UNIT - IV

RANDOM PROCESSES

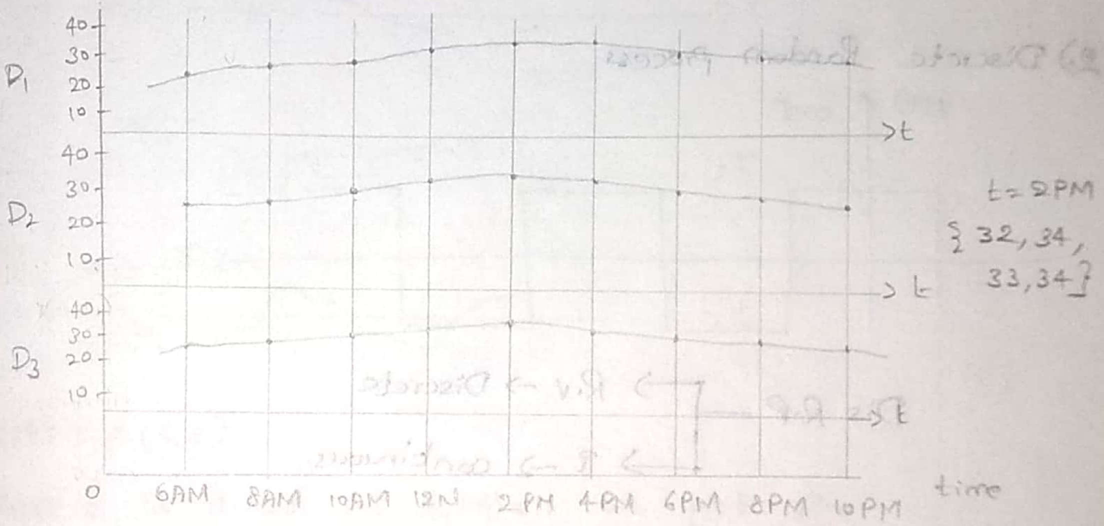
Random process :- Assigning real number and vary w.r.t time

(or)

If Random variable change with time is called as Random process

Representation of Random process is $X(x,t) = X(t) = X(t,x)$

D \ t	6 AM	8 AM	10 AM	12 N	2 PM	4 PM	6 PM	8 PM	10 PM
D_1	26°	28°	29°	31°	32°	32°	30°	29°	28°
D_2	27°	28°	30°	33°	34°	32°	30°	28°	27°
D_3	27°	29°	30°	31°	33°	31°	29°	27°	26°
D_4	28°	29°	31°	32°	34°	32°	30°	29°	28°



Random process :- Variability of Random variable w.r.t time.

Random process

Random variable → make the time constant

$t = t_1$

$X_1 = X(x, t_1)$

sample (signal) → make the sample as constant.

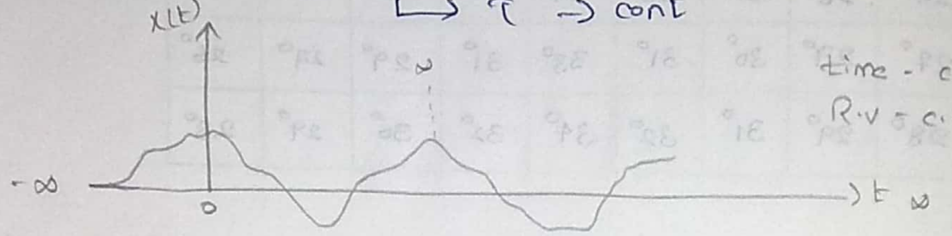
Classification of Random process:-

R.P → R.v → time → continuous / discrete

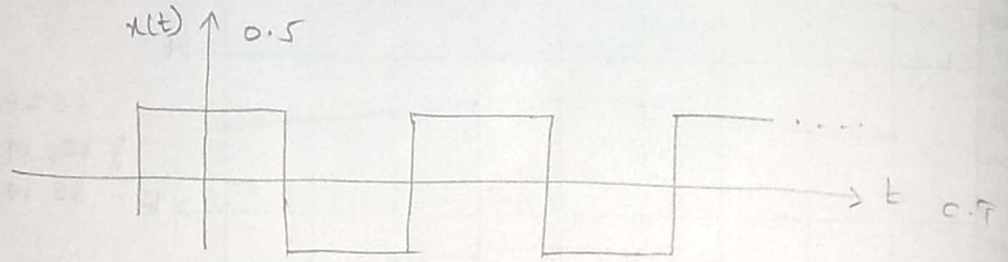
R.v → continuous / Discrete.

1) continuous Random process
2) Discrete Random process
3) Continuous Random sequence
4) Discrete Random sequence.

1) Continuous R.P → R.v → cont ; τ → cont

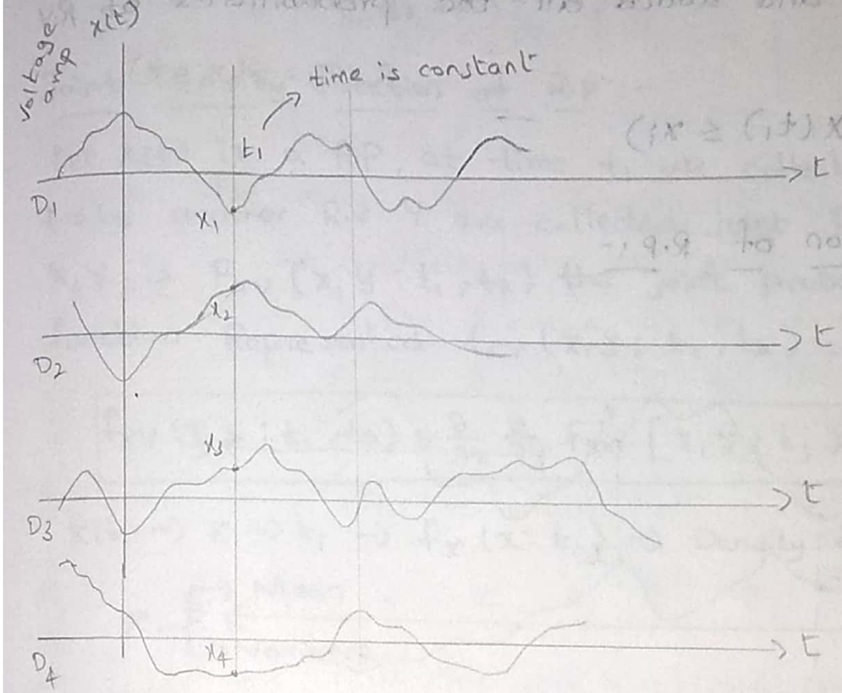


2) Discrete Random process



Dis R.P → R.v → Discrete ; τ → continuous.

Events of Random processes :-

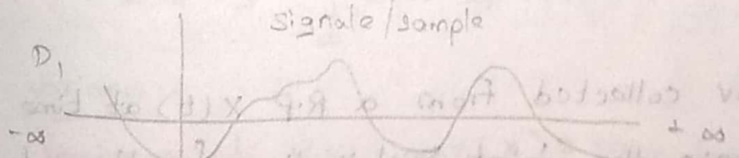


$X(t) = X(s, t)$

time t is fixed at t_1 then we get R.V

Events of Random process

$X(x, t_1) = \{x_1, x_2, x_3, x_4\}$



$X = \{$ R.V. $\}$

- signal operation
- mean and Variance
- multiple Correlation & Covariance
- Density function
- Distribution function.

Probability distribution of Random process:-

→ Collect R.v's from R.P

Let $X(t)$ is the R.P. at time t_1 we collected a R.v $X(x_i)$ then probability Distribution of the R.v upto x_i

$$F_X(x_i, t_1)$$

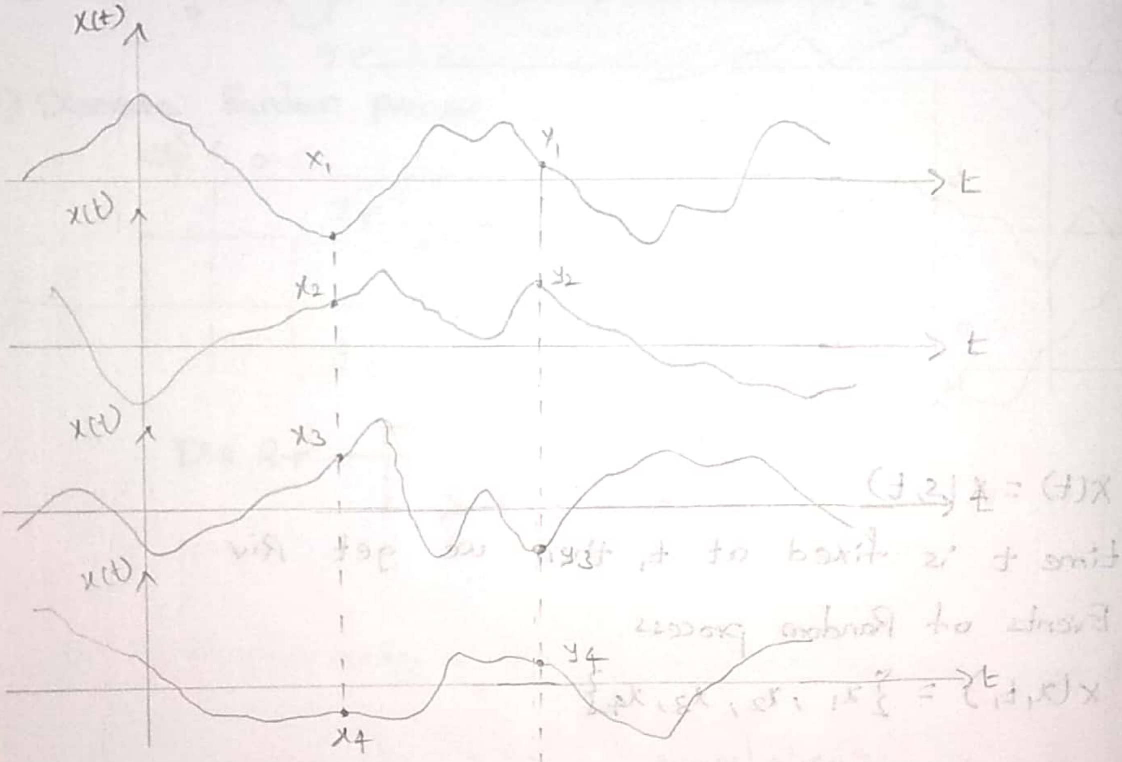
F_X → probability Distribution
x_i → R.v
t_1 → time factor.

* We have R.P $X(t)$, from this we collected R.v's at time factor t_1 and added all the probabilities of R.v upto x_i

$F_X(x) = P(X \leq x)$

$$F_X(x_i, t_1) = P(X(t_1) \leq x_i) \rightarrow$$

Joint Distribution of R.P :-



Let X, Y are two R.V collected from a R.P $X(t)$ at time t_1, t_2 then we obtain the joint Distribution function of the R.V

$$F_{XY}(x_i, y_j : t_1, t_2)$$

$F_{XY}(x,y) = P(X \leq x \cap Y \leq y)$

F_{XY} → joint distribution
x_i, y_j → R.v's
t_1, t_2 → time factors

$$F_{XY}(x_i, y_j : t_1, t_2) = P[X(t_1) \leq x_i \cap Y(t_2) \leq y_j]$$

N-Dimension Random Variables

$$F_{X_1, X_2, \cdots X_N}(x_a, x_b \cdots x_i : t_1, t_2 \cdots t_N)$$
$$= P[X_1(t_1) \leq x_a,\ X_2(t_2) \leq x_b, \cdots X_N(t_N) \leq x_j]$$

Probability Density function of R.P X(t) :-

At time t, we obtain a R.V X the probability Distribution of R.V at t_1 is

$$F_X(x : t_1) = P[X(t_1) \leq x]$$
$$f_X(x, t_1) = \frac{d}{dx} F_X(x : t_1)$$

Joint Density Function of R.P :-

Let x(t) is a R.P, at time t_1 we collected R.V X, at $t = t_2$ another R.V Y are collected joint Distribution of X, Y is $F_{XY}(x, y : t_1, t_2)$ the joint probability density function Represented $f_{XY}(x, y : t_1, t_2) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}(x, y : t_1, t_2)$

$$\boxed{f_{XY}(x, y : t_1, t_2) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{XY}[x, y : t_1, t_2]}$$

$x(t) \rightarrow x \rightarrow t_1 \rightarrow f_X(x : t_1) \rightarrow$ Density function

X → Mean
X → variance.

Mean of a R.V

$$E[x] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

Mean of a R.P

$$E[x(t)] = \int_{-\infty}^{\infty} x(t)\, f_X(x, t)\, dx.$$

Variance :-

Variance of R.V = $var(X) = m_2 - (m_1)^2 = E[X^2] - [E(X)]^2$

Variance of R.P :-

Variance of R.P = $var[x(t)] = E[x^2(t)] - E[E[x(t)]]^2$

Problems

1) If a fair coin experiment is a R.P defined as $x(t) = \cos\pi t$ if the coin show head, $x(t) = t$ if the coin show tail. Find Mean of the R.P.

Sol:- $E[x(t)] = \int_{-\infty}^{\infty} x(t)\, f_x(x:t)\, dx$

Disc R.P $E[x(t)] = \sum x(t)\, P_x(x:t)$

samples	T	H
$x(t)$	t	$\cos\pi t$
$P_x(x,t)$	$1/2$	$1/2$

$E[x(t)] = \sum x(t) \cdot P_x(x:t)$

$= t \cdot \frac{1}{2} + \cos\pi t \cdot \frac{1}{2}$

$E[x(t)] = \frac{\cos\pi t + t}{2}$

2) A R.P $x(t)$ is a deterministic signal work with adding two dice faces gives the given pattern as

$$x(t) = \begin{cases} \sin t & \forall \text{ if sum} = 3 \text{ and } 7 \\ \cos 2t & \forall \text{ if sum} = 2 \text{ and } 12 \\ \sin 2t & \forall \text{ others.} \end{cases}$$

find μ_x, $\mu_x(\pi/4)$, $\mu_x(\pi/2)$, $Var(x(t))$

Sol:-

samples	2	3	4	5	6	7	8	9	10	11	12
$x(t)$	$\cos 2t$	$\sin t$	$\sin 2t$	$\sin 2t$	$\sin 2t$	$\sin t$	$\sin 2t$	$\sin 2t$	$\sin 2t$	$\sin 2t$	$\cos 2t$
$P_x(x,t)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

μ_x:-

Mean $= E[x(t)] = \cos 2t \cdot 1/36 + \sin t \cdot 2/36 + \sin 2t \cdot 3/36 + \sin 2t \cdot 4/36$
$+ \sin 2t \cdot 5/36 + \sin 2t \cdot 6/36 + \sin 2t \cdot 5/36 + \sin 2t \cdot 4/36$
$+ \sin 2t \cdot \frac{3}{36} + \sin 2t \cdot 2/36 + \cos 2t \cdot \frac{1}{36}$

$\mu_x = \sin t\left(\frac{8}{36}\right) + \sin 2t\left(\frac{26}{36}\right) + \cos 2t\left(\frac{2}{36}\right)$

$\mu_x(\pi/4)$:-

$\mu_x = \frac{8}{36}(\sin \pi/4) + \frac{26}{36}\left[\sin \frac{2\pi}{4}\right] + \frac{2}{36}\left[\cos \frac{2\pi}{4}\right]$

$= \frac{8}{36}\left(\frac{1}{\sqrt{2}}\right) + \frac{26}{36}[1] + \frac{2}{36}[0] = \frac{8}{36\sqrt{2}} + \frac{26}{36}$

$\mu_x = \frac{8}{36\sqrt{2}} + \frac{26}{36}$

$\mu_x[\pi/2]$:-

$$\mu_x(\pi/2) = \frac{8}{36}\sin\pi/2 + \frac{26}{36}\sin\frac{2\pi}{2} + \frac{2}{36}\cos\frac{2\pi}{2}$$

$$= \frac{8}{36} + \frac{26}{36}(0) + \frac{2}{36}(-1)$$

$$\mu_x(\pi/2) = \frac{6}{36} = \frac{1}{6}.$$

var(x(t)) :-

$$var[x(t)] = m_2 - (m_1)^2$$

$$= E[x^2(t)] - [E(x)]^2$$

$$E[x^2(t)] = \frac{8}{36}\sin^2 t + \frac{26}{36}\sin^2 2t + \frac{2}{36}\cos^2 2t$$

$$var[x(t)] = \frac{8}{36}\sin^2 t + \frac{26}{36}\sin^2 t + \frac{2}{36}\cos^2 2t - \Big[\frac{8}{36}\sin t + \frac{26}{36}\sin 2t + \frac{2}{36}\cos 2t\Big]^2$$

Stationary R.P:-

Definition:- A R.P is said to be stationary if it won't change w.r.t time. $x(t) = x(t+\tau)$

Types of Stationary R.P:-

1) 1^{st} order Stationary
2) 2^{nd} order stationary / wide sense stationary (WSS)
3) Strictly stationary.

Statistical Parameters of R.P

1.) Mean of R.P / Ensemble ~~Avg~~ Average

2.) Correlation

3.) Covariance

1.) Mean of R.P / Ensemble Average

Let x(t) is a R.P, at time t_1 a R.V X_1 is collected, its density function is $f_X(x_1 : t_1)$ then the mean of the R.P is

$$E[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot f_X(x_1 : t_1) \cdot dx_1$$

$$E[A] = \int_{-\infty}^{\infty} A f_X(x)\,dx.$$

2) ~~Correlation:- X, Y are two R.V. Correlation is R_{XY} =~~

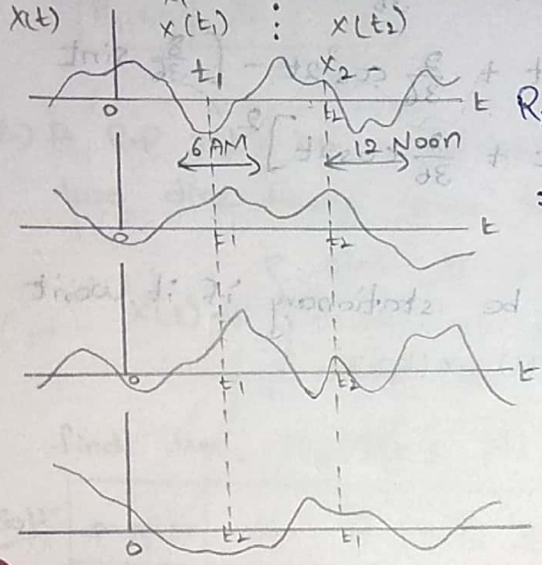
x_1, x_2 two R.V generated from R.P $X(t)$, at t_1, t_2

$$R_{XX}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\,x(t_2)\, f_{XX}(x_1,x_2 : t_1,t_2)\, dx_1\, dx_2$$

① Auto correlation

② Cross correlation.

Auto correlation:- Let 2- R.V $x(t_1)$, $x(t_2)$ are collected from same R.P and find the relation between them is stated as Auto correlation.



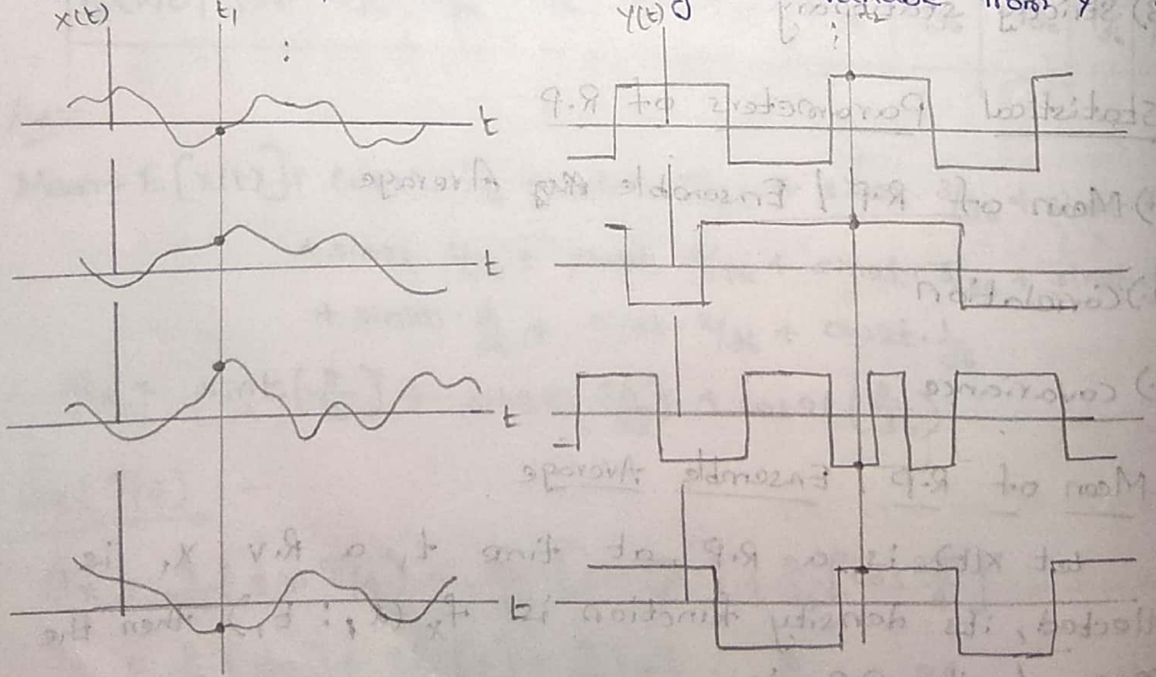
$$R_{XX}(t_1,t_2) = E\left[X(t_1)\,X(t_2)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\cdot x(t_2)\, f_{XX}(x_1,x_2 : t_1,t_2)\, dx_1\, dx_2$$

Let $t_1 = t$, $t_2 = t+\tau$

$$R_{XX}(t,t+\tau) = E\left[x(t)\,x(t+\tau)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\cdot x(t+\tau)\, f_X(x_1,x_2 : t,t+\tau)\, dx_1\, dx_2$$

$\tau \rightarrow$ absolute time.

Cross correlation:- Two different R.P, at t_1 collecting the variable from x, at t_2 collecting the variable from Y.



let $x(t)$, $Y(t)$ are two R.P, from $x(t)$ we collect a R.v at t_1, from $Y(t)$ another R.v at t_2

$$R_{xy}(t_1,t_2) = E[x(t_1)\cdot Y(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t_1)\cdot Y(t_2)\, f_{xy}(x_1,y_1:t_1,t_2)\, dx\,dy$$

$X(t) \to R.P \to$ Temperature

$Y(t) \to R.P \to$ Humidity

$$R_{xy}(t,t+\tau) = E[x(t)\cdot y(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\, y(t+\tau)\, f_{xy}(x,y : t, t+\tau)\, dx\,dy.$$

3) Covariance:- $C_{xy} = E[(x-\bar{x})(y-\bar{y})] = R_{xy} - \bar{x}\cdot\bar{y}$

① Autocovariance

② Cross covariance.

① Autocovariance:- Two R.V are collected from same R.P and finded the covariance

$x(t)$ is the R.P, $x(t_1)$, $x(t_2)$ are two R.V collected from same R.P. $x(t)$.

$$\text{Autocovariance} = C_{xx}(t_1,t_2) = E\left[\left(x(t_1) - \overline{x(t_1)}\right)\left(x(t_2) - \overline{x(t_2)}\right)\right]$$

$\overline{x(t_1)}$ = mean of the R.V collected at t_1 from R.P $x(t)$.

$\overline{x(t_2)}$ = mean of the R.V collected at t_2 from R.P. $x(t)$.

$$C_{xx}(t_1,t_2) = R_{xx}(t_1,t_2) - \overline{x(t_1)}\,\overline{x(t_2)}$$

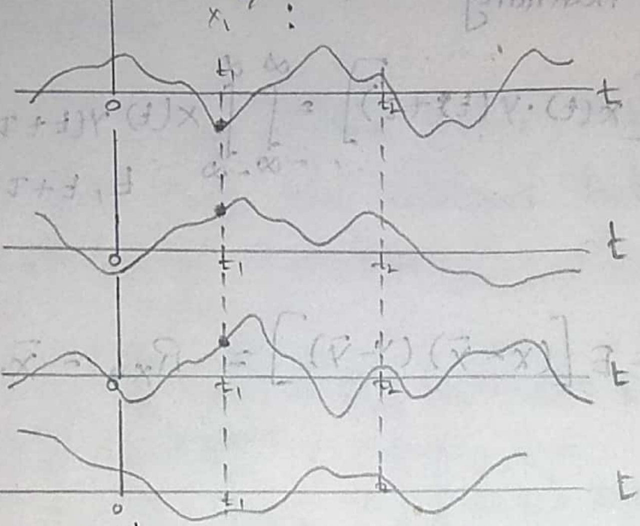
② Crosscovariance :- let $x(t)$, $y(t)$ are two different R.P., at t_1 a R.v collected from $x(t)$, at t_2 another R.v from $Y(t)$ and calculated the covariance.

$$C_{xy}(t_1,t_2) = E[x(t_1) - \overline{x(t_1)}]$$

$$= R_{xy}(t_1,t_2) - \overline{x(t_1)}\cdot\overline{Y(t_2)}$$

Stationary R.P:-

① 1^{st} order Stationary :- let $x(t)$ is the R.P and collected one R.v at time t_1, if the density function of the R.V is independent of time then that R.P is said to be 1^{st} order stationary. density $\rightarrow f_x(x_1 : t_1) = f_x(x_1 : t_1 + \Delta t)$



Conditions for 1^{st} order stationary:-

$$f_x(x_1 : t_1) = f_x(x_1 : t_1 + \Delta t)$$

Single R.v $\rightarrow$ "Mean of the R.P should be independent of time"

Proof:- $f_x(x_1 : t_1) = f_x(x_1 : t_1 + \Delta t)$

Mean of the R.P $E[x(t)] = \int_{-\infty}^{\infty} x(t) \cdot f_x(x_1 : t_1 \cancel{+\Delta t})\, dx_1$

$x(t)$ is stationary $x(t+\Delta t) = x(t)$

$$E[x(t+\Delta t)] = \int_{-\infty}^{\infty} x(t+\Delta t) \cdot f_x(x_1 : t_1)\, dx$$

$$E[x(t+\Delta t)] = \int_{-\infty}^{\infty} x(t) \cdot f_x(x_1 : t_1)\, dx$$

$$E[x(t+\Delta t)] = E[x(t)]$$

2^{nd} order stationary / w.s.s :-

2^{nd} order density function of R.P is independent of time.

Let $x(t)$ is the R.P, Two R.V are collected x_1 at t_1, x_2 at t_2

$$f_X(x_1, x_2 : t_1, t_2) = f_X(x_1, x_2 ; t_1 + \Delta t, t_2 + \Delta t)$$

$x_1 \rightarrow E[x(t_1)]$ mean

$x_2 \rightarrow E[x(t_2)]$ mean

w.s.s con't not

$x(t) = A\cos(\omega t + \theta)$

$f_x(x) \rightarrow$ unit

Mutual one "Auto correlation" $\rightarrow$ Independent of time

$\rightarrow x(t) \rightarrow$ "w.s.s" / 2^{nd} order stationary

Joint wide sence Stationary :-

2^{nd} order Joint density function of two R.P $x(t)$, $y(t)$ is independent of time.

$$f_{XY}(x, y : t_1, t_2) = f_{XY}(x, y : t_1 + \Delta t)$$

$x \rightarrow E[x] = E[x(t_1)]$

$y \rightarrow E[y] = E[y(t_2)]$

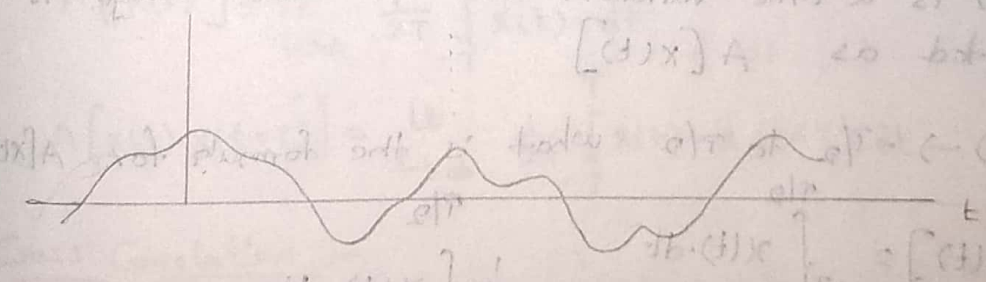
cross correlation

$E[xy] = E[x(t_1) \cdot y(t_2)] \rightarrow$ Independent of time

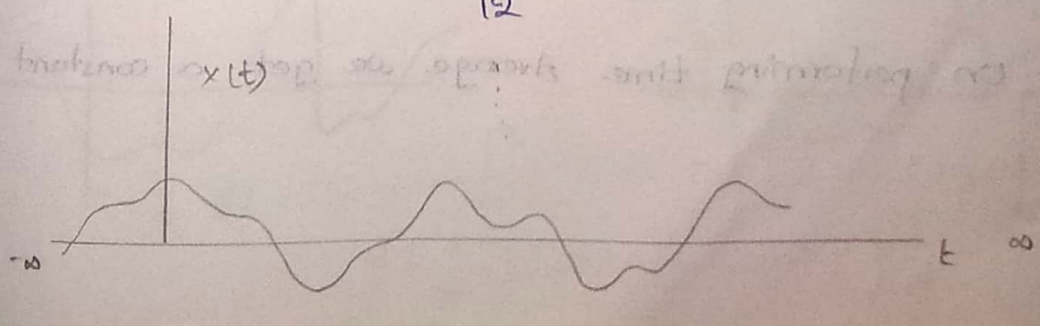
Strictly Stationary R.P :-

n^{th} order density function of R.P $x(t)$ is independent of time.

$$f_X(x_1, x_2, x_3 \ldots x_n : t_1, t_2, t_3 \ldots t_n) =$$
$$f_X(x_1, x_2, x_3 \ldots x_n ; t_1 + \Delta t, t_2 + \Delta t, \ldots t_n + \Delta t)$$



$$\text{Mean} = \frac{T_1 + T_2 + T_3 + \cdots + T_{12}}{12}$$



$$\text{Average } E[x(t)] = \frac{\int_{-\infty}^{\infty} x(t)\,dt}{T}$$

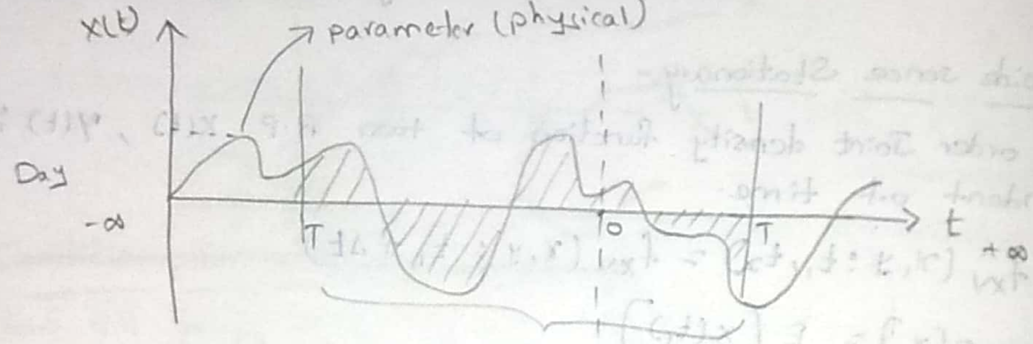
$$\text{Average} = \frac{\int_{T_1}^{T_2} x(t)\,dt}{T_2 - T_1}$$

$$\text{Time average } A[x(t)] = \frac{\int_{-T}^{+T} x(t)\,dt}{T-(-T)} = \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

if the signal distributed from $-\infty$ to ∞

$$A[x(t)] = \lim_{T\to\infty} \frac{\int_{-T}^{T} x(t)\cdot dt}{2T}$$

Time Average:-



$$A[x(t)] = \frac{\int_{-T}^{T} x(t)\cdot dt}{T-(-T)}$$

$$A[x(t)] = \frac{1}{2T}\int_{-T}^{T} x(t)\cdot dt$$

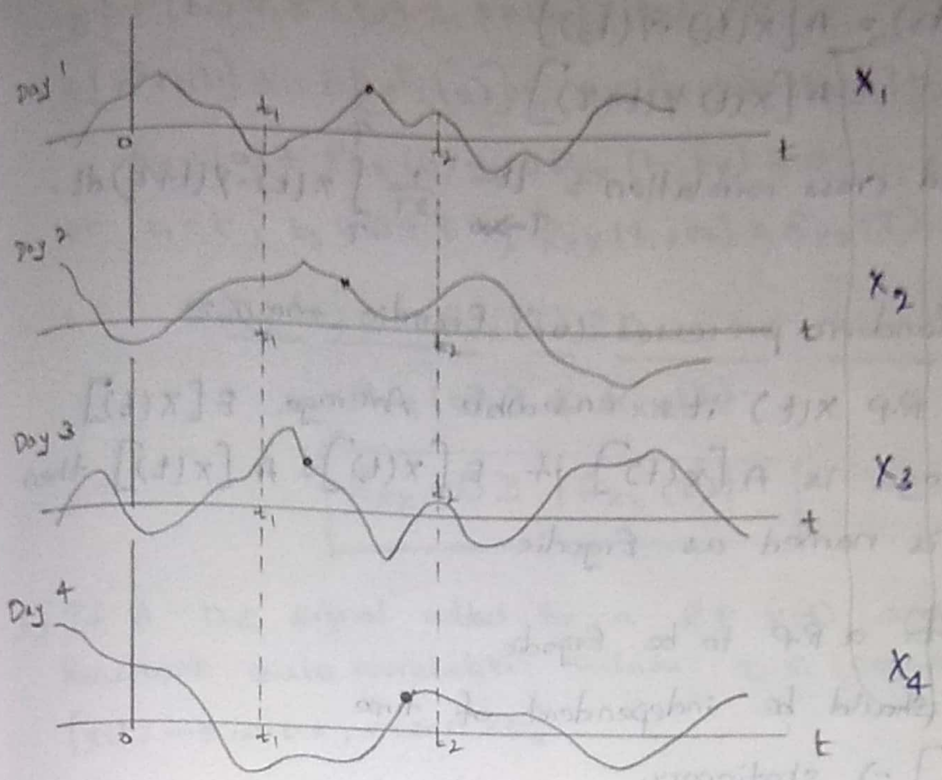
$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)\,dt$$

let $x(t)$ is a time variable and its time Average is represented as $A[x(t)]$

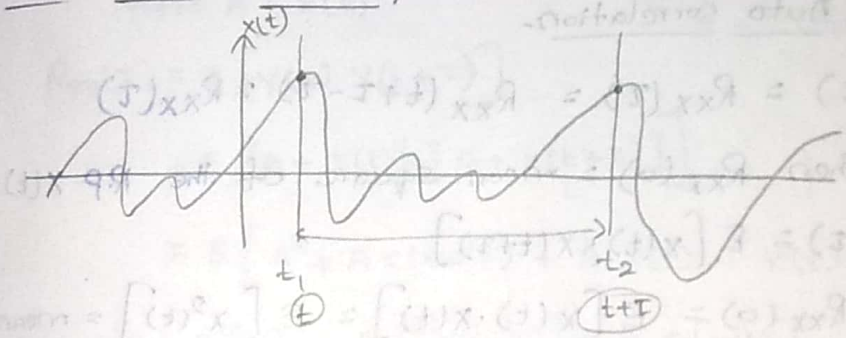
If $x(t) \to -T/2$ to $T/2$ what is the formula for $A[x(t)]$

$$A[x(t)] = \frac{\int_{-T/2}^{T/2} x(t)\cdot dt}{T/2 - (-T/2)} = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$$

Note:- On performing time Average we get a constant



Time based correlation :-



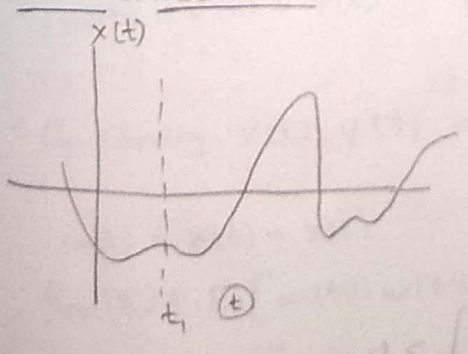
Auto correlation

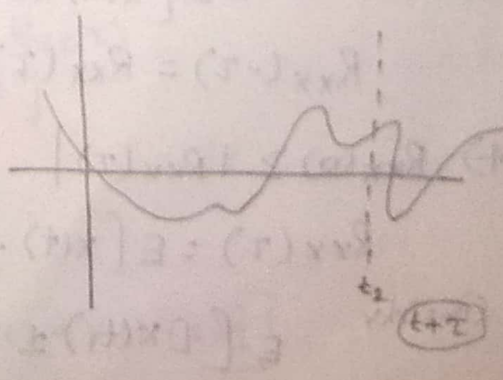
$$R_{XX}(t_1, t_2) = A[x(t_1) \cdot x(t_2)]$$
$$= A[x(t)\, x(t+\tau)]$$

$$A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot dt$$

$$A[x(t) \cdot x(t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot x(t+\tau)\, dt.$$

Cross Correlation :-





$R_{xy}(t_1,t_2) = A[x(t_1) \cdot y(t_2)]$

$= A[x(t)\, y(t+\tau)]$

Time Based cross correlation $= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) \cdot y(t+\tau)\, dt$.

Ergodic Random processes (or) Ergodic mean :-

let a R.P $x(t)$ its ensemble Average $E[x(t)]$ time average is $A[x(t)]$ if $E[x(t)] = A[x(t)]$ then the R.P is named as Ergodic.

Conditions for a R.P to be Ergodic

-> $A[x(t)]$ should be independent of time

-> $E[x(t)]$ -> stationary.

Properties of Auto correlation.

1) $R_{xx}(t, t+\tau) = R_{xx}(\tau) = R_{xx}(t+\tau-\tau) = R_{xx}(\tau)$

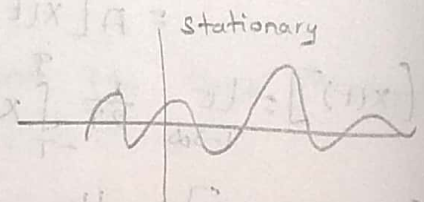
2) if $\tau = 0$ then $R_{xx}(0)$ = mean square of the R.P $x(t)$

Proof:- $R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$

$\tau = 0 \quad R_{xx}(0) = E[x(t) \cdot x(t)] = E[x^2(t)]$ = mean square

3) $R_{xx}(\tau)$ is an even function

$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$



let $\tau = -\tau$

$R_{xx}(-\tau) = E[x(t) \cdot x(t-\tau)]$

$t - \tau = u, \; t = u + \tau$

$= E[x(u+\tau) \cdot x(u)]$

$= E[x(u)\, x(u+\tau)]$

$R_{xx}(-\tau) = R_{xx}(\tau)$

4) $R_{xx}(0) \geq |R_{xx}(\tau)|$

$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$

Consider

$E[[x(t_1) \pm x(t_2)]^2] \geq 0$

$$E\left[x^2(t_1) + x^2(t_2) \pm 2x(t_1)x(t_2)\right] \geq 0$$

$$E\left[x^2(t_1)\right] + E\left[x^2(t_2)\right] \pm 2E\left[x(t_1)x(t_2)\right] \geq 0$$

$$R_{xx}(0) + R_{xx}(0) \pm 2R_{xx}(t_1, t_2) \geq 0$$

let $t_1 = t$, $t_2 = t+\tau$, $R_{xx}(t_1, t_2) = R_{xx}(\tau)$

$$2R_{xx}(0) \pm 2R_{xx}(\tau) \geq 0$$

$$R_{xx}(0) \geq \mp R_{xx}(\tau)$$

$$\boxed{R_{xx}(0) \geq |R_{xx}(\tau)|}$$

5.) If A D.C signal added to a R.P $x(t)$ and the Resultant auto correlation include D.C component $[x(t) \to$ w.s.s, mean $=0]$.

D.C signal $\to 0$ frequency

$x(t) \to$ R.P

$y(t) = A + x(t)$

$A \to$ D.C Signal

$$R_{yy}(\tau) = E\left[y(t) \cdot y(t+\tau)\right]$$

$$= E\left[\{A + x(t)\} \cdot \{A + x(t+\tau)\}\right]$$

$$= E\left[A^2 + A\,x(t+\tau) + A \cdot x(t) + x(t) \cdot x(t+\tau)\right]$$

$$= E\left[A^2\right] + A \cdot E\left[x(t+\tau)\right] + A \cdot E\left[x(t)\right] + E\left[x(t) \cdot x(t+\tau)\right]$$

$\because x(t)$ is w.s.s $x(t+\tau) = x(t)$

$$= E\left[A^2\right] + A \cdot E\left[x(t)\right] + A\,E\left[x(t)\right] + R_{xx}(\tau)$$

$E(x(t)) =$ Mean of R.P

$E[x(t)] = 0$ as per the statement.

$$R_{yy}(\tau) = E\left[A^2\right] + R_{xx}(\tau)$$

$$\boxed{R_{yy}(\tau) = A^2 + R_{xx}(\tau)}$$

6) let $x(t), y(t)$ are two R.V's, sum of the two RV is $w(t) = x(t) + y(t)$, $R_{ww}(\tau) = R_{xx}(\tau) + R_{yy}(\tau) + R_{xy}(\tau) + R_{yx}(\tau)$

$[\because$ Considering $x(t), y(t)$ are w.s.s $]$

$$w(t) = x(t) + y(t)$$

$$R_{ww}(\tau) = E\left[w(t) \cdot w(t+\tau)\right]$$

$$= E\left[\{x(t) + y(t)\} \cdot \{x(t+\tau) + y(t+\tau)\}\right]$$

$= E[x(t)\cdot x(t+\tau) + x(t)\cdot y(t+\tau) + y(t)\cdot x(t+\tau) + y(t)\,y(t+\tau)]$

$= E[x(t)\cdot x(t+\tau)] + E[x(t)\cdot y(t+\tau)] + E[y(t)\cdot x(t+\tau)]$

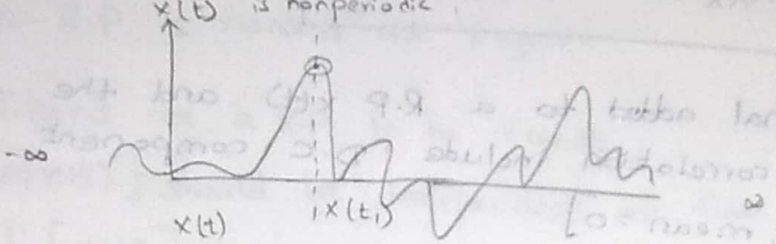
$+ E[y(t)\cdot x(t+\tau)]$

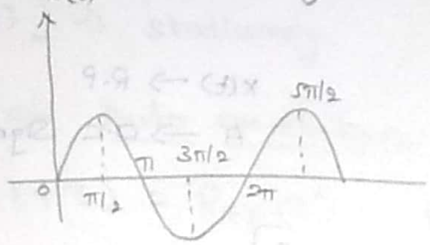
$R_{ww}(\tau) = R_{xx}(\tau) + R_{xy}(\tau) + R_{yy}(\tau) + R_{yx}(\tau)$

7.) $x(t)$ is a non-periodic R.P, with mean value is not equal to zero, and ergodic in nature then

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = \bar{x}^2$$

$x(t)$ is nonperiodic





$R_{xx}(\tau) = E[x(t)\cdot x(t+\tau)]$

we stated, $x(t)$ is non-periodic

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = \underset{\tau\to\infty}{Lt}\ E[x(t)\cdot x(t+\tau)]$$

$x(t)$ and $x(t+\tau)$ are aperiodic so that They are S.I

$= E[x(t)]\cdot E[x(t+\tau)]$

$= E[x(t)]\cdot E[x(t)]$ $\quad$ $x(t+\tau) \to x(t)$ is $x(t)$ is w.s.s

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = \bar{x}\cdot\bar{x}$$

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = \bar{x}^2$$

8.) $x(t)$ is non-periodic, w.s.s and ergodic with mean $= 0$, then

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = 0$$

$$\underset{\tau\to\infty}{Lt}\ R_{xx}(\tau) = \bar{x}^2$$

But $\boxed{\bar{x} = 0}$

$$\lim_{\tau \to \infty} R_{XX}(\tau) = 0$$

a) A R.P $X(t)$ is periodic then $R_{XX}(\tau)$ is also periodic:
[$X(t) \rightarrow$ W.S.S]

$X(t) \rightarrow$ Periodic with period T_0

$$X(t) = X(t + T_0)$$

$$\cancel{R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)] = E[X(t) \cdot X(t+\tau+T_0)]}$$

$$\cancel{= E[X(t) \cdot X(t+T_0+\tau)]}$$

$$R_{XX}(\tau) = R_{XX}(\tau + T_0)$$

$$R_{XX}(\tau) = E[X(t) \cdot X(t+\tau)]$$

$$R_{XX}(\tau + T_0) = E[X(t) \cdot X(t+\tau+T_0)]$$

$$= E[X(t) \cdot X(t+T_0+\tau)]$$

$$= E[X(t) \cdot X(t+\tau)]$$

$$R_{XX}(\tau + T_0) = R_{XX}(\tau)$$

(b) $R_{XX}(\tau)$ is even, having Maximum value at $\tau = 0$.

Based on 2^{nd} & 3^{rd} property we can conclude $R_{XX}(\tau)$ is even & having maximum value at $\tau = 0$.

Problem:-

A R.P is given as $X(t) = At$, where A is uniformly distributed in the range of 0 to 2 find whether $X(t)$ is W.S.S (or) not.

$X(t) \rightarrow$ mean constant

$R_{XX}(\tau) \rightarrow$ Independent of time

Mean of $X(t) = E[X(t)] = \int_{-\infty}^{\infty} X(t) f_X(x,t)\, dx$.

$$f_A(A,t) = \frac{1}{b-a} \;\forall\; a \text{ to } b = \frac{1}{2-0} \;\forall\; 0 \text{ to } 2$$

$$E[X(t)] = \int_0^2 A \cdot t \cdot \tfrac{1}{2}\, dA = \frac{t}{2}\int_0^2 A\, dA = \frac{t}{2}\left[\frac{A^2}{2}\right]_0^2$$

$$= \frac{t}{\cancel{2}}\left[\frac{\cancel{4}}{\cancel{2}}\right]$$

$$= t.$$

Cross Correlation :-

$X(t) \quad , \quad Y(t)$

$t \qquad\qquad t+\tau$

$x(t) \longleftrightarrow y(t+\tau)$

$R_{xy}(\tau) = E[x(t) \cdot y(t+\tau)]$

$E[y(t) \cdot x(t+\tau)] = R_{yx}(\tau)$

$R_{xy}(\tau) \neq R_{yx}(\tau)$ are not same.

shifted $\qquad$ shifted

Cross correlation Properties :-

① $R_{xy}(-\tau) = R_{yx}(\tau)$

let $X(t)$, $Y(t)$ are two R.P, w.s.s

$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$R_{xy}(-\tau) = E[x(t) \cdot y(t-\tau)]$

let $t-\tau = u \Rightarrow t = u+\tau$

$R_{xy}(-\tau) = E[X(u+\tau) \cdot Y(u)]$

$= E[Y(u) \cdot x(u+\tau)] = R_{yx}(\tau)$

② $|R_{xy}(\tau)| \le \sqrt{R_{xx}(0) + R_{yy}(0)}$

Assume $x(t)$, $Y(t)$ are two w.s.s R.P and $E[x^2(t)] \rightarrow$ Mean Square of R.P $x(t)$, $E[y^2(t)]$ is the mean square of R.P $Y(t)$ and

$$\left| E\left[\frac{x(t)}{\sqrt{R_{xx}(0)}} \pm \frac{Y(t)}{\sqrt{R_{yy}(0)}}\right]^2 \right| \ge 0$$

$$\left| E\left[\frac{x^2(t)}{R_{xx}(0)} + \frac{y^2(t)}{R_{yy}(0)} \pm 2\frac{x(t)}{\sqrt{R_{xx}(0)}} \frac{Y(t)}{\sqrt{R_{yy}(0)}}\right] \right| \ge 0$$

③ $R_{xy}(\tau) = \bar{x}\cdot\bar{y}$

Let $x(t), y(t)$ are two R.P, and w.s.s, S.T

$R_{xy}(\tau) = E[x(t)\cdot y(t+\tau)]$

$R_{xy}(\tau) = E[x(t)\cdot y($

④ $\lim_{\tau\to\infty} R_{xy}(\tau) = \bar{x}\cdot\bar{y}$

Let $x(t), y(t)$ are w.s.s and Aperiodic in nature

$\tau \to \infty$

$R_{xy}(\tau) = E[x(t)\cdot y(t+\tau)]$

$\lim_{\tau\to\infty} R_{xy}(\tau) = E[x(t)\cdot y(t+\tau)]$ $\quad y(t+\tau) = y(t) \to$ $y(t) \to$ w.s.s

$= E[x(t)\cdot y(t)] = E[x(t)]\cdot E[y(t)]$

$= \bar{x}\cdot\bar{y}$

⑤ If mean=0 $\lim_{\tau\to\infty} R_{xy}(\tau) = 0$

$x(t), y(t) \longrightarrow$ w.s.s & Aperiodic, mean value of $x(t)$ $y(t)$ is zero

$\lim_{\tau\to\infty} R_{xy}(\tau) = \bar{x}$

Covariance of R.P

- Auto covariance
  - $x(t)$
    - $x(t)$
    - $x(t+\tau)$
- Cross covariance
  - $x(t)$
  - $y(t)\, y(t+\tau)$

Auto covariance :-

$C_{xx}(t, t+\tau) = E\left[\left[x(t) - \overline{x(t)}\right]\left[x(t+\tau) - \overline{x(t+\tau)}\right]\right]$

$= E\left[x(t)\,x(t+\tau) - x(t)\,\overline{x(t+\tau)} - \overline{x(t)}\,x(t+\tau) + \overline{x(t)}\,\overline{x(t+\tau)}\right]$

Let $x(t)$ is stationary

$= E[x(t)\,x(t+\tau)] - E[x(t)\,\underbrace{\overline{x(t+\tau)}}_{\text{Mean constant}}] - E[\overline{x(t)}\,x(t+\tau)]$

$+ E[\overline{x(t)}\,\overline{x(t+\tau)}]$

$$= E\left[x(t)\, x(t+\tau)\right] - E\left[x(t)\right] \overline{x(t+\tau)} - \bar{x}(t)\cdot E\left[x(t+\tau)\right] + \overline{x(t)}\,\overline{x(t+\tau)}$$

$$= R_{xx}(\tau) - \bar{x}(t)\cdot\bar{x}(t) - \cancel{\bar{x}(t)\cdot\bar{x}(t) + \overline{x(t)}\,\overline{x(t)}}$$

$$= R_{xx}(\tau) - \bar{x}^2(t).$$