UNIT- III

Data Transmission

* Gaussian probability density function is the most important density function among all the density functions in the field of science and Engineering. In electronics and communication systems, the distribution of noise signal, either internally generated or externally added, is exactly matches with Gaussian probability density function.

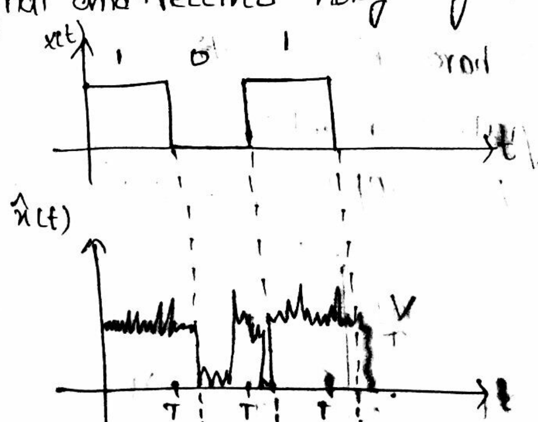
Noise signal is an unwanted quantity which is added to the information signal at each and every point in the signal processing. It is possible to eliminate the Noise completely by knowing its behaviour using Gaussian density function.

- white noise has flat power density spectrum in linear frequency space.
- pink noise is flat in logarithmic frequency space.
- brown noise has power density decreasing in logarithmic frequency space at 6 dB per octave rate.
- blue noise, purple noise, gray noise etc.

→ Noises occur in channel are

- Thermal Noise.
- shot Noise.

During the transmission of data over the channel, it is corrupted by Noise. Hence at the receiver, the noisy signal is received. Therefore correct detection of the transmitted signal is difficult. for example consider the transmitted signal and received noisy signal as shown in below figure.



(a) signal transmitted at the transmitter

(b) The received Noisy signal at the receiver.

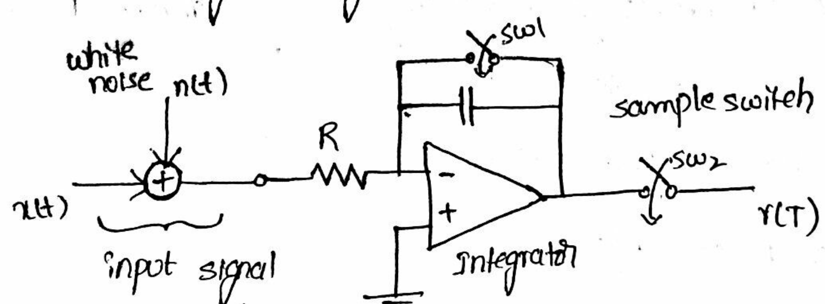
The received signal $\hat{x}(t)$ is a noisy signal at the receiver. Let us consider that, the detector checks $\hat{x}(t)$ at "T" during every bit interval.

In above figure the decision in first interval will be correct i.e symbol 1. At the time when detector checks $\hat{x}(t)$ noise pulse in second interval the decision will be 1 but it is wrong. Thus errors are introduced because of noise.

- The detection method should attenuate noise and amplify signal i.e it should improve signal to Noise ratio of the received signal.
- The detection method should check the received signal at the time instant in the bit interval when SNR is maximum.
- The detection should be performed with minimum error probability.

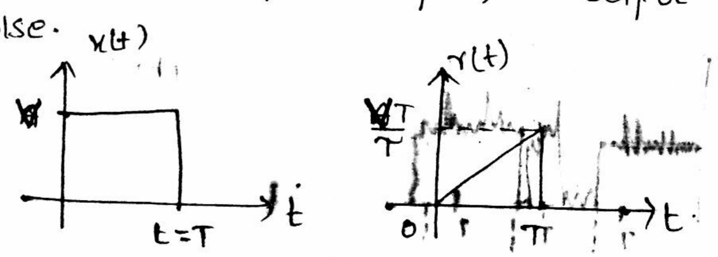
Base Band Signal Receiver:-

consider a very simple and basic detector for the detection of digital signals.



Noisy signal [x(t)+n(t)] is given to the input of integrator and Dump filter. The capacitor is discharged fully at the beginning of the bit interval. The term dump referring to the abrupt discharge of the capacitor after each sampling. The switch sw1 is closed temporarily at the beginning of the bit interval to discharge the capacitor.

The integrator then integrates noisy input signal over one bit period. for the square pulse input, the output of the integrator will be a triangular pulse.



At the end of the bit period i.e at $t=T$, the value of $r(t)$ reaches to its maximum amplitude. Therefore the value of $r(t)$ is sampled at the end of bit period. We will further prove that the signal to noise ratio is maximum at the end of bit period.

Depending upon the value of $r(T)$, the decision is taken. The dump switch sw_1 is then closed momentarily to discharge the capacitor to receive next bit.

Thus integrator generates output independent of the value of previous bit. The output of integrator will decrease after $t>T$.

Signal to Noise Ratio of the Integrator and Dump filter:-

The output of the integrator can be written as.

$$r(t) = \frac{1}{RC}\int_0^T [x(t)+n(t)]\,dt.$$

here the integration is performed over one bit period.

$$r(t) = \frac{1}{RC}\int_0^T x(t)\,dt + \frac{1}{RC}\int_0^T n(t)\,dt.$$

$$r(t) = x_0(t) + n_0(t).$$

where $x_0(t)$ = output signal voltage

$n_0(t)$ = output Noise.

$$x_0(t) = \frac{1}{RC}\int_0^T x(t)\,dt = \frac{1}{RC}\int_0^T V\,dt$$

$$= \frac{V}{RC}\,t\Big]_0^T = \frac{VT}{RC}$$

$$\boxed{x_0(t) = \frac{VT}{\tau}} \qquad (\because RC = \text{time constant} = \tau).$$

The Normalized power in standard 1Ω resistance will be,

$$\text{o/p signal power} = \frac{x_0^2(t)}{1\Omega}$$

$$\boxed{\text{o/p signal power} = \frac{V^2T^2}{\tau^2}}$$

- Noise power:-

The transfer function of integrator for 1 bit duration.

$$|H(f)| = \frac{|1-e^{-j\omega T}|}{|j\omega \tau|}$$

$$1-e^{-j\omega T} = 1-(\cos\omega T - j\sin\omega T)$$

$$= 1-\cos\omega T + j\sin\omega T$$

$$= 2\sin^2\omega T/2 + 2j\sin\omega T/2 \cos\omega T/2.$$

$$= 2\sin\omega T/2 \left[\sin\omega T/2 + j\cos\omega T/2\right].$$

$$|1-e^{-j\omega T}| = 2\sin\omega T/2$$

$$\therefore |H(f)| = \frac{2\sin\omega T/2}{\omega \tau}$$

output Noise power $= \int G_{no}(f)$

The output and input power spectral densities are related as

$$G_{no}(f) = G_{ni}(f)\,|H(f)|^2$$

$$G_{no}(f) = \frac{\eta}{2}\left[\frac{2\sin\omega T/2}{\omega\tau}\right]^2$$

put $\omega = 2\pi f$.

$$G_{no}(f) = \frac{\eta}{2}\left[\frac{2\sin\frac{2\pi f T}{2}}{2\pi f \tau}\right]^2$$

$$= \frac{\eta}{2}\left[\frac{\sin(\pi f T)}{\pi f \tau}\right]^2$$

$$= \frac{\eta}{2}\,\frac{\sin^2(\pi f T)}{(\pi f \tau)^2}$$

$$G_{no}(f) = \frac{\eta}{2}\,\frac{\sin^2(\pi f T)}{(\pi f T)^2}\cdot\left(\frac{T}{\tau}\right)^2.$$

$$\therefore \text{Noise power} = \int G_{no}(f)\,df.$$

$$= \int_0^T \frac{\eta}{2}\,\frac{\sin^2\pi f T}{(\pi f T)^2}\cdot\left(\frac{T}{\tau}\right)^2 df.$$

$$= \frac{\eta}{2}\left(\frac{T}{\tau}\right)^2 \int_0^T \frac{\sin^2\pi f T}{(\pi f T)^2}\,df.$$

Let $\pi f \tau = x$ $\qquad f \Rightarrow 0 \Rightarrow x \Rightarrow 0$

$df = \frac{dx}{\pi\tau}$ $\qquad f \to T \Rightarrow x = \pi T^2$

$$\therefore \text{Noise power} = \frac{\eta}{2}\left(\frac{I}{\tau}\right)^2 \int_0^{\pi T^2} \frac{\sin^2 x}{x^2} \cdot \frac{dx}{\pi T}.$$

$$= \frac{\eta}{2}\left(\frac{I}{\tau}\right)^2 \cdot \frac{1}{\pi T} \int_0^{\pi T^2} \frac{\sin^2 x}{x^2}\, dx.$$

$$= \frac{\eta}{2}\left(\frac{I}{\tau}\right)^2 \frac{1}{\pi T} \cdot \pi.$$

$$\text{Noise power} = \frac{\eta}{2} \cdot \frac{T}{\tau^2}.$$

$$[n_o(t)]^2 = \frac{\eta T}{2\tau^2} \Rightarrow N_o(t) = \frac{\sqrt{\eta}\sqrt{T}}{\sqrt{2}\,\tau}.$$

Signal to Noise power is

$$SNR = \frac{\text{Signal power}}{\text{Noise power}} = \frac{V^2 T^2}{\tau^2} \times \frac{2\tau^2}{\eta T}$$

$$= \frac{2V^2 T}{\eta}$$

$$\boxed{SNR = \frac{2}{\eta} V^2 T} \to \text{figure of Merit.}$$

__To keep SNR as constant:-__

The signal-to-Noise ratio. increases with increasing bit duration τ and that it depends on V^2T which is the Normalized energy of the bit signal, a bit represented by a narrow, high amplitude signal and one by a wide, low amplitude signal are equally effective, provided V^2T is kept constant. The Noise voltage is directly proportional to square root of time. $(\sqrt{T})$.

Probability of error:-

The function of a Receiver is to distinguish the bit '1' from the bit '0' in the presence of Noise, the most important characteristic is the probability that an error will be made in such a deterimination. The error probability P_e for the integrate-and-dump receiver is calculated as follows.

The probability density of the Noise sample $n_o(T)$ is gaussian. The probability density function of the gaussian distributed function is given by standard relation as.

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-(x-m)^2/2\sigma_0^2}.$$

where m is the mean value and

σ_0 is standard deviation.

here we want to evaluate PDF for white gaussian Noise,

$\therefore \quad x = n_o(t)$

white Gaussian Noise has zero mean value, $m=0$.

$$\therefore \quad f_x[n_o(t)] = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-[n_o(t)]^2/2\sigma_0^2}.$$

where σ_0^2 is variance.

$$\sigma_0^2 = \overline{n_o^2(t)}$$

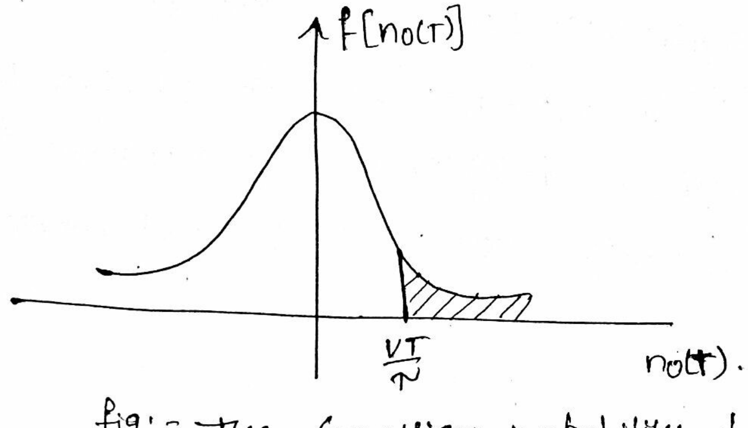


fig:- The Gaussian probability density of the Noise sample $n_o(T)$.

$$P_e = -\frac{1}{2}\left[\operatorname{erfc}(-4)\right]$$

$$\boxed{P_e = \frac{1}{2}\operatorname{erfc}\left[V\sqrt{\frac{T}{\eta}}\right]}$$

P_e decreases rapidly as E_s/η increases. The maximum value of P_e is $1/2$.

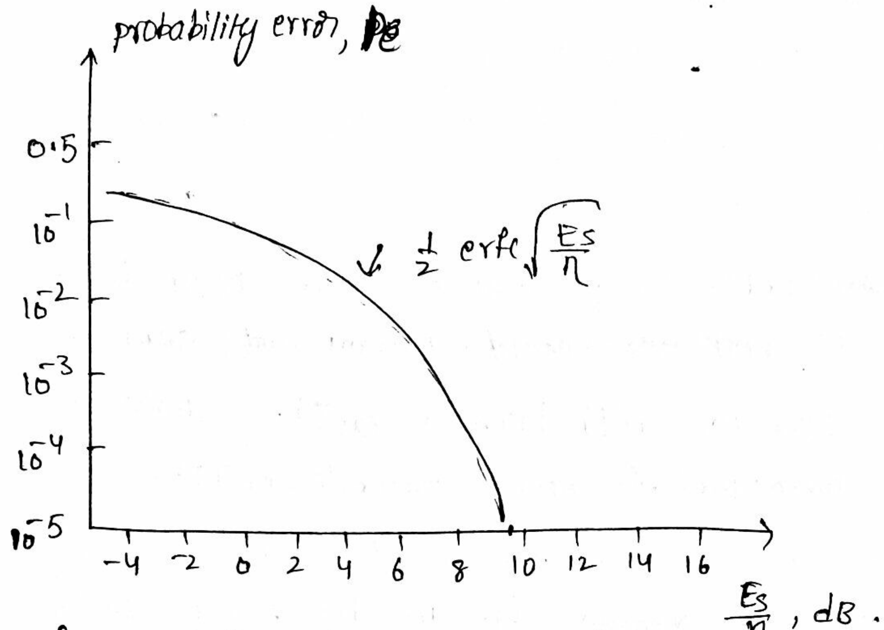


fig:- variation of P_e versus E_s/η.

Optimum filter Receiver for both BaseBand and passband :-

In the Receiver system, the signal was passed through a filter (i.e the integrator), so that at the sampling time the signal voltage might be emphasized in comparision with the noise voltage.

We assume that the received signal is a binary waveform. One binary digit (bit) is represented by a signal waveform $S_1(t)$ which persists for time T, while the other bit is represented by the waveform $S_2(t)$ which also persists for time T.

for example, in the case of transmission at the baseband $S_1(t) = +V$, while $S_2(t) = -V$. for other modulation systems different waveforms are transmitted. for example, for PSK signalling $S_1(t) = A\cos\omega_0 t$. and $S_2(t) = -A\cos\omega_0 t$.

As shown in the following figure the input, which is $S_1(t)$ or $S_2(t)$, is corrupted by the addition of noise $n(t)$. The noise is Gaussian and has a spectral density $G(f)$.

The signal and noise are filtered and then sampled at the end of each bit interval.

The output sample is either $V_o(T) = S_{o1}(T) + n_o(T)$ or

$$V_o(T) = S_{o2}(T) + n_o(T).$$

We assume that immediately after each sample, every energy-storing element in the filter has been discharged.

In the absence of noise the output sample would be $V_o(T) = S_{o1}(T)$ or $S_{o2}(T)$.

When noise is present we have shown that to minimize the probability of error one should assume that $S_1(t)$ has been transmitted if $V_o(T)$ closer to $S_{o1}(T)$ than to $S_{o2}(T)$. Similarly, we assume $S_2(t)$ has been transmitted if $V_o(T)$ is closer to $S_{o2}(T)$.

The decision boundary is therefore midway between $S_{o1}(T)$ and $S_{o2}(T)$. For example in the baseband system, where $S_{o1}(T) = \frac{VT}{\tau}$ and $S_{o2}(T) = -\frac{VT}{\tau}$, the decision boundary $V_o(T) = 0$.

In general, we shall take the decision boundary to be

$$\boxed{V_o(T) = \frac{S_{o1}(T) + S_{o2}(T)}{2}}$$

Gaussian noise, $n(t)$. spectral density, $G_n(f)$.

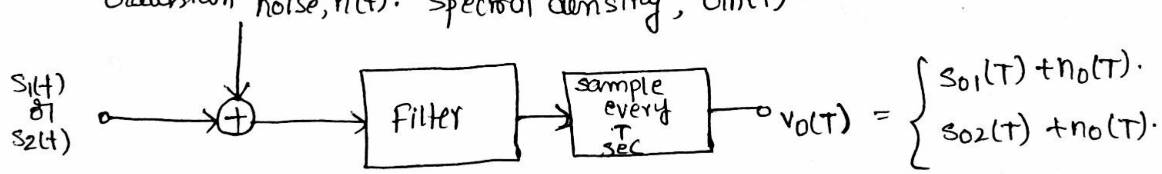


fig:- A Receiver for binary coded signalling.

Suppose that $S_{o1}(T) > S_{o2}(T)$ and that $S_2(t)$ was transmitted. If at the sampling time, the noise $n_o(T)$ is positive and larger in magnitude than the voltage difference $\frac{1}{2}[S_{o1}(T) + S_{o2}(T)] - S_{o2}(T)$, an error will have been made.

That is, an error [we decide that $S_1(t)$ is transmitted rather than $S_2(t)$] will result if

$$n_o(T) \geq \frac{S_{o1}(T) - S_{o2}(T)}{2}.$$

Hence the probability of error is

$$P_e = \int_{[s_{o1}(T)-s_{o2}(T)]/2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_o^2}} e^{-n_o^2(T)/2\sigma_o^2} \, dn_o(T).$$

Let $x = \dfrac{n_o(T)}{\sqrt{2}\sigma_o}$.

$$dx = \frac{dn_o(T)}{\sqrt{2}\sigma_o} \Rightarrow dn_o(T) = \sqrt{2}\sigma_o dx.$$

if $n_o(T) = \dfrac{s_{o1}(T)-s_{o2}(T)}{2} \Rightarrow x \Rightarrow \dfrac{s_{o1}(T)-s_{o2}(T)}{2\sqrt{2}\sigma_o}$

if $n_o(T) = \infty \Rightarrow x \to \infty$.

$$\therefore P_e = \frac{1}{\sqrt{\pi}} \int_{\frac{s_{o1}(T)-s_{o2}(T)}{2\sqrt{2}\sigma_o}}^{\infty} e^{-x^2} dx. = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{\frac{s_{o1}(T)-s_{o2}(T)}{2\sqrt{2}\sigma_o}}^{\infty} e^{-x^2} dx.$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{s_{o1}(T)-s_{o2}(T)}{2\sqrt{2}\sigma_o}\right].$$

P_e decreases as the difference $s_{o1}(T)-s_{o2}(T)$ becomes larger. and as the rms noise voltage σ_o becomes smaller.

The optimum filter, is the filter which maximizes the ratio.

$$\gamma = \frac{s_{o1}(T) - s_{o2}(T)}{\sigma_o}.$$

As a matter of mathematical convenience we shall actually maximize γ^2 rather than γ.

Calculation of the optimum-filter Transfer Function $H(f)$:-

The fondamental requirement of a binary encoded data receiver is that it distinguishes the voltages $s_1(t)+n(t)$ and $s_2(t)+n(t)$. The ability of the receiver depends on how large a particular receiver can make r.

r is proportional not to $s_1(t)$ nor to $s_2(t)$, but rather difference between them.

$s_1(t)$ & $s_2(t)$ is the received signal, the signal which is to be compared with the Noise, i.e. the signal which is relevant in all oor error-probability calculations, is the difference signal.

$$p(t) \equiv s_1(t) - s_2(t).$$

for the purpose of calculating the minimum error probability, we shall assume that the input signal to the optimum filter is $p(t)$. The corresponding output signal of the filter is then

$$P_o(t) \equiv S_{o1}(t) - S_{o2}(t).$$

we shall let $p(f)$ and $P_o(f)$ be the fourier transforms, respectively, of $p(t)$ and $P_o(t)$.

If $H(f)$ is the transfer function of the filter,

$$P_o(f) = H(f)\,p(f).$$

and $$P_o(T) = \int_{-\infty}^{\infty} P_o(f)\, e^{j2\pi f T}\, df = \int_{-\infty}^{\infty} H(f)\,p(f)\, e^{j2\pi f T}\, df.$$

The input noise to the optimum filter is $n(t)$. the output noise is $n_o(t)$ which has a power spectral density $G_{no}(f)$ and is related to the power spectral density of the input noise $G_n(f)$ by

$$G_{no}(f) = |H(f)|^2 G_n(f)$$

the Normalized output noise power, i.e. the noise variance σ_o^2 is

$$\sigma_o^2 = \int_{-\infty}^{\infty} G_{no}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df.$$

we now find that

$$r^2 = \frac{P_o^2(T)}{\sigma_o^2} = \frac{\left| \int_{-\infty}^{\infty} H(f)\,P(f)\, e^{j2\pi f T}\, df \right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df} \rightarrow (1)$$

The schwarz inequality states that given arbitrary complex functions $x(f)$ and $y(f)$ of a common variable f, then

$$\left| \int_{-\infty}^{\infty} x(f)\, y(f)\, df \right|^2 \leq \int_{-\infty}^{\infty} |x(f)|^2 df \int_{-\infty}^{\infty} |y(f)|^2 df.$$

the equal sign applies when

$$x(f) = K y^*(f).$$

where K is an arbitrary constant and $y^*(f)$ is the complex conjugate of $y(f)$.

- we now apply the schwarz inequality to equation (1) by making the identification.

$$x(f) = \sqrt{G_n(f)}\; H(f) \rightarrow (2)$$

$$y(f) = \frac{1}{\sqrt{G_n(f)}}\, P(f)\, e^{j2\pi f T} \rightarrow (3)$$

$$\gamma^2 = \frac{P_0^2(T)}{\sigma_0^2} = \frac{\left| \int_{-\infty}^{\infty} x(f)\, y(f)\, df \right|^2}{\int_{-\infty}^{\infty} |x(f)|^2 df} \leq \frac{\int_{-\infty}^{\infty} |x(f)|^2 df \int_{-\infty}^{\infty} |y(f)|^2 df}{\int_{-\infty}^{\infty} |x(f)|^2 df}$$

$$\leq \int_{-\infty}^{\infty} |y(f)|^2 df.$$

- $$\frac{P_0^2(T)}{\sigma_0^2} = \int_{-\infty}^{\infty} |y(f)|^2 df = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df. \text{ (from (3))} \rightarrow (4)$$

The ratio $P_0^2(T)/\sigma_0^2$ will attain its maximum value when the equal sign in equation (4) may be employed as in the case when $x(f) = K y^*(f)$.

from equation (2) & (3).

$$\sqrt{G_n(f)}\; H(f) = K \left(\frac{P(f)}{\sqrt{G_n(f)}}\, e^{j2\pi T f} \right)^*.$$

$$\boxed{H(f) = K\, \frac{P^*(f)}{G_n(f)}\, e^{-j2\pi T f}} \rightarrow (5).$$

- the maximum ratio is $\left[\frac{P_0^2(T)}{\sigma_0^2} \right]_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} df.$

Optimum Filter Realization using Matched Filter:-

An optimum filter which yields a maximum ratio $P_o^2(T)/\sigma_o^2$ is called a matched filter when the input Noise is white.

In this case $G_n(f) = \eta/2$.

W.K.T

$$H(f) = K \frac{P^*(f)}{G_n(f)} e^{-j2\pi fT}$$

$$\Rightarrow H(f) = K \frac{P^*(f)}{\eta/2} e^{-j2\pi fT}$$

The impulsive response of this filter, i.e. the response of the filter to a unit strength impulse applied at $t=0$ is

$$h(t) = F^{-1}[H(f)] = \frac{2K}{\eta}\int_{-\infty}^{\infty} P^*(f)\, e^{-j2\pi fT} e^{j2\pi ft}\, df.$$

$$h(t) = \frac{2K}{\eta}\int_{-\infty}^{\infty} P^*(f)\, e^{j2\pi f(t-T)}\, df.$$

A physically realizable filter will have an impulse response which is real, i.e not complex. Therefore $h(t) = h^*(t)$.

$$h(t) = \frac{2K}{\eta}\int_{-\infty}^{\infty} P(f)\, e^{j2\pi f(T-t)}\, df.$$

$$\boxed{h(t) = \frac{2K}{\eta} P(T-t)}$$

Finally, since $P(t) \equiv S_1(t) - S_2(t)$, we have

$$h(t) = \frac{2K}{\eta}\left[S_1(T-t) - S_2(T-t)\right]$$

Probability of error of the Matched filter:-

The probability of error which results when employing a matched filter, may be evaluating the maximum signal-to-noise ratio $[P_o^2(T)/\sigma_o^2]_{max}$ given by

$$\left[\frac{P_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{2}{\eta}\int_{-\infty}^{\infty} |P(f)|^2\, df \quad (\text{with } G_n(f) = \eta/2).$$

From parseval's theorem, we have

$$\int_{-\infty}^{\infty} |P(f)|^2\, df = \int_{-\infty}^{\infty} p^2(t)\,dt = \int_0^T p^2(t)\,dt$$

$$\left[\frac{P_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{2}{\eta}\int_0^T [S_1(t) - S_2(t)]^2\,dt$$

$$= \frac{2}{\eta}\left[\int_0^T S_1^2(t)\,dt + \int_0^T S_2^2(t)\,dt - 2\int_0^T S_1(t)S_2(t)\,dt\right]$$

$$\left[\frac{P_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{2}{\eta}\left[E_{S1} + E_{S2} - 2E_{S12}\right]$$

E_{S1} = Signal energy of $S_1(t)$

E_{S2} = Signal energy of $S_2(t)$

E_{S12} = energy due to correlation b/w $S_1(t)$ & $S_2(t)$.

Suppose that we have selected $S_1(t)$ and let $S_2(t)$ have an energy E_{S1}. Then it can be shown that if $S_2(t)$ is to have the same energy, the optimum choice of $S_2(t)$ is

$$S_2(t) = -S_1(t).$$

The choice is optimum in that it yields a maximum output signal $P_o^2(T)$ for a given signal energy.

letting $S_2(t) = -S_1(t)$, we find

$$E_{S1} = E_{S2} = -E_{S12} = E_S.$$

$$\therefore \quad r^2_{max} = \frac{2}{\eta}\left[E_s + E_s + 2E_s\right]$$

$$= \frac{2}{\eta}(4E_s)$$

$$\left[\frac{P_0^2(T)}{\sigma_0^2}\right]_{max} = \frac{8E_s}{\eta}.$$

using $P_0(T) = S_{01}(T) - S_{02}(T)$, we have

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{P_0(T)}{2\sqrt{2}\,\sigma_0}\right] = \frac{1}{2}\text{erfc}\left[\frac{P_0^2(T)}{8\sigma_0^2}\right]^{1/2}.$$

minimum error probability $(P_e)_{min}$ is given by

$$(P_e)_{min} = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{8}\left[\frac{P_0^2(T)}{\sigma_0^2}\right]_{max}\right\}^{1/2}$$

$$(P_e)_{min} = \frac{1}{2}\,\text{erfc}\left(\frac{E_s}{\eta}\right)^{1/2}$$

Optimum Filter Realization using Correlator:-

The input is a binary data waveforms $s_1(t)$ or $s_2(t)$ corrupted by noise $n(t)$. The bit length is T. The received signal plus noise $v_i(t)$ is multiplied by a locally generated waveform $s_1(t)-s_2(t)$. The output of the multiplier is passed through an integrator whose output is sampled at $t=T$.

Immediately after each sampling, at the beginning of each new bit interval, all energy-storing elements in the integrator are discharged. This type of receiver is called a correlator, since we are correlating the received signal and noise with the waveform $s_1(t)-s_2(t)$.

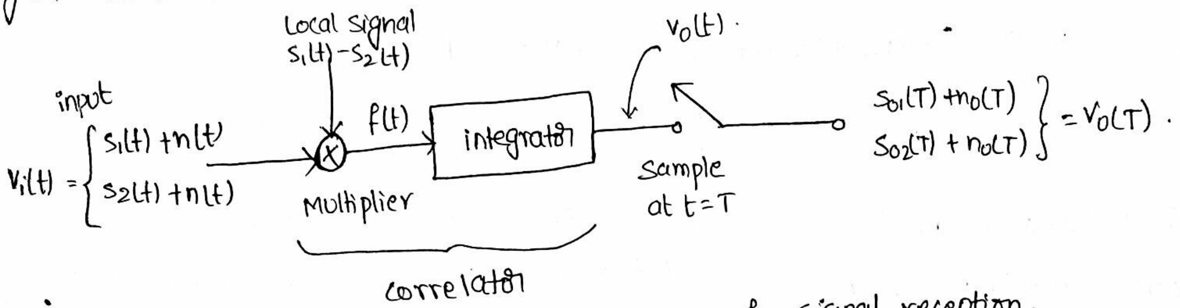


fig:- A coherent system of signal reception.

The output signal and noise of the correlator are

$$s_o(T) = \frac{1}{\tau}\int_0^T s_i(t)\left(s_1(t)-s_2(t)\right) dt \rightarrow ①$$

$$n_o(T) = \frac{1}{\tau}\int_0^T n(t)\left(s_1(t)-s_2(t)\right) dt \rightarrow ②$$

where $s_i(t)$ is either $s_1(t)$ or $s_2(t)$ and

τ is the Time constant of the integrator

(The integrator output is $1/\tau$ times the integral of its output).

We now compare these outputs with the matched filter outputs.

If $h(t)$ is the impulsive response of the matched filter, then the output of the matched filter $v_o(t)$ can be found using the convolution integral. we have

$$v_o(T) = \int_{-\infty}^{\infty} v_i(\lambda)\, h(t-\lambda)\, d\lambda = \int_0^T v_i(\lambda)\, h(t-\lambda)\, d\lambda.$$

The limits on the integral have been changed to 0 and T since we are interested in the filter response to a bit which extends only over that interval.

h(t) for the matched filter is given by.

$$h(t) = \frac{2K}{\eta}\left[s_1(T-t) - s_2(T-t)\right]$$

$$h(t-\lambda) = \frac{2K}{\eta}\left[s_1(T-t+\lambda) - s_2(T-t+\lambda)\right]$$

$$\therefore \quad v_o(t) = \frac{2K}{\eta}\int_0^T v_i(\lambda)\left[s_1(T-t+\lambda) - s_2(T-t+\lambda)\right] d\lambda$$

Since $v_i(\lambda) = s_i(\lambda) + n(\lambda)$, and $v_o(t) = s_o(t) + n(t)$

$$\therefore \; s_o(T) = \frac{2K}{\eta}\int_0^T s_i(\lambda)\left[s_1(\lambda) - s_2(\lambda)\right] d\lambda \rightarrow ③$$

where $s_i(\lambda)$ is equal to $s_1(\lambda)$ or $s_2(\lambda)$

$$n_o(t) = \frac{2K}{\eta}\int_0^T n(\lambda)\left[s_1(\lambda) - s_2(\lambda)\right] d\lambda \rightarrow ④$$

Thus $s_o(t)$ and $n_o(t)$ as calculated from equations ① and ② for the correlation receiver, and as calculated from equations ③ and ④ for the matched filter receiver, are identical. Hence the performances of the two systems are identical.

The matched filter and the correlator are not simply two distinct, independent techniques which happen to (give up) yield the same result. In fact they are two techniques of synthesizing the optimum filter h(t).

probability error of ASK:-

Let $s_1(t) = A\cos\omega_0 t$

$s_2(t) = 0$

$$\therefore \; p(t) = s_1(t) - s_2(t) = A\cos\omega_0 t - 0 = A\cos\omega_0 t.$$

$$\gamma^2_{max} = \frac{P_o^2(T)}{\sigma_o^2} = \int_0^T \frac{p^2(t)\,dt}{G_{ni}(f)}$$

$$P_o^2(T) = \int_0^T A^2\cos^2\omega_0 t = \frac{A^2}{2}\int_0^T 1+\cos 2\omega_0 t \, dt$$

$$= \frac{A^2}{2}\left[t + \frac{\sin 2\omega_0 t}{2\omega_0}\right]_0^T = \frac{A^2}{2}\left[T + \frac{\sin 2\omega_0 T}{2\omega_0}\right]$$

white Gaussian Noise $G_{ni}(f) = \eta/2$

$$\therefore \; r^2_{max} = \frac{\int_0^T p^2(t)\,dt}{\sigma_0^2} = \frac{\frac{A^2}{2}(T) + \frac{A^2}{4\omega_0}\sin(2\omega_0 T)}{\eta/2}$$

$$r^2_{max} = \frac{A^2}{\eta}\left[T + \frac{\sin(2\omega_0 T)}{2\omega_0}\right].$$

put $\omega_0 = \frac{2\pi}{T}$

$$r^2_{max} = \frac{A^2}{\eta}\left[T + \frac{\sin 4\pi}{4\pi/T}\right]$$

$$\boxed{r^2_{max} = \frac{A^2(T)}{\eta}} \quad (\because \sin 4\pi = 0).$$

$$\therefore \text{probability of error} = \frac{1}{2}\operatorname{erfc}\left(\frac{r}{2\sqrt{2}}\right)$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{r^2}{8}\right)^{1/2}$$

$$= \frac{1}{2}\operatorname{erfc}\left[\frac{1}{8}\cdot r^2_{max}\right]^{1/2}$$

$$= \frac{1}{2}\operatorname{erfc}\left[\frac{1}{8}\cdot\frac{A^2 T}{\eta}\right]^{1/2}$$

$$= \frac{1}{2}\operatorname{erfc}\left[\frac{1}{\cancel{8}4\eta}\,\frac{A^2 T}{2}\right]^{1/2}$$

$$\boxed{P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{E_s}{4\eta}\right]^{1/2}}$$

where $E_s = \frac{A^2}{2}\cdot T$

=o=

Error probability of Frequency shift keying:-

In frequency-shift keying (FSK) the received signal is either

$$S_1(t) = A\cos(\omega_0+\Omega)t$$

$$S_2(t) = A\cos(\omega_0-\Omega)t.$$

$$\gamma^2_{max} = \left[\frac{P_0^2(T)}{\sigma_0^2}\right]_{max} = \frac{2}{\eta}\int_0^T p^2(t)\,dt.$$

$$p(t) = S_1(t) - S_2(t)$$

$$\therefore \gamma^2_{max} = \frac{2}{\eta}\int_0^T [S_1(t)-S_2(t)]^2\,dt$$

$$= \frac{2}{\eta}\int_0^T [A\cos(\omega_0+\Omega)t - A\cos(\omega_0-\Omega)t]^2\,dt.$$

$$= \frac{2}{\eta}\int_0^T (A^2\cos^2(\omega_0+\Omega)t + A^2\cos^2(\omega_0-\Omega) - 2A^2\cos(\omega_0+\Omega)t\cos(\omega_0-\Omega)t]\,dt$$

$$\gamma^2_{max} = \frac{2A^2}{\eta}\int_0^T \left[\frac{1+\cos[2(\omega_0+\Omega)]t}{2} + \frac{1+\cos[2(\omega_0-\Omega)]t}{2} - 2\cos(\omega_0+\Omega)t\cos(\omega_0-\Omega)t\right]dt$$

$$2\cos(\omega_0+\Omega)t\cos(\omega_0-\Omega)t = \cos(\omega_0+\cancel{\Omega}+\omega_0-\cancel{\Omega})t + \cos(\cancel{\omega_0}+\Omega-\cancel{\omega_0}+\Omega)t$$

$$= \cos(2\omega_0)t + \cos(2\Omega)t$$

$$\therefore \gamma^2_{max} = \frac{2A^2}{\eta}\int_0^T \left[\frac{1}{2} + \frac{1}{2}\cos 2(\omega_0+\Omega)t + \frac{1}{2} + \frac{1}{2}\cos 2(\omega_0-\Omega)t - \cos(2\omega_0)t - \cos(2\Omega)t\right]dt.$$

$$= \frac{2A^2}{\eta}\left[\int_0^T (1)\,dt + \frac{1}{2}\int_0^T \cos 2(\omega_0+\Omega)t\,dt + \frac{1}{2}\int_0^T \cos 2(\omega_0-\Omega)t\,dt - \int_0^T (\cos 2\omega_0 t)\,dt - \int_0^T \cos(2\Omega t)\,dt\right.$$

$$= \frac{2A^2}{\eta}\left[[t]_0^T + \frac{1}{2}\left.\frac{\sin 2(\omega_0+\Omega)t}{2(\omega_0+\Omega)}\right]_0^T + \frac{1}{2}\left.\frac{\sin(2\omega_0-2\Omega)t}{(2\omega_0-2\Omega)}\right]_0^T - \left.\frac{\sin 2\omega_0 t}{2\omega_0}\right]_0^T - \left.\frac{\sin 2\Omega t}{2\Omega}\right]_0^T\right]$$

$$r_{max}^2 = \frac{2A^2}{\eta}\left[T + \frac{\sin 2(\omega_0+\Omega)T}{4(\omega_0+\Omega)} + \frac{\sin 2(\omega_0-\Omega)T}{4(\omega_0-\Omega)} - \frac{\sin 2\omega_0 T}{2\omega_0} - \frac{\sin 2\Omega T}{2\Omega}\right]$$

$$= \frac{2A^2}{\eta}\left[T - \frac{\sin 2\Omega T}{2\Omega}\right] \qquad \left(\because \omega_0 = \frac{2\pi}{T} \text{ then } \sin 2\pi = 0\right).$$

$$(\because \omega_0 >> \Omega).$$

$$r_{max}^2 = \frac{2A^2 T}{\eta}\left[1 - \frac{\sin 2\Omega T}{2\Omega T}\right].$$

. The quantity $[P_0^2(T)/\sigma_0^2]_{max}$ attains its largest value when Ω is selected so that $2\Omega T = 3\pi/2$

$$r_{max}^2 = \frac{2A^2 T}{\eta}\left[1 - \frac{\sin 3\pi/2}{3\pi/2}\right] = \frac{2A^2 T}{\eta}\left[1 + \frac{2}{3\pi}\right]$$

$$= 2.42\,\frac{A^2 T}{\eta} = 4.84\,\frac{(A^2/2)T}{\eta} = 4.84\,\frac{E_s}{\eta}.$$

$$P_e = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{8}\,r_{max}^2\right\}^{1/2}$$

$$= \frac{1}{2}\,\text{erfc}\left[\frac{1}{8}\cdot(4.84)\,\frac{E_s}{\eta}\right]^{1/2}$$

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(0.6\,\frac{E_s}{\eta}\right)^{1/2}}$$

where E_s = Signal energy

when one of two orthogonal frequencies are transmitted, $2\Omega T = m\pi$ (m is an integer) and

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left[\frac{E_s}{2\eta}\right]^{1/2}}$$

Error probability of phase shift keying:-

An important application of the coherent reception system is its use in phase-shift keying (PSK).

input signal is

$$s_1(t) = A\cos\omega_0 t$$

$$s_2(t) = -A\cos\omega_0 t.$$

$$s_1(t) - s_2(t) = A\cos\omega_0 t + A\cos\omega_0 t = 2A\cos\omega_0 t.$$

$$\gamma^2_{max} = \left[\frac{P_0^2(T)}{\sigma_0^2}\right]_{max} = \frac{2}{\eta}\int_0^T (s_1(t) - s_2(t))^2\, dt$$

$$= \frac{2}{\eta}\int_0^T (2A\cos\omega_0 t)^2\, dt = \frac{8A^2}{\eta}\int_0^T \cos^2\omega_0 t\, dt$$

$$= \frac{8A^2}{\eta}\int_0^T \left[\frac{1+\cos 2\omega_0 t}{2}\right] dt$$

$$= \frac{8A^2}{\eta}\left[\frac{t}{2} + \frac{\sin 2\omega_0 t}{4\omega_0}\right]_0^T$$

$$= \frac{8A^2}{\eta}\left[\frac{T}{2} + \frac{\sin 2\omega_0 T}{4\omega_0}\right] \qquad \left(\because \omega_0 = \frac{2\pi}{T},\ \sin 2\omega_0 T = 0\right)$$

$$= \frac{8A^2}{\eta} \cdot \frac{T}{2}$$

$$\gamma^2_{max} = \frac{4A^2 T}{\eta}.$$

$$\text{probability of error} = \frac{1}{2}\operatorname{erfc}\left[\frac{1}{8}\gamma^2_{max}\right]^{1/2}$$

$$= \frac{1}{2}\operatorname{erfc}\left[\frac{1}{\cancel{8}_2}\cdot\frac{\cancel{4}A^2 T}{\eta}\right]^{1/2}$$

$$= \frac{1}{2}\operatorname{erfc}\left[\frac{(A^2/2)\,T}{\eta}\right]^{1/2} = \frac{1}{2}\operatorname{erfc}\left[\frac{E_s}{\eta}\right]^{1/2}$$

$$\therefore \boxed{P_e = \frac{1}{2}\operatorname{erfc}\left[\frac{E_s}{\eta}\right]^{1/2}}$$

Imperfect phase synchronization:-

$$S_1(t) = A\cos\omega_0 t$$

$$S_2(t) = -A\cos\omega_0 t.$$

the required local waveform for the correlator is given by $S_1(t) - S_2(t) = 2A\cos\omega_0 t.$

when $S_1(t)$ is received, the output of the correlator at the sampling time $t=T$ is $S_{o1}(T)$, similarly when $S_2(t)$ is received the output is $S_{o2}(T)$.

Suppose now that the local signal used at the correlator were not $2A\cos\omega_0 t$ as required, but rather $2A\cos(\omega_0 t+\phi)$ where ϕ is some fixed phase offset.

$S_{o1}(T) = CA^2T$

$S_{o2}(T) = -CA^2T.$

at the sampling instant, the correlator output would becomes

$$S_o(T) = \pm 2CA^2T\cos\phi.$$

i.e the output signal is reduced, being multiplied by the factor ϕ. In this case energy becomes $E_s\cos^2\phi$ and probability of error can be written as

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{E_s}{\eta}\cos^2\phi\right]^{1/2}.$$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{A^2T}{2\eta}\cos^2\phi\right]^{1/2}.$$

* The phase shift ϕ increases the probability of error P_e.
* Typical error probabilities in a communication system range from 10^{-4} to 10^{-7}. In this range, if $\phi=25^\circ$ the probability of error is increased by a factor of 10 as compared with the result obtained for $\phi=0$.

If $\phi = 25^\circ$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{0.82\,A^2T}{2\eta}\right]^{1/2}$$

$\phi = 0^\circ$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{A^2T}{2\eta}\right]^{1/2}.$$

If $p(t) = 2A\cos(\omega_0 t + \phi)$.

then $S_{01}(T) = \frac{1}{\tau}\int_0^T s_1(t)p(t)\,dt$

$$= \frac{1}{\tau}\int_0^T A\cos\omega_0 t\,(2A\cos(\omega_0 t+\phi))\,dt$$

$$= \frac{2A^2}{\tau}\int_0^T \cos\omega_0 t\,(\cos(\omega_0 t+\phi))\,dt$$

$$= \frac{2A^2}{\tau}\cdot\frac{1}{2}\int_0^T 2\cos\omega_0 t\,\cos(\omega_0 t+\phi)\,dt.$$

$$= \frac{A^2}{\tau}\int_0^T [\cos(2\omega_0 t+\phi) + \cos\phi]\,dt.$$

$$= \frac{A^2}{\tau}\left.\frac{\sin(2\omega_0 t+\phi)}{2\omega_0}\right]_0^T + \frac{A^2}{\tau}\cos\phi\, t\Big]_0^T$$

$$S_{01}(T) = cA^2\left[\frac{\sin(2\omega_0 T+\phi)}{2\omega_0} - \frac{\sin\phi}{2\omega_0}\right] + cA^2\cos\phi\,(T).$$

$$= cA^2\left[\frac{\sin(4\pi+\phi)}{2\omega_0} - \frac{\sin\phi}{2\omega_0}\right] + cA^2T\cos\phi.$$

$$= cA^2\left[\frac{\sin\phi}{2\omega_0} - \frac{\sin\phi}{2\omega_0}\right] + cA^2T\cos\phi$$

$$\boxed{S_{01}(T) = cA^2T\cos\phi} \longrightarrow (1).$$

$$S_{02}(T) = \frac{1}{\tau}\int_0^T(-A\cos\omega_0 t)\,2A\cos(\omega_0 t+\phi)\,dt.$$

$$= \frac{-2A^2c}{2}\int_0^T[\cos(2\omega_0 t+\phi) + \cos\phi]\,dt.$$

$$= -A^2c\left.\frac{\sin(2\omega_0 t+\phi)}{2\omega_0}\right]_0^T + \cos\phi\, t\Big]_0^T.$$

$$= -A^2c\left[(0) + \cos\phi\, T\right].$$

$$\boxed{S_{02}(T) = -cA^2T\cos\phi}$$

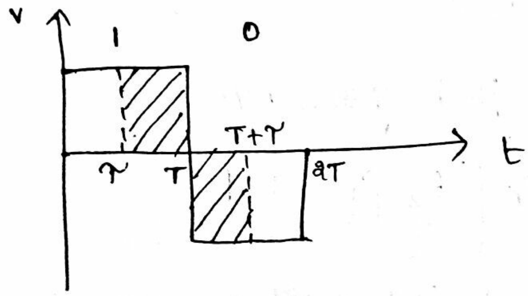
If ϕ increases the outputs $S_{01}(T)$ and $S_{02}(T)$ decreases.

Imperfect Bit synchronization:-

To calculate the probability of error the integration extend exactly over the duration of the bit i.e integration starts at $t=0$ and ends at $t=T$.

As a matter of practice either because of limitations in the synchronizer or because of the influence of noise, the integration may extend not from 0 to T but rather from τ to $T+\tau$.

If the two bits overlapped have the same logic value then again the overlap has no effect on the signal voltage $s_o(T+\tau)$. But, if the overlapped bits are different, the overlap will cause a reduction in $s_o(T+\tau)$.



$0 \to T \to s_1(t) = A\cos\omega_0 t$

$T \to 2T \to s_2(t) = -A\cos\omega_0 t$.

$$p(t) = s_1(t) - s_2(t) = 2A\cos\omega_0 t.$$

$$s_o(T+\tau) = \frac{2K}{\eta}\int_{\tau}^{T+\tau} s_1(t)\left[s_1(t) - s_2(t)\right] dt.$$

$$= \frac{2K}{\eta}\int_{\tau}^{T} s_1(t)\left(s_1(t) - s_2(t)\right) dt + \frac{2K}{\eta}\int_{T}^{T+\tau} s_2(t)\left(s_1(t) - s_2(t)\right) dt$$

$$s_o(T+\tau) = \frac{2K}{\eta}\int_{\tau}^{T} A\cos\omega_0 t\,(2A\cos\omega_0 t)\, dt + \frac{2K}{\eta}\int_{T}^{T+\tau} -A\cos\omega_0 t\,(2A\cos\omega_0 t)\, dt.$$

$$= \frac{2K}{\eta}\, 2A^2 \int_{\tau}^{T} \frac{1+\cos 2\omega_0 t}{2}\, dt + \frac{2K}{\eta}(-2A^2)\int_{T}^{T+\tau} \frac{1+\cos 2\omega_0 t}{2}\, dt$$

$$= \frac{2K}{\eta} A^2 \left[t + \frac{\sin 2\omega_0 t}{2\omega_0}\right]_{\tau}^{T} - \frac{2K}{\eta} A^2 \left[t + \frac{\sin 2\omega_0 t}{2\omega_0}\right]_{T}^{T+\tau}.$$

$$= \frac{2K}{\eta} A^2 \left[T + \frac{\sin 2\omega_0 T}{2\omega_0} - \tau - \frac{\sin 2\omega_0 \tau}{2\omega_0}\right] -$$

$$\frac{2K}{\eta} A^2 \left[T+\tau + \frac{\sin(2\omega_0(T+\tau))}{2\omega_0} - T - \frac{\sin 2\omega_0 T}{2\omega_0}\right].$$

$$S_0(T+\tau) = \frac{2K}{\eta} A^2 \left[T-\tau - \frac{\sin 2\omega_0\tau}{2\omega_0}\right] - \frac{2K}{\eta} A^2 \left[\cancel{T}+\tau + \frac{\sin 2\omega_0\tau}{2\omega_0} - \cancel{f}\right]$$

$$= \frac{2K}{\eta} A^2 (T-\tau) - \frac{2K}{\eta} A^2 (\tau).$$

$$= \frac{2K}{\eta} A^2 T \left[1 - \frac{\tau}{T} - \frac{\tau}{T}\right].$$

$$S_0(T+\tau) = \frac{2K}{\eta} A^2 T \left[1 - \frac{2\tau}{T}\right].$$

It is readily verified that if the overlap is in the other direction, i.e the integration extends from $-\tau$ to $T-\tau$

$$\therefore \quad S_0(T+\tau) = \frac{2K}{\eta} [A^2 T] \left[1 - \frac{2|\tau|}{T}\right]:$$

$\therefore$ probability of error P_e is given by.

$$P_e = \frac{1}{2} \operatorname{erfc} \left[\frac{E_s}{\eta} \left[1 - \frac{2|\tau|}{T}\right]^2\right]^{1/2}.$$

If both phase error and timing error are present then P_e becomes.

$$P_e = \frac{1}{2} \operatorname{erfc} \left[\left(\frac{E_s}{\eta}\right) \cos^2\phi \left(1 - \frac{2|\tau|}{T}\right)^2\right]^{1/2}.$$