UNIT - 1

2217118 Probability Theory And Stochastic Process

→ Experiment:

An experiment is defined as the process which is conducted to get some results. If the same experiment is performed repeatedly under the same conditions, similar results are expected.

Examples: Tossing a coin, throwing a dice, firing a missile.

Experiment can be divided into two types. They are;

(i) Deterministic or Predictable Experiment

(ii) Random or UnPredictable Experiment

(i) Deterministic or Predictable Experiment:

Eg: (1) Ohm's law is a predictable experiment, because we are known that as voltage increases, current also increases and vice versa.

(II) "Throwing a stone upwards" because we are known that after sometime stone falls down due to gravitational force. Here, Results are known in advance

(ii) Random or Unpredictable Experiment:

An experiment whose results are not known in advance is called random or unpredictable experiment

Eg: (1) Tossing a coin, (2) Throwing a dice

Tossing a coin for getting head.

Throwing a dice for getting 5.

Set Theory :

→ **Set:** A set is a well-defined collection of elements.

A set is denoted by capital letters A, B, C, ... etc.

Elements are denoted by small letters a, b, c, etc.

x is an element of A is written as $x \in A$

x is not an element of A then it can be written as $x \notin A$.

→ **Set Representations:** There are two ways to represent a set.

(i) <u>Roster</u> or <u>Tabular form representation</u>:

In this form, all the elements of the set are listed and separated by commas. Elements are enclosed within the braces.

Eg: Set of all decimal digits such as

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

(ii) <u>Rule</u> or <u>Set builder form representation</u>:

In this method, a set is defined by specifying a common property possessed by the elements of the set, in general $X = \{x : P(x)\}$ or $X = \{x / P(x)\}$

Eg: $X = \{x : x = k^2$, where k is a natural number $\leq 5\}$

i, e., 1, 2, 3, 4, 5.

$$X = \{1^2, 2^2, 3^2, 4^2, 5^2\}$$

$$X = \{1, 4, 9, 16, 25\}$$

→ **Finite set:** A set with finite number of elements is called finite set.

Eg: Set of all decimal digits. Here no. of elements in set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} = 10$. Hence it is finite set.

→ Infinite Set: A set with infinite number of elements is called infinite set.

Eg: Set of all positive integers such as

$A = \{1, 2, 3, 4, 5, \ldots\ldots\}$

Here no. of elements in set A $= \infty$ (infinite). Hence it is called infinit set.

→ Null Set: A set which contains no elements is called null set, is represented by 'ϕ'. It is also known as empty set.

Eg: $A = \{x$ is a multiple of 2, x is odd$\} = \phi$

→ Singleton Set: A set containing only one element is called singleton set.

Eg: $A = \{1\}$

→ Sub-set: If every element of set A also belongs to set B, then A is a subset of B, denoted as $A \subseteq B$. I then A is not subset of B, then atleast one element of A doesnot belongs to B, and is written as $A \not\subseteq B$.

* Every set is its subset i.e. $A \subseteq A$.

* Null set is a subset of any set A.. i.e, $\phi \subseteq A$

→ Proper Sub-set: If atleast one element exists in B which is not in A, then A is proper subset of B, is written as $A \subset B$.

→ Universal Set: A set which contains all the elements of a given system is called universal set. It is largest set. It is represented by 'S'. All Sets are subsets of universal set. i.e $A \subseteq S$

* Algebra of Sets or Properties of Set theory:

1. Independent laws:

(i) $A \cup A = A$

(ii) $A \cap A = A$

2. Associative laws:

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

3. Commutative laws:

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \wedge A$

4. Distributive laws:

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. Identity laws:

(i) $A \cup S = S$

(ii) $A \cap S = A$

(iii) $A \cup \phi = A$

(iv) $A \cap \phi = \phi$

6. Complement laws:

(i) $A \cup \overline{A} = S$

(ii) $A \cap \overline{A} = \phi$

(iii) $\overline{(\overline{A})} = A$

(iv) $\overline{S} = \phi$

(v) $\overline{\phi} = S$

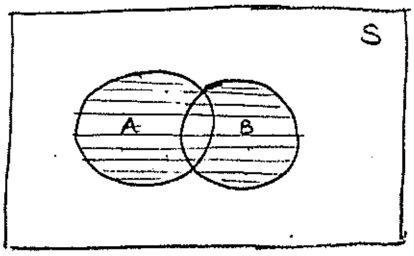
7. De Morgan's laws:

(i) $\overline{A \cup B} = \overline{A} \cap \overline{B}$

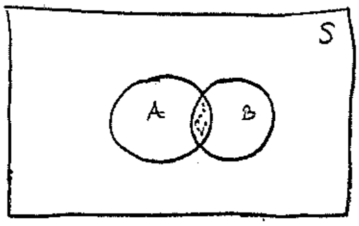
(ii) $\overline{A \cap B} = \overline{A} \cup \overline{B}$

* Set operations with Venn diagrams:

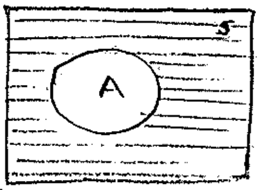
(i) Union operation : $A \cup B$



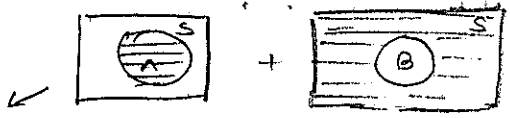
(ii) Intersection operation : $A \cap B$

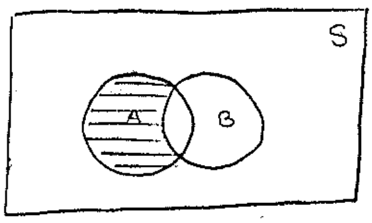


(iii) Complement : $\overline{A}$

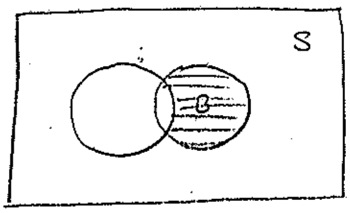


(iv) Difference : $A \sim B = A/B = A \cap \overline{B}$ A $\overline{B}$





$B \sim A = B/A = B \cap \overline{A}$



* Trail: The single performance of an experiment is called trial.

Eg: Tossing a coin, throwing a dice.

* Outcome: The end result of an experiment is called outcome.

***** Sample Space:

The set of all possible outcomes of a random experiment is called sample space.

Eg: 1. Set of all possible outcomes of a tossing coin. The possible outcomes are head or tail. The sample space of tossing coin $S = \{h, t\}$ h-head t-tail.

2. All possible outcomes of throwing a dice. The outcomes 1 or 2 or 3 or 4 or 5 or 6. That is the sample space of throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$.

* Sample space is represented by S.

Sample space is divided into 3 types,

i) Discrete and Finite Sample Space:

If a sample space of any experiment is finite set, then it is called discrete and finite sample space.

Eg: (1) In tossing a coin, sample space $S = \{h, t\}$

(2) In throwing a dice, sample space $S = \{1, 2, 3, 4, 5, 6\}$

(1) Here sample space having elements i.e. finite. Hence, it is discrete and finite sample space experiment.

(2) Sample space $= \{1, 2, 3, 4, 5, 6\}$. Here sample space having six elements i.e. finite set. Therefore It is discrete and finite sample spa

(ii) Discrete and Infinite sample space:

If a sample space of any random experiment is infinite set, then it is called discrete and infinite sample space.

Eg: 1) Consider an experiment, choose randomly a positive integer.

$S = \{1, 2, 3, 4, \ldots\}$

Here S is countably infinite. So, 'S' is discrete and infinite set.

2) Consider an experiment, choose even numbers from natural numbers.

$S = \{2, 4, 6, 8, \ldots\}$

Here S is infinite. So 'S' is discrete and infinite set

(iii) Continuous Sample Space:

If a sample space of the random experiment uncountably infinite set. Then it is called continuous sample space.

Eg: (1) Consider an experiment, choose randomly a rational number b/w 1 and 2 that means $1 \leq S \leq 2$

$S = \{1, 1.11, 1.12, \ldots 2\}$

Here S is uncountably infinite. So 'S' is continuous sample space.

(2) Consider an experiment measuring room temperature between t_1 and t_2 i.e. $S = \{t_1 \leq S \leq t_2\}$.

* Event: The particular outcome of a sample space of a random experiment is called event (or)

The expected subset of sample space is called an event.

Eg: Throwing a dice $S = \{1, 2, 3, 4, 5, 6\}$.

A_1 = An even number occurs = $\{2, 4, 6\}$

i.e. $A_1 \subseteq S$

→ Elementary Events: An elementary event is an event which cannot broken into further events.

Eg: (1) In tossing a coin, getting head, getting tail are elementary events.

(2) Throwing a dice, getting 1, 2, 3, 4, 5, 6 are all elementary events.

→ Compound Events: A compound event is an event which can be broken into further events.

Eg: In throwing a dice, the even number turns up. $S = \{2, 4, 6\}$. This is the combination of elementary events. by getting 2, 4, 6.

→ Impossible Events: The event 'ϕ' is called impossible event

Impossible event $= \phi = \{\}$

Eg: In tossing a coin, getting both head and tail in a single trial is an impossible event.

→ Sure or Certain Event: The sample space itself is an event is known as sure or certain event.

→ Exhaustive Events: The no. of outcomes possible in any trial of a random experiment is known as exhaustive event.

Eg: 1) In tossing a coin, there are two possible outcomes in any trial.

$\therefore$ No. of exhaustive events = 2.

2) Throwing a dice, there are 6 possible outcomes in any trial.

$\therefore$ No of exhaustive events = 6

3) Drawing a card from a pack of well-shuffled cards.

The total no of possible outcomes in any trial is 52

$\therefore$ No of exhaustive events = 52.

→ Equally Likely Events: A set of events of a random experiment are said to be equally likely if no one of them is expected to occur in preference to other in any single trial of random experiment.

Eg: 1) In tossing a coin, random experiment, the head and tail are equally likely events.

2) Consider an experiment of throwing an unbiased dice, then all the possible faces are equally likely events $\{1, 2, 3, 4, 5, 6\}$.

→ Favourable Events: The events which are favourable to a particular event of an random experiments are called favourable events.

Eg: When a dice is rolled "getting 2, 4, 6" are favourable events "getting an even number".

→ Success And Failures: All favourable events are success and remaining events are failures.

Eg: 1) In rolling a dice, getting 2,4,6 are (favourable) success to the event getting an even number and getting 1,3,5 are failures.

→ Mutually Exclusive Events: Set of events of random experiment are said to be mutually exclusive if the happening of one event prevents the happening of the other event.

Eg: 1) In tossing a coin, both head and tail cannot happen at the same time. Therefore head and tail are mutually exclusive events to each other.

2) In throwing a dice, all the six faces are mutually exclusive events to each other.

→ Independent Events: If two events are said to be independent events then happening or failure of one event doesnot effect the happening or failure of another event, other case is called dependent events.

Eg: Consider an exam. A, B, C and D are written exam "A passing in exam" doesnot effect the passing of B, C, and D, so the event of passing is independent.

→ Types of Events based on Sample Space:

1) The sample space is discrete and finite then the events are also discrete and finite.

2) The sample space is discrete and infinite then the events are discrete and finite or discrete and infinite.

Eg: (1) Consider a sample space of random experiment is all natural numbers, so samplespace is list of natural numbers. If the event be "chose a number 10" in the sample space. $S = \{1, 2, 3, 4, 5, \ldots\}$

Event $E_1 = \{10\}$.

$\therefore$ E_1 is discrete and finite event.

(2) Let E_2 be the event "choose positive odd numbers"

$\therefore$ $E_2 = \{1, 3, 5, 7, \ldots\}$

$\therefore$ E_2 is dicrete and infinite events.

3) The sample space is continuous space then the events are continuous or discrete and finite or discrete and infinite.

Eg: Consider sample space of random experiment is $0.5 \leq S \leq 8.5$.

Let E_1 be the event "choose a number 7" in a sample space.

$E_1 = \{\phi, 7\}$

$\therefore$ E_1 is discrete and finite.

Let E_2 be the event "choose a rational b/w 1.5 & 6.5" in the sample space.

$E_2 = \{1.5 \leq S \leq 6.5\}$

$\therefore$ E_2 is ~~discrete~~ ~~and~~ uncountably infinite, continuous events. Thus E_2 is continuous.

Let E_3 be the even where E_3 is $\{0.5001, 0.5002, 0.5003 \ldots\ldots\ldots\}$

Here E_3 is countably infinite. Thus event. E_3 is discrete and infinite event

Probability:

There are 3 approaches to understand probability

(i) Classical or mathematical or apriori probability

(ii) Relative frequency or a posteriori probability

(iii) Axiomatic approach probability.

(i) Classical or Mathematical or Apriori Probability:

If there are 'n' equally likely and mutually exclusive events of a random experiment out of which 'm' events are favourable for a particular event 'A', then we define probability of event A ~~defined~~ as

$$\text{Probability of event } A = P(A) = \frac{\text{no-of favourable outcomes w.r.t } A}{\text{total no. of possible outcomes of expt.}}$$

i.e., $\boxed{P(A) = \frac{m}{n}}$

Here 'm' results are favourable to A, 'n−m' results are unfavourable to A.

The set of unfavourable events of A denoted by $\bar{A}$ or A'.

$$P(A') = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

$$P(\bar{A}) = 1 - P(A)$$

$$\boxed{P(A) + P(\bar{A}) = 1}$$

Notes:

1) If probability of A is equal to zero then A is impossible event i.e., $P(A) = 0$

2) If $P(A) = 1$ then A is sure or certain event.

(ii). ~~Relative Frequency~~

Limitations:

1. Here the events are equally likely and mutually exclusive this need not to be true always.

2. Sample space is finite i.e, 'n' is finite. But sample space of random experiment is infinite, i.e $n=\infty$

(ii) Relative Frequency or A Posteriori Probability:

let m for the frequency of occurence of event A associated with the 'n' independent trials of a random experiment, then probability of event A is defined as $\boxed{P(A) = \lim_{n \to \infty} \frac{m}{n}}$ Here, n is very large.

$\therefore$ '$\frac{m}{n}$' is relative frequency of event A in 'n' trials.

This probability, determined as the limit of relative frequency of occurence, is referred to as a posteriori probability, i.e. probability determined after the event.

Problem: let $X = \{1, 2, 3, 7, 8, 9, 13, 14\}$; $Y = \{1, 3, 5, 7, 9, 11\}$ and $Z = \{2, 4, 6, 8, 10, 12\}$ Here sample space S is {integer number 1 to 15}. Find (i) $X \cup Y$, $X \cup Z$, $Y \cup Z$, $X \cup Y \cup Z$ (ii) $X \cap Y$, $X \cap Z$, $Y \cap Z$, (iii) $\bar{X}$, $\bar{Y}$, $\bar{Z}$ (iv) $\overline{X \cup Y}$, $\overline{X \cup Z}$, $\overline{Y \cup Z}$, $\overline{X \cup Y \cup Z}$.

Sol: (i) $X \cup Y$, $X \cup Z$, $Y \cup Z$ & $X \cup Y \cup Z$

$X \cup Y = \{1, 2, 3, 5, 7, 8, 9, 11, 13, 14\}$

$X \cup Z = \{1, 2, 3, 4, \cancel{5}, 6, 7, 8, 9, 10, \cancel{11}, 12, 13, 14\}$

$Y \cup Z = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

$X \cup Y \cup Z = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

(ii) $X \cap Y$, $X \cap Z$, $Y \cap Z$,

$X \cap Y = \{1, 3, 7, 9\}$

$X \cap Z = \{2, 8\}$

$Y \cap Z = \{\ \} = \phi$

(iii) $\overline{X}$, $\overline{Y}$, $\overline{Z}$

$X = \{1, 2, 3, 7, 8, 9, 13, 14\} \Rightarrow \overline{X} = S - X$

$\overline{X} = \{4, 5, 6, 10, 11, 12, 15\}$

$Y = \{1, 3, 5, 7, 9, 11\} \Rightarrow \overline{Y} = S - Y$

$\overline{Y} = \{2, 4, 6, 8, 10, 12, 13, 14, 15\}$

$Z = \{2, 4, 6, 8, 10, 12\} \Rightarrow \overline{Z} = S - Z$

$\overline{Z} = \{1, 3, 5, 7, 9, 11, 13, 14, 15\}$

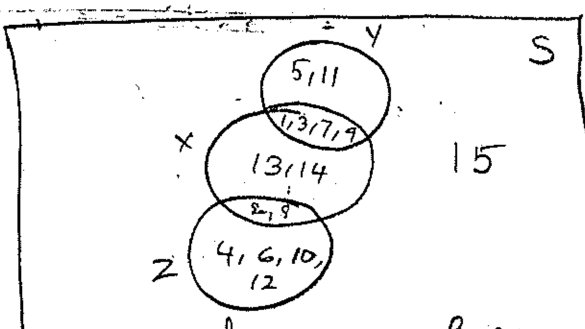
(iv) $\overline{X \cup Y}$, $\overline{Y \cup Z}$, $\overline{X \cup Z}$, $\overline{X \cup Y \cup Z}$

$\overline{X \cup Y} = S - X \cup Y = \{4, 6, 10, 12, 15\}$

$\overline{Y \cup Z} = S - Y \cup Z = \{13, 14, 15\}$

$\overline{X \cup Z} = S - X \cup Z = \{\cancel{4}5, 11, 15\}$

$\overline{X \cup Y \cup Z} = S - X \cup Y \cup Z = \{15\}$



Limitation
Actual practise 'm' is always finite quantity.

(iii) Axiomatic Approach Probability:

The probability of event A, denoted by P(A) is so chosen as to satisfy the following axioms.

Axiom 1 :- The probability of certain event is unity, i.e, $\boxed{P(A)=1}$. It is high possible probability. The probability of impossible event is 0.

Axiom 2 :- The probability of impossible event is 0 i.e

$$\boxed{P(A)=0}$$

Axiom 3 :- The probability of any event is always less than or equal to one and non-negative mathematically

i.e $\boxed{0 \leq P(A) \leq 1}$

Axiom 4 : If A and B are two mutually exclusive events, then probability $\boxed{P(A+B) = P(A)+P(B)}$ $[\because P(A \cap B) = 0]$

For any number of mutually exclusive events, A_1, A_2 .... then $\boxed{P\left(\bigcup_{n=1}^{N} A_n\right) = \bigcup_{n=1}^{N} P(A_n)}$

$$P(A_1 + A_2 + \cdots\cdots A_N) = P(A_1) + P(A_2) + \cdots + P(A_N)$$

Axiom 5 : If A and B are not mutually exclusive

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

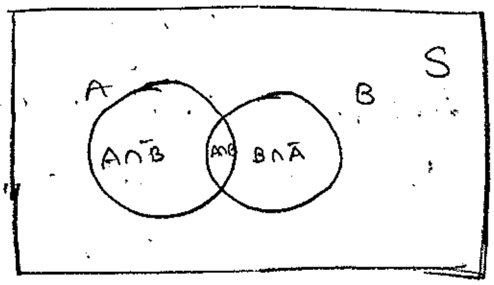
*Joint Probability or ~~Addition Theorem~~ :-

Consider an experiment A whose outcomes are $A = \{A_1, A_2, \dots\dots\}$ and experiment B with values $B = \{B_1, B_2, \dots\dots\}$. If two experiments A and B have some common elements, they are ~~not~~ mutually exclusive, i.e., those elements corresponds to the simultaneous or joint occurence of the experiments A and B.

The probability $P(A \cap B)$ denotes the probability of the simultaneous occurence of the events A and B. This is called joint probability or compound probability of A and B.

$$\boxed{P(A \cap B) = P(A) + P(B) - P(A \cup B)}$$

Proof: $A \cup B = A \cup B \cap \overline{A}$



$P(A \cup B) = P(A \cup B \cap \overline{A})$

Using Venn diagram, we observe that A and $B \cap \overline{A}$ are mutually exclusive events.

$$= P(A) + P(B \cap \overline{A})$$

Add and subtract $P(A \cap B)$

$$= P(A) + P(A \cap B) - P(A \cap B) + P(B \cap \overline{A})$$

$$= P(A) + P(A \cap B) + P(B \cap \overline{A}) - P(A \cap B)$$

$\because P(B) = P(A \cap B) + P(B \cap \overline{A})$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

Hence proved.

* Conditional Probability:

If A and B are two events in an experiment, ~~given that~~ the probability of outcome B given that A is known is called the conditional probability or transition probability, it is written as $P(B/A)$.

Then, the conditional probability of event B given that event A has already happened is defined by

$$\boxed{P(B/A) = \frac{P(A\cap B)}{P(A)}} \longrightarrow (1) \qquad \because P(A\cap B) = P(A\cap$$

$\Updownarrow$

$$P(B/A) = \frac{N_{AB}}{N_A} = \frac{N_{AB}/N}{N_A/N} = \frac{P(AB)}{P(A)} = \frac{P(A\cap B)}{P(A)}$$

where N_A - no. of times event A occurs

N_B - no. of times event B occurs

N_{AB} or N_{BA} - no. of times joint A and B occurs

N - total no. of trials.

IIIly $P(A/B) = \frac{N_{BA}}{N_B} = \frac{N_{BA}/N}{N_B/N} = \frac{P(BA)}{P(B)} = \frac{P(B\cap A)}{P(B)}$

$= \frac{P(A\cap B)}{P(B)}$

$\because P(A\cap B) = P(B\cap A)$

&

$$\boxed{\therefore P(A/B) = \frac{P(A\cap B)}{P(B)}} \longrightarrow (2)$$

The conditional probability of event A given that event B has already happened.

From equation (2), $P(A/B) = \frac{P(A\cap B)}{P(B)}$

$$P(A\cap B) = P(A/B)\,P(B) \longrightarrow (3)$$

From equation ①, $P(B/A) = \frac{P(A \cap B)}{P(A)}$

$$P(A \cap B) = P(B/A)\,P(A) \longrightarrow ④$$

Equation ③ and ④ called as law of multiplication theorem,

* Properties of Conditional Probability:

1. $P(A/B) \geq 0$ & $P(B/A) \geq 0$

Proof: We know $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$P(A \cap B) \geq 0 \quad \& \quad P(B) \geq 0$$

$$P(A \cancel{/} B) = \left(\frac{\geq 0}{\geq 0}\right) \geq 0$$

$$\therefore P(A \cancel{/} B) \geq 0$$

$$P(A/B) \geq 0$$

$|||^{ly}$ $P(B/A) \geq 0$

2. $P(S/A) = 1$

We know that $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$P(S/A) = \frac{P(S \cap A)}{P(A)}$$

$$= \frac{P(A)}{P(A)} \qquad \because A \cap S = S \cap A = A$$

$$\therefore P(S/A) = 1$$

3. If A and B are mutually exclusive to each other and $A \cap C$ and $B \cap C$ are also mutually exclusive to each other then $P(A \cup B/C) = P(A/C) + P(B/C)$

Proof: We know that $P(A/B) = \frac{\quad}{P(B)}$

$$P(A \cup B/C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{P(C \cap (A \cup B))}{P(C)}$$ commutative law $\because P(A \cap B) = P(B \cap A)$

$$= \frac{P(C \cap A) \cup P(C \cap B)}{P(C)}$$

$$= \frac{P(A \cap C) \cup P(B \cap C)}{P(C)}$$

$\because P(A \cup B) = P(A) + P(B)$

$$= \frac{P(A \cap C) + P(B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)}$$

$$P(A \cup B/C) = P(A/C) + P(B/C)$$

Hence proved.

* Statistically Independent Event's Probability:

If A and B are two events in an experiment and possibility of occurence of event B doesnot depend upon occurrence of event A, then these two events A and B are known as statistically independent events. Since occurence of event B doesnot depend on the occurence of event A, then the probability of event B will be the same as conditional probability of event B.

i.e., $P(B/A) = P(B)$ and $P(A/B) = P(A)$.

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A \cap B) = P(B/A) \cdot P(A)$$

$$= P(B) \cdot P(A)$$

$$\boxed{\therefore P(A \cap B) = P(A) \cdot P(B)}$$

or $$P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$P(A\cap B) = P(A/B)\cdot P(B)$$

$$\boxed{\therefore P(A\cap B) = P(A)\cdot P(B)}$$

Therefore if A and B are statistically independent events, then $\boxed{P(A\cap B) = P(A)\cdot P(B)}$

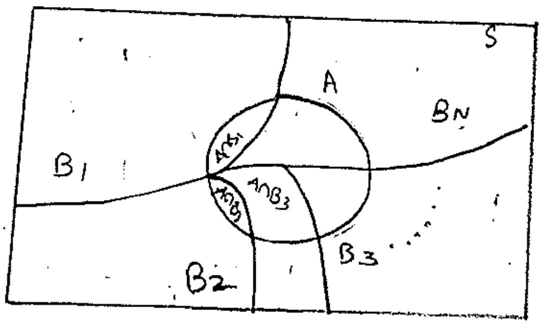
* Law of Addition Theorem or Total Probability:

Statement: Let N mutually exclusive events, B_n ~~where~~ $n = 1, 2, 3 \ldots N$ whose union equals the sample space S. ~~who~~ On the sample space, the probability of any event A, $P(A)$ can be written in terms of conditional probabilities.

$$\boxed{P(A) = \sum_{n=1}^{N} P(B_n)\, P(A/B_n)}$$

Proof: Let a sample space S and events A and B_n as shown in Venn diagram:

$$S = \bigcup_{n=1}^{N} B_n = B_1 + B_2 + \cdots + B_N$$



We know $A = A\cap S$

$$A = A\cap \bigcup_{n=1}^{N} B_n$$

$$P(A) = P\left(A\cap \bigcup_{n=1}^{N} B_n\right) = P\left(\bigcup_{n=1}^{N}(A\cap B_n)\right)$$

$$= P(A\cap B_1 + A\cap B_2 + A\cap B_3 + \cdots + A\cap B_N)$$

Since B_n & $A\cap B_n$ are mutually exclusive events

$\because P(A+B) = P(A) + P(B)$ we have

$$= P(A\cap B_1) + P(A\cap B_2) + \cdots + P(A\cap B_N)$$

From conditional probabilities

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \Rightarrow \text{lly } P(A/B_n) = \frac{P(A \cap B_n)}{P(B_n)}$$

$$P(A \cap B_n) = P(A/B_n)\, P(B_n)$$

$$\Rightarrow P(A) = \sum_{n=1}^{N} P(A \cap B_n)$$

$$= \sum_{n=1}^{N} P(A/B_n)\, P(B_n)$$

$$\therefore P(A) = \sum_{n=1}^{N} P(A/B_n)\, P(B_n).$$

Hence proved.

*****

* Bayes Theorem:

Statement: let N mutually exclusive events B_n, $n = 1, 2, 3 \ldots$ whose union equals to sample space S. On the sample space S, the probability of any event A, P(A) can be written in terms of conditional probabilities,

$$P(A) = \sum_{n=1}^{N} P(B_n)\, P(A/B_n)$$

then

$$\boxed{P(B_n/A) = \frac{P(B_n) \cdot P(A/B_n)}{\sum_{n=1}^{N} P(B_n) \cdot P(A/B_n)}}$$

Proof: Consider sample space S and events A and B_n as shown in venn diagram.

$$\longrightarrow \quad S = \bigcup_{n=1}^{N} B_n = B_1 + B_2 + B_3 + \cdots\cdots + B_N$$

We know that $A = A \cap S$

$$A = A \cap \bigcup_{n=1}^{N} B_n$$

$$P(A) = P\left(A \cap \left(\bigcup_{n=1}^{N} B_n\right)\right) = P\left(\bigcup_{n=1}^{N} (A \cap B_n)\right)$$

$$= P(A \cap B_1 + A \cap B_2 + \cdots\cdots + A \cap B_N)$$

$$= P(A \cap B_1) + P(A \cap B_2) + \cdots\cdots + P(A \cap B_N)$$

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$|||^{ly}$ $$P(A/B_n) = \frac{P(A \cap B_n)}{P(B_n)}$$

$$P(A \cap B_n) = P(A/B_n)\, P(B_n)$$

$$P(A) = \sum_{n=1}^{N} P(A \cap B_n) = \sum_{n=1}^{N} P(A/B_n)\, P(B_n)$$

$$P(A) = \sum_{n=1}^{N} P(A/B_n)\, P(B_n)$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$

$$P(B_n/A) = \frac{P(A \cap B_n)}{P(A)}$$

$$P(A \cap B_n) = P(A) \cdot P(B_n/A) \longrightarrow (2)$$

From (1) and (2)

$$P(A/B_n)\, P(B_n) = P(A) \cdot P(B_n/A)$$

$$\frac{P(B_n)\, P(A/B_n)}{P(A)} = P(B_n/A)$$

$$\therefore P(B_n/A) = \frac{P(B_n)\, P(A/B_n)}{\sum_{n=1}^{N} P(B_n)\, P(A/B_n)}$$

* 2013/14 " Set no: 3 1 (b)

Sol: The given event A is " draw a $47\,\Omega$ resistor"

event B is " draw a resistor with 5% tolerance"

event C is " draw a $100\,\Omega$ resistor".

(i) The probability of drawing 47Ω resistor $= P(A) = \frac{44 \text{ no of resistor}}{\text{Total no of resistor}}$

$$P(A) = \frac{44}{100} = 0.44$$

The probability of drawing a resistor with 5% tolerance $P(B) = \frac{62}{100}$

$$P(B) = 0.62$$

The probability of drawing a 100Ω resistor $P(C) = \frac{32}{100}$

$$P(C) = 0.32$$

(ii) Joint Probabilities: The probability of drawing 47Ω resistor with 5% tolerance $= P(A \cap B) = \frac{\text{No of 47Ω resistors with 5\% tolerance}}{\text{Total no of resistors}}$

$$= \frac{28}{100}$$

$$\therefore P(A \cap B) = 0.28$$

The probability of drawing 100Ω resistor with 5% tolerance

$$P(B \cap C) = \frac{24}{100} = 0.24$$

The probability of drawing 47Ω resistor with 100Ω resistor

$P(C \cap A) = P(\text{empty space}) = 0 = P(A \cap C)$ and $P(A \cap B \cap C) = 0$

(iii) Conditional Probabilities: (i) The probability of drawing a 47Ω resistor given that resistor drawn is 5% $= P(A/B)$

$$= \frac{P(A \cap B)}{P(B)} = \frac{0.28}{0.62} = 0.4516$$

(ii) $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.28}{0.44} = 0.6363$

The probability of drawing a resistor of 5% tolerance given that resistor is 47Ω.

The probabity of $B/C = P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{0.24}{0.32} = 0.75$

The probability of $C/B = P(C/B) = \frac{0.24}{0.62} = 0.3870$

And (ii) The probabity of $C/A = \frac{P(C \cap A)}{P(A)} = \frac{0}{0.44} = 0$

The probability of $A/C = \frac{P(A \cap C)}{P(C)} = 0$

(∵ Here A and C are disjoint events.)

i.e $P(A \cap C) = 0$

30/7/14

* Previous Question Paper Problems *

1. If a box contains 16 red balls, 12 blue balls and 22 green balls, then what is the probability of drawing a ball (a) it is red colour
(b) It is either red or blue
(c) It is not a green ball.

Sol: Given data: No. of red balls in a box $= R = 16$
No. of blue balls in a box $= B = 12$
No. of green balls in a box $= G = 22$
Total no. of balls in a box $= R + B + G$
$= 16 + 12 + 22$
$= 50$

(a) Probability for drawing a red ball from a box

$$P(R) = \frac{\text{No. of favourable outcomes } (R)}{\text{Total no. of possible outcomes}}$$

$$P(R) = \frac{16}{50} = 0.32$$

(b) Probability for drawing either red or blue ball

$$P(R+B) = P(R) + P(B)$$

$\because P(A+B) = P(A) + P(B)$ If A, B are mutual exclusive events

$$= 0.32 + P(B)$$

$$P(B) = \frac{12}{50} = 0.24$$

$$P(R+B) = 0.32 + 0.24$$

$$P(R+B) = 0.56$$

(c) Probability for drawing a ball which is not a green ball

$$P(\overline{G}) = 1 - P(G) = 1 - \frac{22}{50} = 0.56$$

$$P(\overline{G}) = 0.56$$

2. A card is drawn at random from a deck of 52 playing cards. Find the probability of card drawing
(a) an Ace (b) 6 or heart (c) Neither 9 or spade

Sol: Given no. of playing cards = 52

(a) Probability for drawing an ace card from a deck of 52 playing cards

$$P(\text{Ace}) = P(A) = \frac{\text{total no. of a ace cards}}{\text{total no. of playing cards}}$$

$$= \frac{4}{52}$$

$$= 0.0769$$

$$\therefore P(A) \simeq 0.077$$

(b) Probability for drawing a 6 or heart from a deck of 52 playing cards = P(6 or heart)

$$= P(6) + P(\text{heart}) - P(6 \cap \text{heart})$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$

$$= 0.31$$

(Here 6 no. cards = 4, heart symbol cards = 13,
6 & heart cards = 1)

(c) Probability forf drawing neither 9 or spade from a deck of 52 playing cards = P(not either 9 or spade
(i.e, not either 9 or spade")

$$= P(\overline{9 \text{ or spade}})$$

$$= 1 - P(9 \text{ or spade})$$

$$= 1 - (P(9) + P(\text{spade}) - P(9 \cap \text{spade}))$$

$$= 1 - 0.31$$

$$\therefore P(\overline{9 \text{ or spade}}) = 0.69$$

3. When a die is tossed, find the probability of events A
$A = \{$Odd number shown up$\}$; $B = \{$Number less than 3 shown up$\}$:
the find out $A \cap B$ and $A \cup B$.

Sol: Given data: The sample space of tossing a die
$S = \{1, 2, 3, 4, 5, 6\}$.

Probability of odd number shown up $P(A) = \frac{3}{6} = \frac{1}{2} = 0.50$

Probability of numbers less than 3 shown up $P(B) = \frac{2}{6} = \frac{1}{3} = 0.33$

Probability of $A \cap B$ = $P(A \cap B) = \frac{1}{6} = 0.16$

$\because S = \{\boxed{1}, 2, 3, 4, 5, 6\}$

($\because$ Odd number shown up means possible events 1, 3 and 5 and number less than 3 shown up means 1 and 2 only)

Probability of $A \cup B$ = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.5 + 0.33 - 0.16$

$P(A \cup B) = 0.67$

4. A card is drawn from a well shuffled pack of playing cards. What is the probability that in either a spade or ace?

Sol: Probability of either spade or ace $= P(\text{spade}) + P(\text{ace}) - P(\text{spade \& ace})$

$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$

$= 0.307$

$\simeq 0.31$

5. Two boxes are selected randomly. The first box contain 2 white balls and 3 black balls. Second box contains 3 white and 4 black balls. What is the probability of drawing a white ball?

5-Sol: Probability of selecting 1st box = $P(B_1) = \frac{1}{2} = 0.5$

Probability of selecting 2nd box = $P(B_2) = \frac{1}{2} = 0.5$

Given favourable event is selecting white ball.

Conditional probability of white ball from the given box 1

$$P(W/B_1) = \frac{P(W \cap B_1)}{P(B_1)} = \frac{\frac{1}{5}}{1/2} = \frac{2}{5} = 0.4$$

$$P(W/B_2) = \frac{3}{7} = 0.4285 \simeq 0.43$$

Total probability for drawing a white ball is

$$P(W) = P(B_1)\,P(W/B_1) + P(B_2).\,P(W/B_2)$$

$$= 0.5 \times 0.4 + 0.5 \times 0.4285$$

$$= 0.4143$$

$$\therefore P(W) = 0.415$$

*Random Variables:

Random variable is defined as a rule or functional relationship that assigns a real number to each possible outcome of a random experiment.

Random variables are denoted by uppercase letters X, Y etc and the values taken by them are denoted by lowercase letters as with subscripts as x_1, x_2, y_1, y_2 etc

* Types of Random Variables:

There are three types of random variables

(i) discrete random variable

(ii) Continuous random variable

(iii) mixed random variable

(i) Discrete Random Variables:

The assigned values of random variable are finite or countably infinite. Then the random variable is known as discrete random variable.

Eg: 1. In the experiment, tossing two coins, the sample space is 2^2. S = {HH, HT, TH, TT}.

Consider a random variable X which assign real values to each element of S ~~the~~ no. of heads in the experiment. then HH → 2; TH → 1; HT → 1; TT → 0

$$X = \{0, 1, 2\}$$

No. of elements in random variable is finite, hence it is discrete r.v.

ii) let us consider experiment of throwing a dice. Here sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let us define random variable $X(s) = s^2$ then the assigned values of random variables are

$s = 0$; $X(0) = 0^2 = 0$

$s = 1$; $X(1) = 1^2 = 1$

$s = 2$; $X(2) = 2^2 = 4$

$s = 3$; $X(3) = 3^2 = 9$

$s = 4$; $X(4) = 4^2 = 16$

$s = 5$; $X(5) = 5^2 = 25$

$s = 6$; $X(6) = 6^2 = 36$

$$X(s) = X = \{1, 4, 9, 16, 25, 36\}.$$

The assigned values of r.v is finite and hence it is discrete random variable.

(ii) Continuous Random Variables:

The assigned values of random variable ar uncountably infinite. Then it is called continuous random variable.

Eg: Consider an experiment "choose a rational number between 1 and 10", $S = \{1 \leq s \leq 10\}$

Let us define random variable X is $X(s) = s$.

i.e, $X = \{1 \leq x \leq 10\}$

The assigned values are uncountably infinite. Hence it is continuous random variable.

SH. Venugopal Reddy, HOD, NIET

(iii) Mixed Random Variables

A mixed random variable is one for which some of its values are discrete and some are continuous.

Eg: $X = \{15, 16, 17, 18 \leq X \leq 22, 24, 26, 28\}$.

15, 16, 17, 24, 26, 28 are discrete and $18 \leq X \leq 22$ are continous. Hence it is mixed r.v.

* Conditions for a function to be a Random Variable:

1. The function must be single-valued function, i.e., each outcome doesnot have two or more assigned values.

2. The set $\{X \leq x\}$ shall be an event for any real number x. The probability of this event is denoted by $P(X \leq x)$ and is equal to the sum of probability of elementary events, i.e, $P(X \leq x) = P(X=1) + P(X=2) + P(X=3) + \cdots + P(X=x)$

3. The probabilities of the events $\{X = -\infty\}$ and $\{X = \infty\}$ are zero i.e the function must have impossible events.

* Probability Distribution (or) Cummulative Distributive Functions of a R.V. (CDF):

The cumulative distribution function of a random variable X may be defined as the probability that a random variable X takes a value less than or equal to x.

Let us consider a probability of the event $X \leq x$. The probability of this event may be defined as

$$\boxed{\text{CDF of } X = F_X(x) = P(X \leq x)}$$

Example: In an experiment, 3 coins are tossed simultaneously. If the no of heads is the random variable. Find probability function for this random variable. And also $F_X(0), F_X(1), F_X(2), F_X(3)$.

Sol: $S = \{ \underset{x_1}{HHH}, \underset{x_2}{HHT}, \underset{x_3}{HTH}, \underset{x_4}{HTT}, \underset{x_5}{THH}, \underset{x_6}{THT}, \underset{x_7}{TTH}, \underset{x_8}{TTT} \}$

In the three tosses of coins we got 8 possible outcomes, this is the sample space S. Let the random variable, no of heads, be X.

$$X = \{3, 2, 2, 1, 2, 1, 1, 0\}$$

The probability of each of 8 possible outcomes, will be $\frac{1}{8}$. i.e, $P(x_1) = P(x_2) \cdots\cdots = P(x_8) = \frac{1}{8}$

Probability functions:

(i) No heads appearing i.e x_8 $P(X=0) = P(x_8) = \frac{1}{8}$

(ii) One head appearing i.e, $x_4, x_6, x_7 \Rightarrow P(X=1) =$

$$P(x_4) + P(x_6) + P(x_7)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(iii) Two heads appearing i.e, $x_2, x_3, x_5 \Rightarrow P(X=2)$

$$= P(x_2) + P(x_3) + P(x_5)$$
$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

(iv) Three heads appearing i.e, x_1 $P(X=3) = P(x_1) = \frac{1}{8}$

X	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

Cumulative (added) Distribution Function:-

The CDF of $X = F_X(x) = P(X \le x)$

$x=0$; $F_X(0) = P(X \le 0) = P(X=0) = \frac{1}{8}$

$x=1$; $F_X(1) = P(X \le 1) = P(X=0) + P(X=1)$

$= \frac{1}{8} + \frac{3}{8} = \frac{2}{8} = \frac{1}{2}$

$x=2$; $F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}$

$= \frac{7}{8}$

$x=3$; $F_X(3) = P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

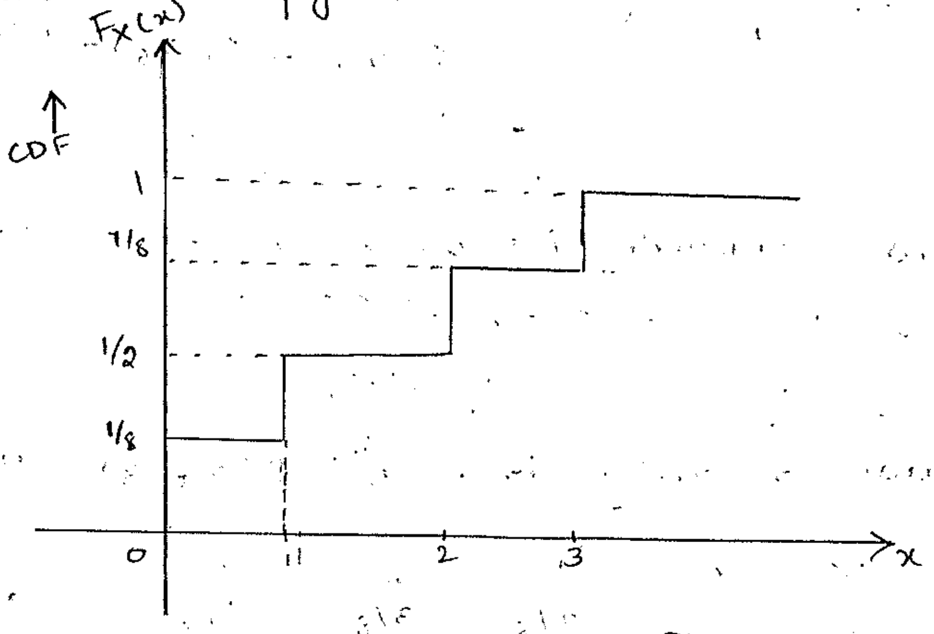
$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$

$= \frac{8}{8} = 1$

$\therefore F_X(3) = 1$

CDF Plot:

A graph plotted between CDF ($F_X(x)$) and x is as shown in fig:



In exams upto here

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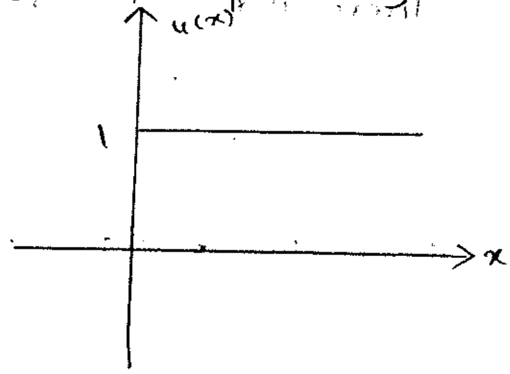
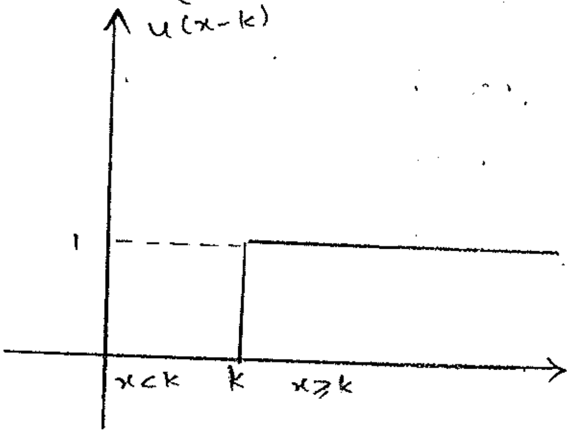
CDF Expression:- The unit step function is defined by:

$$u(x) = \begin{cases} 1 & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$



$$u(x-k) = \begin{cases} 1 & ; x \geq k \\ 0 & ; x < k \end{cases}$$



$X = \{0, 1, 2, 3\}$
$\quad\; x_1, x_2, x_3, x_4$

$$F_X(x) = \frac{1}{8}u(x) + \left(\frac{1}{2} - \frac{1}{8}\right)u(x-1) + \left(\frac{7}{8} - \frac{1}{2}\right)u(x-2) + \left(1 - \frac{7}{8}\right)u(x-3)$$

$$= P(X = x_1)\,u(x - x_1) + P(X = x_2)\,u(x - x_2) + P(X = x_3)\,u(x - x_3) + P(X = x_4)\,u(x - x_4)$$

$$F_X(x) = \sum_{i=1}^{4} P(X = x_i)\, u(x - x_i)$$

In general, for n elements

$$\therefore\; F_X(x) = \sum_{i=1}^{N} P(X = x_i)\, u(x - x_i) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$$

***** (any 4)

* Properties of CDF function:

1. $F_X(-\infty) = 0$ & $F_X(\infty) = 1$

Proof: In a randomvariable X, all the assigned real values varies from $-\infty$ to ∞ i.e, $-\infty$ ~~to~~ $\leq X \leq \infty$

The CDF of $X = F_X(x) = P(X \leq x)$

$$F_X(-\infty) = P(X \leq -\infty)$$

There are no real numbers "less than $-\infty$."

$$\therefore F_X(-\infty) = P(X \le -\infty) = 0$$

Since $X \le -\infty$ is impossible event. The probability of impossible event is 0.

$$F_X(\infty) = P(X \le \infty)$$

There exists all real numbers "less than ∞".

$$\therefore F_X(\infty) = P(X \le \infty)$$

$$F_X(\infty) = P(S) = 1$$

$(\because X \le \infty$ is certain event i.e, $P(S) = 1)$

$$\therefore F_X(-\infty) = 0 \quad \& \quad F_X(\infty) = 1$$

Hence the property is proved.

2. CDF is bounded between 0 and 1 i.e, $0 \le F_X(x) \le 1$.

Proof: Real values varies from $-\infty$ to ∞.

i.e, $-\infty \le x \le \infty$

Hence the range of CDF is

$$F_X(-\infty) \le F_X(x) \le F_X(\infty)$$

We know that $F_X(-\infty) = 0$

$F_X(\infty) = 1$

$$\therefore 0 \le F_X(x) \le 1$$

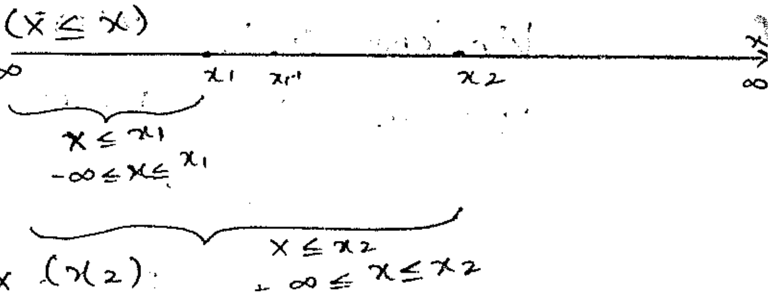
Hence it is proved

3. If $x_1 < x_2$ then $F_X(x_1) \le F_X(x_2)$

Proof: Here the event $X \le x_1$ is subset of the event $X \le x_2$ i.e, $(X \le x_1) \subseteq (X \le x_2)$

$$P(X \le x_1) \le P(X \le x_2)$$

The CDF of $X = F_X(x) = P(X \le x)$



$\therefore F_X(x_1) = P(X \le x_1)$ &

$F_X(x_2) = P(X \le x_2)$

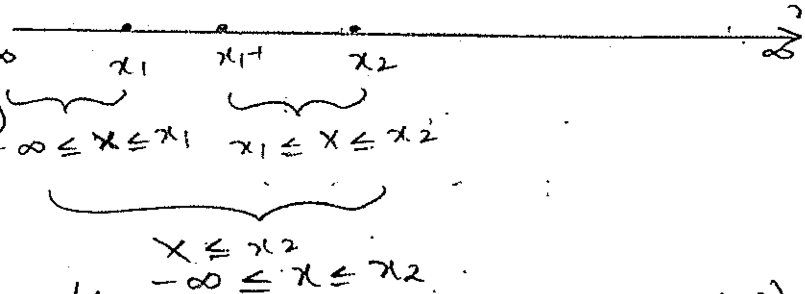
Here $F_X(x_1) \le F_X(x_2)$.

Therefore this property states that CDF is a monotone non decreasing function of x.

***** 4. $P(x_1 \le X \le x_2) = F_X(x_2) - F_X(x_1)$

Proof:

$X \le x_2 \neq X \le x_1 + x_1 \le X \le x_2$



$P(X \le x_2) = P(X \le x_1 + x_1 \le X \le x_2)$

Here $X \le x_1$ & $x_1 < X \le x_2$ are mutually exclusive events $(\because P(A+B) = P(A) + P(B))$

~~CDF of~~ $P(X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$

$\because$ CDF of $X = F_X(x) = P(X \le x)$

$F_X(x_1) = P(X \le x_1)$

$F_X(x_2) = P(X \le x_2)$

$F_X(x_2) = F_X(x_1) + P(x_1 \le X \le x_2)$

$\therefore F_X(x_2) - F_X(x_1) = P(x_1 < X \le x_2)$

5. $P(X > x) = 1 - F_X(x)$

Proof: Sample space $S = X \le x + X > x$

$(S = -\infty \le X \le x + x < X \le \infty)$

$-\infty \le X \le \infty$

$P(S) = P(X \le x + X > x)$

Here $X \le x$ and $X > x$ are mutually exclusive events then $P(S) = P(X \le x) + P(X > x)$

We know that $P(S) = 1$

CDF of $X = F_X(x) = P(X \le x)$

$$1 = F_X(x) + P(X > x)$$

$$\therefore 1 - F_X(x) = P(X > x)$$

6. $F_X(x^+) = F_X(x)$; $x^+ = x + \epsilon$; $\epsilon > 0$

It is infinitely small i.e., $\epsilon \to 0$

This property states that the function $F_X(x)$ is continuous from the right.

7. $P(X = x) = F_X(x) - F_X(\bar{x})$; $\bar{x} = x - \epsilon$

Proof: We know that

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

let $x_1 = \bar{x} = x - \epsilon$ & $x_2 = x$

$$P(x - \epsilon < X \le x) = F_X(x) - F_X(\bar{x})$$

Apply $\epsilon \to 0$, we get

$$P(x - 0 < X \le x) = F_X(x) - F_X(\bar{x})$$

$$P(x < X \le x) = F_X(x) - F_X(\bar{x})$$

$$\therefore P(X = x) = F_X(x) - F_X(\bar{x})$$

8. $P(x_1 \le X \le x_2) = F_X(x_2) - F_X(\bar{x}_1)$

Proof: $P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) + P(X = x_1)$

We know that

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

let $x_1 = \bar{x} = x - \epsilon$ & $x_2 = x$

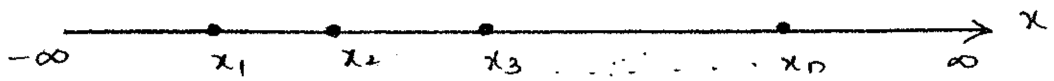
$$P(x - \epsilon < X \le x) = F_X(x) - F_X(\bar{x})$$

Apply $\epsilon \to 0$ we get

$$P(x-0 < X \le x) = F_X(x) - F_X(\bar{x})$$
$$P(x < X \le x) = F_X(x) - F_X(\bar{x})$$
$$P(X = x) = F_X(x) - F_X(\bar{x})$$
$$P(X = x_1) = F_X(x_1) - F_X(\bar{x}_1)$$
$$P(x_1 \le X \le x_2) = F_X(x_2) - \cancel{F_X(x_1)} + \cancel{F_X(x_1)} - F_X(\bar{x}_1)$$
$$= F_X(x_2) - F_X(\bar{x}_1)$$

* CDF function for discrete random variables:-

Let, X is a discrete random variable, then it can take on values as $x_1, x_2, x_3 \cdots x_n$. This means that x be expressed as $x_1, x_2, x_3 \cdots x_n$.



The CDF of random variable $X = F_X(x) = P(X \le x)$

$$F_X(x) = \begin{cases} 0 & ; \ -\infty \le x \le x_1 \\ P(X=x_1) & ; \ x_1 \le x \le x_2 \\ P(X=x_1)+P(X=x_2) & ; \ x_2 \le x < x_3 \\ \vdots & \\ P(X=x_1)+P(X=x_2)+ \cdots + P(X=x_n) = \sum_{i=1}^{N} P(X=x_i) = 1 & ; \ x_n \le x < \infty \end{cases}$$

or.

$$\boxed{F_X(x) = \begin{cases} 0 & ; \ -\infty \le x < x_1 \\ \sum_{i=1}^{N} P(X=x_i) & ; \ x_1 \le x \le x_n \\ 1 & ; \ x_n < x < \infty \end{cases}}$$

* Probability Density Function (PDF):-
The derivative of cummulative distribution function w.r.t some dummy variable x is known as PDF.

The PDF of 'X' $= f_X(x) = \frac{d}{dx}[F_X(x)]$.

or

$$f_X(x) = f(x) = \begin{cases} P(X = x_i) ; & X = x_i \\ 0 ; & X \neq x_i \end{cases}$$

→ Expression for PDF function:
We know expression for CDF function as

The CDF of $X = F_X(x) = P(X \le x) = \sum_{i=1}^{N} P(X = x_i)\, u(x - x_i)$

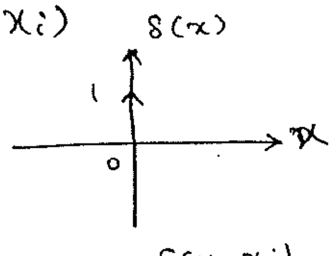
The PDF of $X = f_X(x) = \frac{d}{dx}[F_X(x)]$

$$= \frac{d}{dx}\left[\sum_{i=1}^{N} P(X = x_i)\, u(x - x_i)\right]$$

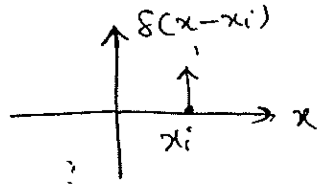
$$f_X(x) = \sum_{i=1}^{N} P(X = x_i) \frac{d}{dx} u(x - x_i)$$

We know relationship b/w unit step & unit impulse functions i.e., $\delta(x) = \frac{d}{dx}[u(x)]$.

$$\delta(x - x_i) = \frac{d}{dx} u(x - x_i)$$



$$\delta(x) = \begin{cases} 1 ; & x = 0 \\ 0 ; & x \neq 0 \end{cases}$$



$$\delta(x - x_i) = \begin{cases} 1 ; & x = x_i \\ 0 ; & x \neq x_i \end{cases}$$

*****

$$\boxed{\therefore f_X(x) = \sum_{i=1}^{N} P(X = x_i)\, \delta(x - x_i)}$$

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* Properties of PDF function:

1. PDF is always non-zero quantity (+ve quantity) for all values of x. i.e. $f_X(x) \geq 0 \ \forall x$.

Proof: We know that CDF values lies b/w 0 & 1.

i.e, $0 \leq F_X(x) < 1$

The PDF of $X = f_X(x) = \frac{d}{dx}[F_X(x)]$

The CDF of $X = F_X(x) = P(X \leq x)$

The differentiation of these values must be +ve number. $0 \leq F_X(x) \leq 1$

i.e, $\frac{d}{dx}[F_X(x)] \geq 0$ (+ve)

$\therefore f_X(x) = \frac{d}{dx}[F_X(x)] \geq 0$

V.imp

***2. Area under PDF function is unity.

i.e., $\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1$

Proof: The PDF of $X = f_X(x) = \frac{d}{dx}[F_X(x)] \longrightarrow (1)$

The CDF of $X = F_X(x) = P(X \leq x)$

Apply $\int_{-\infty}^{\infty} (\ )\, dx$ to the both sides of eq (1), we get

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty} \frac{d}{dx}[F_X(x)]\, dx$$

$$= \int_{-\infty}^{\infty} 1 \cdot d[F_X(x)]$$

$$\boxed{\because \int 1 \cdot d[x] = x}$$

$$= [F_X(x)]_{-\infty}^{\infty}$$

$$= F_X[\infty] - F_X[-\infty]$$

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We know $F_X[-\infty]=0$ & $F_X[\infty]=1$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1-0$$

$$=1$$

3. The CDF may be expressed as integration of PDF, i.e, $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Proof: The PDF of 'X' $= f_X(x) = \frac{d}{dx}[F_X(x)] \rightarrow (1)$

The CDF of 'X' $= F_X(x) = P(X \le x)$

Apply $\int_{-\infty}^{x} (\;)\,dx$ to both sides of eq (1), we have

$$\int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} \frac{d}{\cancel{dx}}[F_X(x)]\,\cancel{dx}$$

$$= \int_{-\infty}^{x} 1 \cdot d[F_X(x)]$$

$$= [F_X(x)]_{-\infty}^{x}$$

$$= F_X(x) - F_X(-\infty)$$

We know that $F_X(-\infty) = 0$

$$\therefore \int_{-\infty}^{x} f_X(x)\,dx = F_X(x) - 0$$

$$= F_X(x).$$

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4. $P(x_1 < X \le x_2) = \int_{x_1} f_X(x)\, dx$ CH. Venugopal

Proof: The PDF of $X = f_X(x) = \frac{d}{dx}[F_X(x)] \rightarrow (1)$

The CDF of $X = F_X(x) = P(X \le x)$

Apply $\int_{x_1}^{x_2} (\ )\, dx$ to the both sides of eq (1), we get

$$\int_{x_1}^{x_2} f_X(x)\, dx = \int_{x_1}^{x_2} \frac{d}{d\cancel{x}}[F_X(x)]\, d\cancel{x}$$

$$= \int_{x_1}^{x_2} 1 \cdot d[F_X(x)]$$

$$= [F_X(x)]_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

We know that

$$F_X(x_2) - F_X(x_1) = P(x_1 < X \le x_2)$$

$$\int_{x_1}^{x_2} f_X(x)\, dx = P(x_1 < X \le x_2)$$

5. Let X be a continuous random variable, then the probability of particular event is 0, i.e., $\boxed{P(X = x) = 0}$

Proof: We know probability of previous statement

$$P(x_1 < X \le x_2) = \int_{x_1}^{x_2} f_X(x)\, dx.$$

If $x_1 = x_2 = x$ then

$$P(x < x \le x) = \int_x^x f_X(x)\, dx$$

$$P(X = x) = 0 \qquad \left[\because \int_a^a f(x)\, dx.\right]$$

$\cancel{P(x_1 < X \le x_2)} = P(x_1 \le X < x_2) = P(x_1 \le X \le x_2) = P(x_1 < X < x_2) = \int_{x_1}^{x_2} f(x)\, dx$

* Problems:

*1. PDF is given by the expression $f_X(x) = a e^{-b|x|}$. Here X is a random variable whose values lies in the range $x = -\infty$ to $x = \infty$. Determine the following:-

(i) The relationship between a and b.

(ii) The CDF function.

(iii) The probability that outcome lies b/w 1 and 2.

Sol: Given PDF expression $f_X(x) = a e^{-b|x|}$

The PDF of 'x' $= f_X(x) = a e^{-b|x|}$

$$= \begin{cases} a e^{-bx} ; & x \geq 0 \\ a e^{bx} ; & x < 0 \end{cases} \qquad \left\{ \because |x| = \begin{cases} x ; & x \geq 0 \\ -x ; & x < 0 \end{cases} \right\}$$

(i) We know area under PDF curve is unity i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_{-\infty}^{0} (a e^{bx})\, dx + \int_{0}^{\infty} (a e^{-bx})\, dx$$

$$1 = \left. \frac{a e^{bx}}{b} \right|_{-\infty}^{0} + \left. \frac{a e^{-bx}}{-b} \right|_{0}^{\infty}$$

$$1 = \frac{a}{b}\left[e^{b(0)} - e^{b(-\infty)}\right] + \frac{a}{-b}\left[e^{-b\infty} - e^{-b0}\right]$$

$$1 = \frac{a(1-0)}{b} - \frac{a}{b}(0-1)$$

$$1 = \frac{a}{b} + \frac{a}{b}$$

$$1 = \frac{2a}{b}$$

$$\boxed{b = 2a}$$

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(ii) We know the CDF of random variable 'X' $= F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Case (i): If $x<0$ i.e. $-\infty < x < 0$

$$F_X(x) = \int_{-\infty}^{x} a e^{bx}\,dx \qquad \because f_X(x) = a e^{bx};\ x<0$$

$$= a \frac{e^{bx}}{b}\Big|_{-\infty}^{x}$$

$$= \frac{a}{b}\left(e^{bx} - e^{-(b\infty)}\right)$$

$$= \frac{a}{b}\left(e^{bx} - 0\right)$$

$$= \frac{a e^{bx}}{b}$$

But $b = 2a \Rightarrow \frac{a}{b} = \frac{1}{2}$

$$\boxed{\therefore F_X(x) = \frac{1}{2} e^{bx} \; ; \quad x<0}$$

Case (ii): If $x \geq 0$ i.e. $0 \leq x < \infty$

$$F_X(x) = \int_{-\infty}^{0} (a e^{bx})\,dx + \int_{0}^{x} (a e^{-bx})\,dx$$

$$= \frac{a}{b}\left[e^{bx}\right] + \frac{a}{-b}\left[e^{-bx}\right]$$

$$\therefore F_X(x) = \begin{cases} \frac{1}{2} e^{bx} \; ; & x<0 \\ 1 - \frac{1}{2} e^{-bx} \; ; & x \geq 0 \end{cases}$$

$$= \frac{a}{b}\left[e^{b(0)} - e^{b(-\infty)}\right] - \frac{a}{b}\left[e^{-b(x)} - e^{-b(0)}\right]$$

$$= \frac{a}{b}\left[1 - 0\right] - \frac{a}{b}\left[e^{-xb} - 1\right]$$

$$= \frac{a}{b} - \frac{a}{b} e^{-bx} + \frac{a}{b}$$

$$= \frac{2a}{b} - \frac{a}{b} e^{-bx}$$

$$F_X(x) = 2 \times \frac{1}{2} - \frac{1}{2} e^{-bx} = 1 - \frac{1}{2} e^{-bx}; \quad x \geq 0.$$

(iii) Probability that outcome lies b/w 1 and 2 i.e., $P(1 < X \leq 2)$

We know $P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)\, dx$

$$P(1 < X \leq 2) = \int_1^2 f_X(x)\, dx$$

$$= \int_1^2 a e^{-bx}\, dx$$

$$= a \left. \frac{e^{-bx}}{-b} \right|_1^2$$

$$= -\frac{a}{b}\left[e^{-2b} - e^{-b}\right]$$

$$= \frac{a}{b}\left[e^{-b} - e^{-2b}\right]$$

$$\therefore P(1 < X \leq 2) = \frac{1}{2}\left[e^{-b} - e^{-2b}\right] \quad \left(\because \frac{a}{b} = \frac{1}{2}\right).$$

or

$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$$P(1 < X \leq 2) = F_X(2) - F_X(1)$$

$$F_X(2) = \left. 1 - \frac{1}{2} e^{-bx} \right|_{x=2} = 1 - \frac{1}{2} e^{-2b}$$

$$F_X(1) = \left. 1 - \frac{1}{2} e^{-bx} \right|_{x=1} = 1 - \frac{1}{2} e^{-b}$$

$$P(1 < X \leq 2) = \cancel{1} - \frac{1}{2} e^{-2b} - \cancel{1} + \frac{1}{2} e^{-b}$$

$$\therefore P(1 < X \leq 2) = \frac{1}{2}\left(e^{-b} - e^{-2b}\right)$$

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** 2. The CDF for a certain random variable is given as

$$F_X(x) = \begin{cases} 0 & ; -\infty < x \le 0 \\ kx^2 & ; 0 < x \le 10 \\ 100k & ; 10 < x < \infty \end{cases}$$

(i) Find the value of k.

(ii) Find the value of $P(X \le 5)$.

(iii) Find the value of $P(5 < X \le 7)$

(iv) Find the expression for PDF function.

Sol: Given CDF of 'X' = $F_X(x) = \begin{cases} 0 & ; -\infty < x \le 0 \\ kx^2 & ; 0 < x \le 10 \\ 100k & ; 10 < x < \infty \end{cases}$ → ①

(i) We know expression for CDF = $F_X(x) = \begin{cases} 0 & ; -\infty < x \le x_1 \\ \sum_{i=1}^{N} P(X=x_i) & ; x_1 \le X \le \\ 1 & ; x_n < x < \infty \end{cases}$ → ②

Comparing ① and ② we get

$$100k = 1$$

$$k = \frac{1}{100}$$

$$\boxed{\therefore k = 0.01}$$

(ii) $P(X \le 5)$

$$\therefore F_X(x) = \begin{cases} 0 & ; -\infty < x \le 0 \\ \frac{x^2}{100} & ; 0 < x \le 10 \\ 1 & ; 10 < x < \infty \end{cases}$$

The CDF of 'X' = $F_X(x) = P(X \le x)$

$$F_X(5) = P(X \le 5) = \left.\frac{x^2}{100}\right|_{x=5}$$

$$= \frac{25}{100}$$

[illegible] $\boxed{\therefore F_X(5) = 0.25}$

(iii) $P(5 < X \leq 7)$

We know $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

$$P(5 < X \leq 7) = F_X(7) - F_X(5)$$

$$= \left.\frac{x^2}{100}\right|_{x=7} - \left.\frac{x^2}{100}\right|_{x=5}$$

$$= \frac{49}{100} - \frac{25}{100}$$

$$= 0.49 - 0.25$$

$$\boxed{\therefore P(5 < X \leq 7) = 0.24}$$

(iv) The PDF of 'X' $= f_X(x) = \frac{d}{dx}[F_X(x)]$

The CDF of 'X' $= F_X(x) = P(X \leq x)$

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} \frac{d}{dx}(0) & ; -\infty < x \leq 0 \\ \frac{d}{dx}\left(\frac{x^2}{100}\right) & ; 0 < x \leq 10 \\ \frac{d}{dx}(1) & ; 10 < x < \infty \end{cases}$$

$$f_X(x) = \begin{cases} 0 & ; -\infty < x \leq 0 \\ \frac{x}{50} & ; 0 < x \leq 10 \\ 0 & ; 10 < x < \infty \end{cases}$$

$$\boxed{\therefore f_X(x) = \begin{cases} \frac{x}{50} & ; 0 < x \leq 10 \\ 0 & ; \text{otherwise} \end{cases}}$$

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** 3. Check whether the following is PDF or not:

(i) $f_X(x) = \begin{cases} 0 & ; x < 2 \\ \frac{1}{18}(3+2x) ; & 2 < x \leq 4 \\ 0 & ; x > 4. \end{cases}$

Sol: We know area under PDF, is unity. i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{2}(0)\,dx + \int_{2}^{4}\frac{1}{18}(3+2x)\,dx + \int_{4}^{\infty}(0)\,dx \quad x>4$$

$$= \frac{1}{18}\left(3x\Big|_2^4 + \frac{2x^2}{2}\Big|_2^4\right)$$

$$= \frac{1}{18}\left(3(4-2) + (4^2-2^2)\right)$$

$$= \frac{1}{18}\left(6 + (16-4)\right)$$

$$= \frac{1}{18}(6+12)$$

$$= \frac{1}{18} \times 18$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \quad \& \quad f_X(x) \geq 0 \ \forall x$$

Hence the given $f_X(x)$ is a PDF.

4. Show that $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a PDF; $-\infty < x < \infty$, $\sigma > 0$.

Sol: We know area under PDF is unity i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

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$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2}\,dx$$

Put $t = \dfrac{x-\mu}{\sqrt{2}\,\sigma}$

$$dt = \frac{dx}{\sqrt{2}\,\sigma} \Rightarrow dx = \sqrt{2}\,\sigma\,dt$$

If $x \to -\infty$ then $t \to -\infty$

If $x \to \infty$ then $t \to \infty$

$$= \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t)^2}\,dt\,\cancel{\sqrt{2}}\,\cancel{\sigma}$$

$$= \frac{1}{\sqrt{\pi}} \times 2 \int_{0}^{\infty} e^{-t^2}\,dt$$

$\because f(t) = e^{-t^2}$

$f(-t) = e^{-(-t)^2} = e^{-t^2}$

$\therefore f(t) = f(-t)$

So it is an even function

$$= \frac{1}{\cancel{\sqrt{\pi}}} \times \cancel{2} \times \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \qquad \left(\because \int_0^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\right)$$

Hence the given function is a PDF

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5. Find a constant $b>0$ so that the function $f_X(x)$ is given as $f_X(x) = \begin{cases} \frac{1}{10} e^{3x} \; ; & 0 \le x \le b \\ 0 \; ; & \text{otherwise} \end{cases}$ is a valid PDF function

Sol: We know area under PDF is unity i.e.,

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} (0)\, dx + \int_{0}^{b} \left(\frac{1}{10} e^{3x}\right) dx + \int_{b}^{\infty} (0)\, dx = 1$$

$$\Rightarrow \frac{1}{10} \frac{e^{3x}}{3} \Big|_0^b = 1$$

$$\Rightarrow \frac{1}{30} \left[e^{3b} - e^{0} \right] = 1$$

$$\Rightarrow \frac{1}{30} \left[e^{3b} - 1 \right] = 1$$

$$\Rightarrow \frac{1}{30} \left[e^{3b} \right] - \frac{1}{30} = 1$$

$$\Rightarrow e^{3b} - 1 = 30$$

$$\Rightarrow e^{3b} = 30 + 1$$

$$\Rightarrow e^{3b} = 31$$

$$\Rightarrow \ln(e^{3b}) = \ln(31)$$

$$\Rightarrow 3b = \ln(31)$$

$$\Rightarrow b = \frac{1}{3} \ln(31)$$

$$\boxed{\therefore b = 1.1446}$$

6. If the PDF is $f_X(x) = k(1-x^2)$ ~~then~~ find k and CDF values.

Sol: $\Rightarrow$ Given PDF of $X = f_X(x) = k(1-x^2) \; ; \; 0 < x < 1$

We know area under PDF is unity i.e.,

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$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{0} (0)\,dx + \int_{0}^{1} k(1-x^2)\,dx + \int_{1}^{\infty} (0)\,dx = 1$$

$$\Rightarrow \quad k\left[x - \frac{x^3}{3}\right]_0^1 = 1$$

$$\Rightarrow \quad k\left[1 - \frac{1}{3}\right] = 1$$

$$\Rightarrow \quad k\left[\frac{2}{3}\right] = 1$$

$$\Rightarrow \quad k = \frac{3}{2} \qquad \boxed{\therefore f_X(x) = \frac{3}{2}(1-x^2),\ 0<x<1}$$

$$\Rightarrow \boxed{k = 1.5}$$

$\Rightarrow$ The CDF of 'X' $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Case (i): If $x<0$ i.e. $-\infty < x < 0$

$$F_X(x) = \int_{-\infty}^{x} (0)\,dx = 0 \quad ; \quad -\infty < x < 0$$

Case (ii): If $x \geq 0$ i.e. $0 < x < 1$

$$F_X(x) = \int_{-\infty}^{0} (0)\,dx + \int_{0}^{x} \frac{3}{2}(1-x^2)\,dx$$

$$= \frac{3}{2}\left(x - \frac{x^3}{3}\right)\Big|_0^x$$

$$= \frac{3}{2}\left((x-0) - \left(\frac{x^3}{3} - 0\right)\right)$$

$$= \frac{3}{2}(x) - \frac{x^3}{2}$$

$$\boxed{F_X(x) = \frac{1}{2}(3x - x^3) \ ; \ 0 < x < 1}$$

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Case (iii): If $x > 1$ then $1 < x < \infty$:

$$F_X(x) = \int_{-\infty}^{0} (0)\, dx + \int_{0}^{1} \frac{3}{2}(1-x^2)\, dx + \int_{1}^{x} (0)\, dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{2}\left[1 - \frac{1}{3}\right]$$

$$= \frac{3}{2}\left[\frac{2}{3}\right]$$

$\therefore F_X(x) = 1$

$$\therefore F_X(x) = \begin{cases} 0 & ; -\infty < x < 0 \\ \frac{1}{2}(3x - x^3) & ; 0 < x < 1 \\ 1 & ; 1 < x < \infty \end{cases}$$

7. A random variable X has probabilities shown in table

(i) Find the value of k.

(ii) Find $F_X(x)$, $f_X(x)$ and draw the plots.

x	$x_1 = -3$	$x_2 = -2$	$x_3 = -1$	$x_4 = 0$	$x_5 = 1$	$x_6 = 2$
$P(x)$	0.2	0.5k	k	0.1	0.3k	k

Sol: We know total probability of a random variable is unity, i.e.,
$$\sum_{i=1}^{6} P(x_i) = 1$$

(i) $P(x_1) + P(x_2) + P(x_3) + P(x_4) + P(x_5) + P(x_6) = 1$

$$0.2 + 0.5k + k + 0.1 + 0.3k + k = 1$$

$$2.8k + 0.3 = 1$$

$$2.8k = 1 - 0.3$$

$$2.8k = 0.7$$

$$k = \frac{0.7}{2.8} \Rightarrow \boxed{k = 0.25}$$

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Therefore ; $P(x_1) = 0.2$; $P(x_2) = 0.5k = 0.125$

$P(x_3) = k = 0.25$

$P(x_4) = 0.1$

$P(x_5) = 0.3k = 0.3 \times 0.25 = 0.075$

$P(x_6) = k = 0.25$

ii) CDF of 'X' $= F_X(x) = P(X \leq x)$

$x = -3$; $F_X(-3) = P(X \leq -3) = P(x_1) = 0.2$

$x = -2$; $F_X(-2) = P(X \leq -2) = P(x_1) + P(x_2) = 0.2 + 0.125$
$= 0.325$

$x = -1$; $F_X(-1) = P(X \leq -1) = P(x_1) + P(x_2) + P(x_3)$
$= 0.2 + 0.125 + 0.25$
$= 0.575$

$x = 0$; $F_X(0) = P(X \leq 0) = P(x_1) + P(x_2) + P(x_3) + P(x_4)$
$= 0.575 + 0.1$
$= 0.675$

$x = 1$; $F_X(1) = P(X \leq 1) = P(x_1) + P(x_2) + P(x_3) + P(x_4) + P(x_5)$
$= 0.675 + 0.075$
$= 0.75$

$x = 2$; $F_X(2) = P(X \leq 2) = P(x_1) + P(x_2) + P(x_3) + P(x_4) + P(x_5) + P(x_6)$
$= 0.75 + 0.25$
$= 1.0$

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X	−3	−2	−1	0	1	2
P(x) (pdf)	0·2	0·125	0·25	0·1	0·075	0·25
$F_X(x)$	0·2	0·325	0·575	0·675	0·75	1

The CDF of 'x' = $F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$

N=6 ⇒

$$= \sum_{i=1}^{6} P(x_i)\, u(x-x_i)$$

$$= P(x_1)\, u(x-x_1) + P(x_2)\, u(x-x_2) + P(x_3)u(x-x_3) + P(x_4)\, u(x-x_4) + P(x_5)\, u(x-x_5) + P(x_6)\, u(x-x_6)$$

$$\boxed{F_X(x) = 0{\cdot}2\, u(x+3) + 0{\cdot}125\, u(x+2) + 0{\cdot}25\, u(x+1) + 0{\cdot}1\, u(x) + 0{\cdot}075\, u(x-1) + 0{\cdot}25\, u(x-2)}$$

The PDF of 'x' = $f_X(x) = \sum_{i=1}^{N} P(x_i)\, \delta(x-x_i)$

$$= \sum_{i=1}^{6} P(x_i)\, \delta(x-x_i)$$

$$= P(x_1)\cdot\delta(x-x_1) + P(x_2)\, \delta(x-x_2) + P(x_3)\, \delta(x-x_3) + P(x_4)\, \delta(x-x_4) + P(x_5)\, \delta(x-x_5) + P(x_6)\, \delta(x-6)$$

$$\boxed{f_X(x) = 0{\cdot}2\, \delta(x+3) + 0{\cdot}125\, \delta(x+2) + 0{\cdot}25\, \delta(x+1) + 0{\cdot}1\, \delta(x) + 0{\cdot}075\, \delta(x-1) + 0{\cdot}25\, \delta(x-2)}$$

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→ CDF of 'X' :-

The CDF of $X = F_X(x) = P(X \le x)$

$x = 0$; $F_X(0) = P(X \le 0) = P(X=0) = \frac{1}{16} = 0.0625$

$x = 1$; $F_X(1) = P(X \le 1) = P(X=0) + P(X=1) = \frac{5}{16} = 0.3125$

$x = 2$; $F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{16} + \frac{4}{16} + \frac{5}{16} = \frac{11}{16} = 0.6875$

$x = 3$; $F_X(3) = P(X \le 3) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16} = 0.9375$

$x = 4$; $F_X(4) = P(X \le 4) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{16}{16} = 1$

X	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F_X(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

* Mathematical Expressions:

The CDF of $X = F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$

For $N = 5$

$F_X(x) = \sum_{i=1}^{5} P(x_i)\, u(x - x_i)$

$= P(x_1)\, u(x - x_1) + P(x_2)\, u(x - x_2) + P(x_3)\, u(x - x_3) + P(x_4)\, u(x - x_4) + P(x_5)\, u(x - x_5)$

$$\boxed{F_X(x) = \frac{1}{16}\, u(x) + \frac{4}{16}\, u(x-1) + \frac{6}{16}\, u(x-2) + \frac{4}{16}\, u(x-3) + \frac{1}{16}\, u(x-4)}$$

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The PDF of $X \Rightarrow f_X(x) = \sum_{i=1}^{N} P(x_i)\,\delta(x-x_i)$

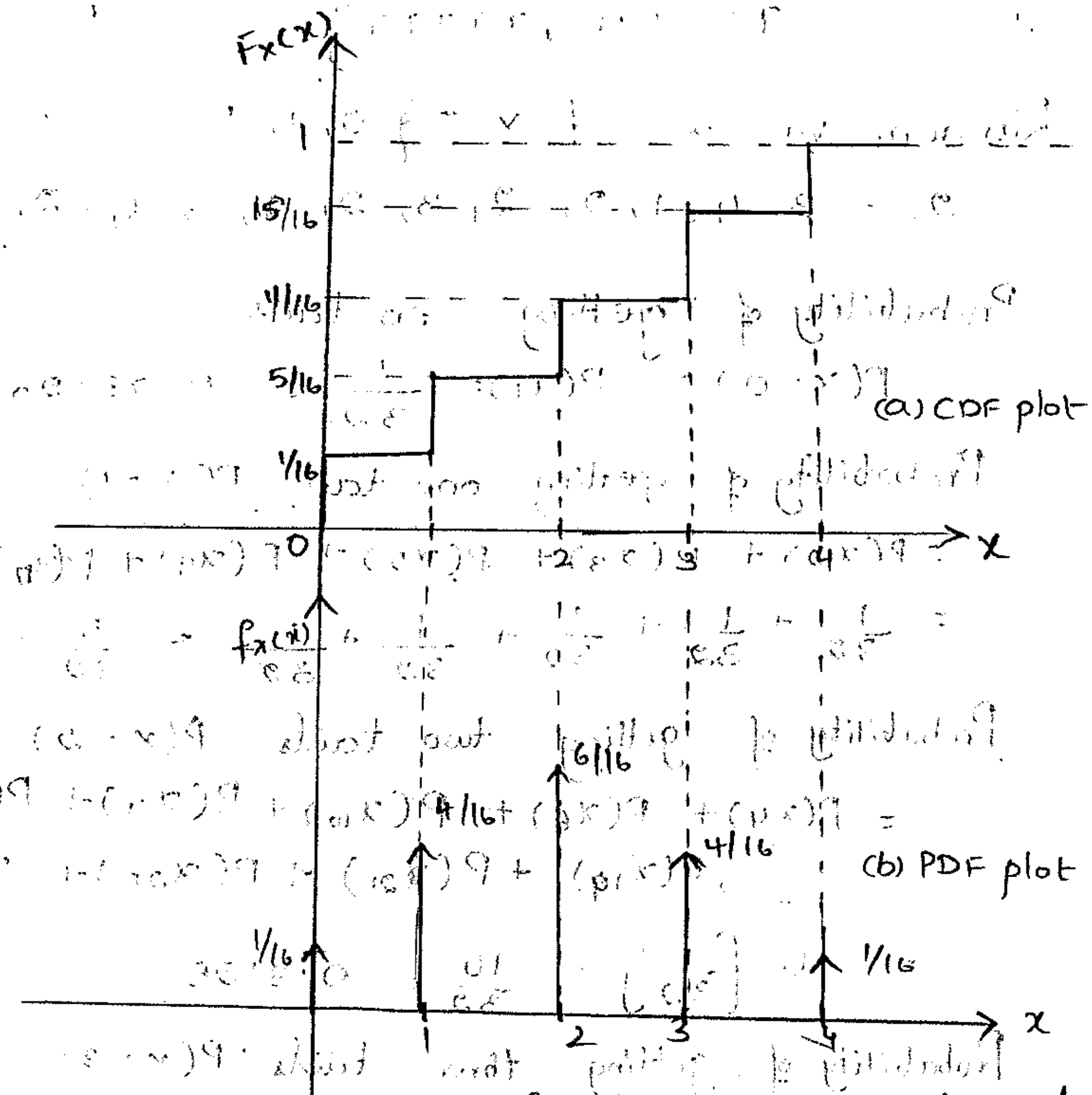
For $N=5$

$$= \sum_{i=1}^{5} P(x_i)\,\delta(x-x_i) = P(x_1)\,\delta(x-x_1) + P(x_2)\,\delta(x-x_2)$$

$$+ P(x_3)\,\delta(x-x_3) + P(x_4)\,\delta(x-x_4) + P(x_5)\,\delta(x-x_5)$$

$$\boxed{f_X(x) = \frac{1}{16}\delta(x) + \frac{4}{16}\delta(x-1) + \frac{6}{16}\delta(x-2) + \frac{4}{16}\delta(x-3) + \frac{1}{16}\delta(x-4)}$$

Graphical Plots:



10. A pair coin tossed 5 times for getting no. of heads. The random variable X is associated with no. of heads. Compute the CDF and PDF values and sketch the plots, respectively.

x	0	1	2	3	4	5
$P(x)$	1/32	5/32	10/32	10/32	5/32	1/32
$F_X(x)$	1/32	6/32	16/32	26/32	31/32	32/32

$X = \{0,1,1,2,1,2,2,3,1,2,2,3,2,3,3,4,1,2,2,3,2,3,3,4,2,3,3,4,$
$3,4,4,5\}$

Probability of each outcome = $\frac{1}{32}$ ($\because$ all are equally likely events)

Sample space of experiment

S = { HHHHH, HHHHT, HHHTH, HHHTT, HHTHH, HHTHT, HHTTH, HHTTT, HTHHH, HTHHT, HTHTH, HTHTT, HTTHH, HTTHT, HTTTH, HTTTT, THHHH, THHHT, THHTH, THHTT, THTHH, THTHT, THTTH, THTTT, TTHHH, TTHHT, TTHTH, TTHTT, TTTHH, TTTHT, TTTTH, TTTTT }

Random variable of X = { 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5 }

Probability of getting no tails

$P(X=0) = P(x_1) = \frac{1}{32} = 0.03125$

Probability of getting one tail $P(X=1)$

$= P(x_2) + P(x_3) + P(x_5) + P(x_9) + P(x_{17})$

$= \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{5}{32} = 0.15625$

Probability of getting two tails $P(X=2)$

$= P(x_4) + P(x_6) + P(x_{10}) + P(x_{11}) + P(x_{13}) + P(x_{18}) + P(x_{19}) + P(x_{21}) + P(x_{25}) + P(x_7)$

$= 10\left[\frac{1}{32}\right] = \frac{10}{32} = 0.3125$

Probability of getting three tails $P(X=3)$

$= P(x_8) + P(x_{12}) + P(x_{14}) + P(x_{15}) + P(x_{20}) + P(x_{22}) + P(x_{23}) + P(x_{26}) + P(x_{27}) + P(x_{29})$

$= 10\left[\frac{1}{32}\right] = \frac{10}{32} = 0.3125$

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9 Sol: Total no of possible outcomes of tossing 4 pair coins experiment = $2^4 = 16$.

Each element's probality (outcome) = $\frac{1}{16}$ ($\because$ all are equally likely events).

$\{H = 0 ; T = 1\}$

Sample space of experiment $S = \{$HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT$\}$

$\{$4 bit binary representation$\}$

Random variable $X = \{\underset{x_1}{0}, \underset{x_2}{1}, \underset{x_3}{1}, \underset{x_4}{2}, \underset{x_5}{1}, \underset{x_6}{2}, \underset{x_7}{2}, \underset{x_8}{3}, \underset{x_9}{1}, \underset{x_{10}}{2}, \underset{x_{11}}{2}, \underset{x_{12}}{3}, \underset{x_{13}}{2}, \underset{x_{14}}{3}, \underset{x_{15}}{3}, \underset{x_{16}}{4}\}$

Probability for getting no tails = Probability of x $(P(X=0))$

$= P(x_1) = \frac{1}{16} = \underline{0.0625}$

Probability for getting 1 tail = $P(X=1) = P(x_2) + P(x_3) + P(x_5) + P(x_9)$

$= \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16}$

$= \frac{4}{16} = \frac{1}{4} = \underline{0.25}$

Probability for getting 2 tails = $P(X=2) = P(x_4) + P(x_6) + P(x_7) + P(x_{10}) + P(x_{11}) + P(x_{13})$

$= \frac{6}{16} = \underline{0.375}$

Probability for getting 3 tails = $P(X=3) = P(x_8) + P(x_{12}) + P(x_{14}) + P(x_{15})$

$= \frac{4}{16} = \underline{0.25}$

Probability for getting 4 tail = $P(X=4) = P(x_{16}) = \frac{1}{16}$

$= \underline{0.0625}$

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→ CDF of X :-

The CDF of $X = F_X(x) = P(X \le x)$

$x = 0$; $F_X(0) = P(X \le 0) = P(X=0) = \frac{1}{16} = 0.0625$

$x = 1$; $F_X(1) = P(X \le 1) = P(X=0) + P(X=1) = \frac{5}{16} = 0.3125$

$x = 2$; $F_X(2) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \frac{1}{16} + \frac{4}{16} + \frac{5}{16} = \frac{11}{16} = 0.6875$

$x = 3$; $F_X(3) = P(X \le 3) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} = \frac{15}{16} = 0.9375$

$x = 4$; $F_X(4) = P(X \le 4) = \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{16}{16} = 1$

X	0	1	2	3	4
P(x)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
$F_X(x)$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

* Mathematical Expressions:

The CDF of $X = F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$

For $N = 5$

$$F_X(x) = \sum_{i=1}^{5} P(x_i)\, u(x - x_i)$$

$$= P(x_1)\, u(x - x_1) + P(x_2)\, u(x - x_2) + P(x_3)\, u(x - x_3) + P(x_4)\, u(x - x_4) + P(x_5)\, u(x - x_5)$$

$$\boxed{F_X(x) = \frac{1}{16} u(x) + \frac{4}{16} u(x-1) + \frac{6}{16} u(x-2) + \frac{4}{16} u(x-3) + \frac{1}{16} u(x-4)}$$

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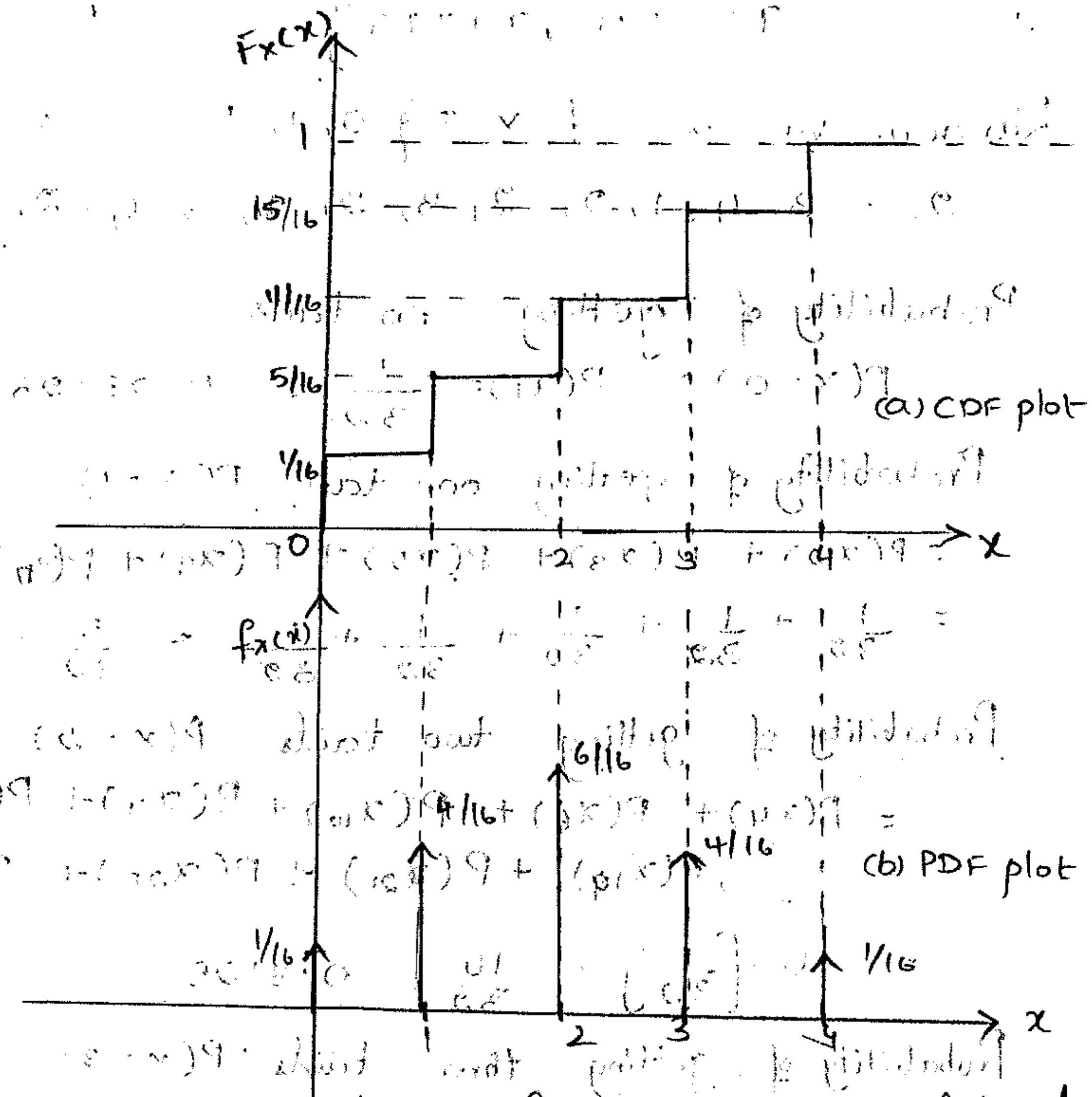
The PDF of X = $f_X(x) = \sum_{i=1}^{N} P(x_i)\,\delta(x-x_i)$

For N=5

$$= \sum_{i=1}^{5} P(x_i)\,\delta(x-x_i) = P(x_1)\,\delta(x-x_1) + P(x_2)\,\delta(x-x_2) + P(x_3)\,\delta(x-x_3) + P(x_4)\,\delta(x-x_4) + P(x_5)\,\delta(x-x_5)$$

$$\boxed{f_X(x) = \frac{1}{16}\delta(x) + \frac{4}{16}\delta(x-1) + \frac{6}{16}\delta(x-2) + \frac{4}{16}\delta(x-3) + \frac{1}{16}\delta(x-4)}$$

Graphical Plots:



(a) CDF plot

(b) PDF plot

10. A pair coin tossed 5 times for getting no. of heads. The random variable X is associated with no. of heads. Compute the CDF and PDF values and sketch the plots, respectively.

x	0	1	2	3	4	5
P(x)	1/32	5/32	10/32	10/32	5/32	1/32
$F_X(x)$	1/32	6/32	16/32	26/32	31/32	32/32

X = { 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5 }

Probability of each outcome = $\frac{1}{32}$ ($\because$ all are equally likely events)

Sample space of experiment

S = { HHHHH, HHHHT, HHHTH, HHHTT, HHTHH,
HHTHT, HHTTH, HHTTT, HTHHH, HTHHT,
HTHTH, HTHTT, HTTHH, HTTHT, HTTTH,
HTTTT, THHHH, THHHT, THHTH, THHTT,
THTHH, THTHT, THTTH, THTTT, TTHHH,
TTHHT, TTHTH, TTHTT, TTTHH, TTTHT,
TTTTH, TTTTT }

Random variable of x = { 0, 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3,
2, 3, 3, 4, 1, 2, 2, 3, 2, 3, 3, 4, 2, 3, 3, 4, 3, 4, 4, 5

Probability of getting no tails
$P(x=0) = P(x_1) = \frac{1}{32} = 0.03125$

Probability of getting one tail $P(x=1)$
$= P(x_2) + P(x_3) + P(x_5) + P(x_9) + P(x_{17})$
$= \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{5}{32} = 0.15625$

Probability of getting two tails $P(x=2)$
$= P(x_4) + P(x_6) + P(x_{10}) + P(x_{11}) + P(x_{13}) + P(x_{18}) +$
$P(x_{19}) + P(x_{21}) + P(x_{25}) + P(x_7)$

$= 10\left[\frac{1}{32}\right] = \frac{10}{32} = 0.3125$

Probability of getting three tails $P(x=3)$
$= P(x_8) + P(x_{12}) + P(x_{14}) + P(x_{15}) + P(x_{20}) + P(x_{22}) + P(x_{23}) +$
$P(x_{26}) + P(x_{27}) + P(x_{29})$

$= 10\left[\frac{1}{32}\right] = \frac{10}{32} = 0.3125$

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(iv) Probability of getting four tails $P(X=4)$

$$= P(x_{16}) + P(x_{24}) + P(x_{28}) + P(x_{30}) + P(x_{31})$$

$$= \frac{5}{32} = 0.15625$$

(v) Probability of getting five tails $P(x=5)$

$$P(x_{32}) = \frac{1}{32} = 0.03125$$

CDF of X:-

The CDF of x is $F_X(x) = P(X \leq x)$

$x=0$; $F_X(0) = P(X \leq 0) = P(X=0) = \frac{1}{32}$

$x=1$; $F_X(1) = P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{32} + \frac{5}{32} = \frac{6}{32}$

$x=2$; $F_X(2) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$$= \frac{6}{32} + \frac{10}{32} = \frac{16}{32}$$

$x=3$; $F_X(3) = P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \frac{16}{32} + \frac{10}{32} = \frac{26}{32}$$

$x=4$; $F_X(4) = P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{26}{32} + \frac{5}{32} = \frac{31}{32}$$

$x=5$; $F_X(5) = P(X \leq 5) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$$= \frac{31}{32} + \frac{1}{32} = \frac{32}{32} = 1$$

x	0	1	2	3	4	5
P(x)	1/32	5/32	10/32	10/32	5/32	1/32
Fx(x)	1/32	6/32	16/32	26/32	31/32	1

Mathematical Expression: The CDF of X

$$F_X(x) = \sum_{i=1}^{N} P(x_i)\, u(x - x_i)$$

$$F_X(x) = \sum_{i=1}^{6} P(x_i)\, u(x - x_i) = P(x_1)\, u(x - x_1) + P(x_2)\, u(x - x_2) + P(x_3)\, u(x - x_3) + P(x_4)\, u(x - x_4) + P(x_5)\, u(x - x_5) + P(x_6)\, u(x - x_6)$$

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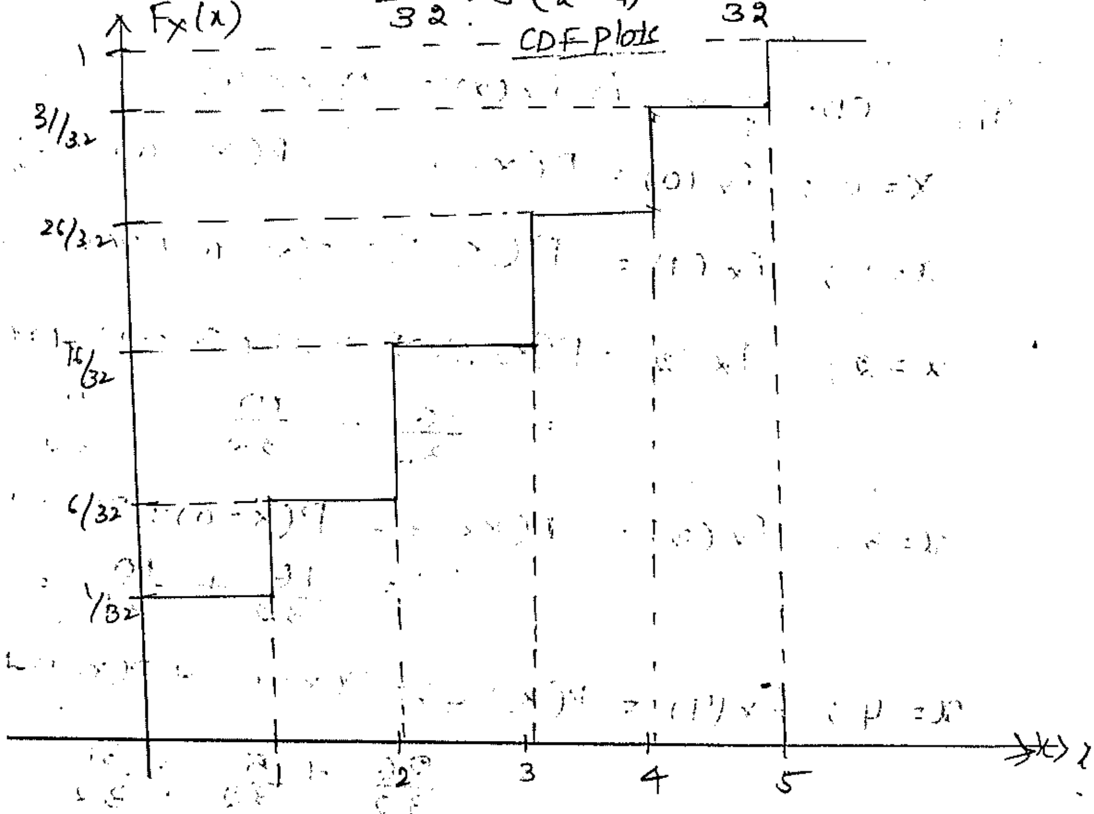
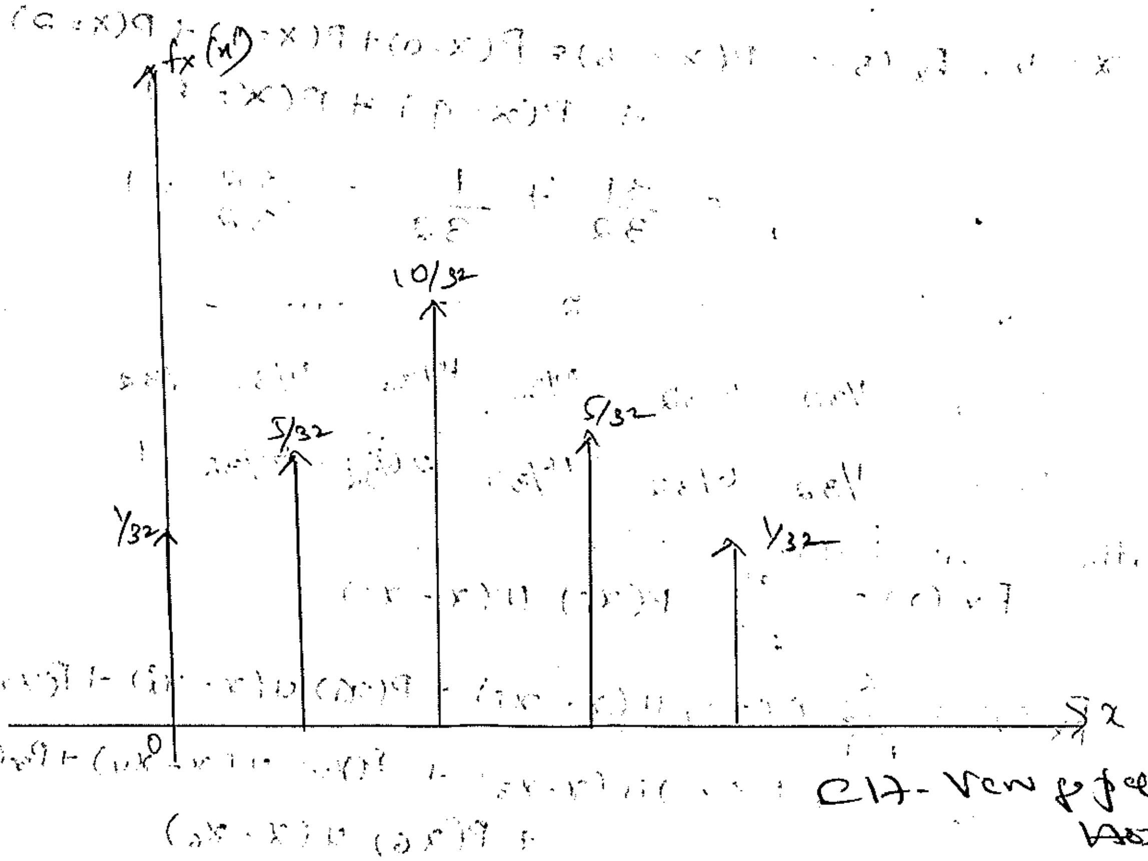
$$F_X(x) = \frac{1}{32}u(x) + \frac{5}{32}u(x-1) + \frac{10}{32}u(x-2) + \frac{10}{32}u(x-3) + \frac{5}{3}u(x-4) + \frac{1}{32}u(x-5)$$

The PDF of $X = f_X(x) = \sum_{i=1}^{N} P(x_i)\,\delta(x - x_i)$

$$f_X(x) = \frac{1}{32}\delta(x) + \frac{5}{32}\delta(x-1) + \frac{10}{32}\delta(x-2) + \frac{10}{32}\delta(x-3) + \frac{5}{32}\delta(x-4) + \frac{1}{32}\delta(x-5)$$

Plots:





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* Standard Distribution Functions:- There are 6 SDFs.

(i) Binomial distribution function

(ii) Poisson distribution function

(iii) Uniform or rectangular distribution function

(iv) Exponential distrubution function

(v) Rayleigh distribution function

(vi) Gaussian or Normal distribution function

(i) Binomial Distribution Function:

Let us consider an experiment which has only two outcomes where one outcome is called as success and another as failure. Let the experiment repeated by the no. of trials. These are nothing but "Bernoulli trials" and let us consider probability of success is 'P' in each trial is same. The trials are independent, then the probability of failure in each trial is "Q = 1 − P" ("q = 1 − p"). The probability of k success in 'n' trials is given by the probability function known as binomial probability density function.

$$\boxed{\text{The Binomial PDF of } 'X' = f_X(x) = \sum_{k=0}^{n} {}^{n}C_{k}\, p^k q^{n-k}\, \delta(x-k)}$$

where n – no. of trials

p – probability of success event

$q = 1-p$ – probability of failure event

k = no. of success out of 'n' trials

$${}^{n}C_{k} = \frac{n!}{k!\,(n-k)!}$$

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The Binomial CDF of $X = F_X(x) = \sum_{k=0}^{n} {}^nC_k \, p^k q^{n-k} \, u(x-k)$

* Apps :

1. The binomial expression can be applied for Bernoulli trials
2. It is applied to many games.
3. It is used to find out the detection problems in radars and solar systems.

Binomial distribution can be applied for the following conditions: under

1. The no. of trials are finite i.e, n is finite $(n < \infty)$
2. The trials are independent to each other.
3. The probability of success in each trial is same
4. Each trials results is two mutually exclusive events, one is success and another one is failure.

(ii) Poisson Distribution Function :

It is a limiting case of binomial distribution under the following conditions:

1. The no. of trials are indefinitely large i.e., $n \to \infty$.
2. Probability of success indefinitely small i.e., $p \to 0$.

and so we will define another distribution function i.e., poisson distribution function.

The Poisson ~~probability~~ PDF of random variable X is defined by

$$\boxed{\text{The Poisson PDF of } `x' = f_x(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\,\delta(x-k)}$$

where constant $= \lambda = np$

n – no. of trials

p = probability of success.

$$\boxed{\text{The poisson CDF of } `x' = F_x(x) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}\,u(x-k)}$$

*Applns :

1. The poisson distribution can be applied for a wide variety counting type applications.
2. It is used to identify no. of defectives in a sample that can be taken from manufacturing company.
3. It is used to describe the no. of telephone calls during the intervals.
4. It is used to identify no. of electrons emitted from cathode in a given interval.

(iii) Uniform or Rectangular Distrubution Function:

A random variable X is said to be uniform distribution if its probability density function is constant over a interval (a, b). The uniform PDF of X is defined by

$$\boxed{\text{The uniform PDF of } `x' = f_x(x) = \begin{cases} c & ; \ a \le x \le b \\ 0 & ; \ \text{elsewhere} \end{cases}}$$

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We know Area under PDF is

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{a} (0)\,dx + \int_{a}^{b} c\,dx + \int_{a}^{\infty} (0)\,dx = 1$$

$$c\; x\Big|_a^b = 1$$

$$c(b-a) = 1$$

$$c = \frac{1}{b-a}$$

$$\therefore \boxed{f_X(x) = \begin{cases} \dfrac{1}{b-a} & ; \; a \le x \le b \\ 0 & ; \; \text{elsewhere.} \end{cases}}$$

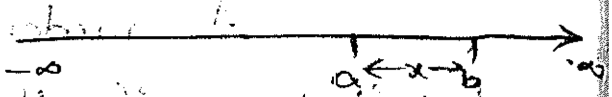
The CDF of $X = F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

(i) Case i: If $x < a$ i.e. $-\infty < x < a$

$$F_X(x) = \int_{-\infty}^{x} (0)\,dx = 0$$

$$F_X(x) = 0 \; ; \; x < a$$

Case (ii): If $a \le x \le b$



$$F_X(x) = \int_{-\infty}^{a} (0)\,dx + \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= 0 + \frac{x}{b-a}\Big|_a^x$$

$$\therefore F_X(x) = \frac{x-a}{b-a} \; ; \; a \le x \le b$$

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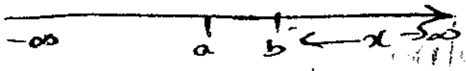
Case (iii): If $x > b$



$$= \int_{-\infty}^{a} (0)\,dx + \int_{a}^{b} \frac{1}{(b-a)}\,dx + \int_{b}^{x} (0)\,dx$$

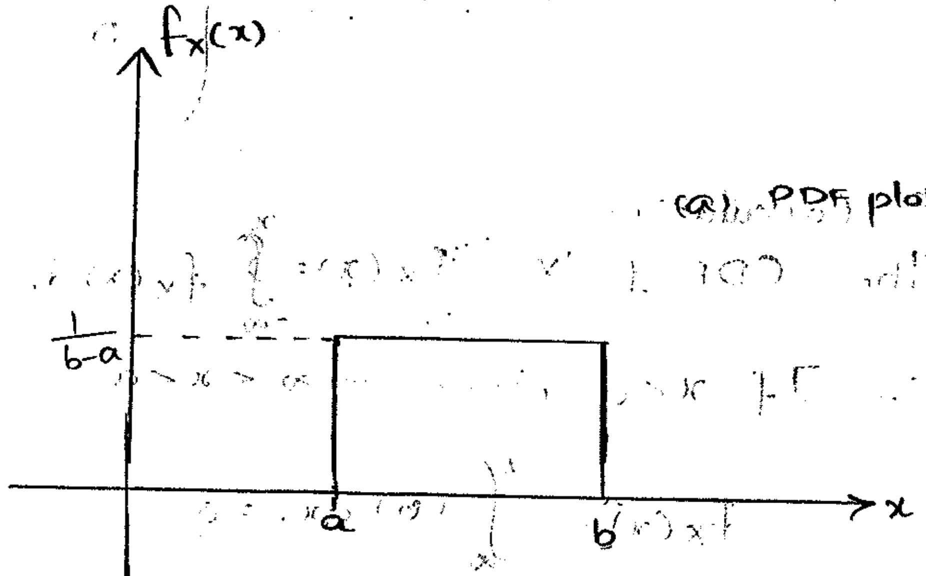
$$= \frac{x}{b-a}\Big|_{a}^{b}$$

$$= \frac{b-a}{b-a}$$

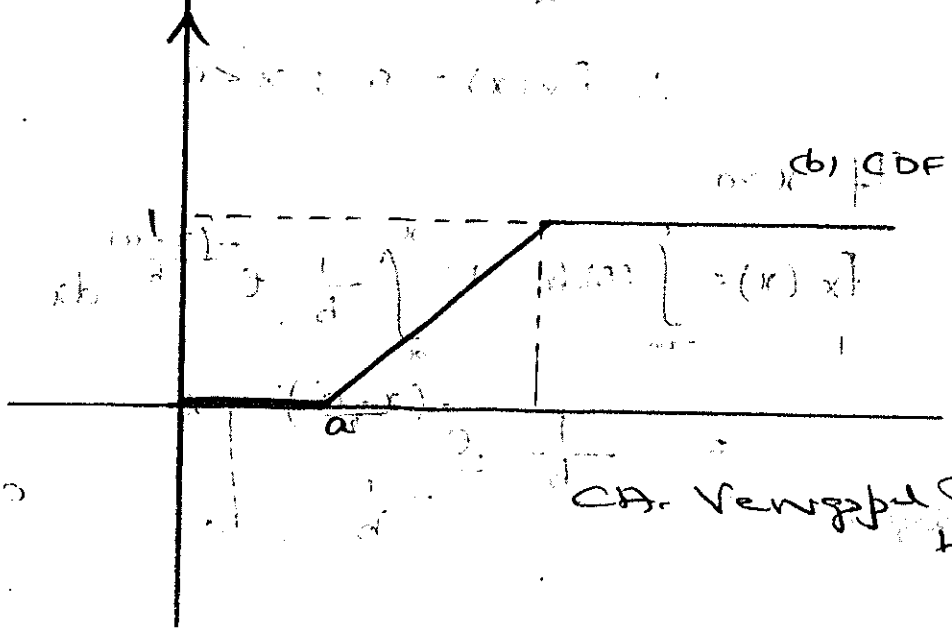
$$= 1$$

$$\therefore F_X(x) = 1 \; ; \; x > b$$

$$\boxed{\therefore F_X(x) = \begin{cases} 0 & ; \; x < a \\ \frac{x-a}{b-a} & ; \; a \le x \le b \\ 1 & ; \; x > b \end{cases}}$$



(a) PDF plot



(b) CDF plot

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* Applns:

1. The random distribution of errors introduced in the roundoff process are uniformly distributed.

2. In digital communications, when a sample of the signal is roundoff to its ~~digit~~ nearest level or in a game, when a real number is converted into an integer the PDF of errors are uniformly distributed.

(iv) Exponential Distribution Function:

The exponential PDF of continuous random variable X is defined by

$$\boxed{\text{Exponential PDF of } 'X' = f_X(x) = \begin{cases} \frac{1}{b} e^{-\left(\frac{x-a}{b}\right)} & : x > a \\ 0 & : x < a \end{cases}}$$

* CDF Calculations:

The CDF of 'X' $= F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Case (i): If $x < a$, i.e., $-\infty < x < a$

$$F_X(x) = \int_{-\infty}^{x} (0)\,dx = 0$$

$$\therefore F_X(x) = 0 \;;\; x < a$$

If $x > a$

$$F_X(x) = \int_{-\infty}^{a} (0)dx + \int_{a}^{x} \frac{1}{b} e^{-\frac{(x-a)}{b}} dx$$

$$= \frac{1}{\cancel{b}} \frac{e^{-\left(\frac{x-a}{b}\right)}}{-\frac{1}{\cancel{b}}} \Bigg|_a^x$$

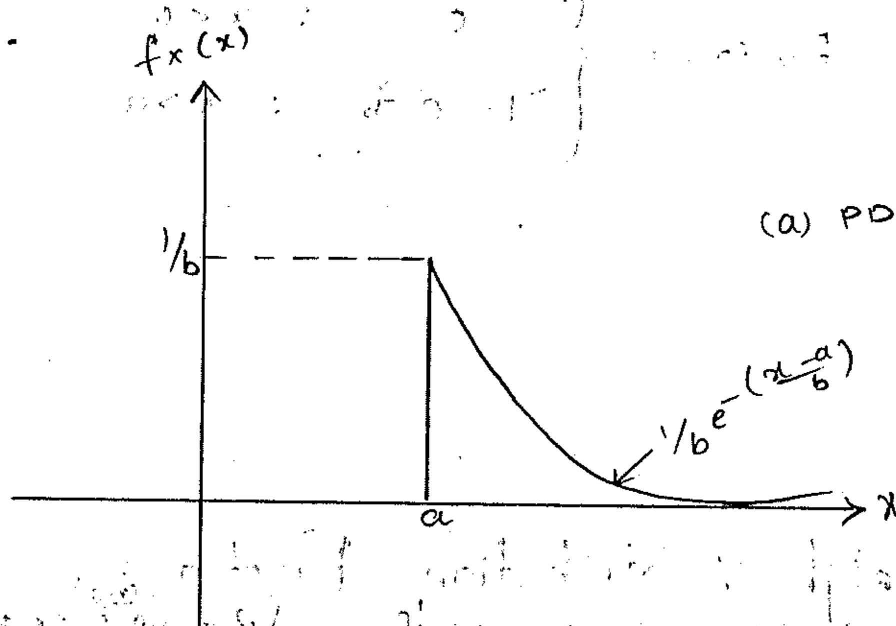
$$= -\left[e^{-\left(\frac{x-a}{b}\right)} - e^{-\left(\frac{a-a}{b}\right)} \right]$$

$$= -\left[e^{-\frac{(x-a)}{b}} - 1 \right]$$

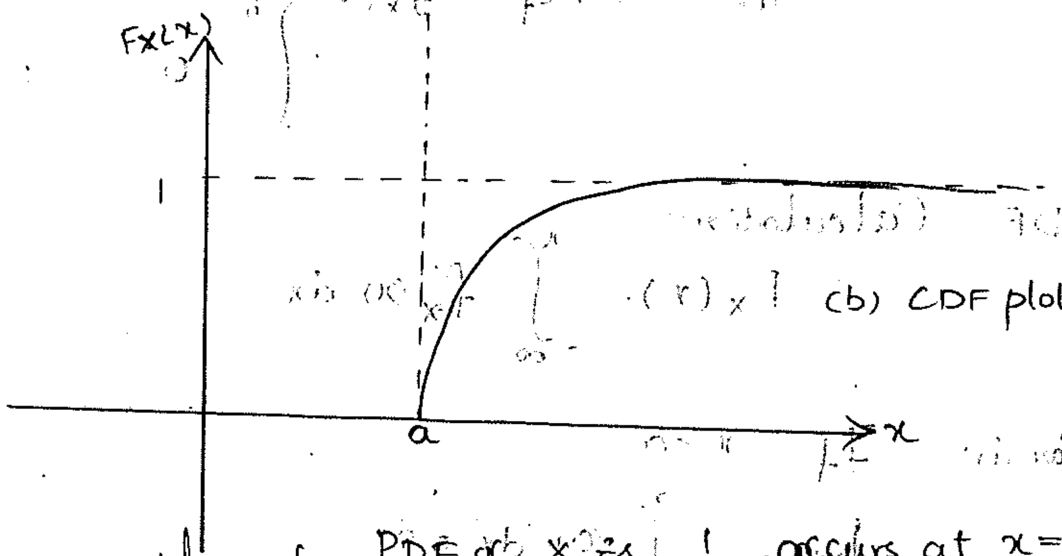
$$= 1 - e^{-\frac{(x-a)}{b}} \; ; \; x > a$$

$$\boxed{F_X(x) = \begin{cases} 0 & : x < a \\ 1 - e^{-\frac{(x-a)}{b}} & : x > a \end{cases}}$$

Plots:



(a) PDF plot



(b) CDF plot

Maximum value of PDF of X is $\frac{1}{b}$ occurs at $x = a$ i.e.,

$$\boxed{f_X(x)\big|_{max} = \frac{1}{b} \text{ at } x = a}$$

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* Applns:

1. The distributions of fluctuations in the signal strength recieved by radar recievers from certain types of targets.

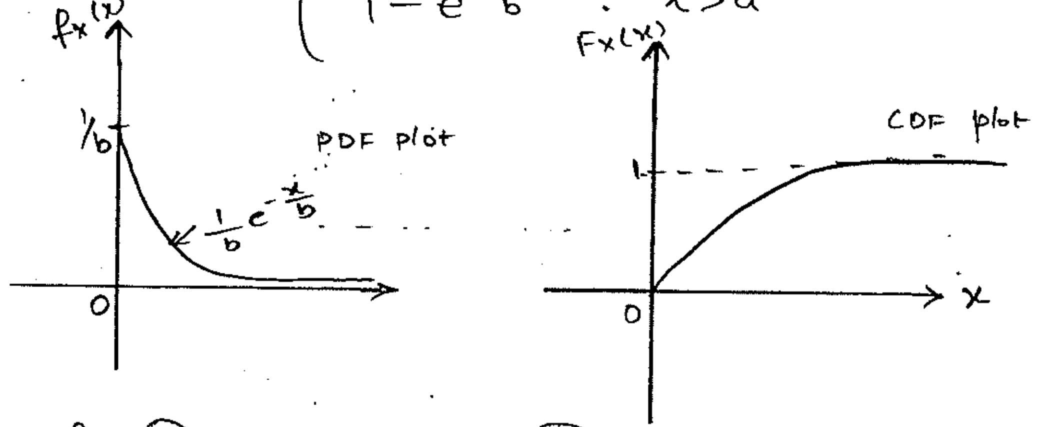
2. The distribution of raindrop sizes when a large number of rainstorm measurements are made.

Note:

If $a = 0$ then

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{x}{b}} & : x > 0 \\ 0 & : x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & : x < 0 \\ 1 - e^{-\frac{x}{b}} & : x > a \end{cases}$$



(V) Rayleigh Distribution Function:

$$\boxed{\text{The PDF of } X \quad f_X(x) = \begin{cases} \frac{2}{b}(x-a)e^{-\frac{(x-a)^2}{b}} & x > a \\ 0 & : x < a \end{cases}}$$

* CDF Calculation:

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$$

Case (i): If $x < a$

$$F_X(x) = \int_{-\infty}^{x} (0)\, dx = 0$$

Case (ii): If $x > a$

$$F_X(x) = \int_{-\infty}^{a} (0)\, dx + \int_{a}^{x} \frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}\, dx$$

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Put $\dfrac{(x-a)^2}{b} = t$

$$\frac{2(x-a)}{b}\,dx = dt$$

$x \to a \;;\; t \to 0$

$x \to x \;;\; t \to \dfrac{(x-a)^2}{b}$

$$\Rightarrow F_X(x) = \int_0^{\frac{(x-a)^2}{b}} e^{-t}\,dt$$

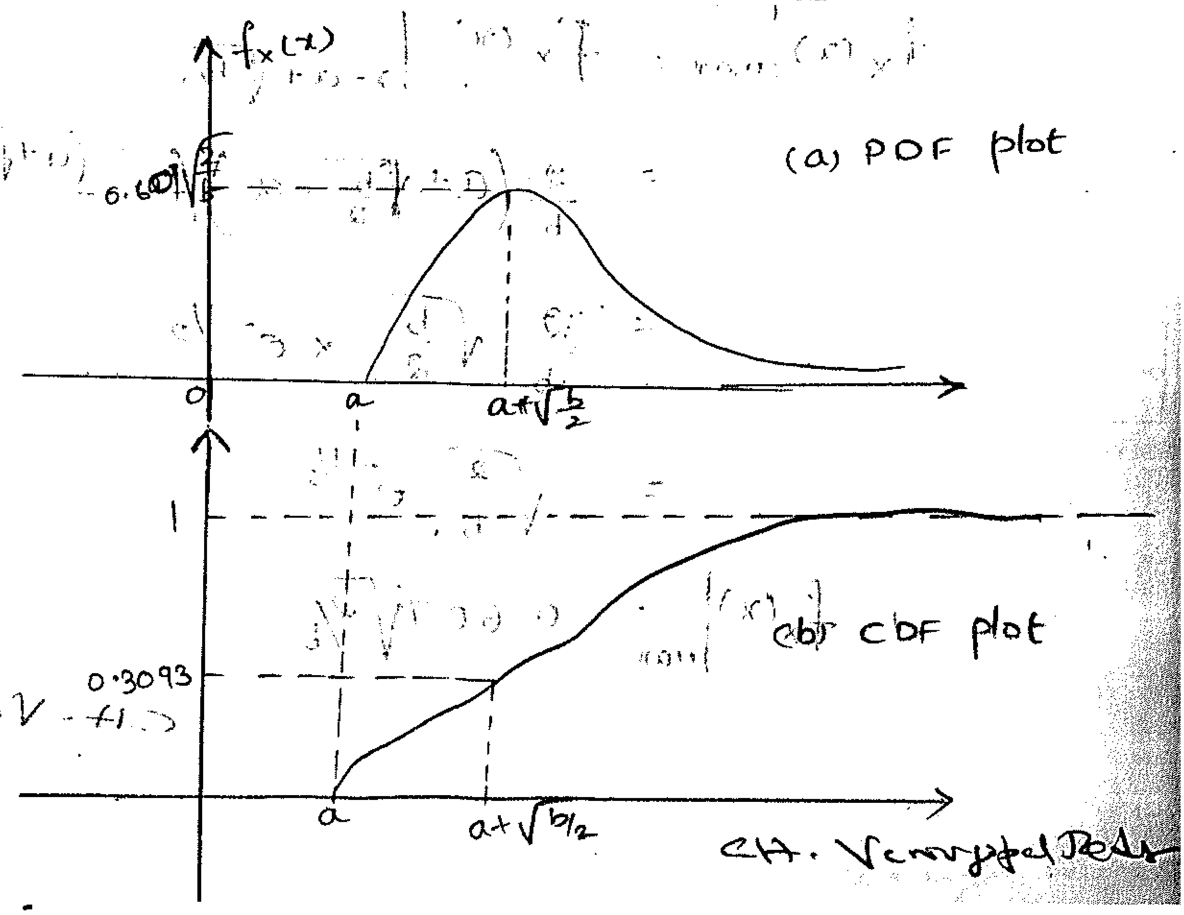
$$= \left[-e^{-t}\right]_0^{\frac{(x-a)^2}{b}}$$

$$= -\left[e^{-\frac{(x-a)^2}{b}} - e^{0}\right]$$

$$F_X(x) = 1 - e^{-\frac{(x-a)^2}{b}}$$

$$\boxed{\therefore\; F_X(x) = \begin{cases} 0 & : x < a \\ 1 - e^{-\frac{(x-a)^2}{b}} & : x > a \end{cases}}$$

Plots:



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Maximum of PDF of X is given by

$$\frac{d}{dx}[f_X(x)] = 0$$

$$\frac{d}{dx}\left[\frac{2}{b}(x-a)\, e^{-\frac{(x-a)^2}{b}}\right] = 0$$

$$\frac{2}{b}\left[(x-a)\, e^{-\frac{(x-a)^2}{b}} \times \left(\frac{-2(x-a)}{b}\right) \times 1 + e^{-\frac{(x-a)^2}{b}} \times 1\right] = 0$$

$$\frac{2}{b}\, e^{-\frac{(x-a)^2}{b}} \left[\frac{-2(x-a)^2}{b} + 1\right] = 0$$

$$\frac{-2(x-a)^2}{b} = -1$$

$$(x-a)^2 \Rightarrow \frac{b}{2}$$

$$x - a = \sqrt{\frac{b}{2}}$$

$$\boxed{\therefore\ x = a + \sqrt{\frac{b}{2}}}$$

$$f_X(x)\Big|_{max} = f_X(x)\Big|_{x=a+\sqrt{b/2}}$$

$$= \frac{2}{b}\left(a+\sqrt{\frac{b}{2}} - a\right)\left(e^{-\frac{(a+\sqrt{b/2}-a)^2}{b}}\right)$$

$$= \frac{2}{b}\sqrt{\frac{b}{2}} \times e^{-1/2}$$

$$= \sqrt{\frac{2}{b}}\, e^{-1/2}$$

$$\boxed{f_X(x)\Big|_{max} = 0.607\sqrt{2/b}}$$

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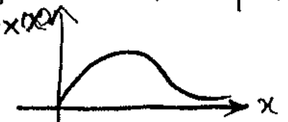
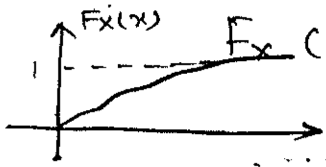
*Applns:

1. It describes the envelope of white noise when the noise is passed through a band pass filter.
2. The Rayleigh density function has a relationship with Gaussian density function.
3. Some types of signal fluctuations recieved by the reciever are modeled as Rayleigh distribution.

Note:

If $a = 0$, $f_X(x) = \begin{cases} \frac{2}{b} x e^{-\frac{x^2}{b}} & ; x > 0 \\ 0 & ; x < 0 \end{cases}$





$F_X(x) = \begin{cases} 0 & ; x < 0 \\ 1 - e^{-\frac{x^2}{b}} & ; x > 0 \end{cases}$

(vi) Gaussian or Normal Distribution Function:

(a) Gaussian Random Variable: A random variable that satisfies gaussian density function is called a gaussian random variable.

The gaussian density function of random variable 'X' is defined by

$$\boxed{\text{The Gaussian PDF of 'X'} = f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; -\infty < x < \infty}$$

(b) $$\boxed{\text{The Gaussian CDF of 'X'} = F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}$$

where

μ (or) m (or) a_x — mean or average value of 'X'

σ_x^2 (or) σ^2 = Variance of 'X'

σ_X (or) σ = Standard deviation (SD) of 'X'.

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4. $P((x_1 < X \le x_2)/B) = F_X(x_2/B) - F_X(x_1/B)$

Proof: Like $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$$P((x_1 < X \le x_2)/B) = F_X(x_2/B) - F_X(x_1/B)$$

5. If $x_1 < x_2$, $F_X(x_1/B) \le F_X(x_2/B)$

Proof: like if $x_1 < x_2$, $F_X(x_1) \le F_X(x_2)$

$$x_1 < x_2,\ F_X(x_1/B) \le F_X(x_2/B)$$

* Conditional Density Function:

The conditional density function of random variable X is defined as differentiation of conditional distribution function of random variable X, i.e.,

$$\boxed{f_X(x/B) = \frac{d}{dx}\left[F_X(x/B)\right]}$$

* Properties:

1. $f_X(x/B) \ge 0$

Proof: We know $0 \le F_X(x/B) \le 1$

$$f_X(x/B) = \frac{d}{dx}\left[F_X(x/B)\right]$$

$$= \frac{d}{dx}\left[F_X(x)\right] \ge 0$$

Hence $f_X(x/B) \ge 0$

2. Area under conditional density function is unity

$$\boxed{\text{i.e., } \int_{-\infty}^{\infty} f_X(x/B)\,dx = 1}$$

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Proof: Consider L.H.S $= \int_{-\infty}^{\infty} f_X(x/B)\,dx$

$$= \int_{-\infty}^{\infty} \frac{d}{dx} F_X(x/B)\,dx$$

$$= \int_{-\infty}^{\infty} 1 \cdot d\left[F_X(x/B)\right] \qquad \because f_X(x/B) = \frac{d}{dx}\left[F_X(x/B)\right]$$

$$= \left[F_X(x/B)\right]_{-\infty}^{\infty} \qquad F_X(-\infty/B) = 0 \;\&\; F_X(\infty/B) = 1$$

$$= F_X(\infty/B) - F_X(-\infty/B)$$

$$= 1 - 0$$

$$\Rightarrow R.H.S = 1$$

3. $\int_{-\infty}^{x} f_X(x/B)\,dx = F_X(x/B)$

Proof: We know that $= \int_{-\infty}^{x} f_X(x/B)\,dx$

$$= \int_{-\infty}^{x} 1 \cdot d\left[F_X(x/B)\right] dx$$

$$= F_X(x/B) - F(-\infty/B)$$

$$= F_X(x/B) - 0$$

$$\int_{-\infty}^{x} f_X(x/B)\,dx = F_X(x/B)$$

4. $\int_{x_1}^{x_2} f_X(x/B)\,dx = F_X(x_2/B) - F_X(x_1/B)$

Proof: $\int_{x_1}^{x_2} f_X(x/B) = \int_{x_1}^{x_2} 1 \cdot d\left[F_X(x/B)\right] dx$

$$= F_X(x_2/B) - F_X(x_1/B)$$

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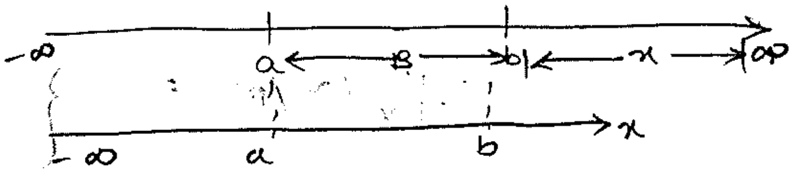
$$F_X(x/B) = \frac{P(X \le x \cap a < X \le b)}{P(a < X \le b)} = \frac{P(a < X \le x)}{P(a < X \le b)} = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} \; ; \; a < x \le b$$

$\because x_1$ intersect x_2

$\therefore P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

Case(iii): If $x \geq b$



$$X \le x \cap a < X \le b = a < X \le b$$

$$F_X(x/B) = \frac{P(X \le x \cap a < X \le b)}{P(a < X \le b)} = \frac{P(a < X \le b)}{P(a < X \le b)} = 1 \; ; \; x \ge b$$

$$F_X(x/B) = 1 \quad ; \quad x \geq b$$

$$\boxed{F_X(x/B) = \begin{cases} 0 & ; \; x < a \\ \dfrac{F_X(x) - F_X(a)}{F_X(a) - F_X(b)} & ; \; a \le x \le b \\ 1 & ; \; x \geq b \end{cases}}$$

Apply $\frac{d}{dx}$ to the above equation will get a conditional density function.

$$\frac{d}{dx} F_X(x/B) = \begin{cases} \frac{d}{dx}(0) & ; \; x < a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} & ; \; a < x \le b \\ 0 & ; \; x \geq b \end{cases}$$

$\int_{x_1}^{x_2} f_X(x) = F_X(x_2) - F_X(x_1)$

$$\boxed{f_X(x/B) = \begin{cases} \dfrac{f_X(x)}{\int_a^b f_X(x)\,dx} & ; \; a \le x \le b \\ 0 & ; \; \text{otherwise} \end{cases}}$$

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* Problems:

1. Let the CDF of random variable X is $F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$. Find the probabilities (i) $P(-\infty < X \leq 6.5)$ (ii) $P(X > 4)$ (iii) $P(6 < X \leq 9)$.

Sol: Given CDF of random variable X is $F_X(x) = \sum_{n=1}^{12} \frac{n^2 u(x-n)}{650}$

The CDF of X = $F_X(x) = \underset{\downarrow}{P(X \leq x)} = \sum_{n=1}^{N} P(x_n) u(x-n)$

$P(-\infty < X \leq x)$

(i) $P(-\infty < X \leq 6.5) = P(X \leq 6.5) = F_X(6.5)$

$$= \sum_{n=1}^{12} \frac{n^2}{650} u(6.5-n)$$

We know for unit step signal $u(n) = \begin{cases} 1 ; & n \geq 0 \\ 0 ; & n < 0 \end{cases}$

$$\Rightarrow u(6.5-n) = \begin{cases} 1 ; & 6.5 - n \geq 0 \Rightarrow 6.5 \geq n \\ & n \leq 6.5 \\ 0 ; & 6.5 - n < 0 \Rightarrow n > 6.5 \end{cases}$$

$$= \sum_{n=1}^{6.5} \frac{n^2}{650}(1) + \sum_{n=6.5+}^{12} \frac{n^2}{650}(0)$$

$$= \frac{1}{650} \sum_{n=1}^{6.5} n^2$$

We know from G.P $\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

$$= \frac{1}{650} \cdot \frac{1}{6} (6)(6+1)(2\times6+1)$$

$$= \frac{1}{650}(7)(12+1) = \frac{1}{650}(7)(13)$$

$$= 0.14$$

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$$P(X \le 6.5) = \overset{(OR)}{P}(X \le 6) = F_X(6)$$

$$= \sum_{n=1}^{6} \frac{n^2}{650}$$

$$= \frac{1 \times 6 \times 7 \times 13}{650 \times 6}$$

$$\therefore F_X(6) = 0.14$$

(ii) Wkt, $P(X \le x) + P(X > x) = 1 = P(S)$

$$P(X > x) = 1 - P(X \le x)$$

$$P(X > 4) = 1 - P(X \le 4)$$

$$= 1 - \sum_{n=1}^{4} \frac{n^2}{650}$$

$$= 1 - \frac{1}{650} \times \frac{4 \times 5 \times 9}{6}$$

$$P(X > 4) = 0.95$$

(iii) $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$$P(6 < X \le 9) = F_X(9) - F_X(6)$$

$$= \sum_{n=1}^{9} \frac{n^2}{650} - \sum_{n=1}^{6} \frac{n^2}{650}$$

$$= \frac{1}{650}\left[\sum_{n=1}^{9} n^2 - \sum_{n=1}^{6} n^2\right]$$

$$= \frac{1}{650}\left[\frac{9 \times 10 \times 19}{6} - \frac{6 \times 7 \times 13}{6}\right]$$

$$= \frac{1}{650}\left[\frac{90 \times 10}{6} - \frac{6 \times 7 \times 13}{6}\right]$$

0.29