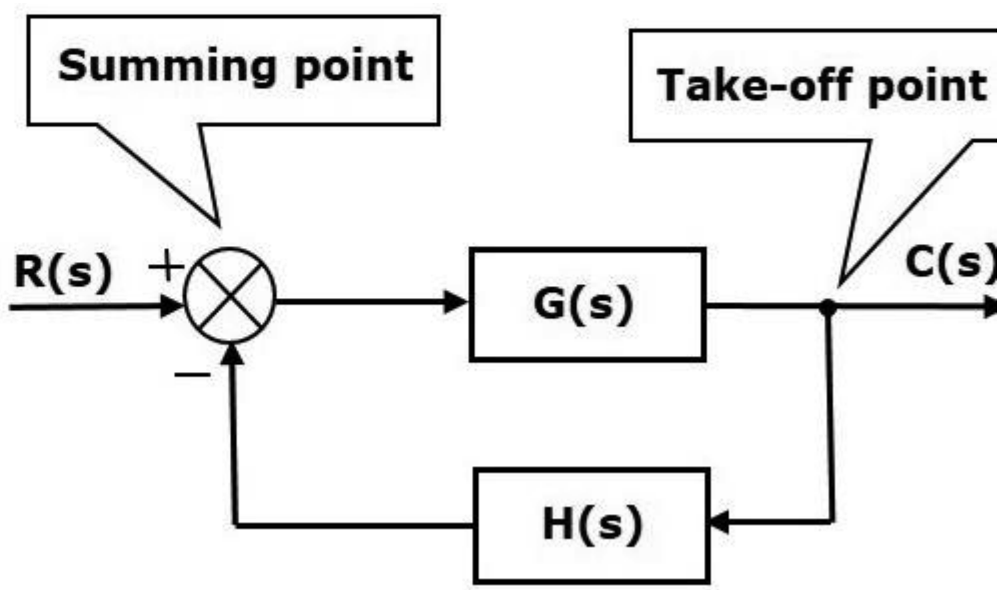
TRANSFER FUNCTION REPRESENTATION

Block Diagrams

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

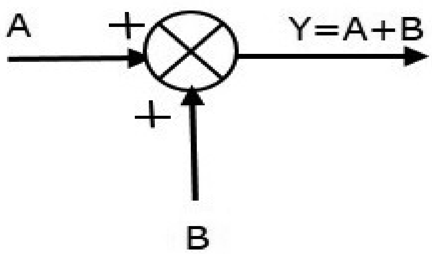
Transfer Function, $G(s) = \frac{Y(s)}{X(s)}$

$$\Rightarrow Y(s) = G(s)X(s)$$

Summing Point

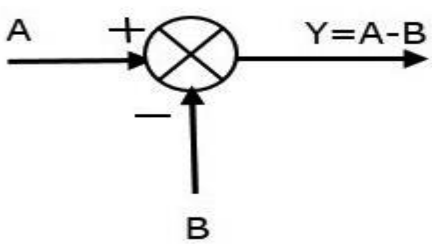
The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B** i.e. = A + B.



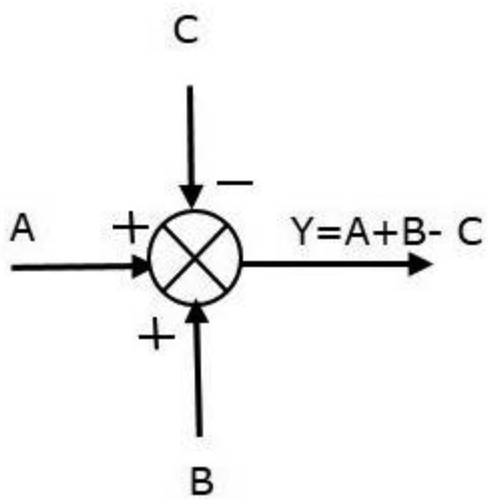
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B** i.e

Y = A + (-B) = A - B.



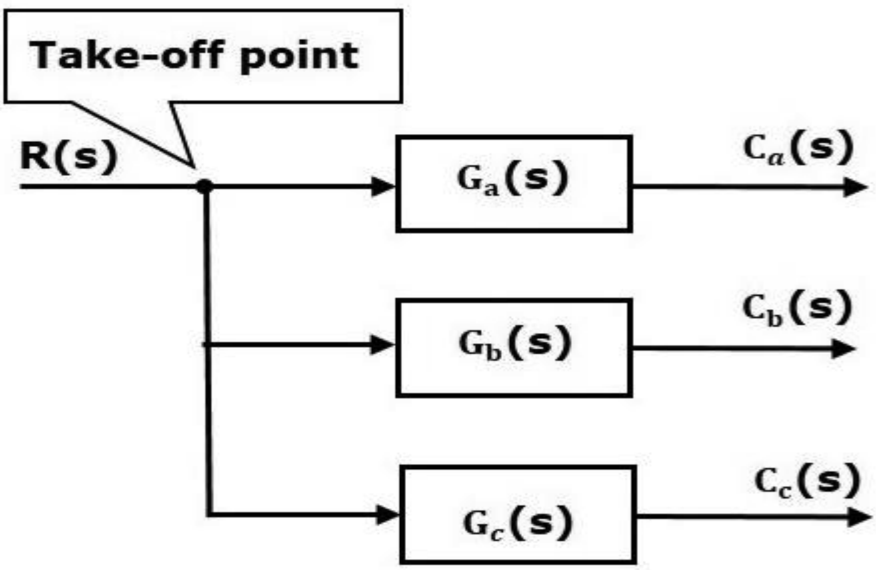
The following figure shows the summing point with three inputs (A, B, C) and one output (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

Y = A + B + (−C) = A + B − C.

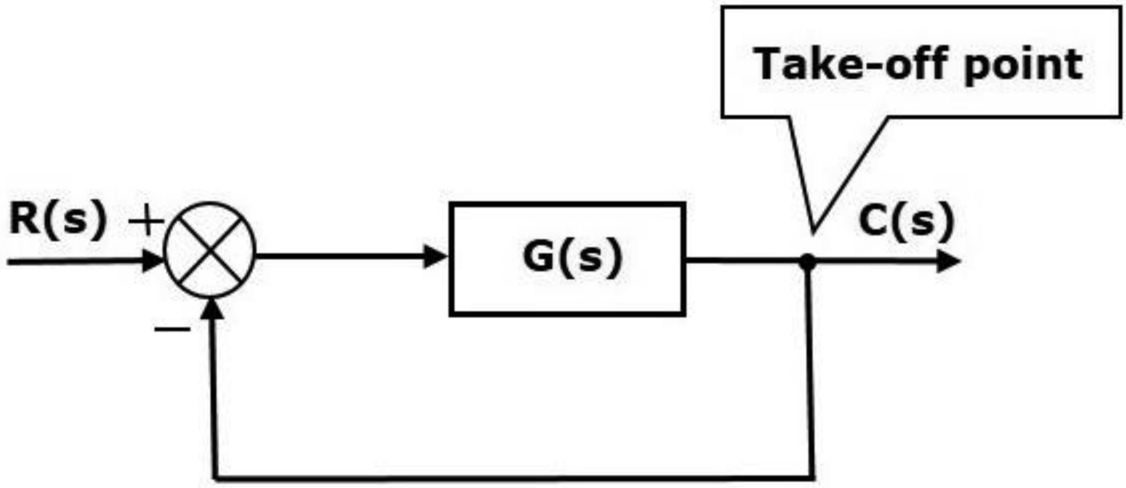


Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points.In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.



In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.



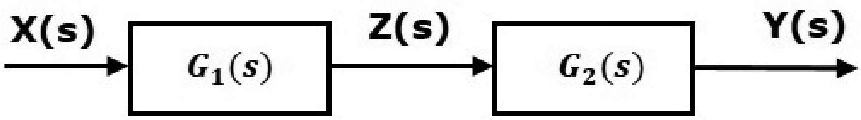
Block diagram algebra is nothing but the algebra involved with the basic elements of the block diagram. This algebra deals with the pictorial representation of algebraic equations.

Basic Connections for Blocks

There are three basic types of connections between two blocks.

Series Connection

Series connection is also called **cascade connection**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in series.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

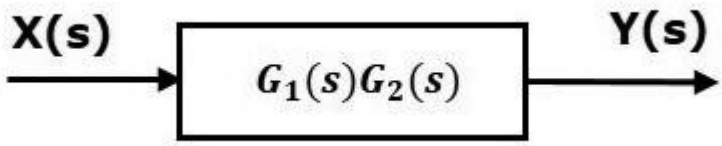
Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$. Where, $G(s) = G_1(s)G_2(s)$.

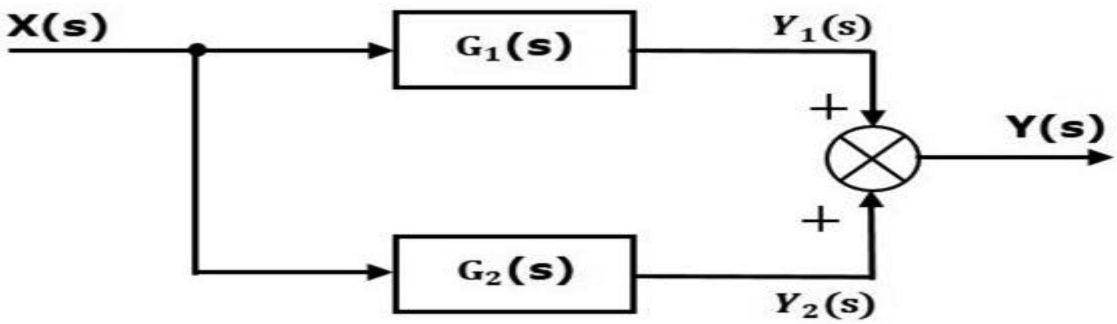
That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent series connection of 'n' blocks with a single block. The transfer function of this single block is the product of the transfer functions of all those 'n' blocks.

Parallel Connection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in parallel. The outputs of these two blocks are connected to the summing point.



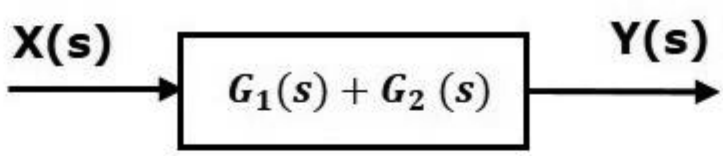
$$Y(s) = Y_1(s) + Y_2(s)$$

$$Y_1(s) = G_1(s)X(s) \text{ and } Y_2(s) = G_2(s)X(s)$$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

$$G(s) = G_1(s) + G_2(s).$$

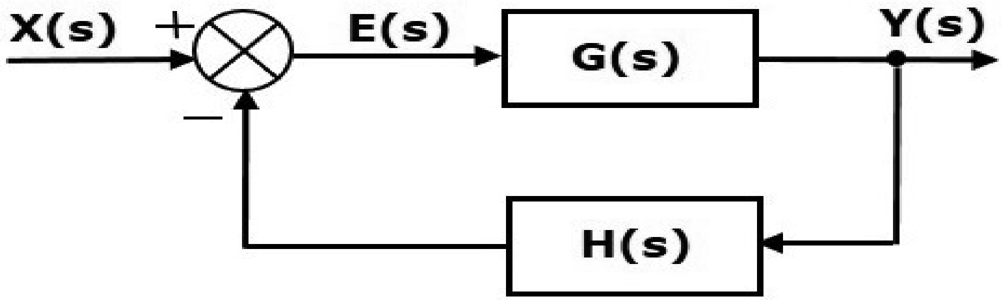
That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

Feedback Connection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions G(s)G(s) and H(s)H(s) form a closed loop.



The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is -

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

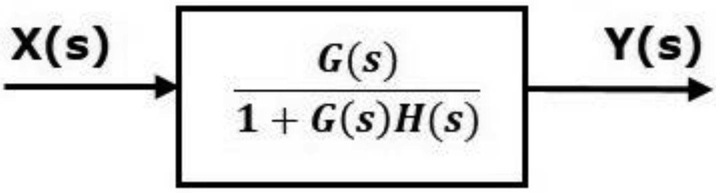
$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore, the negative feedback closed loop transfer function is :

$$\frac{G(s)}{1+G(s)H(s)}$$

This means we can represent the negative feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.



Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e.,

$$\frac{G(s)}{1-G(s)H(s)}$$

Block Diagram Algebra for Summing Points

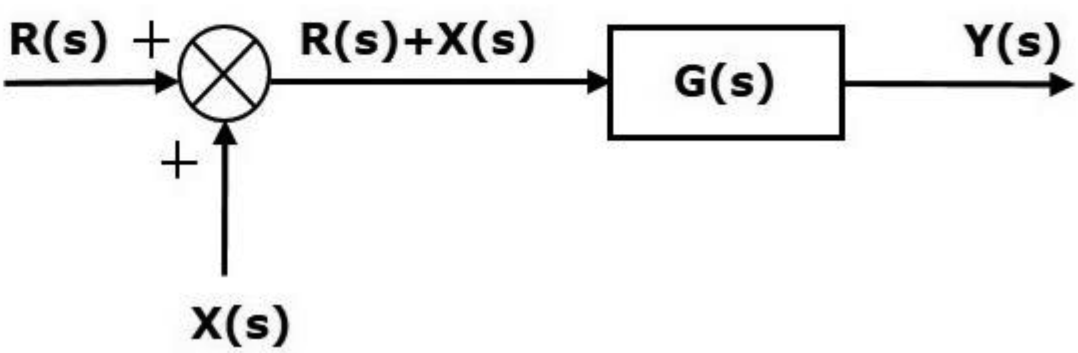
There are two possibilities of shifting summing points with respect to blocks −

- Shifting summing point after the block
- Shifting summing point before the block

Let us now see what kind of arrangements need to be done in the above two cases one by one.

Shifting the Summing Point before a Block to after a Block

Consider the block diagram shown in the following figure. Here, the summing point is present before the block.



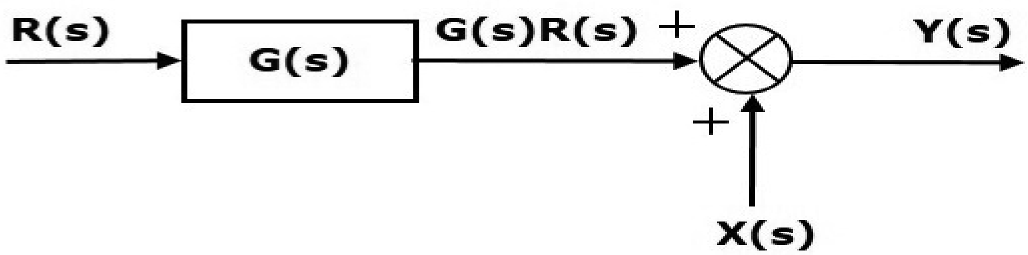
Summing point has two inputs $R(s)$ and $X(s)$

The output of Summing point is $\{R(s) + X(s)\}$.

So, the input to the block $G(s)$ is $\{R(s) + X(s)\}$ and the output of it is –

$$Y(s) = G(s)\{R(s) + X(s)\}$$

$$\Rightarrow Y(s) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 1)}$$



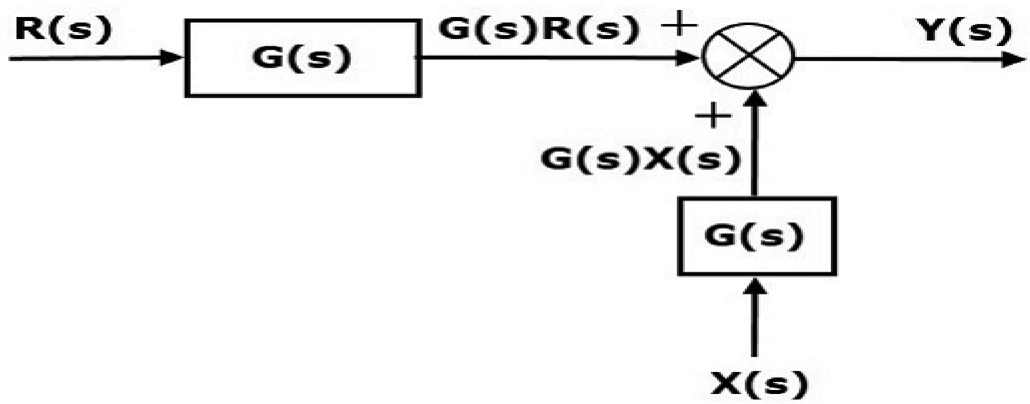
Output of the block $G(s)$ is $G(s)R(s)$.

The output of the summing point is

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 2)}$$

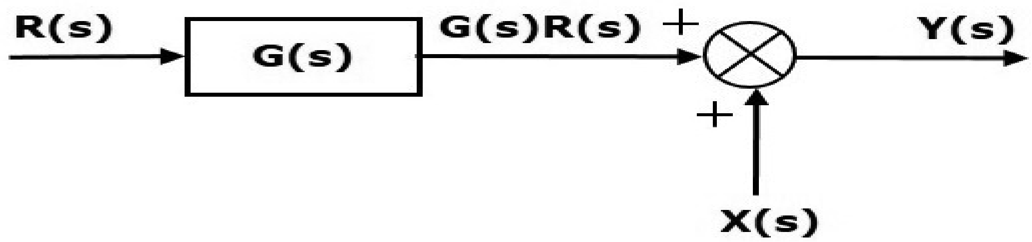
Compare Equation 1 and Equation 2.

The first term 'G(s)R(s)''G(s)R(s)' is same in both the equations. But, there is difference in the second term. In order to get the second term also same, we require one more block G(s)G(s). It is having the input X(s)X(s) and the output of this block is given as input to summing point instead of X(s)X(s). This block diagram is shown in the following figure.



Shifting Summing Point Before the Block

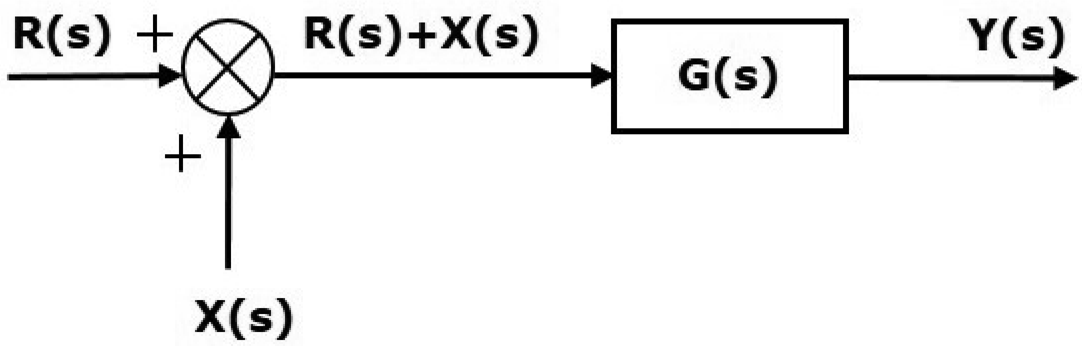
Consider the block diagram shown in the following figure. Here, the summing point is present after the block.



Output of this block diagram is -

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 3)}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.

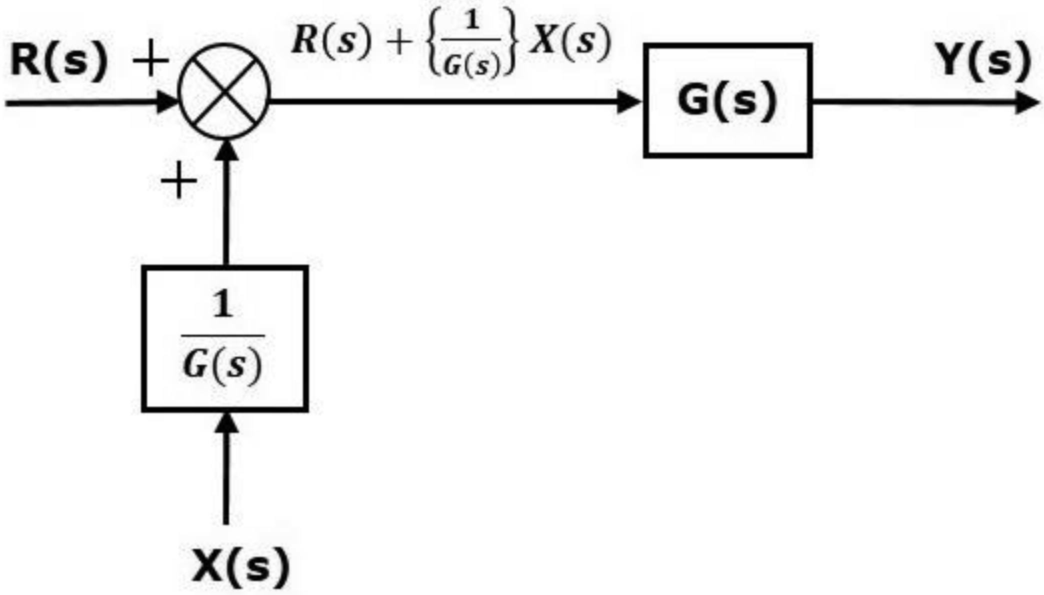


Output of this block diagram is -

$$Y(S) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 4)}$$

Compare Equation 3 and Equation 4,

The first term 'G(s)R(s)' is same in both equations. But, there is difference in the second term. In order to get the second term also same, we require one more block 1/G(s). It is having the input X(s) and the output of this block is given as input to summing point instead of X(s). This block diagram is shown in the following figure.



Block Diagram Algebra for Take-off Points

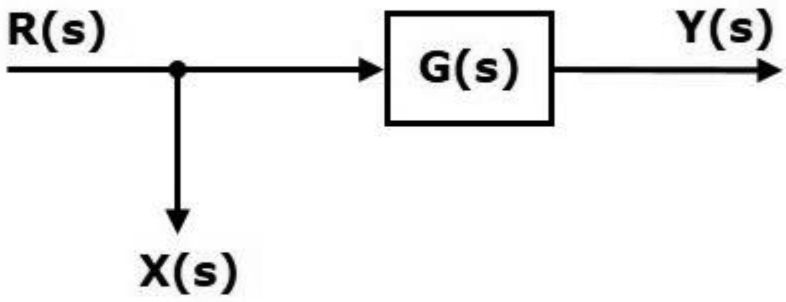
There are two possibilities of shifting the take-off points with respect to blocks −

- Shifting take-off point after the block
- Shifting take-off point before the block

Let us now see what kind of arrangements is to be done in the above two cases, one by one.

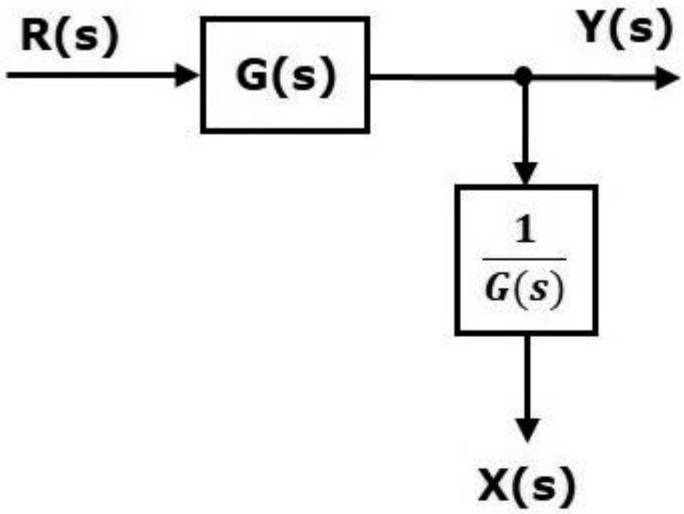
Shifting a Take-off Point form a Position before a Block to a position after the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.



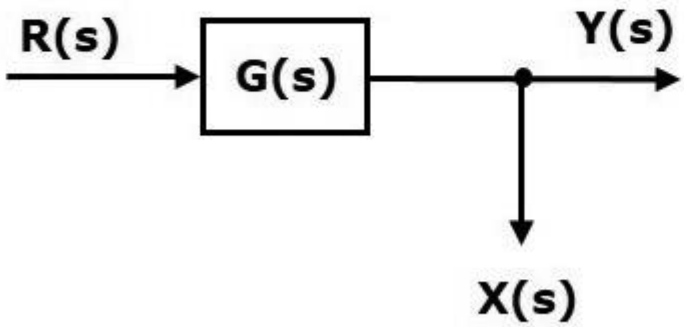
Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

When you shift the take-off point after the block, the output Y(s) will be same. But, there is difference in X(s) value. So, in order to get the same X(s) value, we require one more block 1/G(s). It is having the input Y(s) and the output is X(s) this block diagram is shown in the following figure.



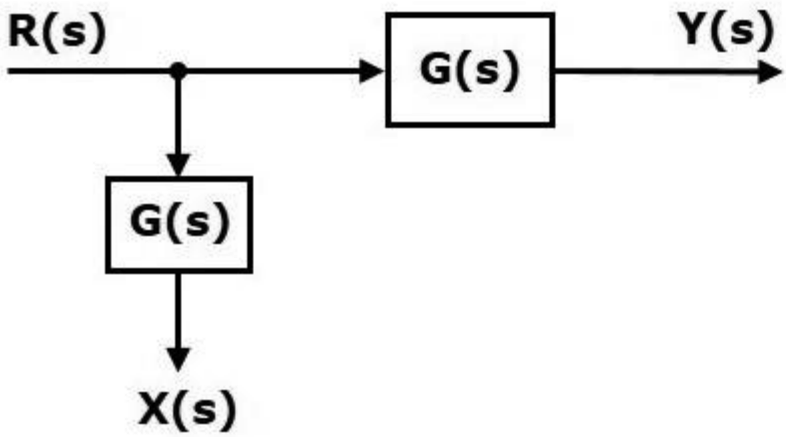
Shifting Take-off Point from a Position after a Block to a position before the Block

Consider the block diagram shown in the following figure. Here, the take-off point is present after the block.



Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block, the output Y(s) will be same. But, there is difference in X(s) value. So, in order to get same X(s) value, we require one more block G(s) It is having the input R(s) and the output is X(s). This block diagram is shown in the following figure.



The concepts discussed in the previous chapter are helpful for reducing (simplifying) the block diagrams.

Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

- **Rule 1** − Check for the blocks connected in series and simplify.
- **Rule 2** − Check for the blocks connected in parallel and simplify.
- **Rule 3** − Check for the blocks connected in feedback loop and simplify.
- **Rule 4** − If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5** − If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6** − Repeat the above steps till you get the simplified form, i.e., single block.

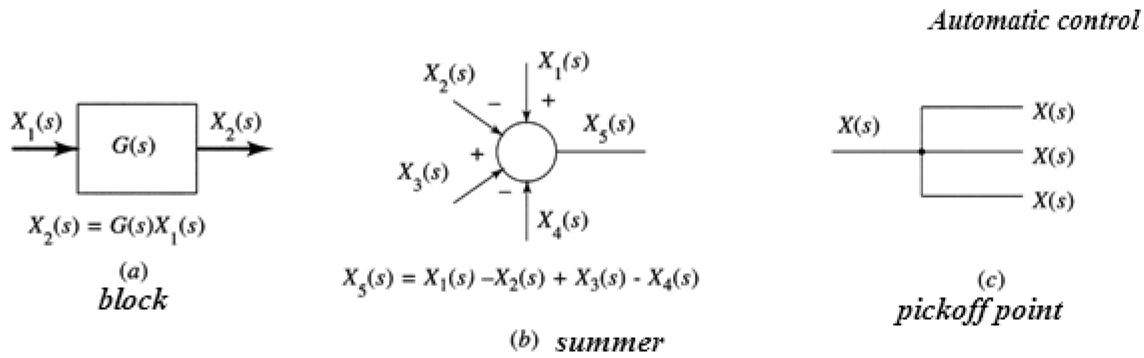
Note − The transfer function present in this single block is the transfer function of the overall block diagram.

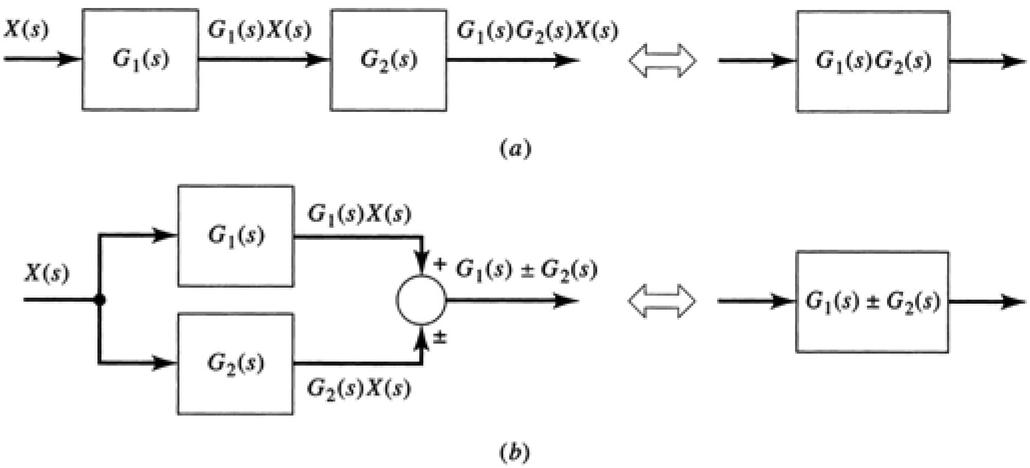
Note − Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

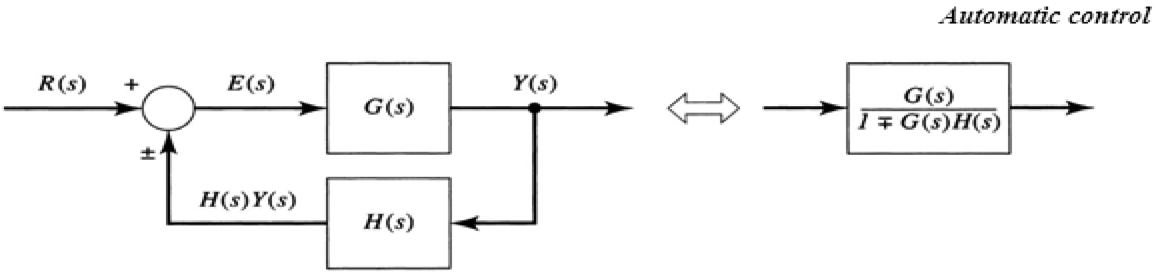
- **Step 1** − Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.
- **Step 2** − Repeat step 1 for remaining inputs.
- **Step 3** − Get the overall transfer function by adding all those transfer functions.

The block diagram reduction process takes more time for complicated systems because; we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

Block Diagram Reduction- Summary





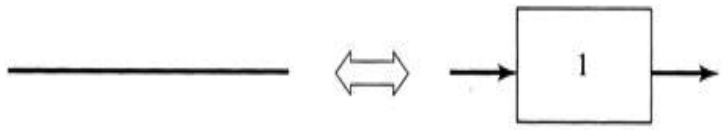


$$Y(s) = G(s)E(s)$$

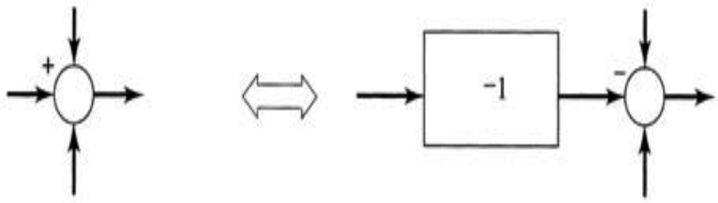
$$E(s) = R(s) \pm H(s)Y(s)$$

$$Y(s) = G(s)[R(s) \pm H(s)Y(s)] = G(s)R(s) \pm G(s)H(s)Y(s)$$

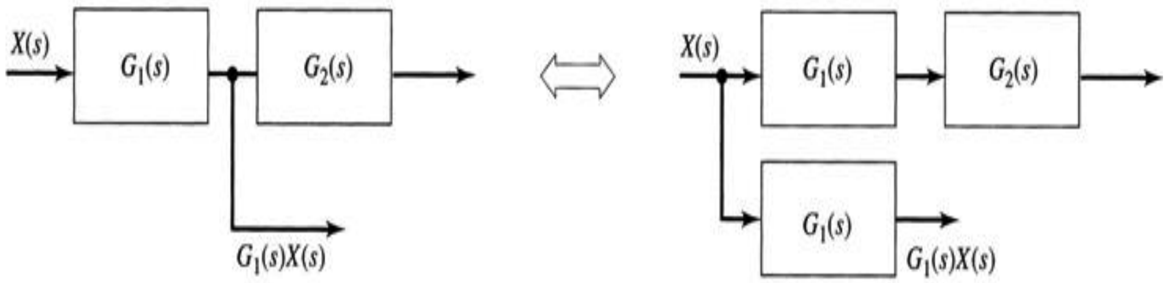
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}$$



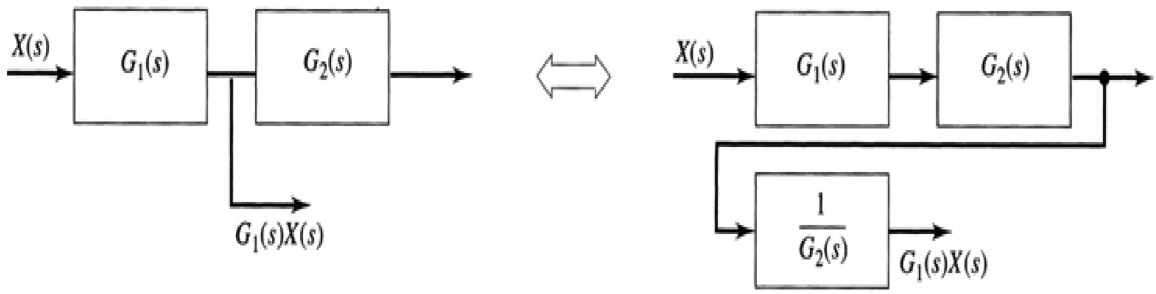
(a) Insertion or removal of unity gain



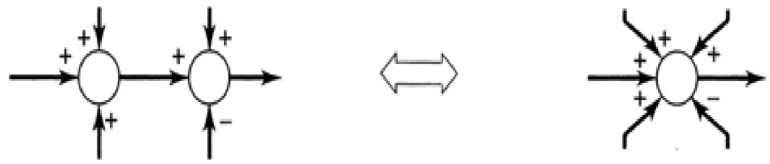
(b) Changing a summer sign



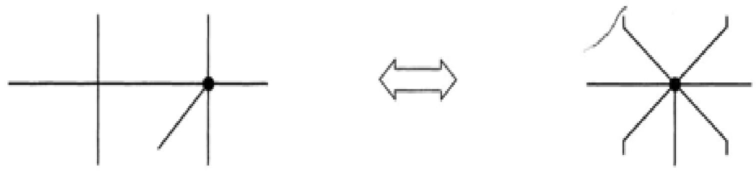
(c) Moving a pickoff point back



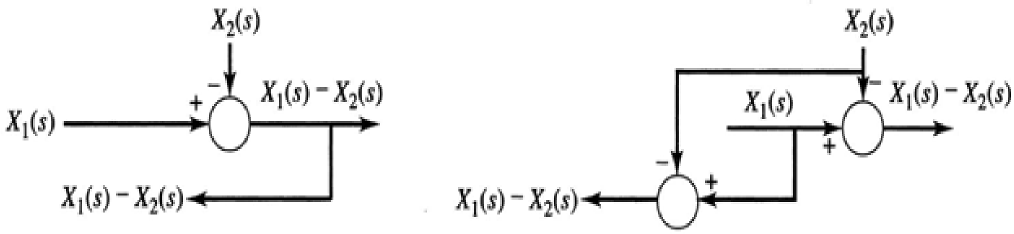
(d) Moving a pickoff point forward



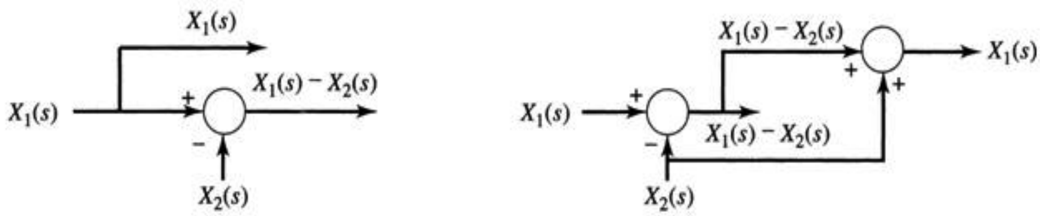
(e) Combining or expanding summations



(f) Combining or expanding junctions



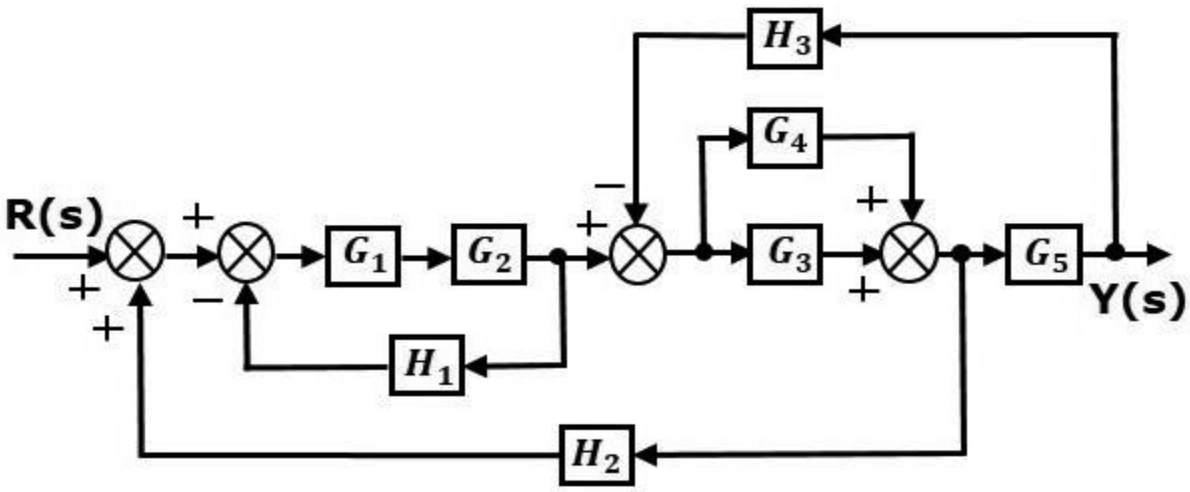
(g) Moving a pickoff point behind a summation



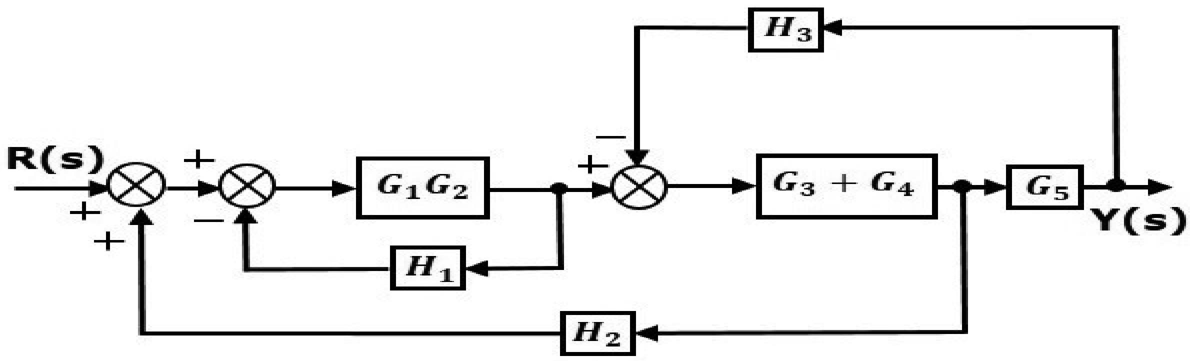
(h) Moving a pickoff point forward of a summation

Examples:

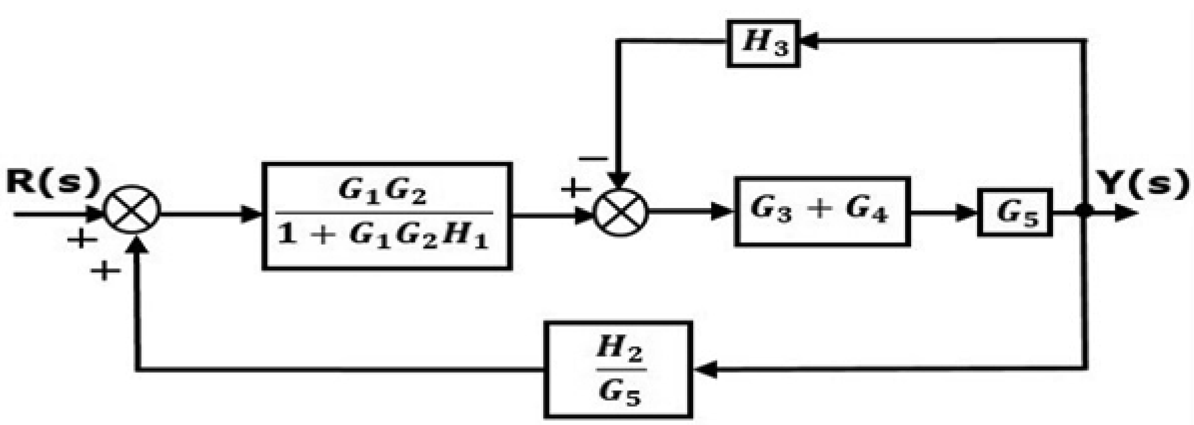
1.Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.



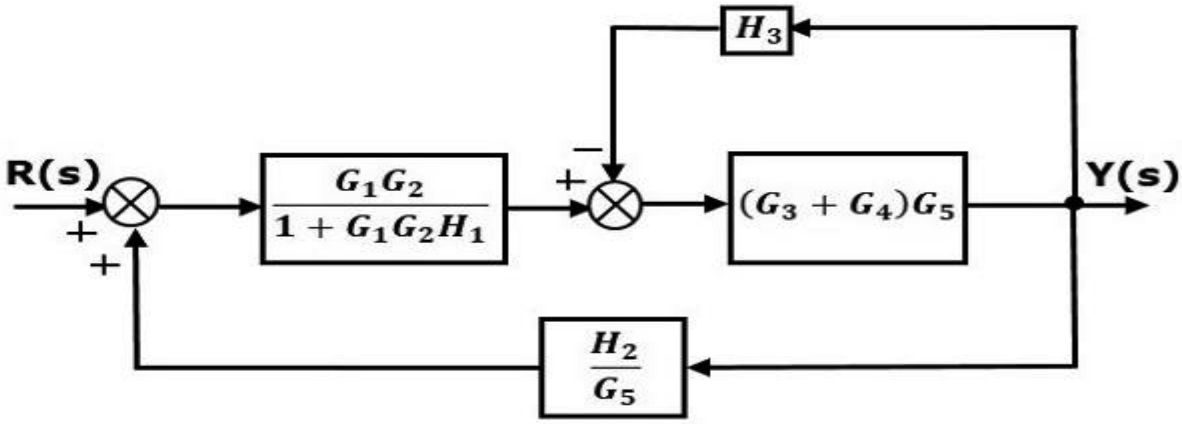
Step 1 – Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks G_3 and G_4. The modified block diagram is shown in the following figure.



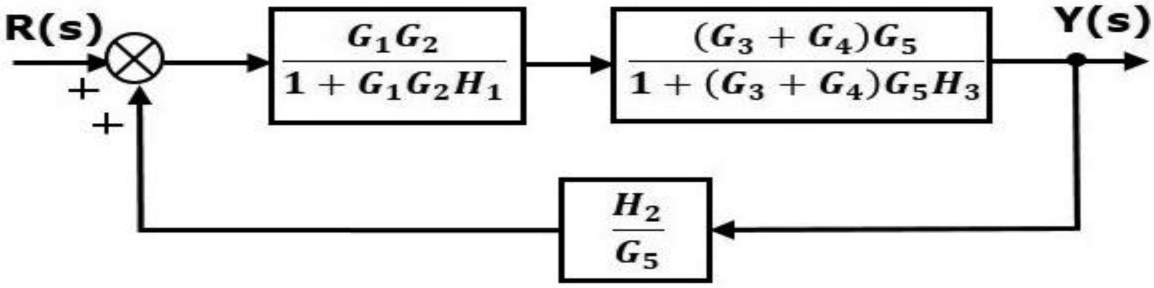
Step 2 – Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.



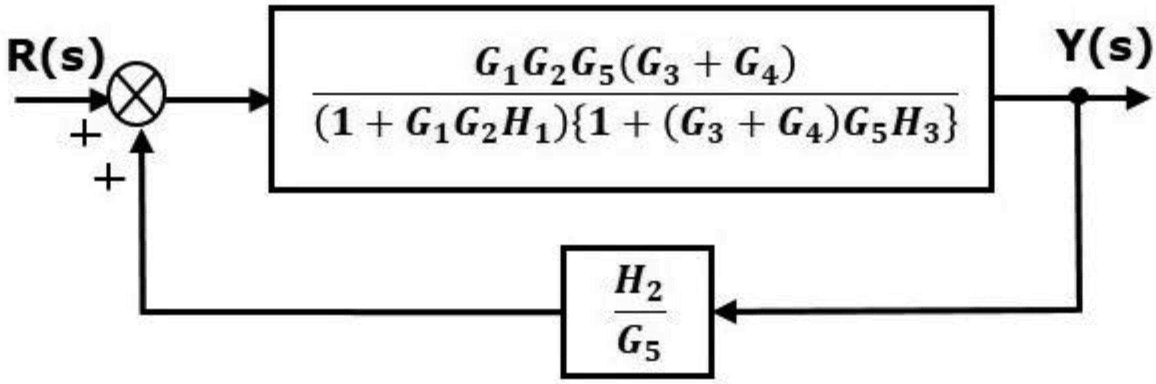
Step 3 – Use Rule 1 for blocks $(G_3 + G_4)$ and G_5. The modified block diagram is shown in the following figure.



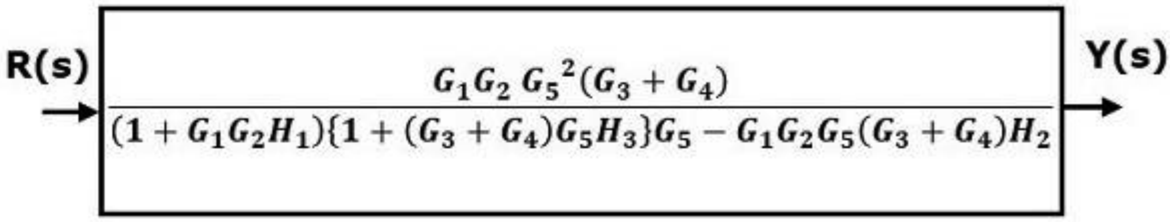
Step 4 – Use Rule 3 for blocks $(G_3+G_4)G_5$ and H_3. The modified block diagram is shown in the following figure.



Step 5 – Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



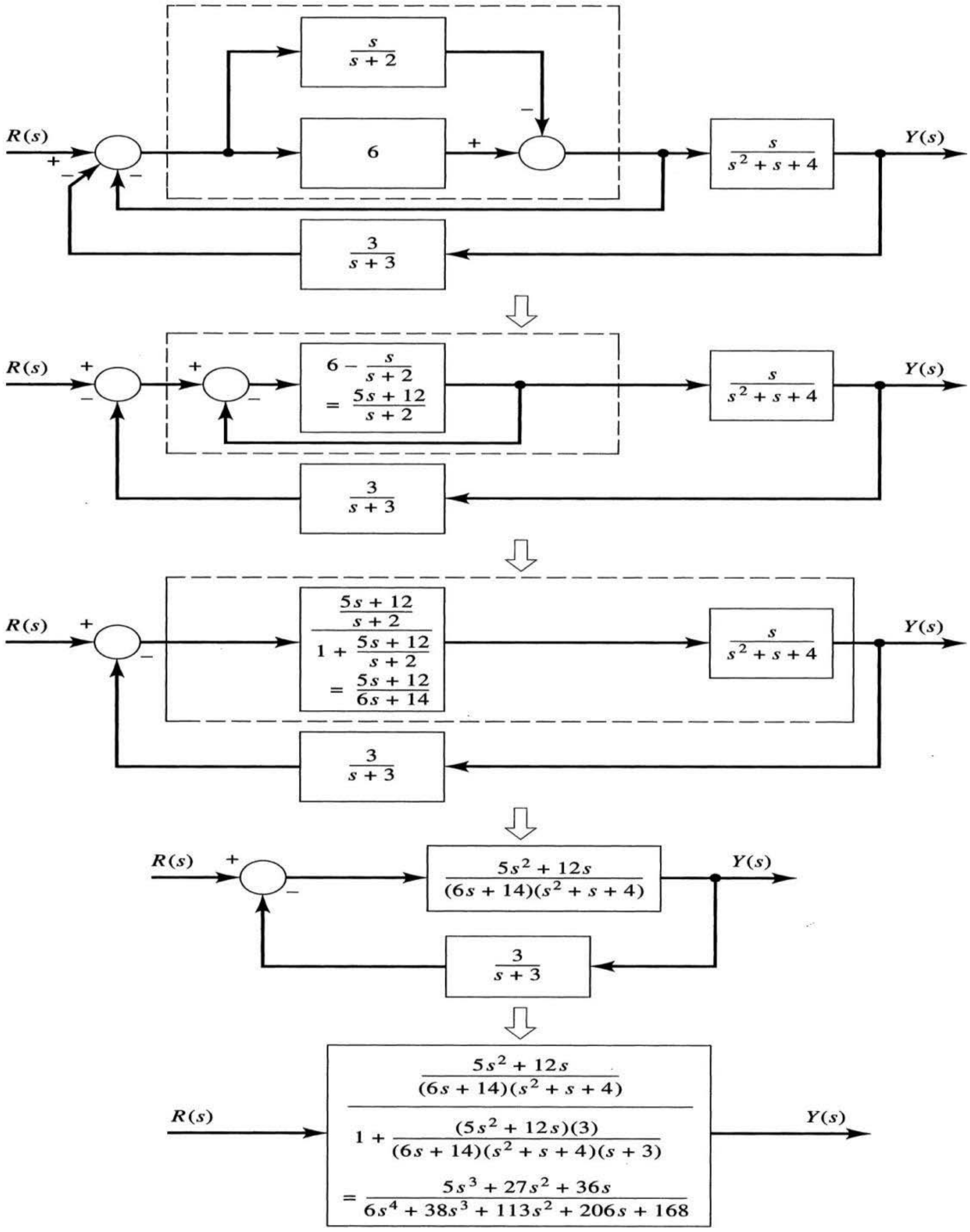
Step 6 – Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



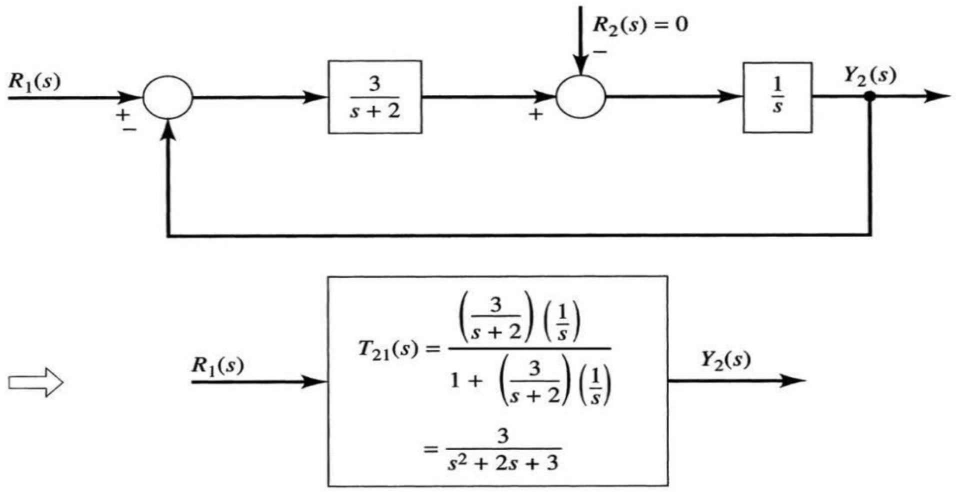
Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

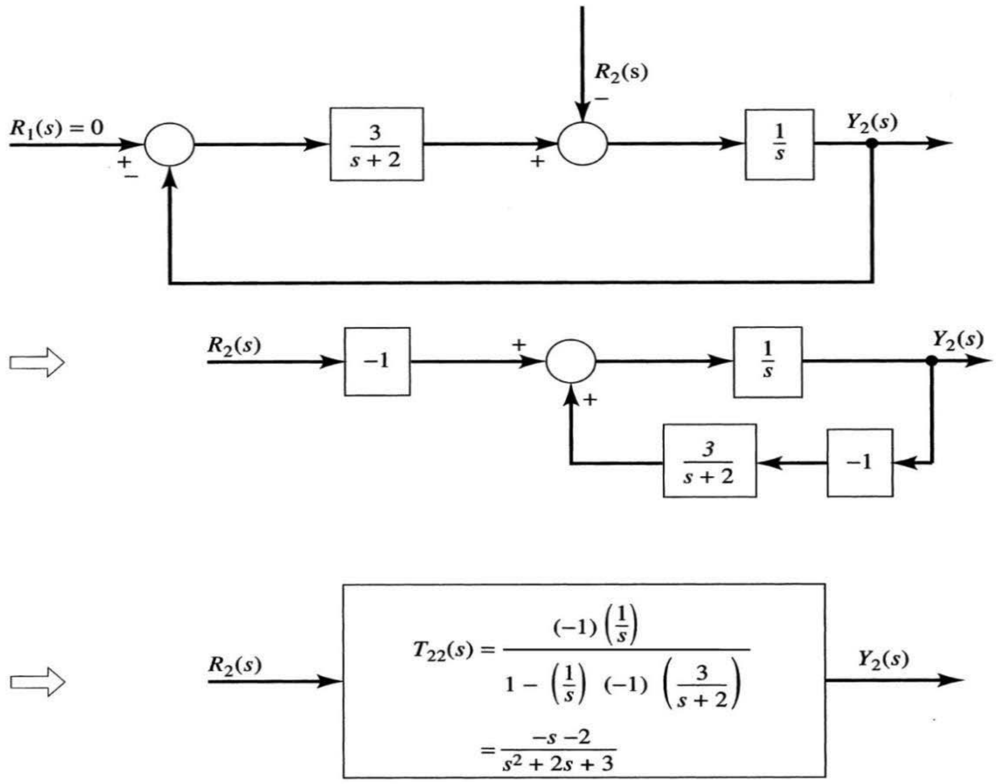
2.Determine the transfer function Y(s)/R(s).



3. Determine the transfer function $Y_2(s)/R_1(s)$.


$R_2(s) = 0$
$R_1(s)$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_1(s)$
$T_{21}(s) = \frac{\left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)}{1 + \left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)} = \frac{3}{s^2 + 2s + 3}$
$Y_2(s)$

(*a*)


$R_2(s)$
$R_1(s) = 0$
$\frac{3}{s+2}$
$\frac{1}{s}$
$Y_2(s)$
$R_2(s)$
-1
$\frac{1}{s}$
$Y_2(s)$
$\frac{3}{s+2}$
-1
$R_2(s)$
$T_{22}(s) = \frac{(-1)\left(\frac{1}{s}\right)}{1 - \left(\frac{1}{s}\right)(-1)\left(\frac{3}{s+2}\right)} = \frac{-s-2}{s^2 + 2s + 3}$
$Y_2(s)$

(*b*)

Signal Flow Graph

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

Basic Elements of Signal Flow Graph

Nodes and branches are the basic elements of signal flow graph.

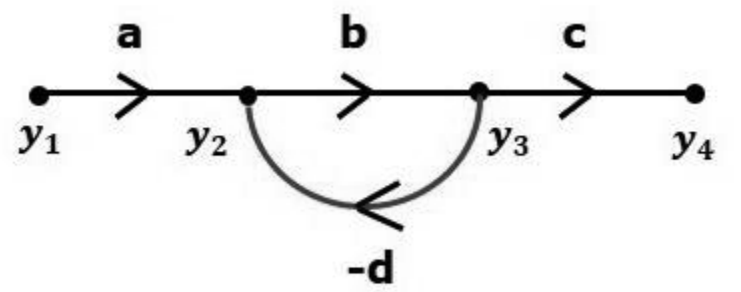
Node

Node is a point which represents either a variable or a signal. There are three types of nodes — input node, output node and mixed node.

- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.

Example

Let us consider the following signal flow graph to identify these nodes.



- The **nodes** present in this signal flow graph are y_1, y_2, y_3 and y_4.
- y_1 and y_4 are the **input node** and **output node** respectively.
- y_2 and y_3 are **mixed nodes**.

Branch

Branch is a line segment which joins two nodes. It has both **gain** and **direction**. For example, there are four branches in the above signal flow graph. These branches have **gains** of **a, b, c** and **-d**.

Construction of Signal Flow Graph

Let us construct a signal flow graph by considering the following algebraic equations −

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

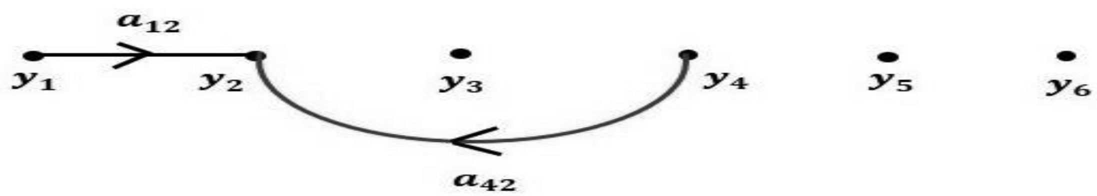
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

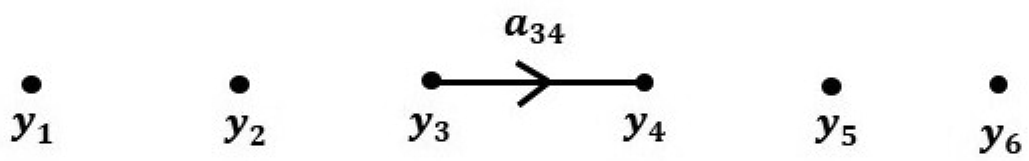
To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below −

Step 1 − Signal flow graph for $y_2 = a_{13}y_1 + a_{42}y_4$ is shown in the following figure.

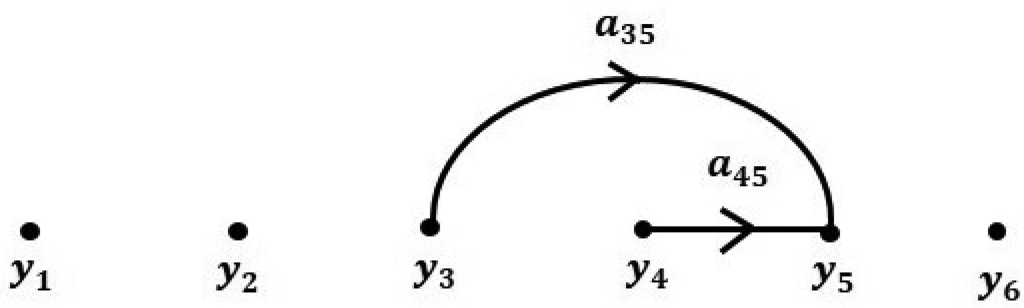


Step 2 – Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.

Step 3 – Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

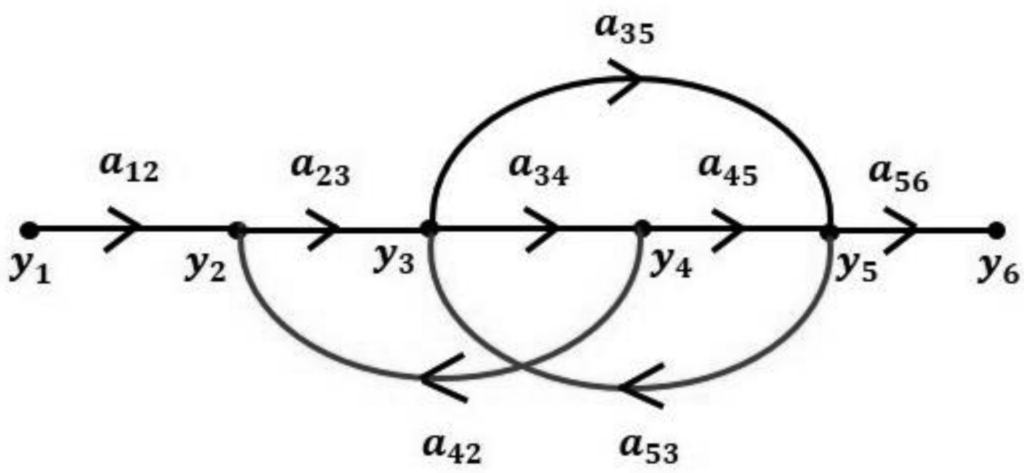


Step 4 – Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 – Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.

Step 6 – Signal flow graph of overall system is shown in the following figure.



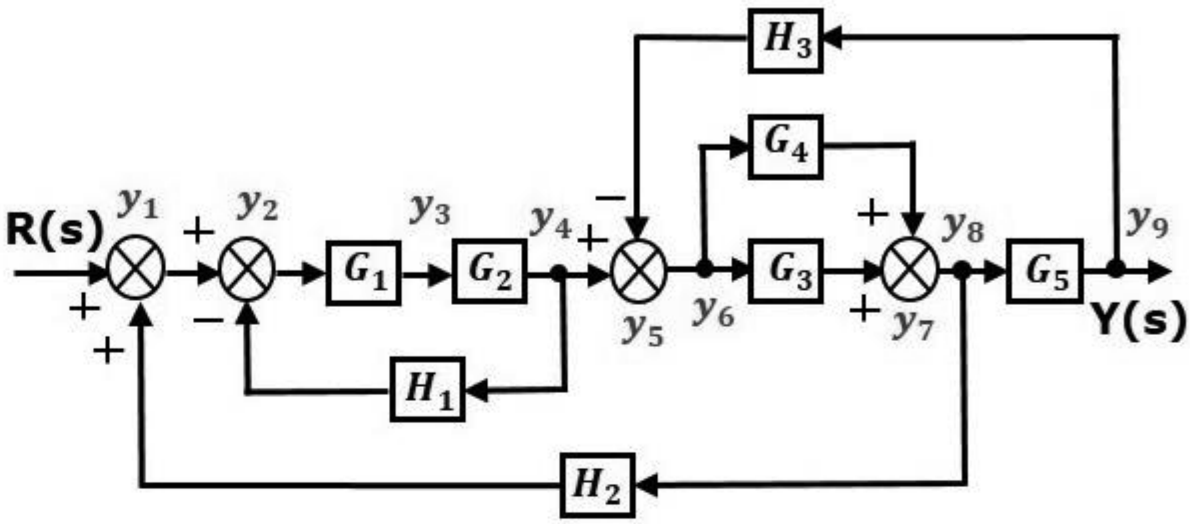
Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Represent the blocks of block diagram as **branches** in signal flow graph.
- Represent the transfer functions inside the blocks of block diagram as gains of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. **For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

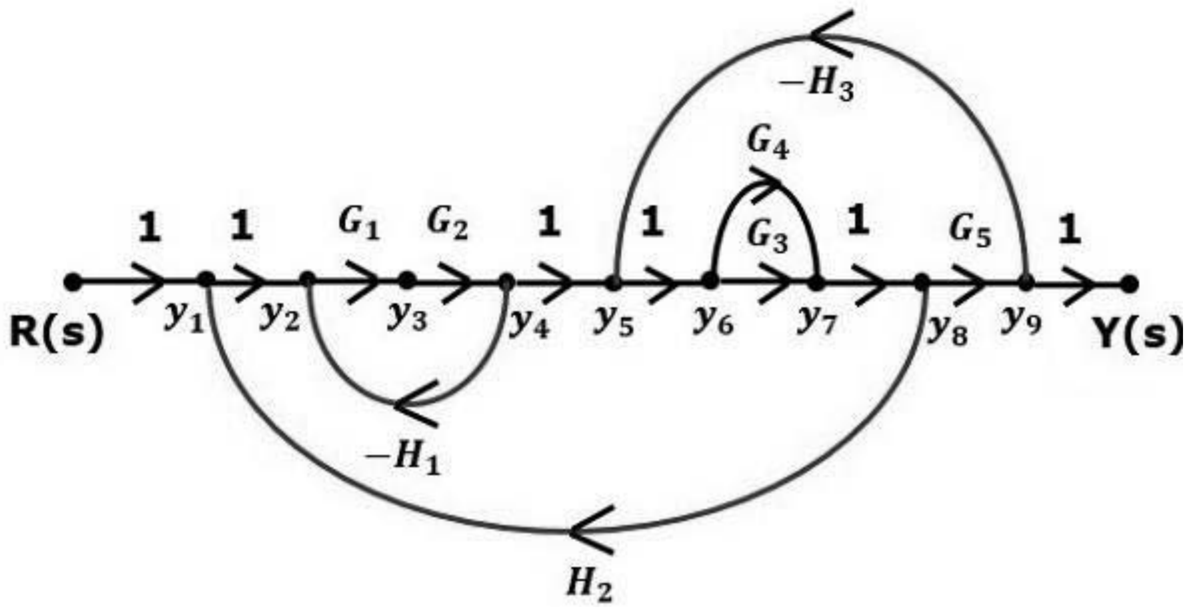
Example

Let us convert the following block diagram into its equivalent signal flow graph.



Represent the input signal R(s) and output signal C(s) of block diagram as input node R(s) and output node C(s) of signal flow graph.
Just for reference, the remaining nodes (y_1 to y_9) are labeled in the block diagram. There are nine nodes other than input and output nodes. That is four nodes for four summing points, four nodes for four take-off points and one node for the variable between blocks G_1 and G_2.

The following figure shows the equivalent signal flow graph.



Let us now discuss the Mason's Gain Formula. Suppose there are 'N' forward paths in a signal flow graph. The gain between the input and the output nodes of a signal flow graph is nothing but the **transfer function** of the system. It can be calculated by using Mason's gain formula.

Mason's gain formula is

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

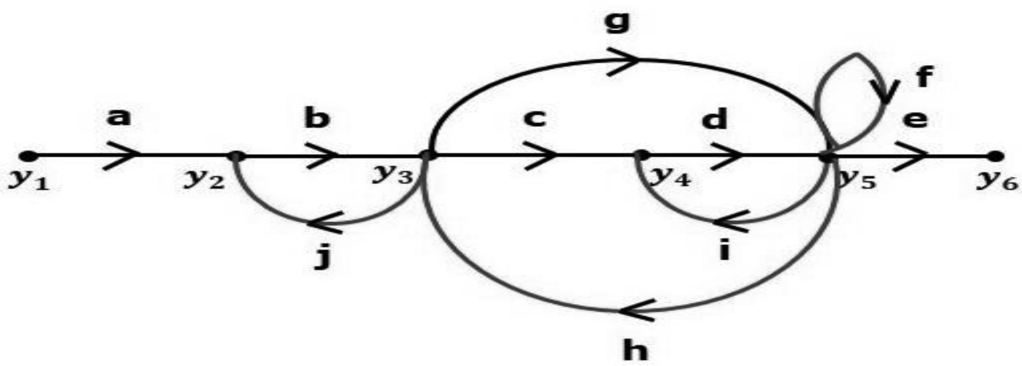
Where,

- **C(s)** is the output node
- **R(s)** is the input node
- **T** is the transfer function or gain between R(s) and C(s)
- **Pi** is the ith forward path gain

  Δ=1−(sum of all individual loop gains) +(sum of gain products of all possible two nontouching loops)−(sum of gain products of all possible three nontouching loops) +....

  Δ_i is obtained from Δ by removing the loops which are touching the ith forward path.

Consider the following signal flow graph in order to understand the basic terminology involved here.



Path

It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples − $y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$ and $y_5 \rightarrow y_3 \rightarrow y_2$

Forward Path

The path that exists from the input node to the output node is known as **forward path**.

Examples − $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples − $abcde$ is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and abge is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Loop

The path that starts from one node and ends at the same node is known as a **loop**. Hence, it is a closed path.

Examples − $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_3 \rightarrow y_5 \rightarrow y_3$.

Loop Gain

It is obtained by calculating the product of all branch gains of a loop.

Examples − b_j is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$ and g_h is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$.

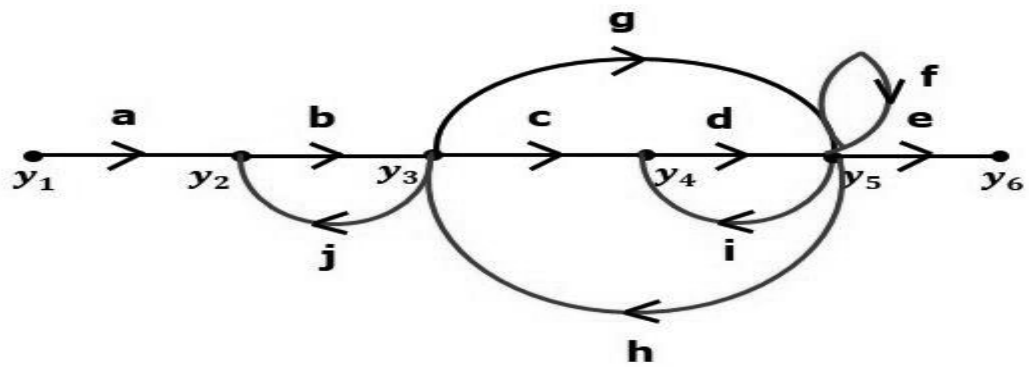
Non-touching Loops

These are the loops, which should not have any common node.

Examples − The loops, $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_4 \rightarrow y_5 \rightarrow y_4$ are non-touching.

Calculation of Transfer Function using Mason's Gain Formula

Let us consider the same signal flow graph for finding transfer function.



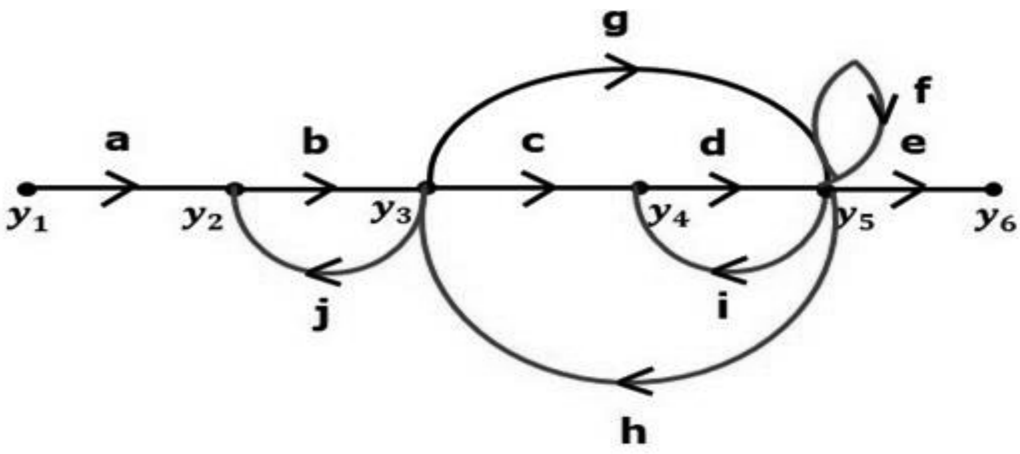
- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- First forward path gain, $p_1 = abcde$
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$
- Second forward path gain, $p_2 = abge$
- Number of individual loops, L = 5.

  Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

  Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

- Number of two non-touching loops = 2.
- First non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_4 \rightarrow y_5 \rightarrow y_4$.
- Gain product of first non-touching loops pair $l_1 l_4 = bjdi$
- Second non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_5 \rightarrow y_5$.
- Gain product of second non-touching loops pair is $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.We know,



- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- First forward path gain, $p_1 = abcde$.
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.
- Second forward path gain, $p_2 = abge$.
- Number of individual loops, L = 5.
- Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.
- Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.
- Number of two non-touching loops = 2.
- First non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_4 \rightarrow y_5 \rightarrow y_4$.
- Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$
- Second non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_5 \rightarrow y_5$.
- Gain product of second non-touching loops pair is - $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$
$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$
$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta_1 = 1$.

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, N = 2 in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{2} P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

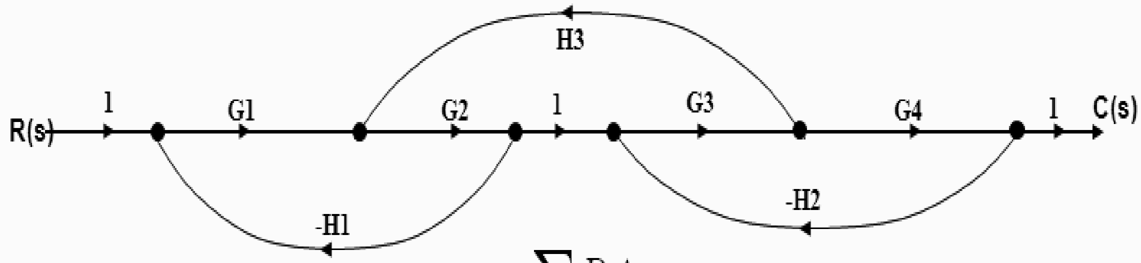
$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is -

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

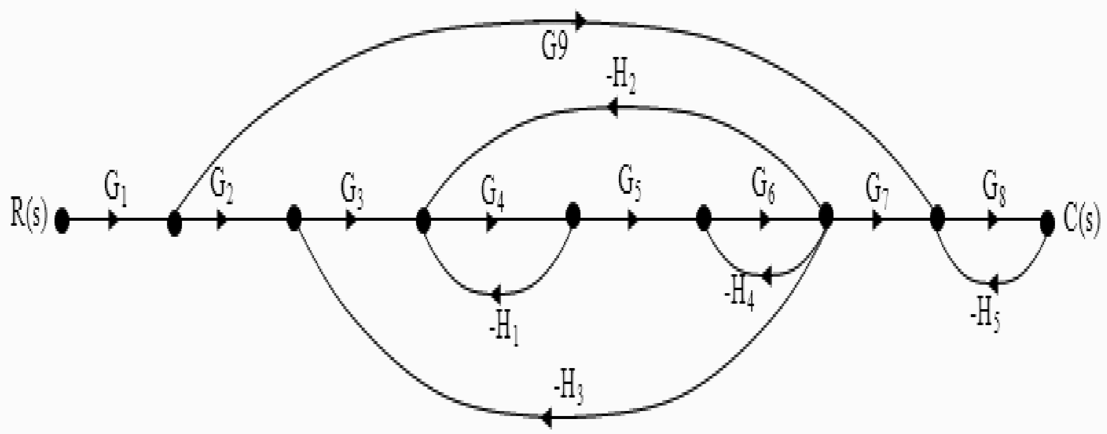
Example-1: Determine the transfer function C(s)/R(s).



$$T(s) = \frac{\sum P_k \Delta_k}{\Delta}$$

- $P_1 = G_1G_2G_3G_4$ $\Delta_1 = 1$ There is no P_2 or Δ_2 or more.
- $\sum L_1 = -G_1G_2H_1 + G_2G_3H_3 - G_3G_4H_2$
- $\sum L_2 = G_1G_2G_3G_4H_1H_2$
- $\Delta = 1 - \sum L_1 + \sum L_2 = 1 + G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2$
- $T(s) = \frac{\sum P_1\Delta_1}{\Delta} = \frac{G_1G_2G_3G_4}{1 + G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2}$

Example-2: Determine the transfer function C(s)/R(s).



$$M_1 = G_1G_2G_3G_4G_5G_6G_7G_8 \qquad \Delta_1 = 1$$

$$M_2 = G_1G_9G_8 \qquad \Delta_2 = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2] + G_4H_1G_6H_4$$

$$= 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4$$

$$\Delta = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2 - G_8H_5]$$

$$+\left[G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3\right]$$

$$+G_4H_1G_6H_4G_8H_5$$

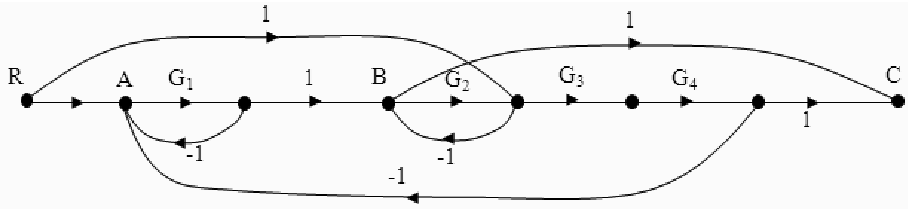
$$\Delta = 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_8H_5$$

$$+G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3$$

$$+G_4H_1G_6H_4G_8H_5$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5G_6G_7G_8 + G_1G_9G_8[1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4]}{\Delta}$$

Example-3: Determine the transfer function C(s)/R(s).



$$M_1 = G_1G_2G_3G_4 \qquad \Delta_1 = 1$$
$$M_2 = G_1 \qquad \Delta_2 = 1$$
$$M_3 = G_3G_4 \qquad \Delta_3 = 1 + G_1$$
$$M_4 = -1 \qquad \Delta_4 = 1 + G_1$$
$$M_5 = -G_3G_4G_1 \qquad \Delta_5 = 1$$

$$\Delta = 1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2 \; = \; 1 + G_1 + G_2 + G_1G_2G_3G_4 + G_1G_2$$

a) $$\frac{C}{M} = \frac{G_1G_2G_3G_4 + G_1 + G_3G_4(1+G_1) - (1)(1+G_1) - G_1G_3G_4}{1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2}$$

UNIT-II

TIME RESPONSE ANALYSIS

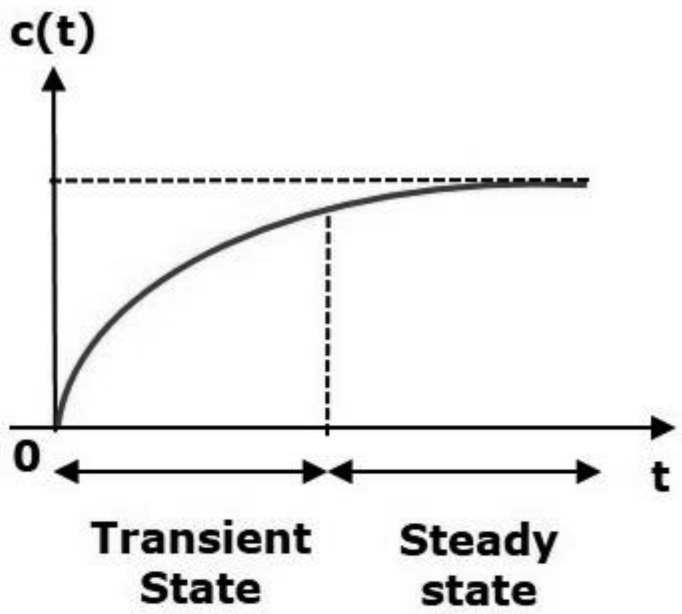
We can analyze the response of the control systems in both the time domain and the frequency domain. We will discuss frequency response analysis of control systems in later chapters. Let us now discuss about the time response analysis of control systems.

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- $c_{tr}(t)$ is the transient response

- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t \to \infty} c_{tr}(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of ‘t’ is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system $c(t) = 10 + 5e^{-t}$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

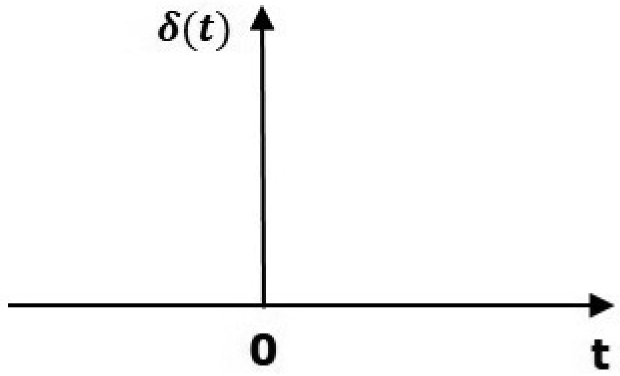
Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$$\delta(t) = 0 \text{ for } t \neq 0$$

$$\text{and } \int_{0^-}^{0^+} \delta(t)dt = 1$$

The following figure shows unit impulse signal.



So, the unit impulse signal exists only at‘t’ is equal to zero. The area of this signal under small interval of time around‘t’ is equal to zero is one. The value of unit impulse signal is zero for all other values of‘t’.

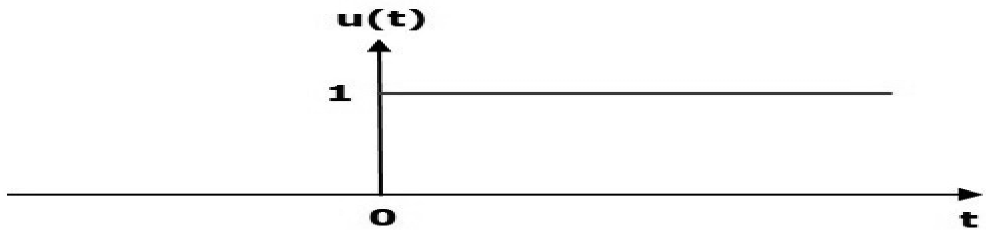
Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t) = 1; t \geq 0$$

$$= 0; t < 0$$

Following figure shows unit step signal.



So, the unit step signal exists for all positive values of‘t’ including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of‘t’.

Unit Ramp Signal

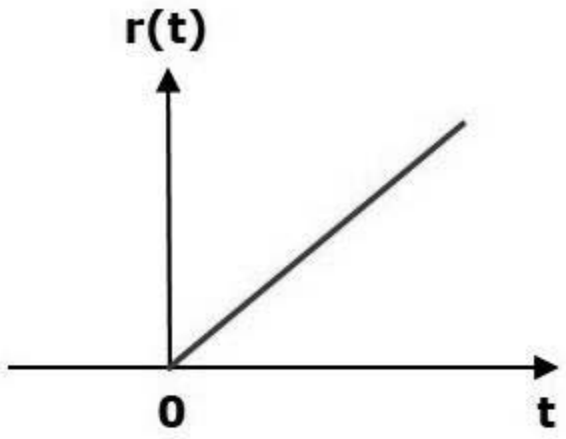
A unit ramp signal, r (t) is defined as

$$r(t) = t; t \geq 0$$
$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of‘t’ including zero. And its value increases linearly with respect to‘t’ during this interval. The value of unit ramp signal is zero for all negative values of‘t’.

Unit Parabolic Signal

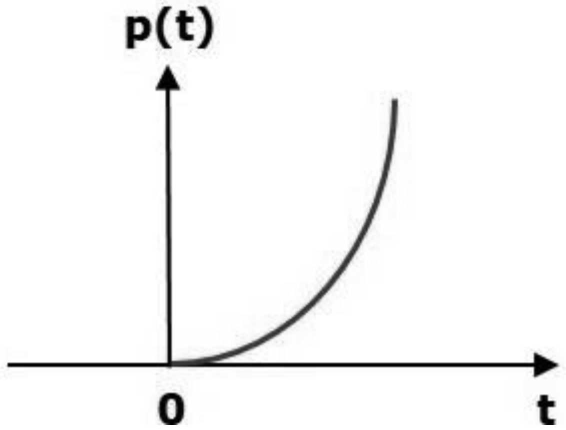
A unit parabolic signal, p(t) is defined as,

$$p(t) = \frac{t^2}{2}; t \geq 0$$
$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

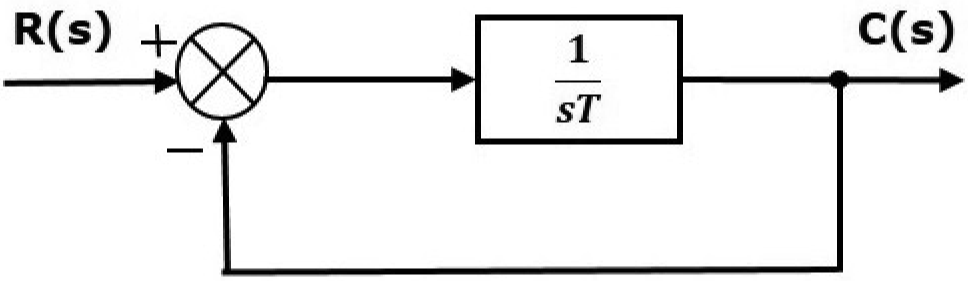
$$p(t) = \frac{t^2}{2}u(t)$$

The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of'‘t’ including zero. And its value increases non-linearly with respect to'‘t’ during this interval. The value of the unit parabolic signal is zero for all the negative values of'‘t’.

In this chapter, let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, 1/sT is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right)R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, r(t)=δ(t)

Apply Laplace transform on both the sides.

R(s) =1

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right)R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

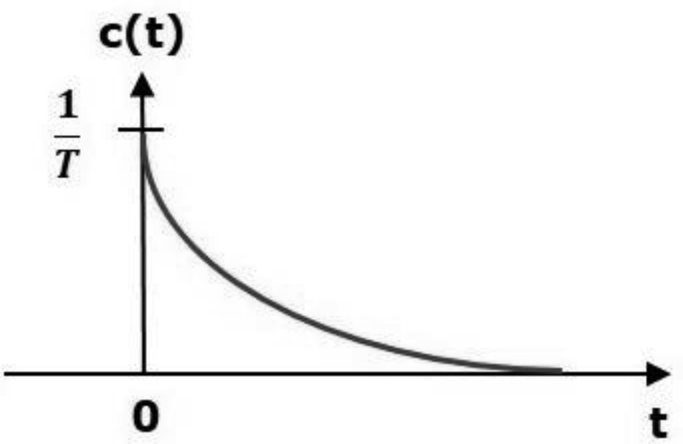
$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

Applying Inverse Laplace Transform on both the sides,

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

$$c(t) = \frac{1}{T}e^{\left(-\frac{t}{T}\right)}u(t)$$

The unit impulse response is shown in the following figure.



The **unit impulse response**, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

So, r(t)=u(t)

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s\,(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\,(sT+1)} = \frac{A\,(sT+1)+Bs}{s\,(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0=T+B$$

$$\Rightarrow B=-T$$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s)

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit step response**, c(t) has both the transient and the steady state terms.

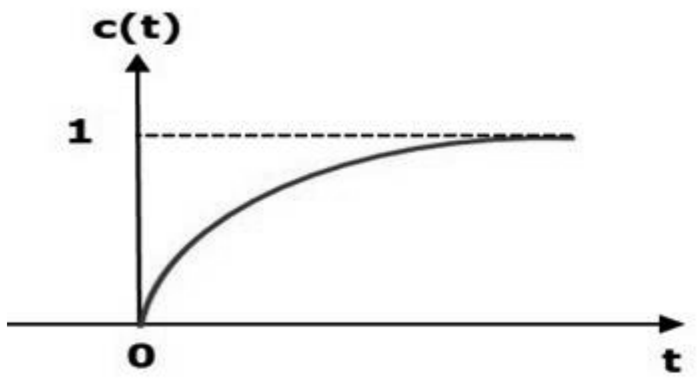
The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit step response is –

$$c_{ss}(t) = u(t)$$

The following figure shows the unit step response



The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

So, r(t)=t u(t)

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1}$$

$$\Rightarrow \frac{1}{s^2(sT+1)} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$$0=T+B \Rightarrow B=-T$$

Similarly, substitute B = −T and equate the coefficient of s^2 terms on both the sides. You will get $C=T^2$

Substitute A = 1, B = −T and $C=T^2$ in the partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)}\right)u(t)$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

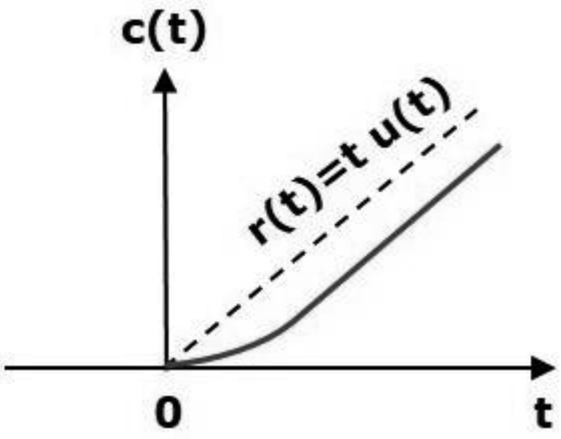
The transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)}u(t)$$

The steady state term in the unit ramp response is –

$c_{ss}(t) = (t - T)u(t)$

The figure below is the unit ramp response:



The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

So, $r(t) = \frac{t^2}{2}u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$ \

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T,\ T^2\ and\ -T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit parabolic response**, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is

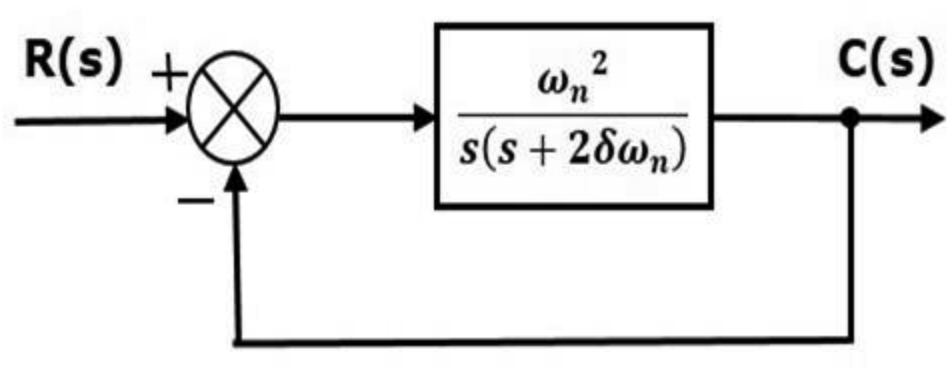
$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit parabolic response is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2\right) u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn't have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

In this chapter, let us discuss the time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\omega_n^2 / s(s+2\delta\omega n)$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\delta\omega_n s+\omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2-1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2-1}$$

b

- The two roots are imaginary when δ = 0.
- The two roots are real and equal when δ = 1.
- The two roots are real but not equal when δ > 1.

- The two roots are complex conjugate when $0 < \delta < 1$.

We can write C(s) equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal, c(t)
- **R(s)** is the Laplace transform of the input signal, r(t)
- **ω_n** is the natural frequency
- **δ** is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain.

Take Laplace transform of the input signal, $r(t)$.

Consider the equation, $C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}\right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$.

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, \ -1 \, and \ -\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1,\ -1\,and\ -2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2 + \omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right)u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right)u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right)u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - \omega_n^2(\delta^2-1)}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$

and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\left(\frac{1}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}}\right) - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)\left(\frac{1}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right)u(t)$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

Impulse Response of Second Orde r System

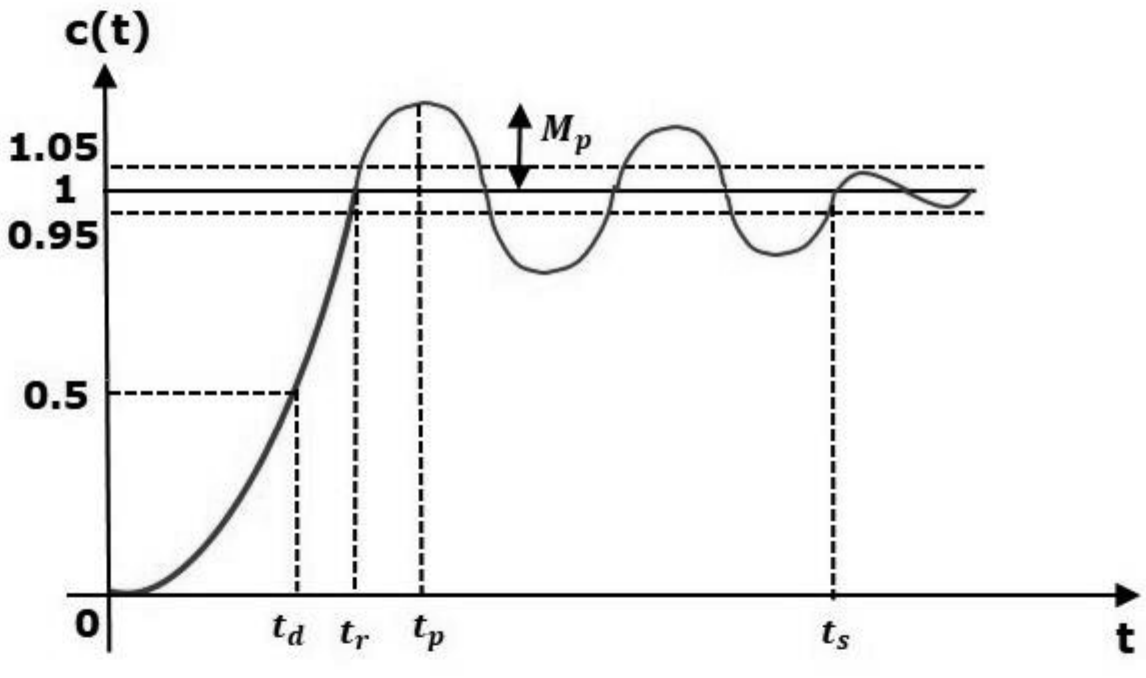
The **impulse response** of the second order system can be obtained by using any one of these two methods.

- Follow the procedure involved while deriving step response by considering the value of R(s) as 1 instead of 1/s.
- Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	Impulse response for t ≥ 0
$\delta = 0$	$\omega_n \sin(\omega_n t)$
$\delta = 1$	$\omega_n^2 t e^{-\omega_n t}$
$0 < \delta < 1$	$\left(\frac{\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}}\right)\left(e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t}\right)$

In this chapter, let us discuss the time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by tdtd.

Consider the step response of the second order system for $t \geq 0$, when 'δ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by $\mathbf{t_r}$.

At $t = t_1 = 0$, $c(t) = 0$.

We know that the final value of the step response is one. Therefore, at t=t2, the value of step response is one. Substitute, these values in the following equation.

$$c(t)=1-\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t+\theta)$$

$$c(t_2)=1=1-\left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2+\theta)$$

$$\Rightarrow\left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2+\theta)=0$$

$$\Rightarrow\sin(\omega_d t_2+\theta)=0$$

$$\Rightarrow\omega_d t_2+\theta=\pi$$

$$\Rightarrow t_2=\frac{\pi-\theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r=t_2-t_1$$

$$\therefore\ t_r=\frac{\pi-\theta}{\omega_d}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by t_p. At $t=t_p$ the first derivate of the response is zero.

We know the step response of second order system for under-damped case is

$$c(t)=1-\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t+\theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t}=-\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t+\theta)-\left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t+\theta)$$

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{\mathrm{d}c(t)}{\mathrm{d}t} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{\mathrm{d}c(t)}{\mathrm{d}t} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\left[\omega_d\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta)\right]$$

$$\Rightarrow \omega_n\sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta)\cos(\omega_d t_p + \theta) - \cos(\theta)\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

Peak Overshoot

Peak overshoot $\mathbf{M_p}$ is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically, we can write it as

$$M_p = c(t_p) - c(\infty)$$

Where, $c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of the response.

At t=tp, the response c(t) is -

$$c(t_p) = 1 - \left(\frac{e^{-\delta \omega_n t_p}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta \omega_n \left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}} \right) \sin\left(\omega_d \left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}} \right) (-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

From the above equation, we can conclude that the percentage of peak overshoot %Mp will decrease if the damping ratio δ increases.

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by ts.

The settling time for 5% tolerance band is –

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is –

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $1/\delta\omega_n$.

- Both the settling time ts and the time constant τ are inversely proportional to the damping ratio δ.
- Both the settling time ts and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time ts and time constant τ will never change.

Example

Let us now find the time domain specifications of a control system having the closed loop transfer function when the unit step signal is applied as an input to this control system.

We know that the standard form of the transfer function of the second order closed loop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

By equating these two transfer functions, we will get the un-damped natural frequency ω_n as 2 rad/sec and the damping ratio δ as 0.5.

We know the formula for damped frequency ω_d as

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1-(0.5)^2}$$

$$\Rightarrow \omega_d = 1.732\ rad/sec$$

Substitute, δ value in following relation

$$\theta = \cos^{-1}\delta$$

$$\Rightarrow \theta = \cos^{-1}(0.5) = \frac{\pi}{3}\ rad$$

Substitute the above necessary values in the formula of each time domain specification and simplify in order to get the values of time domain specifications for given transfer function.

The following table shows the formulae of time domain specifications, substitution of necessary values and the final values

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$	$t_d = \frac{1+0.7(0.5)}{2}$	t_d =0.675 sec
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$	$t_r = \frac{\pi-(\frac{\pi}{3})}{1.732}$	t_r =1.207 sec
Peak time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	t_p =1.813 sec
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}\right) \times 100\%$	$\%M_p = \left(e^{-\left(\frac{0.5\pi}{\sqrt{1-(0.5)^2}}\right)}\right) \times 100\%$	$\%\ M_p$ =16.32%
Settling time for 2% tolerance band	$t_s = \frac{4}{\delta\omega_n}$	$t_S = \frac{4}{(0.5)(2)}$	t_s =4 sec

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} E(s)$$

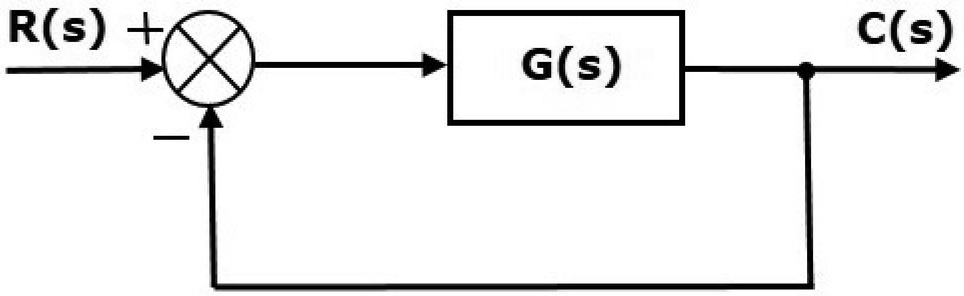
Where,

E(s) is the Laplace transform of the error signal, e(t)

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,

- R(s) is the Laplace transform of the reference Input signal $r(t)$
- C(s) is the Laplace transform of the output signal $c(t)$

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s\to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s\to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s\to 0} s^2G(s)$

Where, Kp, Kv and Ka are position error constant, velocity error constant and acceleration error constant respectively.

Note − If any of the above input signals has the amplitude other than unity, then multiply corresponding steady state error with that amplitude.

Note − We can't define the steady state error for the unit impulse signal because, it exists only at origin. So, we can't compare the impulse response with the unit impulse input as **t** denotes infinity

Example

Let us find the steady state error for an input signal $r(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)$ of unity negative feedback control system with $G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s \to 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s \to 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s \to 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

We will get the overall steady state error, by adding the above three steady state errors.

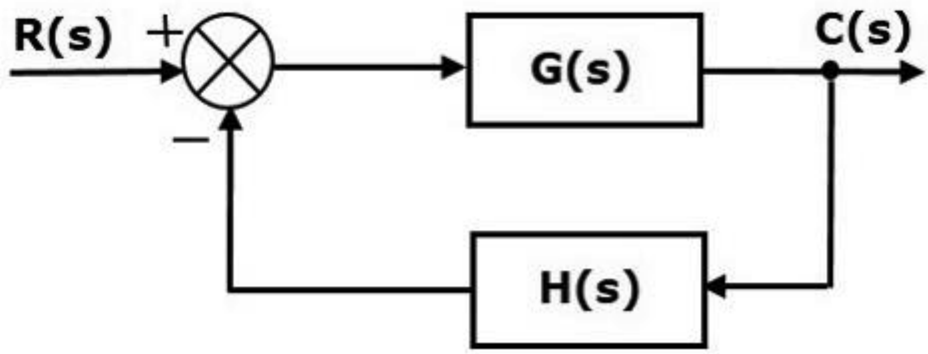
$$e_{ss} = e_{ss1}+e_{ss2}+e_{ss3}$$

$$\Rightarrow e_{ss}=0+0+1=1 \Rightarrow e_{ss}=0+0+1=1$$

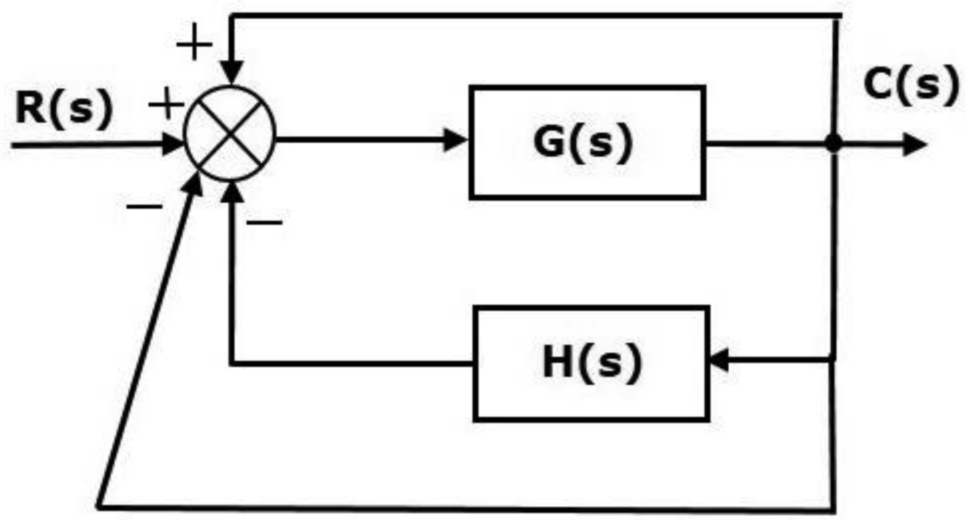
Therefore, we got the steady state error e_{ss} as **1** for this example.

Steady State Errors for Non-Unity Feedback Systems

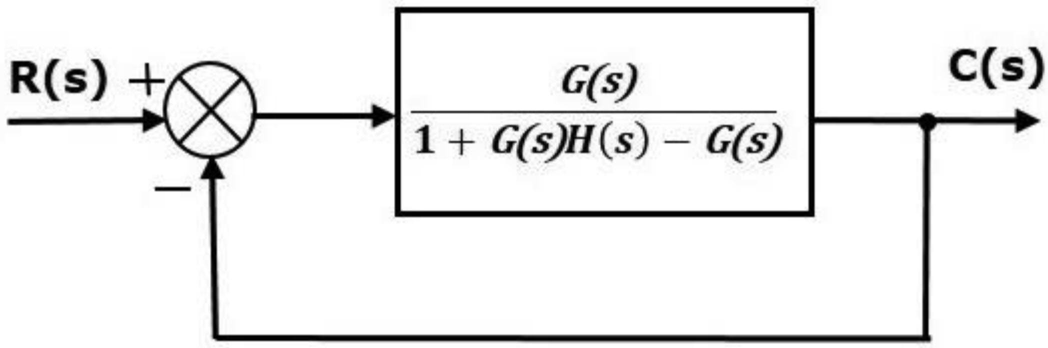
Consider the following block diagram of closed loop control system, which is having non unity negative feedback.



We can find the steady state errors only for the unity feedback systems. So, we have to convert the non-unity feedback system into unity feedback system. For this, include one unity positive feedback path and one unity negative feedback path in the above block diagram. The new block diagram looks like as shown below.



Simplify the above block diagram by keeping the unity negative feedback as it is. The following is the simplified block diagram



This block diagram resembles the block diagram of the unity negative feedback closed loop control system. Here, the single block is having the transfer function G(s) / [1+G(s)H(s)−G(s)] instead of G(s).You can now calculate the steady state errors by using steady state error formula given for the unity negative feedback systems.

Note − It is meaningless to find the steady state errors for unstable closed loop systems. So, we have to calculate the steady state errors only for closed loop stable systems. This means we need to check whether the control system is stable or not before finding the steady state errors. In the next chapter, we will discuss the concepts-related stability.

The various types of controllers are used to improve the performance of control systems. In this chapter, we will discuss the basic controllers such as the proportional, the derivative and the integral controllers.

Proportional Controller

The proportional controller produces an output, which is proportional to error signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

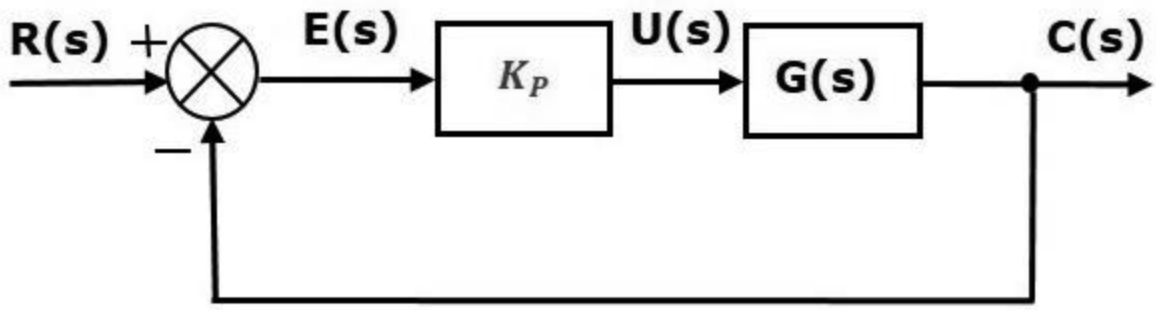
Therefore, the transfer function of the proportional controller is KPKP.
Where,

U(s) is the Laplace transform of the actuating signal u(t)

E(s) is the Laplace transform of the error signal e(t)

K_P is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure.



Derivative Controller

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides.

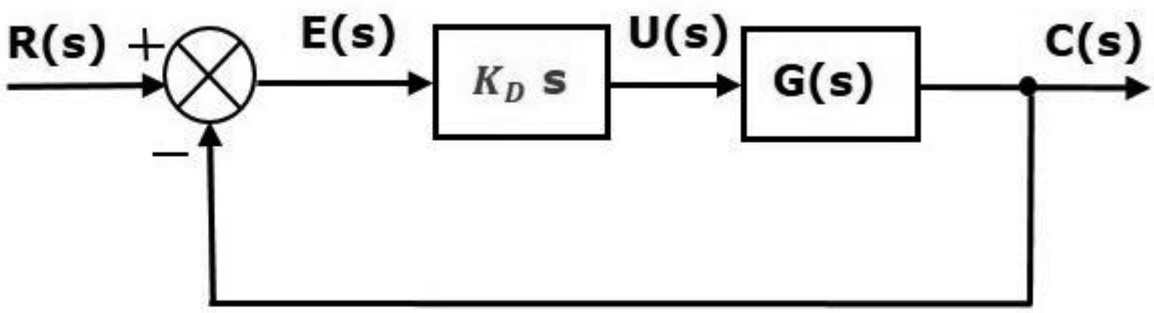
$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is $K_D s$.

Where, KD is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.



The derivative controller is used to make the unstable control system into a stable one.

Integral Controller

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t)dt$$

Apply Laplace transform on both the sides -

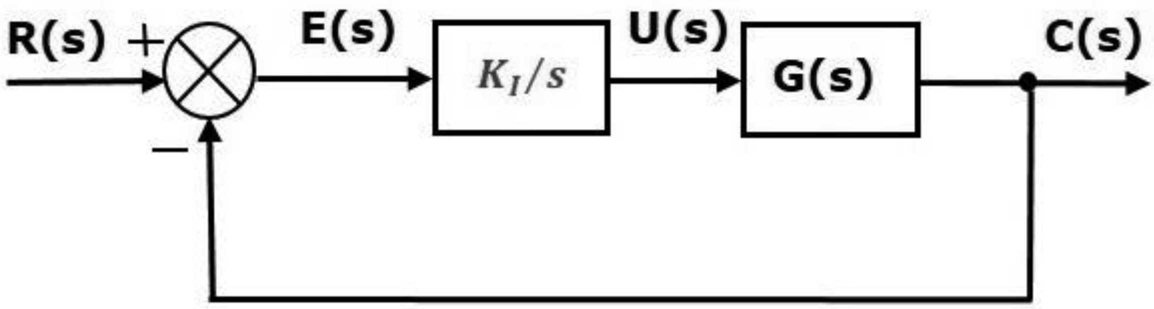
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where, KIKI is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.



The integral controller is used to decrease the steady state error.

Let us now discuss about the combination of basic controllers.

Proportional Derivative (PD) Controller

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

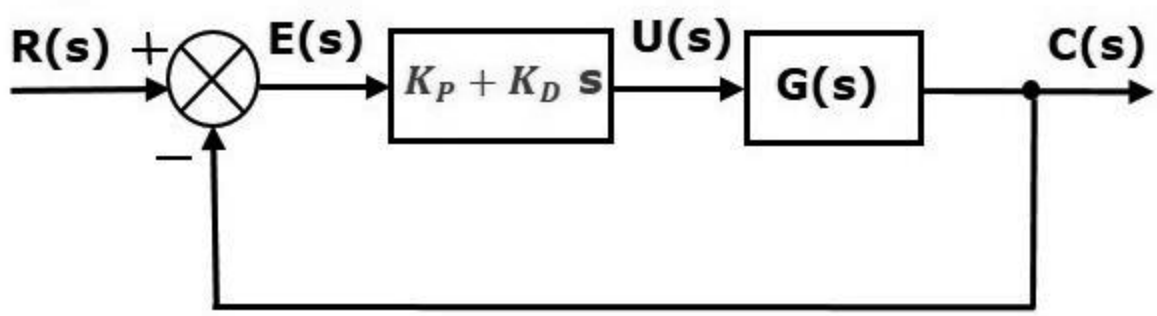
$$u(t) = K_P e(t) + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is K_P+K_Ds.
The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt$$

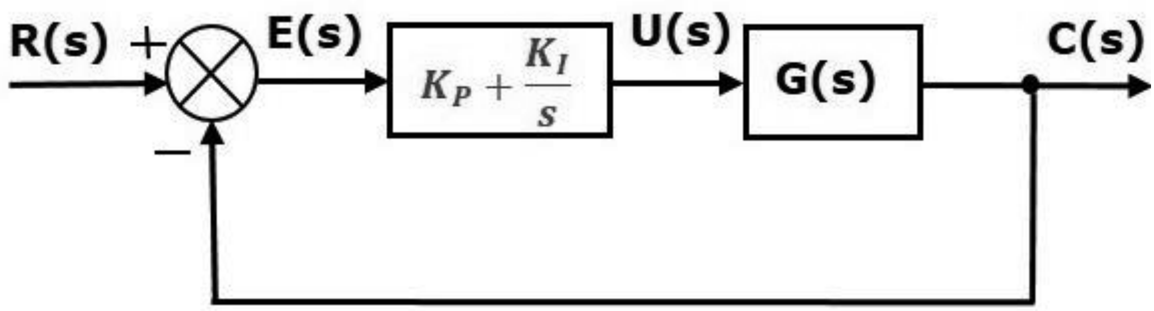
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s}\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral controller is shown in the following figure.



The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

Proportional Integral Derivative (PID) Controller

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t)dt + K_D \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s\right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.