=8$

∴ the limits of y are $0<y<8$

Here this is monotonic transformation

$f_Y(y) = f_X(x)\cdot\left|\frac{\partial x}{\partial y}\right| \rightarrow ①$

$x = T^{-1}(y)$

$y = 8x^3$

$x = \frac{y^{1/3}}{8}$

3.6

$x = \left(\frac{y}{2}\right)^3 = \frac{\sqrt[3]{y}}{2} = \frac{y^{1/3}}{2}$

$\frac{dx}{dy} = \frac{\partial}{\partial y}\left(\frac{y^{1/3}}{2}\right) = \frac{1}{3} y^{1/3-1} = \frac{1}{3} y^{-2/3} = \frac{1}{6 \cdot \sqrt[3]{y^2}}$

$f_X(x) = f_X\left(\frac{y^{1/3}}{2}\right) = \frac{2 \cdot y^{1/3}}{2} = y^{1/3} = \sqrt[3]{y}$

Sub "x" and $\left|\frac{dx}{dy}\right|$ in eq ①

$$\boxed{\begin{aligned} f_Y(y) &= \frac{1}{6}\frac{1}{\sqrt[3]{y}} ; \quad 0 < y < 8 \\ &= 0 \quad ; \text{ otherwise} \end{aligned}}$$

(40) If "x" is a normal r.v with "0" mean and varience σ^2 $(N(0,\sigma^2))$ then find the density function of $y = e^x$

Sol: $Y = e^x$

Here "x" is a gaussian random variable with mean $a_x = 0$.

and varience $\sigma_x^2 = \sigma^2$

$\therefore f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-a_x)^2/2\sigma_x^2}$

$= a_x = 0$;

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma_x^2}$

Here the transformation is monotonic transfer function for monotonic transformations

$f_Y(y) = f_X(x)\left|\frac{\partial x}{\partial y}\right| \rightarrow ①$

$y = e^x \Rightarrow e^x = y \Rightarrow x = \log y$

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\log y)/2\sigma^2}$

$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\log y)^2/2\sigma^2} \frac{1}{|y|}$

$= \frac{1}{|y|\sqrt{2\pi\sigma^2}} e^{-(\log y)^2/2\sigma^2}$

(41) Let $Y = ax+b$. Show that if $X = N(\mu,\sigma^2)$ then $Y = N(a\mu+b, \sigma^2 a^2)$

Sol: Given that

$Y = ax+b$.

Here "x" is a r.v with $N(\mu,\sigma^2)$

$\therefore f_X(x) = N(\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$

Here It is a monotonic transferfunction for monotonic transfer function

$f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right| \rightarrow ①$

$ax+b = y$

$x = \frac{y-b}{a}$

$\frac{\partial x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y-b}{a}\right) = 1/a \quad \left|\frac{\partial x}{\partial y}\right| = \frac{1}{|a|}$

$f_X(x) = f_X\left(\frac{y-b}{a}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\left(\frac{y-b}{a}\right)^2/2\sigma^2}$

$f_Y(y) = f_Y = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{\left(\frac{y-b}{a}\right)^2-\mu^2}{2\sigma^2}}$

$= \frac{1}{|a|\sqrt{2\pi\sigma^2}} e^{-\frac{(y-b/a)^2-\mu^2}{2\sigma^2}}$

$= \frac{1}{|a|\sqrt{2\pi\sigma^2}} e^{-\frac{(y-(b+a\mu))^2}{2\sigma^2 a^2}}$

$= N(b+a\mu, \sigma^2 a^2)$

(42) A random variable "x" is uniformly distributed in the interval (-5, 5). another r.v. $Y = e^{-X/5}$ is formed find $E(Y)$ and $f_Y(y)$. (3.64)

Sol: Given the r.v is "x" is uniformly distributed over the interval $(-5, 15)$

w.k.t the density function of uniform distribution is

$$f_X(x) = \frac{1}{b-a} \; ; \; a \le x \le b$$
$$= 0 \; ; \text{ otherwise}$$

The density function of given r.v "x" is $= \frac{1}{15+5} = \frac{1}{20}$

$\therefore$ mean of $Y = E(Y) = E(e^{-X/5})$

$$= \int_{-\infty}^{\infty} e^{-X/5} f_X(x)\, dx$$
$$= \int_{-5}^{15} e^{-X/5} \frac{1}{20}\, dx$$
$$= \frac{1}{20} \int_{-5}^{15} e^{-X/5}\, dx$$
$$= \frac{1}{20} \left(\frac{e^{-X/5}}{-1/5} \right)_{-5}^{15}$$
$$= \frac{1}{20} \left(\frac{e^{-3}}{-1/5} - \frac{e^{1}}{-1/5} \right)$$
$$= \frac{1}{20} (5 e^{-3} + 5 e^{1})$$
$$= \frac{1}{4} (e^{-3} + e)$$
$$= 0.667$$

density function of y is $y = e^{-X/5}$

The limits of 'y' are if $x = -5$; $y = e = 2.718$

if $x = 15$; $y = e^{-3} = 0.049$

$\therefore$ The limits of y are $0.0049 \le y \le 2.718$.

Here the transformation is monotonic transformation. for monotonic transformation

$$f_Y(y) = f_X(x) \cdot \left| \frac{\partial x}{\partial y} \right|$$
$$y = e^{-X/5}$$
$$e^{-X/5} = y$$
$$-X/5 = \ln(y)$$
$$X = -5 \ln(y)$$
$$\frac{\partial x}{\partial y} = \frac{-5}{y}$$
$$\left| \frac{\partial x}{\partial y} \right| = \frac{5}{|y|}$$

$$\therefore f_X(x) = f_X(-5 \log y)$$
$$= \frac{1}{20}$$

$$\therefore f_Y(y) = \frac{1}{20} \times \frac{5}{|y|} = \frac{1}{4|y|} \; ; 0.049 \le y \le 2.718$$
$$= 0 \; ; \text{ otherwise}$$

(43) It is given that the r.v "x" is a gaussian with mean of "zero", variance of "1". The r.v. "y" is obtained from "x" with the relation $y = 5x - 6$. find the PDF of "y."

Sol: Given $y = 5x - 6$

Here "x" is a gaussian random variable with mean $a_X = 0$ and variance $\sigma^2 = 1$,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_X)^2/2\sigma_X^2}$$

$$a_X = 0 \; ; \; \sigma_X^2 = 1, \quad = \frac{1}{\quad} e^{-x^2/2}$$

Here the transformation is monotonic transformation for monotonic

$$f_Y(y) = f_X(x)\left|\frac{\partial x}{\partial y}\right| \qquad (3.65)$$

$$y = 5x-6$$

$$y+6 = 5x$$

$$x = \frac{y+6}{5}$$

$$\frac{\partial x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y+6}{5}\right) = 1/5 \quad ; \left|\frac{\partial x}{\partial y}\right| = \frac{1}{|5|}$$

$$f_X(x) = f_X\left(\frac{y+6}{5}\right) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\left(\frac{y+6}{5}\right)^2/2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(y+6)^2/50}$$

$$\therefore f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(y+6)^2}{50}} \left|\frac{1}{5}\right|$$

$$= \frac{1}{|5|\sqrt{2\pi}} e^{-(y+6)^2/50}$$

(SP) 44. A r.v "X" undergoes the transformation $Y = \frac{a}{X}$, where a is a real number. Find the density function of "Y"

Sol: Given that $Y = \frac{a}{X}$

Here 'X' a random variable with dencity function $f_X(x)$

$$y = a/x$$

$$xy = a$$

$$x = a/y$$

$$\frac{\partial x}{\partial y} = -a/y^2 \qquad \left|\frac{\partial x}{\partial y}\right| = \left|\frac{-a}{y^2}\right| = \left|\frac{a}{y^2}\right|$$

$$\therefore \boxed{f_Y(y) = \frac{a}{y^2} f_X(a/y)}$$

45. The gaussian random variable x having a mean ... and varience "1" transformed to another random variable 'Y' by a square law transformation. find the density function 'Y'

(or)

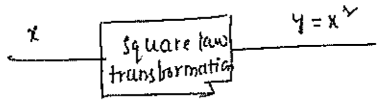
Sol: $Y = X^2$ then find PDF of 'Y' If $X = N(0,1)$

Given "X" is a r.v. with "0" mean and varience '1'

ie $a_X = 0$; $\sigma_X^2 = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi(1)}} e^{-x^2/2}$$

the r.v 'Y' is formed by the square law transformation of 'X' is



Here the transformation is non monotonic transformation for non monotonic transformation

$$f_Y(y) = \sum_{\text{all } x} f_X(x_n) \frac{\partial x_n}{\partial y}$$

Here $x_n = T^{-1}(y_n)$

$$x^2 = y$$

$$x = \sqrt{y} = \pm\sqrt{y}$$

$$x_1 = \sqrt{y}, \quad x_2 = -\sqrt{y},$$

$$\frac{\partial x_1}{\partial y} = \frac{1}{2\sqrt{y}} \; ; \quad \frac{\partial x_2}{\partial y} = \frac{-1}{2\sqrt{y}}$$

$$\left|\frac{\partial x_1}{\partial y}\right| = \frac{1}{|2\sqrt{y}|} \quad ; \quad \left|\frac{\partial x_2}{\partial y}\right| = \frac{1}{|2\sqrt{y}|}$$

$$f_X(x_1) = f_X(\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{y})^2/2} = \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$f_X(x_2) = f_X(-\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-(-\sqrt{y})^2/2} = \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$\therefore \quad f_Y(y) = \sum_{x=1}^{2} f_X(x_n) \cdot \left|\frac{\partial x_n}{\partial y}\right|$$

(3.66)

$$= f(x_1) \cdot \left|\frac{\partial x_1}{\partial y}\right| + f(x_2) \cdot \left|\frac{\partial x_2}{\partial y}\right|$$

$$= \frac{1}{2\sqrt{y}} \times \frac{1}{\sqrt{2\pi}} e^{-y/2} + \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} e^{-y/2}$$

$$= \frac{\cancel{2}}{\sqrt{2\pi}} \cdot e^{-y/2} \frac{1}{2\sqrt{y}}$$

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}}$$

(46) A r.v "X" uniformly distributed in the interval $(-\pi/2, \pi/2)$. X is transformed into new r.v. $Y = T(X) = a \tan x$. where $a > 0$, find the probability density function of 'Y'

Sol: Given. $Y = a \tan x$.

Here "X" is a r.v. uniformly distributed over the interval $(-\pi/2, \pi/2)$

$$f_X(x) = \frac{1}{\pi/2 + \pi/2} = \frac{1}{\pi} \quad ; -\pi \le x \le \pi/2$$

$$= 0 \quad ; \text{ otherwise}$$

The range of 'y' are If $x = -\pi/2$ then $y = a \tan(-\pi/2) = -\infty$

If $x = \pi/2$ then $y = a \tan(\pi/2) = \infty$

Here the transformation is monotonic transformation

NOTE: (All the trigonometric functions are monotonic with in a particular interval otherwise it is not a monotonic)

For monotonic transformation

$$y = a \tan(x)$$

$$y = a \tan(x)$$

$$\frac{y}{a} = \tan x$$

$$x = \tan^{-1}(y/a)$$

$$\left|\frac{\partial x}{\partial y}\right| = \left(\frac{a}{a^2 + y^2}\right).$$

$$f_X(x) = f_X\left[\tan^{-1}(y/a)\right] = 1/\pi$$

$$f_Y(y) = f_X(x) \cdot \frac{\partial x}{\partial y}$$

$$= \frac{1}{\pi} \frac{a}{a^2 + y^2} = \frac{a}{\pi(a^2 + y^2)} \quad ; -\infty \le y < \infty$$

(47) Let us consider the square law transmission $y = cx^2$, then find the density function of 'y'

Sol: Given $y = cx^2$

let us consider here 'x' is a random variable with density function of $f_X(x)$

Here the transmission is non monotonic transmission

$$f_Y(y) = \sum_{\text{all } x} f_X(x_n) \cdot \left|\frac{\partial x_n}{\partial y}\right|$$

$$y = cx^2$$

$$x^2 = y/c$$

$$x = \pm\sqrt{y/c}$$

$$x_1 = \pm\sqrt{y/c} \quad ; \quad x_2 = -\sqrt{y/c}$$

$$\frac{\partial x_1}{\partial y} = \frac{1}{2\sqrt{y/c}} (1/c) = \frac{1}{2\sqrt{yc}} \qquad \frac{dx_2}{dy} = \frac{-1}{\sqrt{2(y/c)}} \left(\frac{1}{c}\right) = \frac{-1}{2\sqrt{yc}}$$

$$\left|\frac{\partial x_1}{\partial y}\right| = \frac{1}{2\sqrt{yc}} \;;\; \left|\frac{\partial x_2}{\partial y}\right| = \frac{1}{2\sqrt{yc}}$$ (3.67)

$$f_Y(y) = \sum_{n=1}^{2} f_X(x_n)\left(\frac{\partial x_n}{\partial y}\right)$$

$$= f_X(x_1)\left(\frac{\partial x_1}{\partial y}\right) + f_X(x_2)\left(\frac{\partial x_2}{\partial y}\right)$$

$$= f_X(\sqrt{y/c})\cdot\frac{1}{2\sqrt{yc}} + f_X(-\sqrt{y/c})\cdot\frac{1}{2\sqrt{yc}}$$

$$= \frac{1}{2\sqrt{yc}}\left[f_X(\sqrt{y/c}) + f_X(-\sqrt{y/c})\right]$$

(48) A random variable "X" is uniformly distributed in the interval $(-a, a)$. It is transmitted to a new r.v "Y" by the transformation $Y = cX^2$. find the density function of y and sketch?

Sol: Given the two the new r.v. $Y = cX^2$

Here X is a r.v uniformly selected distributed over the interval $(-a, a)$

$$\therefore f_X(x) = \frac{1}{2a} \;;\; -a \le x < a$$
$$= 0 \;;\; \text{elseware}$$

The range of y are If $x = -a$, then $y = ca^2$

If $x = a$, then $y = ca^2$

Here only one interval is existing for finding of other interval. let us consider the "X" value is equals to the average value of given interval

$$\therefore \text{If } x = \frac{-a+a}{2} = 0 \Rightarrow y = c(0) = 0.$$

$\therefore$ the range of 'Y' is $0 \le y \le ca^2$

$y = cx^2$

$x^2 = y/c$

$x = \pm\sqrt{y/c} \Rightarrow x_1 = \sqrt{y/c} \;;\; x_2 = -\sqrt{y/c}$.

$$\frac{dx_1}{dy} = \frac{1}{2\sqrt{yc}} \;;\; \frac{dx_2}{dy} = \frac{-1}{2\sqrt{yc}} \;;\; \left|\frac{dx_1}{dy}\right| = \frac{1}{2\sqrt{yc}} \;;\; \left|\frac{\partial x_2}{\partial y}\right| = \frac{1}{2\sqrt{yc}}$$

$$f_Y(y) = \sum_{i=1}^{2} f_X(x_n)\left|\left(\frac{dx_n}{dy}\right)\right|$$

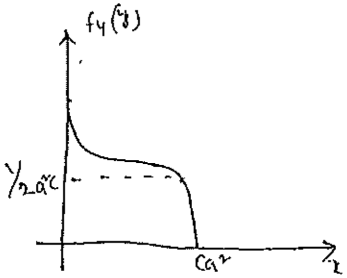
$$= f_X(x_1)\frac{dx_1}{dy} + f_X(x_2)\frac{dx_2}{dy}$$

$$= \frac{1}{2\sqrt{yc}}\left\{ f_X\frac{dx_1}{dy} + f_X(x_2) - \sqrt{y/c}\right\}$$

$$= \frac{1}{2\sqrt{yc}}\left(\frac{1}{2a} + \frac{1}{2a}\right)$$

$$= \frac{2}{2a}\cdot\frac{1}{2\sqrt{yc}}$$

$$f_Y(y) = \frac{1}{2a\sqrt{yc}} \;;\; 0 \le y < ca^2$$
$$= 0 \;;\; \text{elsewhere}$$



(49) A r.v X is uniformly distributed on $(0,6)$. If "X" is transformed to a new r.v $Y = 2(X-3)^2 - 4$, find the density function of 'y', $\bar{Y}$ and σ_Y^2

Sol: Given $Y = 2(X-3)^2 - 4$

Here X is a r.v uniformly distributed over the interval $(0,6)$

$$f_X(x) = 1/6 \;;\; 0 \le x \le 6$$
$$= 0 \;;\; \text{otherwise}$$

The range of "y" is. if $x = 0 \Rightarrow y = 2(-3)^2 - 4 = 14$

If $x = 6 \Rightarrow y = 2(3-3)^2 - 4 = -4$

$$= \frac{1}{6\sqrt{2}} \int_{-4}^{14} \frac{y}{\sqrt{y+4}}\, dy$$

(5.68)

$$= 2$$

$$\overline{y^2} = \int_{-7}^{7} y^2 f_y(y)\, dy$$

$$= \frac{1}{6\sqrt{2}} \int_{-4}^{14} \frac{y^2}{\sqrt{y+4}}\, dy$$

$$= 32.8$$

$$\sigma_x^2 = 32.8 - 4$$

$$\boxed{\sigma_y^2 = 28.8}$$

(50) The charecterstic function for a gaussian r.v "X" having a mean value of "0" is $\phi_x(\omega) = e^{-\frac{\sigma_x^2 \omega^2}{2}}$ find all moments of "x" using $\phi_x(\omega)$

Sol:- Given $\phi_x(\omega) = e^{-\sigma_x^2 \omega^2/2}$

Here r.v "x" is a gaussian r.v with mean $a_x = 0$.

W.K.T $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$e^{-\sigma_x^2 \omega^2/2} = \sum_{k=0}^{\infty} \frac{\left(\frac{-\sigma_x^2 \omega^2}{2}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \sigma_x^{2k} \cdot \omega^{2k}}{2^k \cdot k!} \longrightarrow (1)$$

From the definition of charecterstic function of $\phi_x(\omega) =$

$E(e^{j\omega x})$ Here x is G.r.v. ie continuous

$\therefore$ The range of "y" is "$-4 < y < 14$"

given. $y = 2((x-3)^2 - 4)$

$$y+4 = 2(x-3)^2$$

$$(x-3)^2 = \left(\frac{y+4}{2}\right)$$

$$(x-3) = \left(\frac{y+4}{2}\right)^{1/2}$$

$$x = \sqrt{\frac{y+4}{2}} + 3$$

If $x_1 = \sqrt{\frac{y+4}{2}}$, $x_2 = 3 - \sqrt{\frac{y+4}{2}}$

$$\frac{dx_1}{dy} = 0 + \tfrac{1}{2}\sqrt{\frac{y+4}{2}} = \frac{1}{\sqrt{y+4}}$$

$$\frac{dx_2}{dy} = \frac{-1}{4\sqrt{\frac{y+4}{4}}} = \frac{-1}{\sqrt{y+4}} = \frac{1}{2\sqrt{2(y+4)}}$$

$$f_x(x_1) = f_x\left(3 + \sqrt{\frac{y+4}{2}}\right) = 1/6$$

$$f_x(x_2) = f_x\left(3 - \sqrt{\frac{y+4}{2}}\right) = 1/6$$

$$\therefore f_y(y) = \sum_{n=1}^{2} f_x(x_n)\left(\frac{dx_n}{dy}\right)$$

$$= f_x(x_1)\cdot\frac{\partial x_1}{\partial y} + f_x(x_2)\cdot\frac{\partial x_2}{\partial y}$$

$$= \frac{1}{6} \times \frac{1}{2\sqrt{2(y+4)}} + \frac{1}{6} \times \frac{1}{2\sqrt{2(y+4)}}$$

$$f_y(y) = \frac{1}{6\sqrt{2(y+4)}} \;;\; -4 \le y \le 14$$

$$= 0 \;;\; \text{otheswise}$$

mean of $y = E(y) = \int_{-\infty}^{\infty} y \cdot f_y(y)\, dy$

$$= \frac{(-1)^{n/2}}{j^n} \cdot \frac{n!}{(n/2)!} \frac{\sigma_x^n}{2^{n/2}}$$ (3.70)

The all moments of r.v "X" are $m_n = 0$ if $n = $ odd

$$m_n = \frac{(-1)^{n/2}}{j^n} \cdot \frac{n!}{(n/2)!} \frac{\sigma_x^n}{2^{n/2}} \; ; \; n = \text{even}$$

Here for $n =$ even. $(-1)^{n/2} = j^n$

$m_n = 0$, $n =$ odd.

$$= \frac{n!}{(n/2)!} \cdot \frac{\sigma_x^n}{2^{n/2}} \; ; \; n \text{ is even}$$

(51) Let us consider the moment generating function of a r.v. with "0" mean is $M_x(t) = e^{\sigma_x^2 t^2/2}$ find the all moments about origin from its moment generating function!

Sol:
Given $M_x(t) = e^{\sigma_x^2 t^2/2}$

Here random variable "x" is a gaussian r.v. with mean $a_x = 0$

W.K.T $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$e^{\sigma_x^2 t^2/2} = \sum_{k=0}^{\infty} \frac{\left(\frac{\sigma_x^2 t^2}{2}\right)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{\sigma_x^{2k} t^{2k}}{2^k k!}$$

From the definition of characteristic function of $\phi_x(\omega)$

i.e $\phi_x(\omega) = E(e^{j\omega x})$ Here x is g.r.v. is continuous

$$\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(e^{j\omega x})^n}{n!} f_x(x)\,dx \rightarrow (1)$$

$$\phi_x(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_x(x)\,dx = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(e^{j\omega x})^n}{n!} f_x(x)\,dx$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} x^n \cdot f_x(x)\,dx = \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \int_{-\infty}^{\infty} x^n f_x(x)\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} m_n$$

$$= \phi_x(\omega) = \sum_{n=0}^{\infty} \frac{(j^n) \cdot \omega^n m_n}{n!} \rightarrow (2)$$

equate (1) and (2), we get

$$\sum_{n=0}^{\infty} \frac{(j)^n \omega^n m_n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot \sigma_x^{2k} \cdot \omega^{2k}}{2^k k!} \rightarrow (3)$$

$$\frac{(j)^0 m_0 \omega^0}{0!} + \frac{(j)^1 m_1 \omega}{1!} + \cdots = \frac{(-1)^0 \sigma_x^0 \omega^0}{0!} + \frac{(-1)^1 \sigma_x^2 \omega^2}{2!} + \cdots$$

$$m_0 + \frac{j m_1 \omega}{1!} + \frac{j^2 m_2}{2!} \omega^2 + \cdots = 1 + \frac{(-1)^1 \sigma_x^2}{2!} \omega^2$$

For $n =$ odd $m_n = 0$.

For n is even $\Rightarrow$ n is $2k$

$k = n/2$

From eq (3). $$\sum_{n=0}^{\infty} \frac{j^n \omega^n m_n}{n!} = \sum_{N=0}^{\infty} \frac{(-1)^{n/2} \sigma_x^n \omega^n}{2^{n/2} (n/2)!}$$

By for solving of m_n. we will neglect the summation

$$\frac{j^n \omega^n m_n}{n!} = \frac{(-1)^{n/2} \sigma_x^n \omega^n}{2^{n/2} (n/2)!}$$

$$m_n = \frac{(-1)^{n/2} \cancel{\omega^n} \cdot \sigma_x^n}{} \times \frac{n!}{}$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(jt)^n}{n!} x^n f_x(x)\, dx \qquad (3.71)$$

$$= \sum_{n=0}^{\infty} \frac{(jt)^n}{n!} \int_{-\infty}^{\infty} x^n f_x(x)\, dx \qquad \left[\because m_n = E(x^n) = \int_{-\infty}^{\infty} x^n f_x(x)\, dx\right]$$

$$\phi_x(\omega) = \sum_{n=0}^{\infty} \frac{(jt)^n}{n!} m_n$$

$$\phi_x(\omega) = \sum_{n=0}^{\infty} \frac{j^n t^n m_n}{n!} \rightarrow ②$$

from ① and ②

$$\sum_{n=0}^{\infty} \frac{(j)^n t^n m_n}{n!} = \sum_{k=0}^{\infty} \frac{\sigma_x^{2k} t^{2k}}{2^k k!} \rightarrow ③$$

$$= \frac{(j)^0 m_0 t^0}{0!} + \frac{(j)^1 m_1 t^1}{1!} + \frac{(j)^2 m_2 t^2}{2!} + \cdots$$

$$= \frac{\sigma_x^0 t^0}{0!} + \frac{\sigma_x^2 t^2}{2 \cdot 1!} + \frac{\sigma_x^4 t^4}{2^2 \cdot 2!} + \cdots$$

$$= m_0 + \frac{j m_1}{1!} t^1 + \frac{j^2 m_2}{2!} t^2 + \frac{j^3 m_3}{3!} t^3 + \cdots \longrightarrow ④$$

$$= 1 + \frac{\sigma_x^2}{2 \cdot 1!} t^2 + \frac{\sigma_x^4}{2^2 \cdot 2!} t^4 + \cdots$$

for n = odd. $m_n = 0$. ($\because$ from eq ④)

for n is even. then n is $2k$

$$\boxed{k = n/2}$$

$\therefore$ from eq ③

$$\sum_{n=0}^{\infty} \frac{j^n t^n m_n}{n!} = \sum_{n=0}^{\infty} \frac{(1)^{n/2} \sigma_x^n t^n}{2^{n/2} (n/2)!}$$

for solving of m_n. 't' will neglect the summations

$$\frac{j^n t^n m_n}{n!} = \frac{(1)^{n/2} \sigma_x^n t^n}{2^{n/2} (n/2)!}$$

$$\therefore m_n = \frac{(1)^{n/2} t^n \sigma_x^n}{\ldots} \times \frac{n!}{\ldots}$$

$$m_n = \frac{(1)^{n/2}}{j^n} \cdot \frac{n!}{(n/2)!} \frac{\sigma_x^n}{2^{n/2}}$$

The all moments of r.v "x" are $m_n = 0$; n = odd.

$$m_n = \frac{(1)^{n/2}}{j^n} \frac{n!}{(n/2)!} \frac{\sigma_x^n}{2^{n/2}} \quad ; n = \text{even}$$

Here for n = even $\frac{(-1)^{n/2}}{j^2} = j^n$

$$\boxed{\begin{aligned} m_n &= 0 \ ; \quad \text{for } n = \text{odd.} \\ &= \frac{n!}{(n/2)!} \frac{\sigma_x^n}{2^{n/2}} \ ; \quad n \text{ is even} \end{aligned}}$$

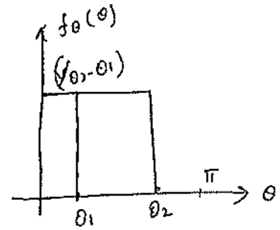
(52) A random variable θ is uniformly distributed over the interval (θ_1, θ_2) where θ_1 an θ_2 are real and satisfy $0 \le \theta_1 < \theta_2 < \pi$ Find and sketch the probability density function of the transformed r.v $y = \cos\theta$.

Sol:- given a random variable "θ" is uniformly distributed

$$\therefore f_\theta(\theta) = \frac{1}{\theta_2 - \theta_1}$$

$$f_\theta(\theta) = \frac{1}{\theta_2 - \theta_1} \ ; \quad \theta_1 < \theta < \theta_2$$

$$= 0 \ ; \text{ elsewhere}$$



$y = \cos\theta$

and also given that

$$y = \cos\theta$$

Here the r.v "θ" is varies then y can also varies

ie $y = y_2 < y < y_1$

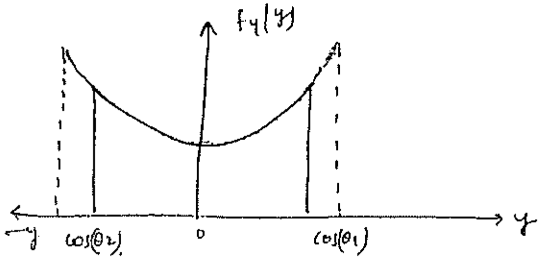
$$y = \cos\theta$$

$$\frac{d\theta}{dy} = \frac{d}{dy}(\cos^{-1} y)$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$$\therefore f_Y(y) = \frac{1}{\theta_2-\theta_1}\frac{1}{\sqrt{1-y^2}} \quad ; y_2 < y < y_1$$

$$= \frac{1}{\theta_2-\theta_1}\frac{1}{\sqrt{1-y^2}} \quad ; \cos(\theta_2) < y < \cos(\theta_1)$$



prepared by
M.V.M. REDDY sir

(53) Find the charecterstic function of the following probability function $f_X(x) = \frac{\lambda}{\pi(\lambda^2+x^2)}$

Sol: Given pdf is $\frac{\lambda}{\pi(\lambda^2+x^2)}$

w.k.t The charecterstic function is the fourier transform of the density function

i.e $\phi_X(\omega) = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$

$$= \int_{-\infty}^{\infty} \frac{\lambda}{\pi(\lambda^2+x^2)} e^{j\omega x}\, dx$$

w.k.t $e^{-\lambda|\omega|} \xleftrightarrow{FT} \frac{\lambda}{\pi(\lambda^2+x^2)}$

$$\therefore \boxed{\phi_X(\omega) = e^{-\lambda|\omega|}}$$