UNIT - I

PROBABILITY AND RANDOM VARIABLE

Definition:

Set (or) Class : It is defined as "a well defined group of elements or objects or members"

- Even a set of sets called set.
- Represented by Capital letters A, B, C, D ....

Elements: The members of a set are elements represented using small letter as a, b, c, d ...

→ Let $A = \{a, b, c, d\}$, Where a is the Element of A it means $a \in A$.

→ Sets are represented in two ways as tabular and rule method.

Example of set

→ Tabular method:

$A = \{$ Lion, Tiger, Camel, deer, Kangaroo ... $\}$

$B = \{$ Car, Truck, School bus, Bicycle, Motor Cycle $\}$.

→ Rule method:

$A = \{x : x = \text{Wild animals}\}$

$B = \{y : y = \text{Vehicles}\}$.

Types of Sets

→ Countable

→ Uncountable

→ Null set

→ Single ton

→ Finite set

→ Infinite set

→ Sub set, Proper set

→ Disjoint sets

→ Universal set.

→ Equivalent sets

→ Equal sets

Countable sets: If the elements in a set are one to one correspondence of numbers.

Examples:

$A = \{0, 1, 2, 3\}$

$B = \{0, 1, 2, 3, 4 \ldots\}$

$C = \{x : x = \text{ECE students}\}$

$D = \{x : x = \text{vowels}\}$

Uncountable sets: If a set is not having any end count that set is defined as uncountable sets. Instead of one to one mapping there is a range of value representation.

Examples:

$A = \{x : 1 < x < 2\}$

$B = \{-1 \leq x < 1\}$

Null set (or) Empty Set: If no element in a set that is named as Null set, denoted by using ϕ.

-> $\{\ \}$ is the null set

-> $\{0\}$ is not a null set.

Examples:

- No whole number less than 0
- let $A = \{x : 2 < x < 3, x \text{ is a natural number}\}$
- $B = \{x : x \text{ is the month have 32 days}\}$
- $C = \{x : x \text{ is the student of ME in ECE Branch}\}$

Singleton Set: A set which contains only one element is called a singleton set.

Examples:

- $A = \{x : x \text{ is a whole number}, x < 1\}$
- let $B = \{x : x \in N \text{ and } x^2 = 4\}$
- let $C = \{x : x \text{ is a even prime number}\}$.

Finite Set : If a set with countable number of elements and having termination that set is finite set.

Example :

- The set of all colors in the rainbow
- $\{x : x$ is the number of students in II year ECE-A section$\}$
- $N = \{x : x \in N, x < 7\}$
- $P = \{2, 3, 5, 7, 11, 13, 17, \dots, 97\}$

Infinite set : If a set have uncountable number of elements such as the count is not terminating that set named as Infinite set.

Example :

- Set of all points in a plane
- $A = \{x : x \in N, x > 1\}$
- $B = \{x : x = 2n, n \in W\}$

Sub set and Proper sub set :

→ let A, B are two sets, if every element of A is in B then, A is the subset of B, $A \subseteq B$.

→ If at least one element in B is not in A that is named as proper sub set.

Examples :

- let $A = \{1, 2, 3, 4\}$, $B = \{4, 2, 1, 3\}$ then $A \subseteq B$ (A is the subset of B)
- Let $A = \{a, e, i, o, u\}$, $B = \{$All Alphabets$\}$ $A \subset B$, A is the proper set of B.

Disjoint sets (or) Mutually exclusive : Two sets are said to be disjoint if they have no common element.

Example :

- $A = \{a, b, c, d\}$, $B = \{e, f, g, h\}$ A, B are said to be disjoint or Mutually exclusive.
- $A = \{$II Year ECE students$\}$, $B = \{$I Year EEE Students$\}$
- $A = \{x : x$ is two wheelers$\}$, $B = \{x : x$ is set of 4 wheelers$\}$

Universal sets :

- A set which contains all the elements of other given sets is called a universal set. The symbol for denoting a universal set is U (or) C
- All sets are sub sets of universal set.

Examples :

- If $A = \{1,2,3\}$, $B = \{2,3,4\}$, $C = \{3,5,7\}$ then $U = \{1,2,3,4,5,7\}$, [Here $A \subseteq U$, $B \subseteq U$, $C \subseteq U$ and $U \supseteq A$, $U \supseteq B$, $U \supseteq C$]
- Let $A = \{$Set of II ECE students$\}$, $B = \{$III Year ECE Students$\}$, $C = \{$IV Year ECE Students$\}$, $D = \{$I Year ECE Students$\}$, $U = \{$set of All ECE students$\}$.

Equivalent sets :

- Two sets A and B are said to be equivalent if their cardinal number is same, i.e., $n(A) = n(B)$. The symbol for denoting an equivalent set is $\leftrightarrow$.

Example :

- $A = \{1, 2, 3\}$ Here $n(A) = 3$
- $B = \{p, q, r\}$ Here $n(B) = 3$
- Therefore, $A \leftrightarrow B$.

Equal sets :

- Two sets A and B are said to be equal if they contain the same elements. Every element of A is an element of B and every element of B is an element of A.

Example :

- $A = \{p, q, r, s\}$
- $B = \{p, s, r, q\}$
- Therefore, $A = B$

Exercise - 1

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3, \dots\}$, $C = \{0.5 < c \leq 8.5\}$,
$D = \{0.0\}$, $E = \{2, 4, 6, 8, 10, 12\}$, $F = \{-5.0 < f \leq 12.0\}$.

- Describe about what type of sets these are ?

	Countable	Uncountable	Finite	Infinite	Null	Singleton	Universal	Equivalent Sets	Equal sets
A	✓		✓						
B	✓			✓					
C		✓		✓					
D	✓		✓			✓			
E	✓		✓						
F		✓		✓					

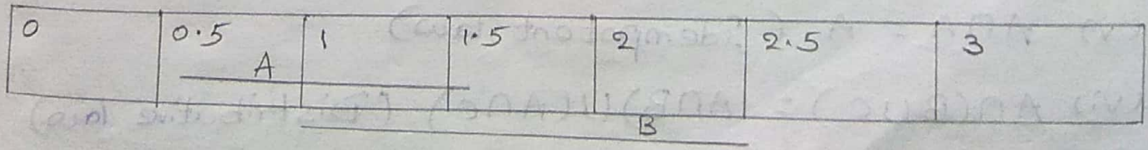
Operations on Sets

→ Difference
→ Union
→ Intersection
→ Complement.

Difference :-

→ let A, B are two sets A - B is the set containing all the elements of A that are not in B.

→ Example :
- Let $A = \{0.6 < a \leq 1.6\}$,
  $B = \{1.0 \leq b \leq 2.5\}$ then A - B and B - A are ?



$A - B = \{0.6 < a - b < 1\}$,

$A - B = \{1.6 < b - a < 2.5\}$

Union of two sets:

→ Union of two given sets is the smallest set which contains all the elements of both the sets. The symbol for denoting union of sets is '$\cup$'.

Some properties of the operation of union:

(i) $A \cup B = B \cup A$ (Commutative law)

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative law)

(iii) $A \cup \phi = A$ (Law of identity element, ϕ is the identity of $\cup$)

(iv) $A \cup A = A$ (Idempotent law)

(v) $U \cup A = U$ (Law of $\cup$) U is the universal set)

Example of Union:

→ If $A = \{1, 3, 7, 5\}$ and $B = \{3, 7, 8, 9\}$. Find union of two sets A and B.

- $A \cup B = \{1, 3, 5, 7, 8, 9\}$ No element is repeated in the union of two sets. The common elements 3, 7 are taken only once.

Intersection of Two Sets

→ Intersection of two given sets is the largest set which contains all the elements that are common to both the sets.

The symbol for denoting intersection of sets is '$\cap$'.

→ Some properties of the operation of intersection.

(i) $A \cap B = B \cap A$ (Commutative law)

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative law)

(iii) $U \cap A = A$ (Law of U)

(iv) $\phi \cap A = \phi$ (Law of ϕ)

(v) $A \cap A = A$ (Idempotent law)

(vi) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)

Here $\cap$ distributes over $\cup$.

(vii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive law)

Here $\cup$ distributes over $\cap$.

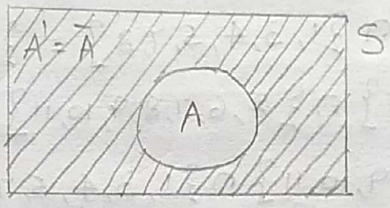
(viii) $A \cap \phi = \phi \cap A = \phi$ i.e. intersection of any set with the empty set is always the empty set.

Complement :

-> It is defined as set of all the elements not in A.

Denoted as $\bar{A}$

-> $\bar{A} = S - A$, where S is universal set.



Important Glimpse on set operations

-> Let S or U represent as Universal Set, ϕ is null set.

-> Any set $\cup$ S is S, $A \cup S = S$, $A \cup \phi = A$

-> Any set $\cap$ S is the same set, $A \cap S = A$, $A \cap \phi = \phi$.

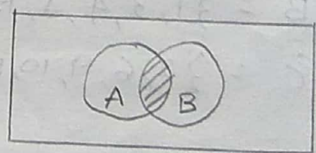
-> Let there are N number sets are get performed union then $(A_1, A_2, A_3, A_4, \dots A_n)$

-> $(A_1 \cup A_2 \cup A_3 \cup A_4 \dots A_n) = \cup_{n=1}^{N} A_n$

-> $(A_1 \cap A_2 \cap A_3 \cap A_4 \dots A_n) = \cap_{n=1}^{N} A_n$

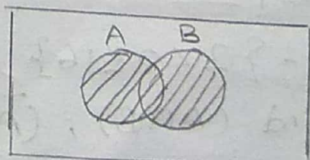
Venn Diagrams

-> Intersection of A and B

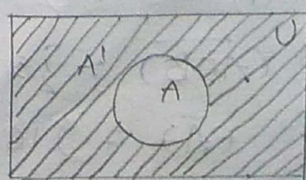


$A \cap B$

-> Union of A and B



-> Complement of set A.



Example-1:- Illustrate Intersection, Union and Complement of the given sets

$S = \{1 \le i \le 12\}$, $A = \{1, 3, 5, 12\}$, $B = \{2, 6, 7, 8, 9, 10, 11\}$,

$C = \{1, 3, 4, 5, 7, 8\}$

A:- Intersection:-

$S \cap A = \{1 \le i \le 12\} \cap \{1, 3, 5, 12\} = \{1, 3, 5, 12\}$

$S \cap B = \{1 \le i \le 12\} \cap \{2, 6, 7, 8, 9, 10, 11\} = \{2, 6, 7, 8, 9, 10, 11\}$

$S \cap C = \{1 \le i \le 12\} \cap \{1, 3, 4, 5, 7, 8\} = \{1, 3, 4, 5, 7, 8\}$

$A \cap B = \{1, 3, 5, 12\} \cap \{2, 6, 7, 8, 9, 10, 11\} = \{\ \} = \phi$

$B \cap C = \{2, 6, 7, 8, 9, 10, 11\} \cap \{1, 3, 4, 5, 7, 8\} = \{7, 8\}$

$A \cap C = \{1, 3, 5, 12\} \cap \{1, 3, 4, 5, 7, 8\} = \{1, 3, 5\}$

Union :-

$S \cup A = \{1 \le i \le 12\} \cup \{1, 3, 5, 12\} = \{1 \le i \le 12\} = S$

$S \cup B = \{1 \le i \le 12\} \cup \{2, 6, 7, 8, 9, 10, 11\} = \{1 \le i \le 12\} = S$

$S \cup C = \{1 \le i \le 12\} \cup \{1, 3, 4, 5, 7, 8\} = \{1 \le i \le 12\} = S$

$A \cup B = \{1, 3, 5, 12\} \cup \{2, 6, 7, 8, 9, 10, 11\} = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12\}$

$A \cup C = \{1, 3, 5, 12\} \cup \{1, 3, 4, 5, 7, 8\} = \{1, 3, 4, 5, 7, 8, 12\}$

$B \cup C = \{2, 6, 7, 8, 9, 10, 11\} \cup \{1, 3, 4, 5, 7, 8\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

Complement

$\bar{A} = \{2, 4, 6, 7, 8, 9, 10, 11\}$

$\bar{B} = \{1, 3, 4, 5, 12\}$

$\bar{C} = \{2, 6, 9, 10, 11, 12\}$

Example-2:-

$A = \{2 < a \le 16\}$, $B = \{5 < b \le 22\}$, $S = \{2 < s \le 24\}$,

Find $(A \cap B)$, $(\overline{A \cap B})$, $\bar{A}$, $\bar{B}$

A:- $(A \cap B) = \{5 \le (a \cap b) \le 16$

$(\overline{A \cap B}) = \{2 < (\overline{a \cap b}) \le 5,\ 16 < (\overline{a \cap b}) \le 24$

$(\bar{A}) = \{16 < (\bar{a}) \le 24$

$(\bar{B}) = \{2 < (\bar{b}) \le 5,\ 22 < (\bar{b}) \le 24$

Probability Theory

Experiment :- Performing an Operation is named as experiment

Sample space :- All the Out comes of an Experiment named as Sample space.

Ex:- Tossing a coin : $S = \{H, T\}$

Tossing a Dice : $S = \{1, 2, 3, 4, 5, 6\}$

Tossing 2 Coins :

$S = \{TT, TH, HT, HH\}$

Types of Sample space:-

- Sample Space
  - Discrete Sample Space
  - Continuous Sample Space
  - Mixed Sample Space

Discrete Sample Space :- If an Experiment gives a sample space with a finite number of values and not having any range representation.

Examples:-

→ Tossing a Coin $S = \{T, H\}$

→ Through 2 dice $S =$

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

→ Tossing 2 coins $S = \{TT, TH, HT, HH\}$

→ Through a dice $S = \{1, 2, 3, 4, 5, 6\}$

Continuous Sample Space:- If a Sample space gives a range of values then that is named as Continuous sample space.

Example :- $S = \{0 \le s \le 12\}$

Mixed Sample Space :- If a sample space is a Combination of both Discrete and continuous sample space that named as mixed sample space.

Ex:- $S = \{1, 2, 3, 4 \le 5 \le 12\}$

Prob :- In a laboratory measuring the amount of bacteria develop is measuring what type of sample it is ?

It is a Continuous sample space.
Because bacteria growth increases time to time till the end of the experiment.

Event :- Outcome of an experiment at particular instant of the time.

→ Event is an element of sample space

→ Let A is the event then $A \subseteq S$.

Types of Events :-

1.) Exhaustive Events / Sure Events

2.) Mutually Exclusive Events

3.) Equally likely Events

4.) Independent Events

5.) Non-Mutually Exclusive Events.

Exhaustive Events / Sure Events :- A set of events is said to be exhaustive, if it includes all the possible events.

Example :-

→ In tossing a coin, the outcome can be either Head or Tail and there is no other possible outcome. So, the set of events {H, T} is exhaustive.

→ Throwing a dice is also gives exhaustive events such as we cant expect other than {1, 2, 3, 4, 5, 6}.

Mutually Exclusive Events :- Two events, A and B are said to be mutually exclusive if they cannot occur together.

-> **Note** :- $P(A \cup B)$ of mutually exclusive events is $P(A) + P(B)$

i.e; $P(A \cup B) = P(A) + P(B)$.

Equally Likely Events :-

-> If one of the events cannot be expected to happen in preference to another, then such events are said to be Equally Likely Events.

-> Each outcome of the random experiment has an equal chance of occurring.

Examples :-

- The occurrence of Head and Tail are same
- Occurrence of Pass and fail in a subject are same.

Independent Events :-

-> Two events are said to be independent, if happening or failure of one does not affect the happening or failure of the other: Otherwise, the events are said to be dependent.

-> If two events, A and B are independent then the joint probability is

-> $P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B)$.

Non-Mutually Exclusive Events :-

-> If some events are having mutual elements then we named as non mutually exclusive.

-> $P(A \cap B) = P(A) + P(B) - P(A \cup B)$, $P(A \cap B)$ is also represented as $P(A \text{ or } B)$.

Types of Experiments.

Deterministic experiment :- A deterministic experiment is one whose outcome may be predicted with certainty beforehand, such as combining Hydrogen and Oxygen, or adding ~~to~~ two numbers such as 2 ? 3

Random experiment :- A random experiment is one whose outcome is determined by chance.

Probability Definition :-

1.) Classical
2.) Relative frequency
3.) Axiomatic.

Classical Definition :-

→ If the events are equally likely then the probability is defined using classical definition as.

→ $$P(A) = \frac{\text{Number of times event A had occured}}{\text{Number of events in the Sample space}}$$

→ Let the sample space S has all possible distinct outcomes then, probability of A is

Examples:-

→ If a dice is rolled probability of getting 2 on its face is 1/6.

→ Note:- The Classical definition is valid for equally likely events.

Limitations of Classical Definition :-

→ In the definition we used equally likely, but the sample space won't have equally likely events so the definition get wrong and not used for most of the applications.

→ Is this possible to get same amount of rain in every session of monsoon?

→ So equally likely cant happen in every situation.

Relative frequency :-

→ In this the experiment is performed "N" number of times and observed the event A, let the event A had occured "n(A)" times, then the probability of event "A" is P(A).

→ $$P(A) = \frac{n(A)}{N}$$

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}$$

→ Properties of Relative frequency:

→ 1.) $0 \le P(A) \le 1$

→ 2.) $P(A) = 1$ if all the time we get the our required event.

Limitations of Relative Frequency:-

→ Conducting the experiment infinite times at same conditions is the one of the limitation.

→ Example:-

- The probability of rain today can't really be obtained by the relative frequency definition since today can't be repeated again.

Axiomatic definition (Most Important):-

→ Each event in a sample space have the probability of event is non negative and no where it become negative.

→ In axiomatic, 3 definitions are given

- Axiom 1: $P(A) \ge 0$
- Axiom 2: $P(S) = 1$
- Axiom 3:
  → If in a sample space "S" there are two events A, B both the events are mutually exclusive $(A \cap B) = \phi$

  Then

  $P(A \cup B) = P(A) + P(B)$.

→ Let N number of events in a sample space S then An

$A_n = A_1, A_2, A_3, A_4, A_5 \dots A_n$.

Then

$P(A_1 \cup A_2 \cup A_3 \dots \cup A_n) = P(A_1) + P(A_2) + P(A_3) + \dots P(A_n)$

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A_n)$$

Properties of Axiomatic Definition

1.) $P(A) \geq 0$

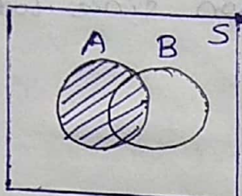
2.) $P(\bar{A}) = 1 - P(A)$

3.) $P(\phi) = 0$

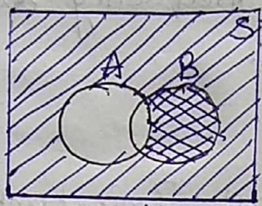
4.) If $A \subseteq B$ then $P(A) \leq P(B)$

5.) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, if A, B are not mutually Exclusive

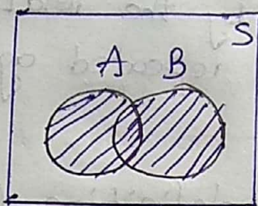
Venn diagrams



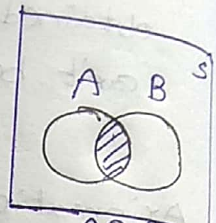
A



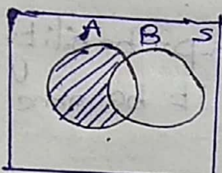
A'



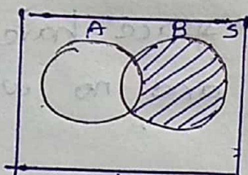
$A \cup B$



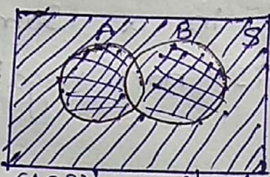
$A \cap B$



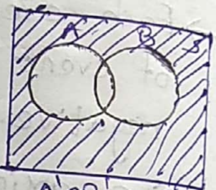
$A \cap B'$



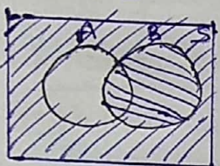
$A' \cap B$



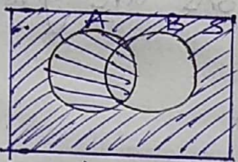
$(A \cap B)'$ or $A' \cup B'$



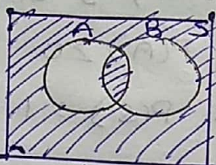
$A' \cap B'$ or $(A \cup B)'$



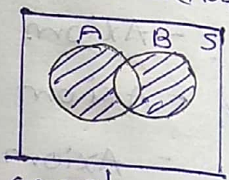
$A' \cup B$ or $(A \cap B')'$



$A \cup B'$ or $(A' \cap B)'$

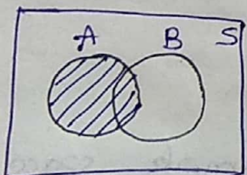


$(A' \cap B') \cup (A \cap B)$

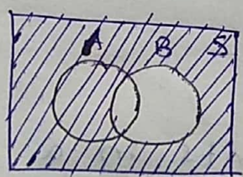


$(A \cap B') \cup (A' \cup B)$

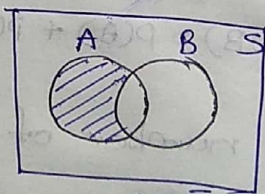
1) $P(A) = P(A \cap \bar{B}) \cup P(A \cap B)$



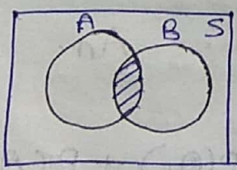
A



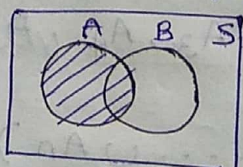
$\bar{B}$



$A \cap \bar{B}$



$A \cap B$

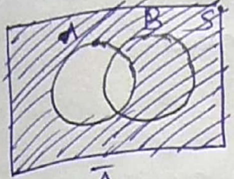


$(A \cap \bar{B}) \cup (A \cap B)$

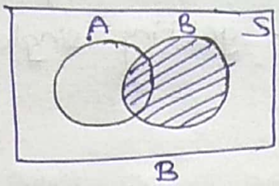
$\therefore P(A) = P[(A \cap \bar{B}) \cup (A \cap B)]$

$= P[A \cap \bar{B}] + P[A \cap B]$

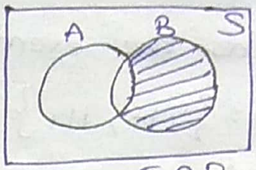
2:) $P(B) = P(\bar{A} \cap B) + P(A \cap B)$



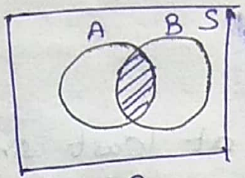
$\bar{A}$



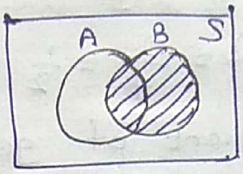
B



$\bar{A} \cap B$



$A \cap B$



$(\bar{A} \cap B) \cup (A \cap B)$

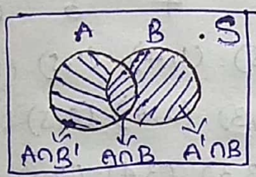
$B = (\bar{A} \cap B) \cup (A \cap B)$

Applying probability on both sides

$P(B) = P[(\bar{A} \cap B) \cup (A \cap B)]$

$P(B) = P(\bar{A} \cap B) + P(A \cap B)$

3:) Prove $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



$A \cup B = (A \cap B') \cup (A \cap B) \cup (\bar{A} \cap B)$

From Axiom-3

$P(A \cup B) = P(A \cap B') + P(A \cap B) + P(A' \cap B)$

$P(A \cup B) + P(A \cap B) = P(A \cap B') + P(A \cap B) + P(A' \cap B) + P(A \cap B)$

$P(A \cup B) + P(A \cap B) = P(A) + P(B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B).$

4:) Find the probability of getting 5, when a fair dice is thrown.

Sol: Sample space $S = \{1, 2, 3, 4, 5, 6\}$

Total number of results in the experiment $N = 6$

$P(5) = \frac{n(5)}{N} = \frac{1}{6}$

5:) An experiment is performed by tossing 2 fair coins generate sample space, find the probability of getting single head, probability of getting at least single head.

Sol: Sample space $S = \{TT, TH, HT, HH\}$

Total number of results in the experiment $N = 4$

Let A be the event of getting single head.

$A = \{TH, HT\}$

$n(A) = 2$

$P(A) = \frac{n(A)}{N} = \frac{2}{4} = \frac{1}{2}$

Let B be the event of getting at least single head

$B = \{TH, HT, HH\}$

$n(B) = 3$

$P(B) = \frac{n(B)}{N} = \frac{3}{4}$.

6.) Two fair dice are thrown find the sample space, Event A is given as getting sum as 6, event B is getting sum as greater than 9, Event C is getting 1st dice face is odd and total sum is 8.

Sol: Total no. of results in the experiment $N = 36$.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

→ Let A be the event of getting sum as 6.

$A = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

$n(A) = 5$

$P(A) = \frac{n(A)}{N} = \frac{5}{36}$

→ Let B be the event of getting sum as greater than 9

$B = \{(4,6), (5,6), (5,5), (6,4), (6,5), (6,6)\}$

$n(B) = 6$

$\therefore P(B) = \frac{n(B)}{N} = \frac{6}{36} = \frac{1}{6}$

→ Let C be the event of getting 1st dice face is odd and total sum is 8.

$C = \{(3,5), (5,3)\}$

$n(C) = 2$

$P(C) = \frac{n(C)}{N} = \frac{2}{36} = \frac{1}{18}$.

7) In a box containing 16 red balls, 12 blue balls, 22 green balls are there what is the probability of getting red ball, probability of getting blue ball, probability of not getting green ball.

Sol:- Total no. of results in the experiment $N = 50$

Let A be the event of getting a red ball

$n(A) = 16$

$$P(A) = \frac{n(A)}{N} = \frac{16}{50} = \frac{8}{25}$$

Let B be the event of getting a blue ball

$n(B) = 12$

$$P(B) = \frac{n(B)}{N} = \frac{12}{50} = \frac{6}{25}$$

Let C be the event of not getting green ball.

$$P(\bar{C}) = 1 - P(C) = 1 - \frac{n(C)}{N} = 1 - \frac{22}{50} = \frac{28}{50} = \frac{14}{25}$$

Balls are mutually exclusive

$$P(A \cup B) = P(A) + P(B)$$

$$P(A \cup B) = \frac{8}{25} + \frac{6}{25} = \frac{14}{25}$$

8) In a box there are 3 types of resistors given in the table event A is picking 47 Ω resistor, event B is getting 10% tolerance resistor, event C is picking 100Ω resistor find the individual probability of the events.

Resistance value	Tolerance 5%	Tolerance 10%	Total
22 Ω	15	10	25
47 Ω	21	19	40
100 Ω	18	52	70
Total	54	81	135

Total no. of results in the experiment $N = 135$

→ Let A be the event of picking 47 Ω resistor

$n(A) = 40$

$$P(A) = \frac{n(A)}{N} = \frac{40}{135} = \frac{8}{27}$$

→ Let B be the event of getting 10% tolerance resistor.

$n(B) = 81$

$$P(B) = \frac{n(B)}{N} = \frac{81}{135} = \frac{9}{15} = \frac{3}{5}$$

→ Let C be the event of picking 100Ω resistor.

$n(C) = 70$

$$P(C) = \frac{n(C)}{N} = \frac{70}{135} = \frac{14}{27}$$

Joint Probability:- A & B are two events having mutual components. Probability of finding these mutual components is a joint probability

Ex:- Event A is cloudy sky and event B is raining. Event A and B having same common relation. Hence then joint probability of A & B is represents as ~~P(A)~~ P(A and B)

$$\left.\begin{aligned} P(A\cap B) &= P(A) + P(B) - P(A\cup B) \\ P(A\cup B) &= P(A) + P(B) - P(A\cap B) \end{aligned}\right\} A, B \rightarrow \text{Mutual}$$

A, B are statistically independent

$$P(A\cap B) = P(A) \cdot P(B).$$

Conditional probability:-

Conditional event:- If A and B are any two events in a random experiment then B occurs after the occurence of A is called conditional event. It is denoted by "B/A". Similarly A occurs after the occurence of 'B' is called conditional event. It is denoted by "A/B".

Conditional probability:-

If A and B are two events in a sample space 'S' and $P(A) \neq 0$, then the probability of B occurs after the occurence of probability of A.

It is denoted by $P(B/A)$

$$P(B/A) = \frac{n(A,B)}{n(A)} \times \frac{N}{N}$$

$$= \frac{\frac{n(A,B)}{N}}{\frac{n(A)}{N}}$$

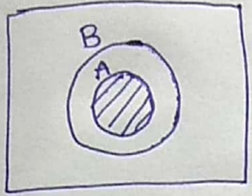
$$\boxed{P(B/A) = \frac{P(A\cap B)}{P(A)}}$$

Similarly $\boxed{P(A/B) = \frac{P(A\cap B)}{P(B)}}$

Properties :-

1.) $P(A/B) \Rightarrow P(B) \neq 0$ $\quad P(B|A) \Rightarrow P(A) \neq 0$

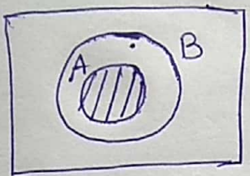
2.) $P(A/B)$ if $A \subseteq B$



$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A/B) = \frac{P(A)}{P(B)} \quad [\because \text{from venn diagram}]$$

3.) If $A \subseteq B$ $P(B|A) = 1$



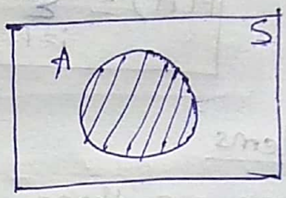
$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$$= \frac{P(A)}{P(A)} \quad [\because \text{from venn diagram}]$$

$$\boxed{P(B|A) = 1}$$

4.) If A is event, S is the sample space.

$$P(A/S) = \frac{P(A \cap S)}{P(S)} = \frac{P(A)}{P(S)}$$



$$P(S/A) = \frac{P(A \cap S)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

5.) A, B are statistically independent

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$$

$$\boxed{P(A/B) = P(A)}$$

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\cancel{P(A)}P(B)}{\cancel{P(A)}}$$

$$\boxed{P(B|A) = P(B)}$$

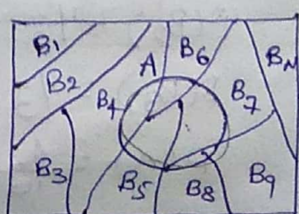
Total Probability Theorem :-

Statement :- If $B_1, B_2, B_3 \cdots B_N$ are 'N' number of mutually exclusive · $B_i \cap B_j = \phi$ and S is the sample space and A is the common term in all the events.

$$P(A) = \sum_{i=1}^{N} P(A/B_i) \cdot P(B_i)$$

Proof :-

From given $B_1 \cup B_2 \cup B_3 \cdots \cup B_N = S$ — ①



$$S = \bigcup_{i=1}^{N} B_i \quad — ②$$

A interms of S

$S \cap A = A$ — ③

Apply probability on both sides

$P(A) = P(S \cap A)$ — ④

sub ② in ④

$P(A) = P\left[\bigcup_{i=1}^{N} B_i \cap A\right]$

$P(A) = \sum_{i=1}^{N} P(B_i \cap A)$

$P(A) = \sum_{i=1}^{N} P(A \cap B_i)$ — ⑤

Conditional probability

$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)}$

$P(A \cap B_i) = P(A/B_i) \cdot P(B_i)$ — ⑥

Sub ⑥ in ⑤

$$\boxed{P(A) = \sum_{i=1}^{N} P(A/B_i) \cdot P(B_i)}$$

Problems

1.) There are three boxes, each box containing a different number of light bulbs. The first box has 10 bulbs, of which four are dead, the second box has six bulbs of which one is dead, and third box has eight bulbs of which three are dead. What is the probability of a dead bulb being selected when a bulb is choosen at random from one of the three boxes?

Sol:- let A be the event of selected bulb is dead bulb from one of the three boxes.

10	6	8
4-D	1-D	3-D

$P(A) = \sum_{i=1}^{3} P(A/B_i) \cdot P(B_i)$

$P(A) = P(A/B_1) \cdot P(B_1) + P(A/B_2) \cdot P(B_2) + P(A/B_3) \cdot P(B_3)$

$P(A) = \frac{4}{10} P(B_1) + \frac{1}{6} P(B_2) + \frac{3}{8} P(B_3)$

$P(B_1) = 1/3$

$P(B_2) = 1/3$

$P(B_3) = 1/3$

$P(A) = \frac{4}{10} \cdot \frac{1}{3} + \frac{1}{6} \cdot \frac{1}{3} + \frac{3}{8} \cdot \frac{1}{3}$

$$P(A) = \frac{\cancel{4}^{2}}{\cancel{30}_{15}} + \frac{1}{18} + \frac{1}{8}$$

$$P(A) = \frac{6+5}{90} + \frac{1}{8} = \frac{11}{90} + \frac{1}{8} = \frac{44+45}{360} = \frac{89}{360}.$$

Baye's Theorem :-

If $B_1, B_2, B_3 \ldots B_N$ are N mutually exclusive events $B_i \cap B_j = \phi$. S is the sample space, Feature extracted and probability of the feature Belong to B_i

$$P(B_i/A) = \frac{P(A/B_i) \cdot P(B_i)}{P(A)}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{From conditional probability}$$

$$P(A/B_i) = \frac{P(A \cap B_i)}{P(B_i)} \quad — ①$$

$$P(B_i/A) = \frac{P(A \cap B_i)}{P(A)}$$

$$P(A \cap B_i) = P(B_i/A)\, P(A) \quad — ②$$

$$② = ①$$

$$P(B_i/A)\, P(A) = P(A/B_i) \cdot P(B_i)$$

$$\boxed{P(B_i/A) = \frac{P(A/B_i) \cdot P(B_i)}{P(A)}}$$

1.) In a box there are 3 types of resistors given in the table. Event A is picking 47 Ω resistor, event B is getting 10% tolerance resistor, event C is picking 100 Ω resistor. Find the joint probability & Conditional probability.

Resistance value	Tolerance 5%	Tolerance 10%	Total
22 Ω	15	10	25
47 Ω	21	19	40
100 Ω	18	52	70
Total	54	81	135

Joint probability :-

$$P(A \cap B) = \frac{n(A,B)}{N} = \frac{19}{135}$$

$$P(B \cap C) = \frac{n(B,C)}{N} = \frac{52}{135}$$

$$P(A \cap C) = \frac{n(A,C)}{N} = \frac{0}{135} = 0$$

Conditional probability :-

$$* \; P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{19}{135}}{\frac{81}{135}} = \frac{19}{81}$$

$$\boxed{P(A/B) = \frac{19}{81}}$$

$*$ $P(A/C) = \frac{P(A \cap C)}{P(C)} = 0 \Rightarrow \boxed{P(A/C) = 0}$

$*$ $P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{19}{135}}{\frac{40}{135}} = \frac{19}{40}$

$$\boxed{P(B/A) = \frac{19}{40}}$$

$*$ $P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{\frac{52}{135}}{\frac{70}{135}} = \frac{52}{70} = \frac{26}{35}$

$$\boxed{P(B/C) = \frac{26}{35}}$$

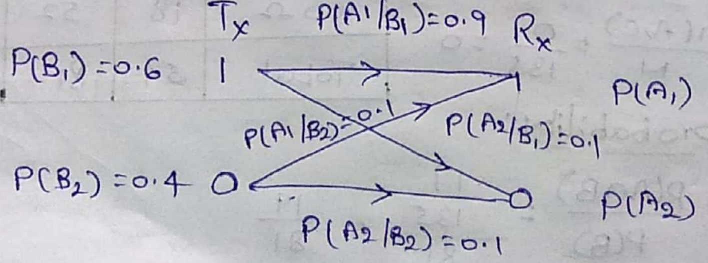
$*$ $P(C/A) = \frac{P(A \cap C)}{P(A)} = 0$

$*$ $P(C/B) = \frac{P(B \cap C)}{P(B)} = \frac{\frac{52}{135}}{\frac{81}{135}} = \frac{52}{81}$

$$\boxed{P(C/B) = \frac{52}{81}}$$

2) An elementary Binary communication system consist of a Transmitter that sends one of two possible symbols (1 or 0) over a channel to receiver, the channel occasionally values error to occur so that a 1 shown up as 0 and vice versa. Probability of selecting 1 and 0 at transmitter are assumed to be $P(B_1) = 0.6$ and $P(B_2) = 0.4$, the conditional probability describes the effect the channel has on the transmitted symbols. The reception probability given a 1 was transmitted are assume to be $P(A1/B_1) = 0.9$, $P(A2/B_1) = 0.1$, the channel is presumed to affect 0's in the same manner so $P(A1/B_2) = 0.1$, $P(A2/B_2) = 0.9$. Find probability of receiving zero, one at receiver side, Find the probability of getting the bits from respective transmitters.

Sol:-



From given
The probability of selecting 1 at transmitter $P(B1) = 0.6$.
The probability of selecting 0 at transmitter $P(B_2) = 0.4$
& $P(A1/B_1) = 0.9$, $P(A_2/B_1) = 0.1$
$P(A1/B_2) = 0.1$, $P(A_2/B_2) = 0.1$

* probability of receiving '1' is

$$P(A_1) = \sum_{i=1}^{2} P(A1/B_i)$$
$$= P(A_1/B_1)\,P(B_1) + P(A_1/B_2)\cdot P(B_2)$$
$$= 0.9(0.6) + 0.1(0.4) = 0.54 + 0.04$$
$$\boxed{P(A1) = 0.58}$$

* probability of receiving '0' is

$$P(A_2) = \sum_{i=1}^{2} P(A_2/B_i) = P(A_2/B_1)\cdot P(B_1) + P(A_2/B_2)P(B_2)$$
$$= 0.1(0.6) + (0.9)(0.4)$$
$$= 0.06 + 0.36$$
$$\boxed{P(A_2) = 0.42}$$

* The probability of getting the bits from respective transmitters

$$\Rightarrow P(B_1/A1) = \frac{P(A1/B1)\,P(B1)}{P(A1)} = \frac{(0.9)(0.6)}{0.58} = \frac{0.54}{0.58} = 0.931$$
$$\boxed{P(B1/A1) = 0.931}$$

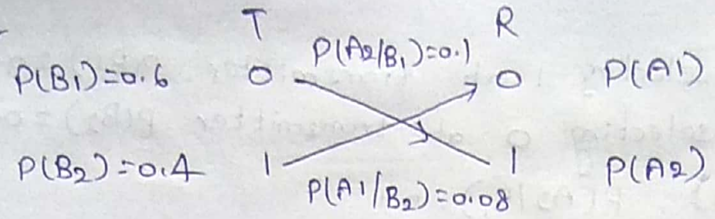
$$\Rightarrow P(B_2/A_1) = \frac{P(A1/B_2)\,P(B_2)}{P(A_1)} = \frac{(0.1)(0.4)}{0.58} = 0.0689$$
$$\boxed{P(B_2/A_1) = 0.0689}$$

$$\Rightarrow P(B_1/A_2) = \frac{P(A_2/B_1)\,P(B_1)}{P(A_2)} = \frac{(0.1)(0.6)}{0.42} = 0.1428$$
$$\boxed{P(B_1/A_2) = 0.1428}$$

$$\Rightarrow P(B_2/A_2) = \frac{P(A_2/B_2)\,P(B_2)}{P(A_2)} = \frac{(0.9)(0.4)}{0.42} = 0.8571$$
$$\boxed{P(B_2/A_2) = 0.8571}$$

3) In a binary communication system a zero and a one is transmitted with probability 0.6 and 0.4 respectively. Due to error in the communication system a zero becomes a one with a probability 0.1 and a one becomes a zero with a probability 0.08. Determine the probability (i) of receiving a one and (ii) that a one was transmitted when the received message is one.

Sol:



(i) Probability of receiving 1 is

$$P(A_2) = \sum_{i=1}^{2} P(A_2/B_i) \cdot P(B_i)$$

$$= P(A_2/B_1) \cdot P(B_1) + P(A_2/B_2)\, P(B_2)$$

$$= (0.1)(0.6) + (0.92)(0.4)$$

$$P(A_2) = 0.06 + 0.368$$

$$\boxed{P(A_2) = 0.428}$$

$$\left\{ \begin{array}{l} P(A_2/B_2) = 0.92 \\ P(A_1/B_1) = 0.9 \end{array} \right\}$$

(ii) $$P(B_2/A_2) = \frac{P(A_2/B_2) \cdot P(B_2)}{P(A_2)}$$

$$= \frac{(0.92)(0.4)}{0.428}$$

$$= \frac{0.368}{0.428}$$

$$\boxed{P(B_2/A_2) = 0.8598}$$

4) In three boxes there are capacitor shown in table. An experiment consist of first randomly selecting a box, assuming each has the same likelihood of selection and the selection a capacitor from the choosen box.

a) What is the probability of selecting a 0.01 micro F capacitor, given that box 2 is selected?

b) If a 0.01-micro Faraday capacitor is selected, what is the probability it came from box 3? use Bayes & total probability theorem.

Value in micro F	Number in box 1	2	3	Total
0.01	20	95	25	140
0.1	55	35	75	165
1.0	70	80	145	295
Total	145	210	245	600

Sol:- The probability of picking Box B_1 from 3 boxes $P(B_1) = \frac{1}{3}$

The probability of picking Box B_2 from 3 boxes $P(B_2) = \frac{1}{3}$

The probability of picking Box B_3 from 3 boxes $P(B_3) = \frac{1}{3}$

(a) * Let A Be the event of selecting 0.01 micro F capacitor

$$P(A/B_2) = \frac{P(A \cap B_2)}{P(B_2)} = \frac{\frac{95}{600}}{\frac{1}{3}} = \frac{\overset{19}{\cancel{95}} \times \cancel{3}}{\underset{\underset{40}{\cancel{120}}}{\cancel{600}}} = \frac{19}{40}$$

$\boxed{P(A/B_2) = 0.475}$

(b) $$P(B_3/A) = \frac{P(A/B_3) \cdot P(B_3)}{P(A)} = \frac{\frac{25 \times 3}{600} \times \frac{1}{3}}{P(A)} \longrightarrow ①$$

$$P(A) = \sum_{i=1}^{3} P(A/B_i)\, P(B_i)$$

$$= P(A/B_1)\, P(B_1) + P(A/B_2)\, P(B_2) + P(A/B_3) \cdot P(B_3)$$

$$= \frac{20 \times \cancel{3}}{600} \times \frac{1}{\cancel{3}} + \frac{95 \times \cancel{3}}{600} \times \frac{1}{\cancel{3}} + \frac{25 \times 3}{600} \times \frac{1}{\cancel{3}}$$

$$= \frac{\overset{7}{\cancel{140}}}{\underset{30}{\cancel{600}}} = \frac{7}{30}$$

from ① $$P(B_3/A) = \frac{\frac{25 \times \cancel{3}}{600} \times \frac{1}{\cancel{3}}}{\frac{7}{30}} = \frac{\overset{5}{\cancel{25}} \times \cancel{30}}{\underset{\underset{4}{\cancel{20}}}{\cancel{600}} \times 7} = \frac{5}{28}$$

$\boxed{P(B_3/A_1) = 0.1785}$

5.) Two dices are thrown find the probability that the sum of on faces are divided by 3 or 4

Sol:- Total number of results in the experiment $N = 36$

Let A be the event of the sum of ~~tn~~ on faces are divided by 3 or 4.

$A = \{(2,1), (5,1), (1,2), (4,2), (3,3), (6,3), (2,4), (15,4), (1,5), (4,5), (3,6), (6,6), (3,1), (2,2), (6,2), (1,3), (5,3), (4,4), (3,5), (2,6)\}$

$n(A) = 20$

$$\therefore P(A) = \frac{n(A)}{N} = \frac{20}{36} = \frac{5}{9}$$

$\boxed{\therefore P(A) = \frac{5}{9}}$

6.) A fair dice is thrown the events are shown as

$A = \{1, 3, 5\}$, $B = \{1, 2\}$, $C = \{1, 3, 4, 5\}$

(i) $P(A/B)$ (ii) $P(B/A)$ (iii) $P(A/C)$ (iv) $P(C/A)$ (v) $P\left(\frac{A \cup B}{C}\right)$

Sol:- (i) $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$

(ii) $P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{1/6}{3/6} = \frac{1}{3}$

(iii) $P(A/C) = \frac{P(A\cap C)}{P(C)} = \frac{3/6}{4/6} = \frac{3}{4}$

(iv) $P(C/A) = \frac{P(A\cap C)}{P(A)} = \frac{3/6}{3/6} = 1$

(v) $P\left(\frac{A\cup B}{C}\right) = \frac{P[(A\cup B)\cap C]}{P(C)} = \frac{3/6}{4/6} = \frac{3}{4}.$

7.) If $P(A) = 0.75$, $P(B) = 0.4$, $P(A\cap B) = 0.3$ are the events A, B are statistically independent or not.

Sol:- We know that any two events are statistically independent

$\therefore \boxed{P(A\cap B) = P(A).P(B)}$

From given $P(A) = 0.75$, $P(B) = 0.4$, $P(A\cap B) = 0.3$

$P(A\cap B) = P(A).P(B)$

$= 0.75(0.4)$

$\boxed{P(A\cap B) = 0.3}$

$\therefore$ The given two events are statistically independent.

8.) A typist make an error of 0.3% per a letter, what is the probability of no error in 10 letters.

Sol:- A typist make an error per a letter is $= 0.3\% = \frac{0.3}{100}$

$= 0.003$

A typist make an error per 10 letters $= 10 \times 0.003 = 0.03$

The probability of no error in 10 letters

$= 1 -$ error in 10 letters

$= 1 - 0.03 = 0.97$

9.) In a single thrown 3 dice are thrown find the probability of getting sum of the faces is 7.

Sol:- Total number of results in the experiment $N = 6^3 = 216$

let A be the event of sum of the faces is 7.

$A = \{(1,1,5), (1,2,4), (1,3,3), (1,4,2), (1,5,1), (2,1,4), (2,2,3),$
$(2,3,2), (2,4,1), (3,1,3), (3,2,2), (3,3,1), (4,4,2), (4,2,1),$
$(5,1,1)\}$

$n(A) = 15$

$\therefore \boxed{P(A) = \frac{n(A)}{N} = \frac{15}{216}}$

10) A coin is tossed twice, if the second thrown is heads a dice is thrown what will be the sample space.

Sol:- A coin is tossed twice $S = \{TT, HT, TH, HH\}$

Suppose if the second thrown is head a dice is thrown then

$S = \{TT, HT, TH, TH2, TH2, TH3, TH4, TH5, TH6, HH1, HH2, HH3, HH4, HH5, HH6\}$.

11) If a dice is rolled what is the probability that the outcome is odd given that the number is prime number.

Sol:- Total number of results in the experiment $N = 6$

$S = \{1, 2, 3, 4, 5, 6\}$

let A be the event of getting prime number

$A = \{2, 3, 5\}$

$n(A) = 3$, $P(A) = \frac{3}{6}$

Let B be the event of getting odd number

$B = \{1, 3, 5\}$

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{2/6}{3/6} = 2/3$.

12) If a pair dice are thrown find the probability that the sum of dice is 7 given that the second dice shown odd.

Sol:- Total number of results in the experiment $N = 36$

let A be the event of getting second dice is odd.

$p(A) = 18/36$

let B be the event of getting the sum of dice is 7

$B = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$

$P(A \cap B) = 3/36$

$\therefore P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3/36}{18/36} = 3/18 = 1/6$.

13.) In a class 60% students are Boys and remaining are girls, it is known that probability of boy getting distribution is 0.30, girl distribution is 0.35. Find the probability if a student is selected and the student get distribution.

Sol:- Total number of boys in a class is $B = 60\%$, $P(B) = \frac{1}{2}$

Total number of girls in a class is $G = 40\%$, $P(G) = \frac{1}{2}$.

Probability of boy getting distribution $P(S/B) = 0.3$

Probability of girl getting distribution $P(S/G) = 0.35$

[S → all students]

∴ The probability of selected student get distinction is

$P(S) = P(S/B)\,P(B) + P(S/G)\,P(G)$

$= 0.3\left(\frac{1}{2}\right) + 0.35\left(\frac{1}{2}\right) = 0.15 + 0.175$

$\boxed{P(S) = 0.325}$

14.) Bag A contains 5 red and 4 Black balls, Bag B contains 3 red and 5 black balls, one bag is selected randomly, from the selected bag one ball is selected. what the probability that the ball is black ball.

Sol:- Probability of getting bag A is $P(A) = 1/2$

Probability of getting bag B is $P(B) = 1/2$

'L' is event of selected ball is black ball

$P(L) = P(L/A)\,P(A) + P(L/B) \cdot P(B)$

$= \frac{4}{9} \times \frac{1}{2} + \frac{5}{8} \times \frac{1}{2}$

$= 0.222 + 0.312 = 0.534$

RANDOM VARIABLES

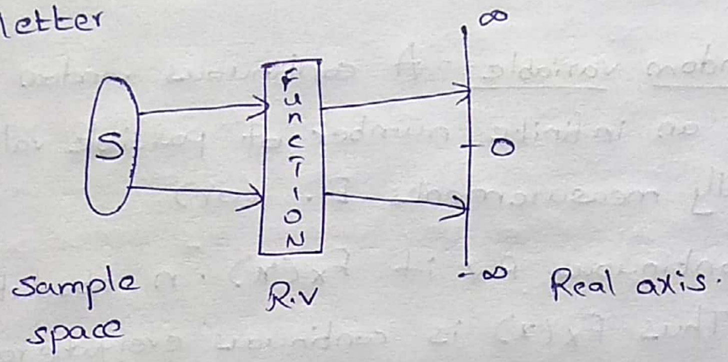
Random variable :- A random variable is a function that maps outcomes of a random experiment to real numbers

(or)

A random variable associate the points in the sample space with real numbers.

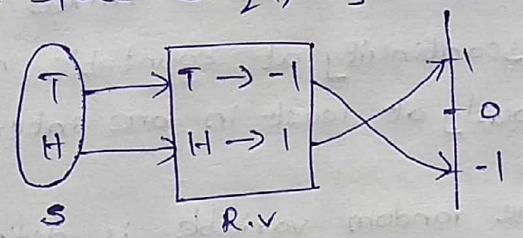
→ A random variable is denoted by a capital letter.

→ Elements of a random variable can be denoted by small letter



1.) An experiment is performed by tossing a coin, Random variable X is defined as assigning 1 if (H), -1 if (T).

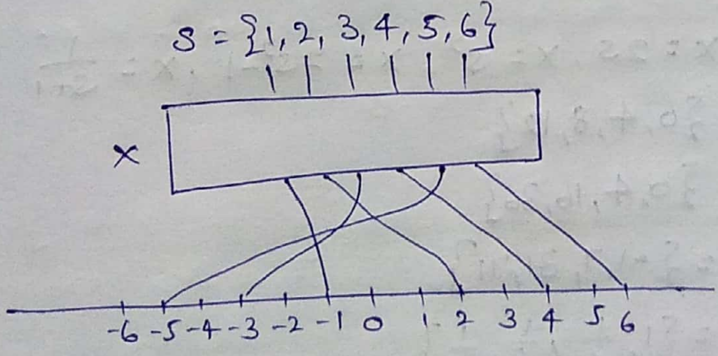
Sample space $S = \{T, H\}$



$X = \{-1, 1\}$.

2.) A dice is rolled X is the random variable; if the element is even assign to same +ve value and if the element is odd assign to -ve of the same value.

Sample space $S = \{1, 2, 3, 4, 5, 6\}$



$X = \{-5, -3, -1, 2, 4, 6\}$

Types of Random Variables:-

1) **Discrete Random Variable:-** A discrete random variable is one which may take on only a countable number of distinct values such as 0, 1, 2, 3 . . . . (or)

A random variable X is called a discrete random variable if $F_X(x)$ is piece-wise constant. Thus $F_X(x)$ is flat except at the points of jump discontinuity.

Eg:- Binomial, poisson.

2.) **Continuous random variable:-** A continuous random variable is one which take an infinite number of possible values. continuous R.Vs are usually measurements E. (or).

X is called continuous R.V if $F_X(x)$. n absolutely continuous function of x. Thus $F_X(x)$ is continuous everywhere on R and $F_X(x)$ exists everywhere except at finite or countably infinite points.

3.) **Mixed random variable:-** X is called a mixed random variable if $F_X(x)$ has jump discontinuity at countable no. of points and increases continuously at least in one interval of X.

1.) Tossing a coin, if the random variable is defined as assigning value -1 to 0 for "T" and assigning 0 to 1 for "H".

$$S = \{T, H\}$$

$$\boxed{X}$$

$$X = \{-1 \le x < 0,\ 0 \le x \le 1\}.$$

2.) $S = \{0, 2, 4, 6\}$ $X = 2s$, $X = s^2$; $X = 2s^2 - 1$, $X = \frac{1}{s+1}$

$X = 2s \rightarrow X = \{0, 4, 8, 12\}$

$X = s^2 \rightarrow X = \{0, 4, 16, 36\}$

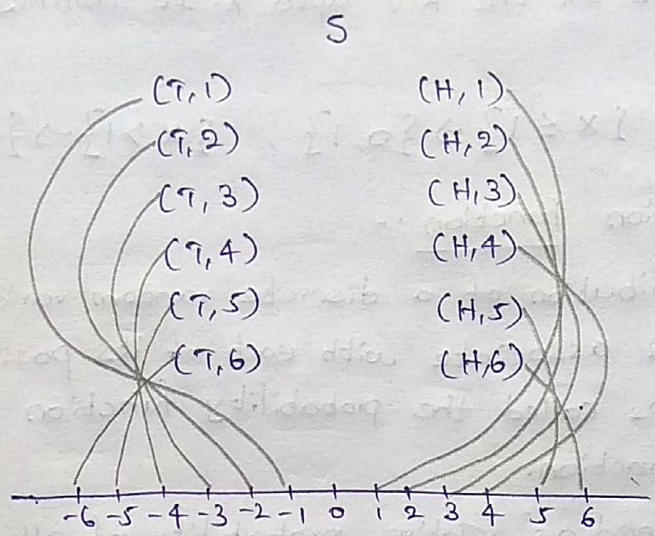
$X = 2s^2 - 1 \rightarrow X = \{-1, 7, 31, 71\}$

$X = \frac{1}{s+1} \rightarrow X = \{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\}$

3) In an experiment of rolling a dice and flipping a coin, the random variable X is choosen such that

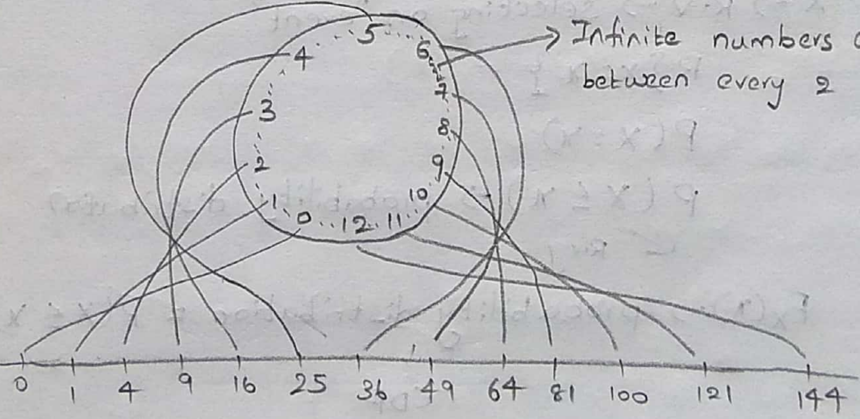
(a) A coin heads (H) outcome - corresponding positive value of X that are equal to the number shown upon the dices and

(b) A coin tails (T) outcome - corresponding negative value of X that are equal to the number shown upon the dices



$X = \{-6, -5, -4, -3, -2, -1, 1, 2, 3, 4, 5, 6\}$.

4) In an experiment the point on a wheel of chance is spin the possible outcome are the no. from 0 to 12 marked on the wheel, the sample space consist of the number in the set. $\{0 < s \leq 12\}$ and if the R.V is defined as $X = X(s) = s^2$. Map the elements of the variable on the real line and explain.



→ Infinite numbers are present in between every 2 numbers.

Conditions to be a random variable :-

if 2 coins are tossed - $\{TT, TH, HT, HH\}$

$X \rightarrow$ count no. of heads

$\{0, 1, 1, 2\}$

$\therefore X = \{0, 1, 2\}$

1.) Each value in the sample space assigned to only one real number

2.) Sample element should not assigned to more than one real value.

3.) One or more sample element may assigne to one real value.

4.) $X \to$ Random variable, $\{X = x\}$ represents element in the R.V also named as "event"

5.) $\{X \leq x\}$ represents all the R.V upto x. It defines all the R.V upto x.

$$X = \{0, 1, 2\}, \quad \{X \leq 1\} \to \{0, 1\}, \quad \{X > 1\} \to \{2\}.$$

Probability Distribution Function :-

The probability distribution of a discrete random variable is a list of probabilityes associate with each of its possible values. It is also sometimes called the probability function or the probability mass function.

PDF :- PDF is defined as relative probability of all the R.V upto x.

(or)

The probability $P(\{X \leq x\}) = P[\{s : X(s) \leq x, s \in S\}]$ is called the probability distribution function {It is also called cumulative distribution function [CDF]} of X and denoted by $F_X(x)$.

$$\therefore F_X(x) = P[\{X \leq x\}]$$

$X \to$ R.V $\to$ selecting an "event"

$P\{X = x\}$

$P(X = x)$

$P(X \leq x) \to$ probability distribution

R.V

$F_X(x) \to$ probability distribution $= P(X \leq x)$

CDF

1.) 2 fair coins are tossed a random variable X is generated based on the number of heads carried in the experiment find probability Distribution of random variable X.

Sol:- Sample space $S = \{TT, TH, HT, HH\}$

$$X = \{0, 1, 1, 2\}$$

X	0	1	2
p(x)	1/4	2/4	1/4
Fx(x)	1/4	3/4	1

$F_X(x) = P(X \le x)$

$\downarrow$

sum of probabilities upto x.

$F_X(0) = P(X \le 0) = P(0) = 1/4$

$F_X(1) = P(X \le 1) = P(0) + P(1) = 1/4 + 2/4 = \frac{3}{4}$

$F_X(2) = P(X \le 2) = P(0) + P(1) + P(2) = 1/4 + 2/4 + 1/4 = 1$

$\therefore P(X=2) = \frac{1}{4}$

2) An experiment is performed by throwing 2 dice random variable is assigned based on sum of the faces, find the $P(X=7)$, $P(X \le 5)$, $P(X>8)$.

Sample space S =

$D_1 \backslash D_2$	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$X = \{2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 11, 11, 12\}$

X	2	3	4	5	6	7	8	9	10	11	12
P(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36
Fx(x)	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

(i) $P(X=7) = 6/36$

(ii) $P(X \le 5) = F_X(5) = 10/36$

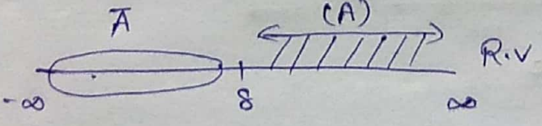
(iii) $P(X>8) = P(9) + P(10) + P(11) + P(12)$

$= 4/36 + 3/36 + 2/36 + 1/36 = 10/36.$

$P(X>8) = 1 - P(X \le 8)$

$= 1 - F_X(8)$

$= 1 - 26/36 = 10/36$



$P(\bar{A}) = 1 - P(A)$

$P(A) = 1 - P(\bar{A})$

Properties of probability Distribution :-

1.) $F_X(-\infty)=0$

$F_X(x) = P(X \leq x)$

Minimum value

$\{-\infty \ldots \ldots 0 \ldots \ldots \infty\}$

$F_X(-\infty) = P(X \leq -\infty)$

$F_X(-\infty) = P(\phi) = 0$

2.) $F_X(\infty)=1$

$F_X(x) = P(X \leq x)$

$F_X(\infty) = P(X \leq \infty)$

$F_X(\infty) = P(S) = 1$

3.) $0 \leq F_X(x) \leq 1$

Based on property 1 & 2 we come to know $F_X(x)$ have min = 0 and max = 1

4.) $F_X(x_1) \leq F_X(x_2)$ if $x_1 < x_2$

$X = \{x_a, x_b, x_c \ldots x_1, x_i, x_j, x_k \ldots x_2, x_l, x_m \ldots\}$

$F_X(x_2) = P(X \leq x_2)$

$= P(X \leq x_1) + P(x_i \leq X \leq x_2)$

$F_X(x_2) = F_X(x_1) + \boxed{P(x_i \leq X \leq x_2)}$ → Added component

$F_X(x_2) \geq F_X(x_1)$

$F_X(x_1) \leq F_X(x_2)$.

5.) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

$X = \{x_a, x_b, x_c, x_d \ldots x_1, x_i, x_j, x_k \ldots x_2, x_l, x_m \ldots\}$

$F_X(x_2) = P(X \leq x_2)$

$= P(X \leq x_1) + P(x_1 < X \leq x_2)$

$F_X(x_2) = F_X(x_1) + P(x_1 < X \leq x_2)$

$F_X(x_2) - F_X(x_1) = P(x_1 < X \leq x_2)$

$$\boxed{P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)}$$

6.) $F_X(x)$ is non negative value

$0 \leq F_X(x) \leq 1$ → 3rd property $F_X(x)$ is non negative

7:) $F_X(x)$ is non Decreasing function

$x = \{x_1, x_2\}$ $\quad x_1 < x_2$

$F_X(x_2) \geq F_X(x_1)$

$F_X(x_1) = F_X(x_2)$

Based on 4th property $F_X(x)$ is nondecreasing

<u>Note</u> :- $F_X(x^+) = F_X(x)$

$+ \rightarrow$ small additive component

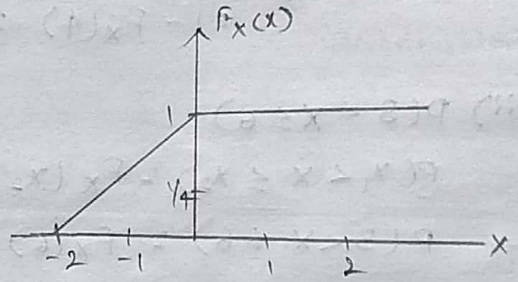
$F_X(x^+) = P(X \leq x + \underbrace{dx}_{\text{small component}}) = F_X(x).$

1:) Consider the random variable X defined by

$F_X(x) = 0 \qquad x < -2$

$F_X(x) = \frac{1}{8}x + \frac{1}{4} \qquad -2 \leq x < 0$

$F_X(x) = 1 \qquad x \geq 0$, Find $P(X=0)$, $P(X \leq 0)$, $P(X > 2)$



(i) $P(X=0) = 1$

(ii) $P(X \leq 0) = F_X(0^+)$

$= P(X=0) - F_X(0^-)$

$= 1 - \left[\frac{1}{8}(0) + \frac{1}{4}\right]$

$= 1 - \frac{1}{4} = 3/4$

(iii) $P(X > 2) = 1$

(iv) $P(-1 < X \leq -1)$

$\therefore P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

$P(-1 < X \leq 1) = F_X(1) - F_X(-1)$

$= 1 - \left[\frac{1}{8}(-1) + \frac{1}{4}\right]$

$= 1 - \frac{1}{4} = 3/4$

2:) The probability mass function of X is given in the table

X	0	1	2	3	4	5	6
P(X)	K	3K	5K	7K	9K	11K	13K

Find $P(X<4)$, $P(X \geq 5)$ and $P(3 < X \leq 6)$, calculate the minimum value of k such that $P(X \geq 2) > 0.3$

Sol:- To find unknown value use 2nd property

$F_X(\infty) = 1$

$F_X(6) = 1$, $P(X \le 6) = K + 3K + 5K + 7K + 9K + 11K + 13K = 1$

$49K = 1$

$\boxed{K = 1/49}$

X	0	1	2	3	4	5	6
P(x)	1/49	3/49	5/49	7/49	9/49	11/49	13/49
$F_X(x)$	1/49	4/49	9/49	16/49	25/49	36/49	49/49

(i) $P(X < 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= P(X \le 3) = F_X(3) = 16/49$

(ii) $P(X \ge 5) = 1 - P(X < 5)$

$= 1 - P(X \ge 4)$

$= 1 - F_X(4) = 1 - \frac{25}{49} = \frac{24}{49}.$

(iii) $P(3 < X \le 6)$

$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$P(3 < X \le 6) = F_X(6) - F_X(3)$

$= 49/49 - 16/49$

$= 33/49$

(iv) $P(X \ge 2) = 1 - P(X < 2) > 0.3$

$= 1 - (K + 3K) > 0.3$

$= 1 - 4K > 0.3$

$0.7 > 4K$

$\frac{0.7}{4} > K$

$K < \frac{0.7}{4}$

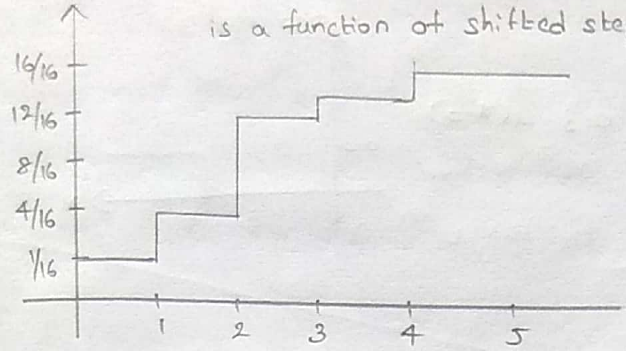
3) Consider an experiment of tossing 4 fair coins the random variable X is associated with the number of tails showing. Compute and sketch the CDF of X.

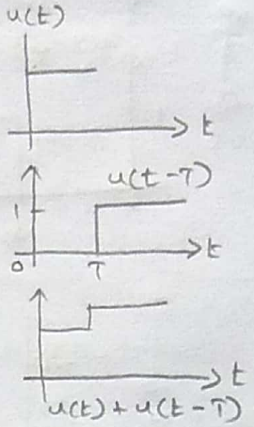
Sol:- S = {TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH, HTTT, HTTH, HTHT, HTHH, HHTH, HHHT, HHHH, HHTT}

X = {4, 3, 3, 2, 3, 2, 2, 1, 3, 2, 2, 1, 1, 1, 0, 2}

x	0	1	2	3	4
$p(x)$	$1/16$	$5/16$	$6/16$	$3/16$	$1/16$
$F_X(x)$	$1/16$	$6/16$	$12/16$	$15/16$	$16/16$

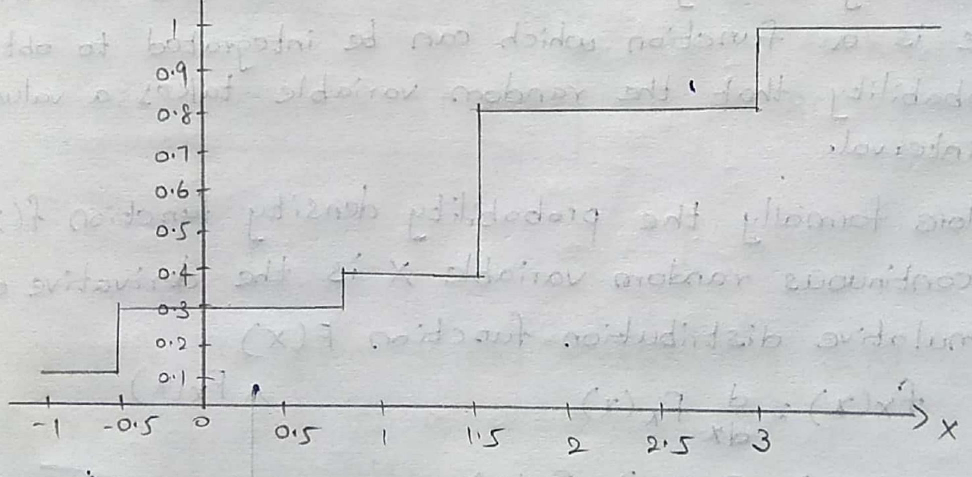
is a function of shifted steps



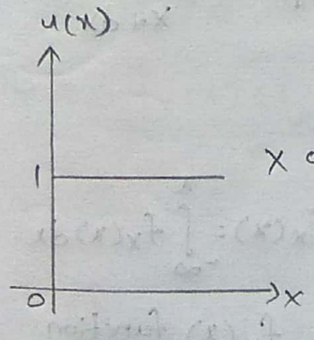


4) The R.V X has the discrete variable in the set {-1, -0.5, 0.7, 1.5, 3} the corresponding probabilities are assumed to be {0.1, 0.2, 0.1, 0.4, 0.2} plot the distribution function and state whether it is discrete distribution or continuous distribution.

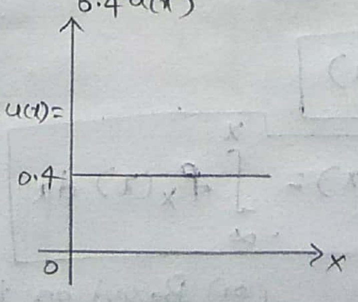
X	-1	-0.5	0.7	1.5	3
$p(x)$	0.1	0.2	0.1	0.4	0.2
$F_X(x)$	0.1	0.3	0.4	0.8	1.0

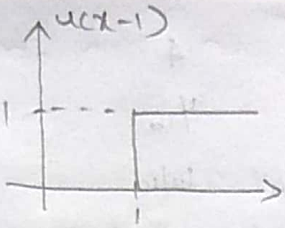


Unit step $u(x) = \begin{cases} 1 & ; \forall\ x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

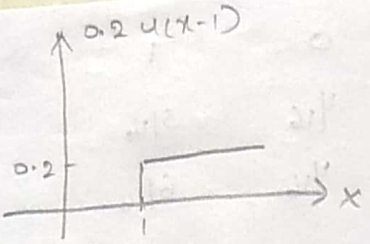


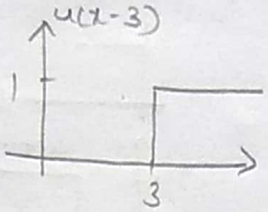
$\times\ 0.4\ u(x) =$



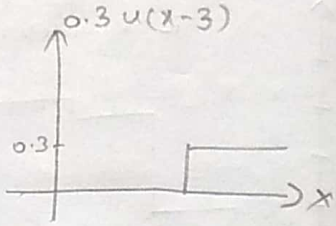


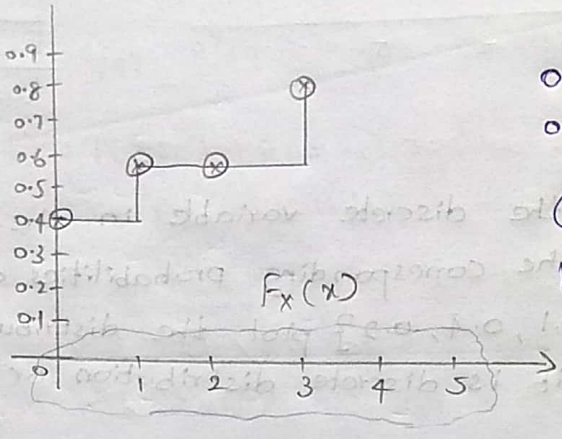
$\times\ 0.2\, u(x-1)$





$\times\ 0.3\, u(x-3)$





$0.4\,u(x) + 0.2\,u(x-1) + 0.3\,u(x-3)$

General formate for probability distribution function of CDF

$$\boxed{F_X(x) = \sum_x P(x)\, u(a-x)}$$

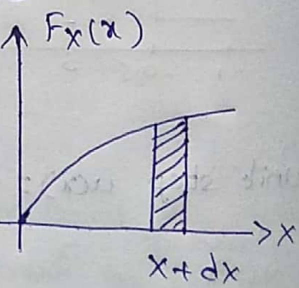
graphical

Probability Density function (Pdf) / probability mass function:-

The probability density function of a continuous random variable is a function which can be integrated to obtain the probability that the random variable takes a value in given interval.

More formally the probability density function $f(x)$ of a continuous random variable X is the derivative of the cumulative distribution function $F(X)$

$$f_X(x) = \frac{d}{dx} F_X(x)$$



$$P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$$

$$f_X(x) = \lim_{dx \to 0} \frac{F_X(x+dx) - F_X(x)}{dx}$$

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Distribution :- $\boxed{F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx}$ $\quad F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx.$

(or) Based on the $f_X(x)$ function.

Properties of Density function :-

1.) Probability density function is a non-negative

$$f_X(x) \geq 0$$

2.) $0 \leq f_X(x) \leq 1$

3.) Adding all the probability density functions from $-\infty$ to $+\infty$ the value becomes unity.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

4.) $$\int_{a_1}^{a_2} f_X(x)\,dx = F_X(a_2) - F_X(a_1)$$

$$= P(a_1 \leq X \leq a_2)$$

5.) Probability distribution upto a random variable x is calculated by adding all the probability density function upto the value x. $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

$$\boxed{F_X(x) = \sum_X P(x)\,u(X-x)}$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$= \frac{d}{dx} \sum_X P(x)\,u(X-x)$$

$$= \sum_X P(x) \frac{d}{dx} u(X-x)$$

$$\boxed{f_X(x) = \sum_X P(x)\,\delta(X-x)}$$

w.k.T

$$\delta(x) = \begin{cases} 1 & \text{for } x=0 \\ 0, & \text{otherwise.} \end{cases}$$

Mass function

Histogram

1.) Find that the given density function is valid or not

$$f_X(x) = \begin{cases} 1/5 \ ; & \text{for } 0 \leq x \leq 5 \\ 0 \ ; & \text{otherwise} \end{cases}$$

(i) $f_X(x) \geq 0$

$1/5 \geq 0$ ✓

(ii) $0 \leq f_X(x) \leq 1$ ✓

(iii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$= \int_0^5 \frac{1}{5}\,dx \Rightarrow \frac{1}{5}[x]_0^5$$

$$= \frac{1}{5}[5-0] \Rightarrow \frac{5}{5}$$

$$= 1$$

Given $f_X(x)$ is valid density function

2.) Find that the given density function is valid or not

$$f_X(x) = \begin{cases} e^{-x} & \text{for } 0 \le x \le \infty \\ 0 & \text{otherwise.} \end{cases}$$

(i) $f_X(x) \ge 0$

(ii) $0 \le f_X(x) \le 1$

at $x = 0$ — at $x = \infty$

$f_X(0) = e^{-0} = 1$ — $f_X(\infty) = e^{-\infty} = 0$

(iii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$$\int_0^{\infty} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -(0-1) = 1$$

Given $f_X(x)$ is valid density function.

3.) Find that the given density function is valid or not

$$f_X(x) = \begin{cases} xe^{-x^2} & \text{for } 0 \le x \le \infty \\ 0 & \text{otherwise} \end{cases}$$

Sol:- $\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} xe^{-x^2}\,dx$

$$= \int_0^{\infty} e^{-t}\frac{dt}{2}$$

let $x^2 = t$

$2x\,dx = dt$

$x\,dx = dt/2$

L.L :- $x = 0 \Rightarrow t = 0$

U.L :- $x = \infty \Rightarrow t = \infty$

$$= \frac{1}{2}\left[\frac{e^{-t}}{-1}\right]_0^{\infty}$$

$$= \frac{-1}{2}[0-1] = \frac{1}{2} \ne 1$$

Hence given $f_X(x)$ is not valid density function.

4.) Find Pdf of given CDF $F_X(x) = \begin{cases} 1 - e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{; otherwise} \end{cases}$

PDF

$f_X(x) = \frac{d}{dx} F_X(x) \rightarrow$ CDF

$$f_X(x) = \frac{d}{dx}(1 - e^{-2x})$$

$$= 0 - e^{-2x}(-2)$$

$$\boxed{f_X(x) = 2e^{-2x}}$$

5.) $F_X(x) = \sin x$ find PDF of the given distribution function

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}\sin x = \cos x.$$

6.) A PDF density function is given as

$$f_X(x) = \begin{cases} Cx & \text{for } 0 \le x \le 3 \\ C(5-x) & \text{for } 3 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$ find C value.

Note:- When he asked to find a unknown constant use 2nd property.

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$\underbrace{\int_{-\infty}^{0} 0\, dx}_{0} + \int_{0}^{3} Cx\,dx + \int_{3}^{5} C(5-x)\,dx + \underbrace{\int_{5}^{\infty} 0.dx}_{0} = 1$$

$$\int_{0}^{3} Cx\,dx + \int_{3}^{5} C(5-x)dx = 1$$

$$C\left[\frac{x^2}{2}\right]_0^3 + C\left[5x - \frac{x^2}{2}\right]_3^5 = 1$$

$$\frac{9C}{2} + C\left[25 - \frac{25}{2} - 15 + \frac{9}{2}\right] = 1$$

$$\frac{9C}{2} + C\left[\frac{50-25-30+9}{2}\right] = 1$$

$$\frac{9C}{2} + C\left[\frac{59-55}{2}\right] = 1$$

$$\frac{9C}{2} + 2C = 1$$

$$C\left(\frac{13}{2}\right) = 1$$

$$\boxed{C = \frac{2}{13}}$$

7.) $f_X(x) = \begin{cases} A(3x - x^2) & \text{for } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

(a) what is the value of A (b) Find $P(X > 1)$

Sol:- $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\int_{0}^{2} A(3x - x^2)dx = 1$$

$$A\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$A\left[\frac{\cancel{12}^{6}}{\cancel{2}} - \frac{9}{3}\right] = 1$$

$$A\left[\frac{10}{3}\right] = 1 \Rightarrow \boxed{A = \frac{3}{10}}$$

$$\int_1^{\infty} f_x(x)dx = \int_1^2 \frac{3}{10}(3x-x^2)dx$$

$$= \frac{3}{10}\left[3\frac{x^2}{2} - \frac{x^3}{3}\right]_1^2 = \frac{3}{10}\left[6 - \frac{8}{3} - \frac{3}{2} + \frac{1}{3}\right]$$

$$= \frac{3}{10}\left[\frac{36-16-9+2}{6}\right] = \frac{3}{10}\left[\frac{38-25}{6}\right]$$

$$= \frac{\cancel{3}}{10}\left[\frac{13}{\cancel{6}_2}\right] = \frac{13}{20}$$

8.) If PDF is given by $f_x(x) = \begin{cases} K(1-x^2) & \forall\ 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

find K, $F_x(x)$, $F_x(0.2)$

(i) $\int_{-\infty}^{\infty} f_x(x)dx = 1$

$$\int_0^1 K(1-x^2)dx = 1 \Rightarrow K\left[x - \frac{x^3}{3}\right]_0^1 = 1 \Rightarrow K\left[1 - \frac{1}{3}\right] = 1$$

$$\Rightarrow \boxed{K = 3/2}$$

(ii) $F_x(x) = \int_{-\infty}^{x} f_x(x)dx$

$$= \int_0^x \frac{3}{2}(1-x^2)dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]$$

(iii) $F_x(0.2)$

$$F_x(0.2) = \int_{-\infty}^{0.2} f_x(x)dx$$

$$= \frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^{0.2} = \frac{3}{2}\left[0.2 - \frac{(0.2)^3}{3}\right]$$

$$= 0.296.$$

Bernoulli trail :- It is used to find out the probability of independent events

Let consider an experiment is performed 'N' of times and it is having two possibilities such as pass-fail, Hit-miss, yes-No, True-False, Tails-heads among 'N' samples 'k' are the positive result with the probability of p. N-K are negative result with the probability of N-P is taken into consideration then we can write the probability of getting 'K' events

$$P(K) = {}^{N}C_{K}\, P^{K} (1-P)^{N-K}$$

Conditions of Bernouli trail :-

1.) The trail are finite - N

2.) The outcome ~~have~~ cant have probability

3.) The outcomes are independent

4.) Have only 2 - possibilities.

Binomial Distribution :-

$$F_X(x) = P(X=x) = \sum_{x=0}^{K} {}^{N}C_K \, p^K (1-P)^{N-K} u(X-K)$$ → staircase.

Binomial density function

$$f_X(x) = \frac{d}{dx} F_X(x).$$

$$= \frac{d}{dx} \sum_{x=0}^{K} ({}^{N}C_K) P^K (1-P)^{N-K} u(X-x)$$

$$= \sum_{x=0}^{K} ({}^{N}C_K) p^K (1-p)^{N-K} \frac{d}{dx} u(X-x)$$

$$= \sum_{x=0}^{K} ({}^{N}C_K) p^K (1-p)^{N-K} \delta(X-x)$$

1) Binomial distribution function $F_X(x) = P(X \leq x)$

$$= \sum_{x=0}^{K} {}^{N}C_K \, P^K (1-P)^{N-K} u(X-x)$$

2.) Binomial density function $f_X(x)$

$$= \sum_{x=0}^{K} {}^{N}C_K \, P^K (1-P)^{N-K} \delta(X-x)$$

1.) In chirala during last week rain fall on an average

of

Sol:-

$P \rightarrow$ Getting rain fall $= \frac{3}{7}$

$1-P \rightarrow$ Not getting rain fall $= \frac{4}{7}$

$$P(X \geq 2) = P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)$$

$$P(X=K) = {}^{N}C_K \, P^K (1-P)^{N-K}$$

$$P(X \geq 2) = {}^{7}C_2 \left(\frac{3}{7}\right)^2 \left(\frac{4}{7}\right)^5 + {}^{7}C_3 \left(\frac{3}{7}\right)^3 \left(\frac{4}{7}\right)^{7-3} + {}^{7}C_4 \left(\frac{3}{7}\right)^4 \left(\frac{4}{7}\right)^{7-4}$$

$$+ {}^{7}C_5 \left(\frac{3}{7}\right)^5 \left(\frac{4}{7}\right)^{7-5} + {}^{7}C_6 \left(\frac{3}{7}\right)^6 \left(\frac{4}{7}\right)^{7-6} + {}^{7}C_7 \left(\frac{3}{7}\right)^7 \left(\frac{4}{7}\right)^{7-7}$$

$${}^{N}C_K = \frac{N!}{K!(N-K)!}$$

(or)

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - [P(X=0) + P(X=1)]$$
$$= 1 - \left[{}^7c_0 \left(\tfrac{3}{7}\right)^0 \left(\tfrac{4}{7}\right)^7 + {}^7c_1 \left(\tfrac{3}{7}\right)^1 \left(\tfrac{4}{7}\right)^6 \right]$$
$$= 1 - \left[1 \cdot (1) \left(\tfrac{4}{7}\right)^7 + 7 \left(\tfrac{3}{7}\right) \left(\tfrac{4}{7}\right)^6 \right]$$
$$= 1 - [0.0198 + 0.1044] = 1 - 0.1242$$
$$= 0.8758.$$

Types of Density functions and Distribution functions:-

Two types

Based on R.V

- Discrete R.V
  - Binomial distribution & density function
  - poission's distribution & density function.
- Continious R.V
  - uniform density & dis func
  - Exponential density & dist func
  - Gaussian density & dist func
  - Relay density & dist funct

Binomial distribution & Density function:-

Distribution $F_X(x) = \sum_{x=0}^{K} {}^nc_k \, p^K (1-p)^K u(x-x)$

Density $f_X(x) = \frac{d}{dx} F_X(x)$

$$= \sum_{x=0}^{K} {}^nc_k \, p^K (1-p)^{n-K} \delta(x-x)$$

Poission's Distribution and density function:-

It is mainly used in traffic engineering such as number of telephone calls carried in 1 sec (or) no. of vehicles passing through the tollgate in 1 hour, Number of bits pass through the channel in 1 sec and their respective probability calculated by using poission's probability density & distribution function.

* Time, date, Rate, state the density function is poissions Distribution function.

Distribution function

$$P(X \le k) = F_X(k) = \sum_{x=0}^{k} \frac{e^{-\lambda}\lambda^{x}}{x!} u(x-x)$$

Density function

$$f_X(x) = \frac{d}{dx} F_X(x) = \sum_{x=0}^{k} \frac{e^{-\lambda}\lambda^{x}}{x!} \delta(x-x)$$

Poissions Distribution function

$$P(X=k) = \frac{e^{-\lambda}\lambda^{k}}{k!}$$ λ is the rate representation.

Binomial	Poission's
1.) Two outcomes	1.) Infinite, discrete outcomes
2.) $F_X(x) = \sum_{x=0}^{k} {}^{N}C_{k}\, p^{k}(1-p)^{N-k} u(x-x)$	2.) $F_X(k) = \sum_{x=0}^{N} \frac{e^{-\lambda}\lambda^{x}}{x!} u(x-x)$
3.) $f_X(x) = \sum_{x=0}^{k} {}^{N}C_{k}\, p^{k}(1-p)^{N-k} \delta(x-x)$	3.) $f_X(x) = \sum_{x=0}^{k} \frac{e^{-\lambda}\lambda^{x}}{x!} \delta(x-x)$.

1.) A R.V modelled as a Binomial density function with $n=8$, $P=1/2$ find $P(X=4)$, $P(X<2)$.

Sol:- $P(X=k) = {}^{n}C_{k}\, p^{k}(1-p)^{n-k}$

$$P(X=4) = {}^{8}C_{4} \left(\tfrac{1}{2}\right)^{4}\left(1-\tfrac{1}{2}\right)^{8-4}$$

$$= \frac{\cancel{8}^{2}\times 7\times \cancel{6}^{2}\times 5}{\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}\left(\tfrac{1}{16}\right)\left(\tfrac{1}{16}\right)$$

$$= \frac{35}{128} = 0.2734$$

$$P(X<2) = P(X=0) + P(X=1)$$

$$= {}^{8}C_{0}\left(\tfrac{1}{2}\right)^{0}\left(\tfrac{1}{2}\right)^{8} + {}^{8}C_{1}\left(\tfrac{1}{2}\right)^{1}\left(\tfrac{1}{2}\right)^{7}$$

$$= \left(\tfrac{1}{2}\right)^{8} + 8\left(\tfrac{1}{2}\right)^{8}$$

$$= 9\left(\tfrac{1}{2}\right)^{8} = 0.0351$$

2.) DRDO had designed a missile that hit the target 3 times when it fixes 5 times, what is the probability that the missile hit the target at least 3 times.

Sol:-

$$P(X \ge 3) = 1 - P(X<3)$$ $p = \frac{3}{5}$

$$P(X<3) = P(X=0) + P(X=1) + P(X=2)$$

$$P(X=0) = {}^{5}C_{0}\, p^{0}(1-p)^{5-0} = {}^{5}C_{0}\left(\tfrac{3}{5}\right)^{0}\left(1-\tfrac{3}{5}\right)^{5}$$

$$= \left(\tfrac{2}{5}\right)^{5}.$$

$P(X=1) = {}^5C_1\left(\frac{3}{5}\right)\left(1-\frac{3}{5}\right)^4 = 5\times\frac{3}{5}\times\left(\frac{2}{5}\right)^4 = 3\times\left(\frac{2}{5}\right)^4$

$P(X=2) = {}^5C_2\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3 = 10\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3$

$P(X<3) = \left(\frac{2}{5}\right)^2 + 3\times\left(\frac{2}{5}\right)^4 + 10\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^3$

$= 0.01024 + 0.768 + 0.2304$

$= 0.31744$

$P(X\geq 3) = 1 - P(X<3)$

$= 1 - 0.31744$

$= 0.68256.$

$P(X=4) = {}^5C_4\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1$

$= 5\times\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^1$

$= 0.2592$

3.) A random variable X is known to be a poissions with $\lambda=4$ find the probability of occurance of an event $P(X=2)$

Sol:- $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

$P(X=2) = \frac{e^{-4}(-4)^2}{2!} = \frac{e^{-4}(16)}{2} = 8\times e^{-4}$

$= 0.1465.$

4.) People enter a club at the rate of 2 members for every 4 minutes what is the probability no person enter between 10:00 to 10:10. what is the probability at least 4 persons enter into the club.

Sol:- $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

4 min — 2

10 min — λ ; $\lambda = \frac{20}{4} = 5$

(i) probability of no person enter into the club is

$P(X=0) = \frac{e^{-5}(5)^0}{0!} = e^{-5} = 0.0067$

(ii) Probability of at least 4 persons enter into the club

$P(X\geq 4) = 1 - P(X<4)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$$= 1 - \left[e^{-5} + \frac{e^{-5}(5)^1}{1!} + \frac{e^{-5}(5)^2}{2!} + \frac{e^{-5}(5)^3}{3!}\right]$$

$$= 1 - \left[e^{-5}\left\{1 + 5 + \frac{25}{2} + \frac{125}{6}\right\}\right]$$

$$= 1 - \left[e^{-5}\left[\frac{36 + 75 + 125}{6}\right]\right]$$

$$= 1 - e^{-5}\left(\frac{236}{6}\right) = 1 - e^{5}(39.33) = 1 - 0.2650$$

$$= 0.735$$

5) Assume auto mobile arrived at a gasoline station are poission's in nature and occur at an average rate of 50/hour if all vehicles are assumed to required 1 minute to obtain fuel what is the probability of form a waiting line.

Sol:- 50 - 1 hr - 60 min

λ - 1 min $\Rightarrow \lambda = \frac{50}{60} = \frac{5}{6}$.

We know that ; A waiting line formed by at least 2 vehicles.

$$\therefore P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{e^{-(5/6)}\left(\frac{5}{6}\right)^0}{0!} + \frac{e^{-5/6}\left(\frac{5}{6}\right)}{1!}\right]$$

$$= 1 - [0.4345 + 0.3621]$$

$$P(X \geq 2) = 0.2034.$$

6) Let a lot contain 2% defective items no. of items in a random sample space are 114, what is the probability at least one item is defective.

Sol:- $n = 114$ $\quad p = 2\% = \frac{2}{100}$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

$$= 1 - {}^{114}C_0 \left(\frac{2}{100}\right)^0 \left(1 - \frac{2}{100}\right)^{114-0}$$

$$= 1 - (1)(1)\left(\frac{98}{100}\right)^{114}$$

$$= 1 - 0.0999 = 0.900$$

Poission's

$100 \rightarrow 2$

$114 \rightarrow \lambda \quad \lambda = \frac{114 \times 2}{100}$

$\lambda = 2.28$

$$P(X \geq 1) = 1 - P(X < 1)$$

$$= 1 - P(X=0)$$

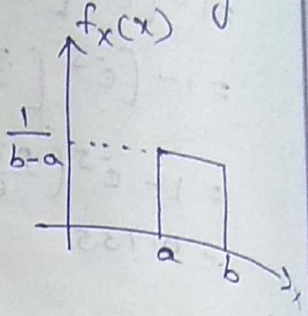
$$= 1 - e^{-2.28}$$

$$= 1 - 0.1022$$

$$= 0.892.$$

Uniform density function & Distribution function:-

A continuous random variable x is called uniformly distributed over the interval $[a, b]$.



Uniform Density function:-

$$f_X(x) = \frac{1}{b-a} \quad \forall \; a \le x \le b$$

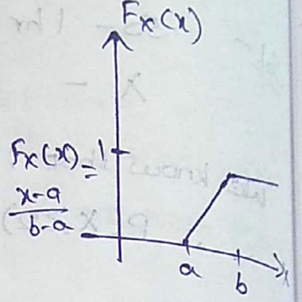
Uniform distribution function:-

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{b-a}\,dx = \int_{a}^{x} \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[x\right]_a^x$$

$$= \frac{1}{b-a}\left[x-a\right]$$

$$\boxed{F_X(x) = \frac{x-a}{b-a} \quad \forall \; a \le x \le b}$$



Exponential Density & Distribution function:-

$$f_X(x) = \frac{1}{b} e^{\frac{-(x-a)}{b}} \quad \forall \; x > a$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\cdot dx = \int_{-\infty}^{x} \frac{1}{b} e^{-\frac{(x-a)}{b}}\,dx$$

$$= \frac{1}{b}\int_{a}^{x} e^{-\frac{(x-a)}{b}}\,dx$$

Let $\frac{x-a}{b} = t$

$x-a = bt$

$dx = bdt$

LL $\Rightarrow x = a;\; t = 0$

UL $\Rightarrow x = x;\; t = \frac{x-a}{b}$

$$= \frac{1}{b}\int_{0}^{x-a/b} e^{-t}\, b\,dt$$

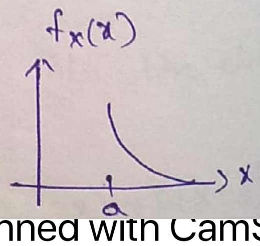
$$= \left[\frac{e^{-t}}{-1}\right]_0^{x-a/b}$$

$$= \left[-e^{\frac{-(x-a)}{b}} - (-1)\right]$$

$$F_X(x) = 1 - e^{\frac{-(x-a)}{b}} \quad \forall \; x > a$$

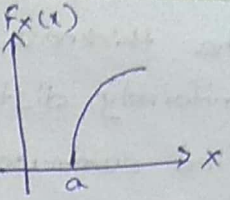
Exponential density function:-

$$f_X(x) = \frac{1}{b} e^{\frac{-(x-a)}{b}} \quad \forall \; x > a$$



ii) <u>Exponential</u> <u>distribution</u> <u>function</u>:-

$$F_X(x) = 1 - e^{-\frac{(x-a)}{b}} \quad \forall \quad x > a$$



1.) If x is uniform distributed over $(0,5)$ find the probability density function & distribution function find $P(X>2)$, $P(X>4)$, $P(1<X<3)$

<u>Sol</u>:- $f_X(x) = \frac{1}{b-a} \quad \forall \quad a \le x \le b$

x is uniform distributed over $(0,5)$

$a=0, \; b=5$

$$f_X(x) = \frac{1}{b-a} = \frac{1}{5-0} = \frac{1}{5} \quad \forall \quad 0 \le x \le 5$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x) \cdot dx$$

$$= \int_{0}^{x} \frac{1}{5} dx = \frac{1}{5}[x] = \frac{1}{5}x \quad \forall \quad 0 \le x \le 5$$

(i) $P(X>2) = 1 - P(X \le 2)$

$= 1 - F_X(2)$

$= 1 - \int_{-\infty}^{2} f_X(x)$

$= 1 - \int_{0}^{2} \frac{1}{5} dx = 1 - \frac{1}{5}[x]_0^2 = 1 - \frac{2}{5}$

$P(X>2) = \frac{3}{5}$

(ii) $P(X>4) = 1 - P(X \le 4)$

$= 1 - F_X(4)$

$= 1 - \int_{-\infty}^{4} f_X(x) dx = 1 - \int_{0}^{4} \frac{1}{5} dx = 1 - \frac{1}{5}[x]_0^4$

$= 1 - \frac{4}{5}$

$P(X>4) = \frac{1}{5}$

(iii) $P(1<X<3) = F_X(3) - F_X(1)$

$= \int_{0}^{3} \frac{1}{5} dx - \int_{0}^{1} \frac{1}{5} dx$

$= \frac{1}{5}[x]_0^3 - \frac{1}{5}[x]_0^1 \Rightarrow \frac{1}{5}[3] - \frac{1}{5}[1]$

$P(1<X<3) = \frac{2}{5}$

2.) The thickness of a sheet used in automobile industry is uniformly distributed in the range of 0.9 to 1.10 mm. Find commutative distribution.

$$f_X(x) = \frac{1}{b-a} \quad \forall \; a \leq x \leq b$$

x is uniformly distributed in the range of 0.9 to 1.10 mm.

$a = 0.9, \; b = 1.10$

$$f_X(x) = \frac{1}{b-a} = \frac{1}{1.10 - 0.9} = \frac{1}{0.2} = 5$$

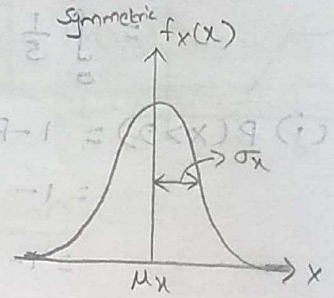
$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} 5\,dx = \int_{0.9}^{x} 5\,dx$$

$$= 5\left[x\right]_{0.9}^{x} = 5\left[x - 0.9\right] = 5x - 4.5$$

Gaussian Density function:-

It is defined as a bell shaped symmetrical function about mean. It is formulated as.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}$$



$\mu_x \rightarrow$ mean

$\sigma_x^2 \rightarrow$ variance

$\sigma_x \rightarrow$ standard deviation, $\mu = \frac{\text{Sum of all points}}{\text{Total no. of points.}}$

-> The spreading of width of the corresponding curve is σ_x

-> Maximum value obtained at $f_X(x) = \mu_x$

=> $\sigma_x < \sigma_x^2$

-> symmetric $x(-t) = x(t) \rightarrow$ even

-> Asymmetric $x(-t) = -x(t) \rightarrow$ odd

Normal Density Function:- $[N(0,1)]$:-

The normal distribution is the most important distribution used to model natural and man made phenomena. Particularly when the random variable is the result of the addition of large number of independent random variable, it can be modelled as a normal random variable.

=> In any gaussian mean = 0, Variance = 1 that the particular gaussian function is called normal density function.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \quad \forall \; -\infty \le x \le \infty$$

Mean, variance

$$N(0,1) = f(x) = \frac{1}{\sqrt{2\pi(1)}} e^{\frac{-(x-0)^2}{2(1)}} \quad \forall \; -\infty \le x \le \infty$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \forall \; -\infty \le x \le \infty}$$

Gaussian distribution / Relation between Gaussian Distribution & Normal distribution

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx.$$

let $\frac{x-\mu_x}{\sigma_x} = t$

$x-\mu_x = t\sigma_x$

$dx = \sigma_x dt$

L.L :- $x = -\infty$; $t = -\infty$

U.L :- $x = x$; $t = \frac{x-\mu_x}{\sigma_x} = t$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{t} e^{-t^2/2} \sigma_x\, dt.$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} \sigma_x \int_{-\infty}^{t} e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} f(t)\,dt$$

Here $f(t)$ is the normal density function with variable t.

$$F_X(x) = F(t)$$

$$F_X(x) = F\left(\frac{x-\mu_x}{\sigma_x}\right)$$

Steps to be follow carry the Gaussian problem:-

1.) Check the problem for mean, variance, standard deviation $(\mu_x, \sigma_x, \sigma_x^2)$

2.) Check the probability distribution $F_X(x) = P(X \le x)$ or $P(X \ge x)$

3.) Using $F_X(x), \mu_x, \sigma_x$ calculate $F\left(\frac{x-\mu_x}{\sigma_x}\right)$ → a

4.) $F(a) = 1 - Q(a)$. (Normal distribution function into error function)

$Q(a)$ = error function.

$$Q(a) = \frac{1}{0.669a + 0.331\sqrt{a^2+5.51}} \left[\frac{e^{-a^2/2}}{\sqrt{2\pi}}\right]$$

Problems:-

1.) Find the probability of an event $X \leq 5.5$ for gaussian random variable with mean $= 3$, standard deviation $= 2$.

Sol:- ① Given data is mean $(\mu_X) = 3$,

standard deviation $(\sigma_X) = 2$.

② Finding $P(X \leq 5.5) = F_X(5.5)$

③ $F_X(x) = F\left(\frac{x-\mu_X}{\sigma_X}\right)$

$$F_X(5.5) = F\left(\frac{5.5-3}{2}\right) = F(1.25)$$

④ $F(a) = 1 - Q(a)$

$$F(1.25) = 1 - Q(1.25)$$

W.k.T

$$Q(a) = \frac{1}{0.669a + 0.331\sqrt{a^2+5.51}} \times \left[\frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}\right]$$

$$Q(1.25) = \frac{1}{0.669 \times 1.25 + 0.331\sqrt{(1.25)^2+5.51}} \left[\frac{e^{-\frac{(1.25)^2}{2}}}{\sqrt{2\pi}}\right]$$

$$= \frac{1}{0.83625 + 0.88026}\left[\frac{0.45783}{2.50662}\right]$$

$$= 0.58257[0.18264]$$

$$= 0.1064$$

$$F(1.25) = 1 - Q(1.25) = 1 - 0.1064 = 0.8938$$

$$F_X(5.5) = F(1.25) = 0.8938$$

$$P(X \leq 5.5) = 0.8938.$$

2.) A Gaussian random variable X with mean $= 4$, variance $= 9$ Find $F_X(8)$.

Sol:- ① Given parameters $\mu_X = 4$, $\sigma_X^2 = 9$, $\sigma_X = \sqrt{\sigma_X^2} = 3$.

② Finding $F_X(8) = F\left(\frac{x-\mu_X}{\sigma_X}\right)$

$$F_X(8) = F\left(\frac{8-4}{3}\right) = F(1.33)$$

③ $F(a) = 1 - Q(a)$:

$F(1.33) = 1 - Q(1.33)$

W.K.T $Q(a) = \dfrac{1}{0.669\,a + 0.331\sqrt{a^2+5.51}} \times \left[\dfrac{e^{-a^2/2}}{\sqrt{2\pi}}\right]$

$Q(1.33) = \dfrac{1}{0.669 \times 1.33 + 0.331\sqrt{(1.33)^2+5.51}} \times \left[\dfrac{e^{-\frac{(1.33)^2}{2}}}{\sqrt{2\pi}}\right]$

$= \dfrac{1}{0.88977 + 0.89301} \times \left[\dfrac{0.41294}{2.50662}\right]$

$= 0.56092(0.16473) = 0.09240$

$F(1.33) = 1 - Q(1.33) = 1 - 0.09240$

$F(1.33) = 0.9076.$

$F_X(8) = F(1.33) = 0.9076$

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3) A Gaussian density function with mean $= 2$, $\sigma_x^2 = 9$

Find $P(X \le 1)$

Sol:- ① Given data mean $(\mu_x) = 2$, Variance $(\sigma_x^2) = 9$, $\sigma_x = \sqrt{\sigma_x^2} = 3$

② $P(X \le 1) = F_X(1)$

$F_X(x) = F\left(\dfrac{x - \mu_x}{\sigma_x}\right)$

$F_X(1) = F\left(\dfrac{1-2}{3}\right) = F(-0.33)$

③ $F(a) = 1 - Q(a)$

$F(-0.33) = 1 - Q(-0.33)$

W.K.T $Q(a) = \dfrac{1}{0.669a + 0.331\sqrt{a^2+5.51}} \times \left[\dfrac{e^{-a^2/2}}{\sqrt{2\pi}}\right]$

$Q(-0.33) = \dfrac{1}{(0.669)(-0.33) + 0.331\sqrt{(-0.33)^2+5.51}} \times \left[\dfrac{e^{-\frac{(-0.33)^2}{2}}}{\sqrt{2\pi}}\right]$

$= \dfrac{1}{-0.22077 + 0.78460} \times \left[\dfrac{0.94700}{2.50662}\right]$

$= 1.77358(0.37779)$

$= 0.67004$

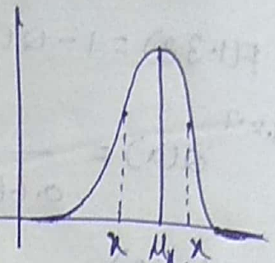
$F(-0.33) = 1 - Q(-0.33) = 1 - 0.67004$

$F(-0.33) = 0.32996 \simeq 0.33$

$F(-0.33) = 0.33$

$P(X \le 1) = 0.33$

Properties of Gaussian:-



1.) It is symmetric about mean

$$(x-\mu_x) = (x+\mu_x)$$

2.) It has maximum value of $\frac{1}{\sqrt{2\pi\sigma_x^2}}$ at $x=\mu_x$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}$$

substitute $x=\mu_x$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}(e^{-(0)})$$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}$$

3.) First derivative of Gaussian is +ve for $x<\mu_x$

First derivative of Gaussian is 0 for $x=\mu_x$

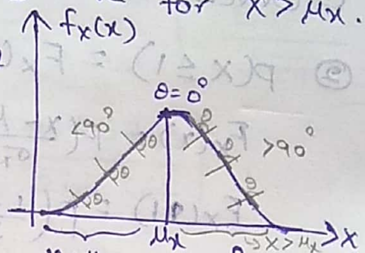
First derivative of Gaussian is -ve for $x>\mu_x$.

$\mathrm{Tan}(<90^\circ) = +ve$

$\mathrm{Tan}(0) = 0$

$\mathrm{Tan}(>90^\circ)$ to

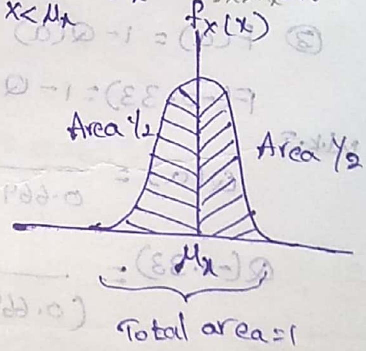
$\mathrm{Tan}(180^\circ) = -ve$



4.) Total area under $f_x(x)=1$

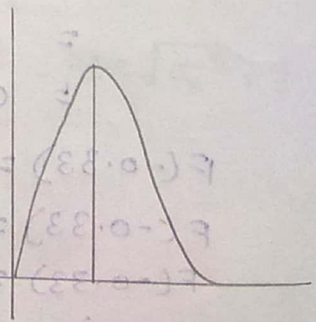
$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

Area of Gaussian divided equally over μ_x.



Rayleigh Density and Distribution function:-

$$f_x(x) = \frac{2}{b}(x-a)e^{\frac{-(x-a)^2}{b}} \quad \forall x>a$$



Distribution function:-

$$F_X(x) = \int_{-\infty}^{\infty} f_x(x)\,dx$$

$$F_X(x) = \int_{a}^{x} \frac{2}{b}(x-a)e^{\frac{-(x-a)^2}{b}}\,dx$$

$$F_X(x) = \int_{t=0}^{\frac{(x-a)^2}{b}} e^{-t}\,dt$$

let

$$\frac{(x-a)^2}{b} = t$$

$$\frac{2(x-a)}{b}dx = dt$$

L.L $x = a \Rightarrow t = 0$

U.L. $x = x \Rightarrow t = \frac{(x-a)^2}{b}$

$$= \left[\frac{e^{-t}}{-1}\right]_0^{(x-a)^2/b}$$

$$= -\left[e^{-(x-a)^2/b} - 1\right]$$

$$\boxed{F_X(x) = 1 - e^{-(x-a)^2/b}}$$

Find the maximum value of Gaussian.

Sol:- Maximum of any function its derivation $= 0$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \quad \forall -\infty \text{ to } \infty$$

$$\frac{d}{dx} f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \cdot -2\left(\frac{x-\mu_x}{2\sigma_x^2}\right) = 0.$$

$$-2\frac{(x-\mu_x)}{2\sigma_x^2} = 0$$

$$x - \mu_x = 0$$

$$\boxed{x = \mu_x}$$

At $x = \mu_x \rightarrow$ Gaussian producing maximum value.

$\rightarrow$ To get maximum value we need to submit $x = \mu_x$ in the gaussian deriving function.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \quad \forall -\infty \text{ to } \infty$$

To get the maximum value substitute $x = \mu_x$ in the above equation.

$$f_X(x)\Big/_{x=\mu_x} = \frac{1}{\sqrt{2\pi\sigma_x^2}}.$$

Rayleigh Density Maximum value :-

$$f_X(x) = \frac{2}{b}(x-a)\, e^{\frac{-(x-a)^2}{b}} \quad \forall \quad x > a$$

$$\frac{d}{dx} f_X(x) = 0$$

$$\frac{d}{dx}\,\frac{2}{b}(x-a)\cdot e^{\frac{-(x-a)^2}{b}} = 0$$

$$\frac{2}{b}(x-a)\left[e^{\frac{-(x-a)^2}{b}}\cdot\frac{-2(x-a)}{b}\right] + e^{\frac{-(x-a)^2}{b}}\cdot\frac{2}{b} = 0$$

$$+\frac{\cancel{4}^{2}}{b^{\cancel{2}}}(x-a)^2\,\cancel{e^{\frac{-(x-a)^2}{b}}} = +\frac{2}{\cancel{b}}\,\cancel{e^{\frac{-(x-a)^2}{b}}}$$

$$\frac{2}{b}(x-a)^2 = 1$$

$$(x-a)^2 = \frac{b}{2}$$

$$x-a = \sqrt{\frac{b}{2}}$$

$$\boxed{x = a+\sqrt{\frac{b}{2}}}$$

To get maximum value $x = a+\sqrt{\frac{b}{2}}$ in $f_x(x)$

$$\frac{2}{b}\left(\cancel{a}+\sqrt{\frac{b}{2}}-\cancel{a}\right)e^{\frac{-(\cancel{a}+\sqrt{\frac{b}{2}}-\cancel{a})^2}{b}}$$

$$= \frac{2}{b}\times\sqrt{\frac{b}{2}}\times e^{\frac{-\frac{\cancel{b}}{2}}{\cancel{b}}}$$

$$= \sqrt{\frac{2}{b}}\cdot e^{-1/2}$$

$$= 0.606\sqrt{\frac{2}{b}}.$$

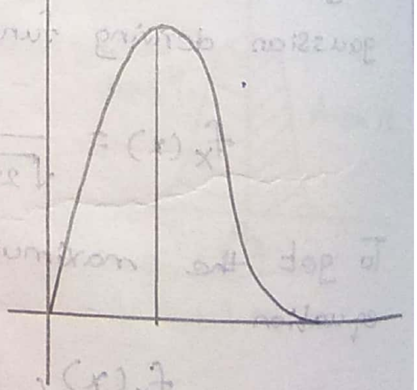
Properties of Rayleigh :-

1.) It is not symmetric

2.) The total area is equal to unity

Total area is $= 1$

But the distribution of area over center value on both sides not same.



3.) Maximum value at $x = a+\sqrt{\frac{b}{2}}$

4.) Maximum value is $0.606\sqrt{\frac{2}{b}}$

Conditional Distribution and Density function :-

We know, conditional probability $P(A/B) = \frac{P(A\cap B)}{P(B)}$

Conditional distribution function:-

$$F_X(x/B) = P(\underbrace{X \le x}_{A}/B), \quad P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$\boxed{F_X(x/B) = P(X \le x/B) = \frac{P(X \le x \cap B)}{P(B)}}$$

Properties :-

1) $F_X(-\infty/B) = 0$

2.) $F_X(\infty/B) = 1$

3.) $0 \le F_X(x/B) \le 1$

4.) $P(x_1 < X \le x_2/B) = F_X(x_2/B) - F_X(x_1/B)$.

5.) if $x_1 < x_2 \Rightarrow F_X(x_1/B) \le F_X(x_2/B)$

6.) $F_X(x/B)$ is non negative.

7.) $F_X(x/B)$ is non decreasing.

Conditional probability Density function :-

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$f_X(x/B) = \frac{d}{dx} F_X(x/B) \rightarrow$ Conditional density function.

Properties of conditional density function:-

1.) $f_X(x/B) \ge 0$

2.) $F_X(x/B) = \int_{-\infty}^{x} f_X(x/B)\,dx$

3.) $P((x_1 \le X \le x_2)/B) = \int_{x_1}^{x_2} f_X(x/B)\,dx$.

4.) $\int_{-\infty}^{\infty} f_X(x/B)\,dx = 1$.

***

1) $F_X(x) = \sum_{n=1}^{12} \frac{n^2}{650} u(x-n)$ Find the probability of $P(-\infty < X \le 6.5)$, $P(X>4)$, $P(6 < X \le 9)$

$\int$ → continuous, Σ → discrete

(i) $P(-\infty < X \le 6.5) = F_X(6.5) - F_X(-\infty)$ $\quad [P(a < X \le b) = F_X(b) - F_X(a)]$

$= F_X(6.5) - 0$

$F_X(6.5) = F_X(6) = \sum_{n=1}^{6} \frac{n^2}{650} u(x-n)$

$= \frac{1}{650} \sum_{n=1}^{6} n^2$

$= \frac{1}{650} [1+4+9+16+25+36]$

$P(-\infty < X \le 6.5) = 0.140$

(ii) $P(X>4) = 1 - P(X \le 4) = 1 - F_X(4) = 1 - \sum_{n=1}^{4} \frac{n^2}{650}$

$= 1 - \frac{1}{650} [1+4+9+16] = 0.953.$

(iii) $P(6 < X \le 9) = F_X(9) - F_X(6)$

$= \frac{1}{650} \sum_{n=1}^{9} n^2 - \frac{1}{650} \sum_{n=1}^{6} n^2$

$= \frac{1}{650} [1+4+9+16+25+36+49+64+81-1-4-9-16-25-36]$

$= \frac{1}{650} (49+64+81) = 0.2984.$

2) $f_X(x) = f(x) = \begin{cases} A(2x-x^2), & 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

What is the value of A, find $P(X>1)$, evaluate CDF function.

Sol:- To find any constant in given equation we

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$\int_{-\infty}^{\infty} A(2x-x^2)\,dx = 1$

$\int_{0}^{2} A(2x-x^2)\,dx = 1$

$A\left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_0^2 = 1$

$A\left[4 - \frac{8}{3}\right] = 1$

$A\left(\frac{4}{3}\right)=1$

$$\boxed{A = 3/4}$$

(i) $P(X>1) = 1-P(X\le 1)$

$= 1 - F_X(1)$

$$F_X(1) = \int_{-\infty}^{1} f_X(x)dx = \int_0^1 \frac{3}{4}(2x-x^2)dx$$

$$= \frac{3}{4}\left[2\left(\frac{x^2}{2}\right)-\frac{x^3}{3}\right]_0^1 \Rightarrow \frac{3}{4}\left[1-\frac{1}{3}\right]$$

$$= \frac{1}{2}$$

$P(X>1) = 1-F_X(1) = 1-\frac{1}{2} = \frac{1}{2}$

(iii) CDF = probability distribution function

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{3}{4}(2x-x^2)dx = \int_0^x \frac{3}{4}(2x-x^2)dx.$$

$$= \frac{3}{4}\left[2\left(\frac{x^2}{2}\right)-\frac{x^3}{3}\right]_0^x$$

$$F_X(x) = \frac{3}{4}\left[x^2-\frac{x^3}{3}\right] \quad \forall\ 0<x<2$$

3.) Two boxes are selected randomly. The first box contain 2 white balls and 3 black balls second box contain 3 white and 4 black balls, what is the probability of drawing a white ball, what is the probability that the white ball is from box 2.

	B_1	B_2
white (wi)	2	3
black (Bl)	3	4

Probability of getting box B_1 from the two boxes $P(B_1)=1/2$

Probability of getting box B_2 from the two boxes $P(B_2)=1/2$

$$P\left(\frac{wi}{B_1}\right) = \frac{2}{5}$$

$$P\left(\frac{wi}{B_2}\right) = \frac{3}{7}$$

* The probability of drawing a white ball from any one of the box

$$P(wi) = P\left(\frac{wi}{B_1}\right)P(B_1) + P\left(\frac{wi}{B_2}\right)P(B_2)$$

$= \frac{2}{5}\cdot\frac{1}{2} + \frac{3}{7}\cdot\frac{1}{2}$

$= \frac{1}{5} + \frac{3}{14} = 0.2 + 0.2142$

$\boxed{P(wi) = 0.4142}$

* The probability that the white ball is from box B_2 is

$$P\left(\frac{B_2}{wi}\right) = \frac{P\left(\frac{wi}{B_2}\right)P(B_2)}{P(wi)} = \frac{\frac{1}{2}\cdot\frac{3}{7}}{0.4142} = \frac{\frac{3}{14}}{0.4142}$$

$\boxed{P\left(\frac{B_2}{wi}\right) = 0.5173}$

4.) In a Binary communication system the probability of bit error is 0.01. If a block of 8 bits are transmitted, find the probability that

a) Exactly 2 bit error will occur $= 2$

b) Atleast 2 bit error will occur ≥ 2

c) More than 3 bit errors will occur > 3

d) All the bits will be errorneous $= 8$.

Sol:- From given, $n = 8$, $p = 0.01$, K is depends on given questions

By using Binomial function

(a) Exactly 2 bit error will occur

$$P(X=2) = {}^nc_K\, p^K (1-p)^{n-K}$$

$$= {}^8c_2 (0.01)^2 (1-0.01)^6$$

$$= \frac{8\times 7}{2\times 1} (0.01)^2 (0.99)^6$$

$\boxed{P(X=2) = 0.002636}$

(b) At least 2 bit error will occur

$$P(X\geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[{}^8c_0 (0.01)^0 (0.99)^8 + {}^8c_1 (0.01)^1 (0.99)^7\right]$$

$$= 1 - [0.9227 + 0.0745]$$

$\boxed{P(X\geq 2) = 0.0028}$

(c) More than 3 bit errors will occur

$$P(X>3) = 1 - P(X\leq 3) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - [0.9227 + 0.0745 + 0.002636 + P(X=3)]$$

$P(X=3) = {}^8C_3\,(0.01)^3(0.99)^5 = 0.000005$

$P(X>3) = 1-[0.9227+0.0745+0.002636+0.000005]$

$= 1-0.9998$

$\boxed{P(X>3) = 0.0002}$

(d) All bits will be erroneous

$P(X=8) = {}^8C_8\,(0.01)^8(0.99)^0$

$= 1\cdot(0.01)^8\cdot 1 = 1\times10^{-16}$

By using poission's function

$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$

1 bit - 0.01 error

8 bit $\rightarrow \lambda$

$\boxed{\lambda = 0.08}$

(a) Exactly 2 bit error will occur

$P(X=2) = \frac{e^{-0.08}(0.08)^2}{2!} = \frac{0.9231\,(0.08)^2}{2}$

$\boxed{P(X=2) = 0.00295}$

(b) At least 2 bit error will occur

$P(X\geq 2) = 1-P(X<2) = 1-[P(X=0)+P(X=1)]$

$= 1-\left[\frac{e^{-0.08}(0.08)^0}{0!}+\frac{e^{-0.08}(0.08)^1}{1!}\right]$

$= 1-[0.9231+0.0738] = 0.0031$

$\boxed{P(X\geq 2) = 0.0031}$

(c) More than 3 bit error will occur

$P(X>3) = 1-P(X\leq 3) = 1-[P(X=0)+P(X=1)+P(X=2)+P(X=3)]$

$= 1-e^{-0.08}\left[\frac{(0.08)^0}{0!}+\frac{(0.08)^1}{1!}+\frac{(0.08)^2}{2!}+\frac{(0.08)^3}{3!}\right]$

$= 1-(0.9231+0.0738+0.00295+0.000007)$

$= 1-0.9998 = 0.0002$

$\boxed{P(X>3) = 0.0002}$

(d) All bits will be errorneous

$P(X=8) = \frac{e^{-0.08}(0.08)^8}{8!} = \frac{0.9231\,(0.08)^8}{8!}$

$= 1.5487\times10^{-9}$

$= 3.841\times10^{-14}$

5) For real constants $b>0, c>0$ and any a find the condition on constant a and relation between a and c such that the function is a valid probability density

$$f_x(x) = \begin{cases} a(1-(x/b)) & \text{for all } 0 \le x \le c \\ 0 & \text{otherwise} \end{cases}$$

Sol:- $f_x(x) \rightarrow$ unknown value = area under the value = 1

$$\int_{-\infty}^{\infty} f_x(x)dx = 1$$

$$\int_0^c a\left(1-\frac{x}{b}\right)dx = 1 \Rightarrow a\left[x - \frac{x^2}{2b}\right]_0^c = 1$$

$$\Rightarrow a\left[c - \frac{c^2}{2b}\right] = 1$$

$$ac\left(1-\frac{c}{2b}\right) = 1 \Rightarrow 1-\frac{c}{2b} = \frac{1}{ac}$$

$\boxed{\frac{c}{2b} = \frac{1}{ac}+1} \rightarrow$ This is the relation between a & c.

6) A random variable X has pdf shown below
(i) find the value of K (ii) find $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$

$$f_x(x) = \begin{cases} Kx & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(i) $$\int_{-\infty}^{\infty} f_x(x) = 1$$

$$\int_0^1 Kx\,dx = 1$$

$$K\left[\frac{x^2}{2}\right]_0^1 = 1 \Rightarrow K\left[\frac{1}{2}\right] = 1$$

$$\boxed{K = 2}$$

(ii) W.K.T $P(a < X < b) = \int_a^b f_x(x)dx$

$$P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_{1/4}^{1/2}$$

$$= \frac{1}{4} - \frac{1}{16}$$

$$\boxed{P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \frac{3}{16}}$$