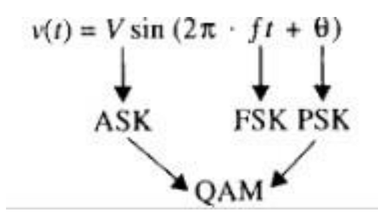
UNIT-2

DIGITAL MODULATION TECHNIQUES

Digital Modulation provides more information capacity, high data security, quicker system availability with great quality communication. Hence, digital modulation techniques have a greater demand, for their capacity to convey larger amounts of data than analog ones.

There are many types of digital modulation techniques and we can even use a combination of these techniques as well. In this chapter, we will be discussing the most prominent digital modulation techniques.

if the information signal is digital and the amplitude (lV of the carrier is varied proportional to the information signal, a digitally modulated signal called amplitude shift keying (ASK) is produced.
If the frequency (f) is varied proportional to the information signal, frequency shift keying (FSK) is produced, and if the phase of the carrier (0) is varied proportional to the information signal,
phase shift keying (PSK) is produced. If both the amplitude and the phase are varied proportional to the information signal, quadrature amplitude modulation (QAM) results. ASK, FSK, PSK, and QAM are all forms of digital modulation:



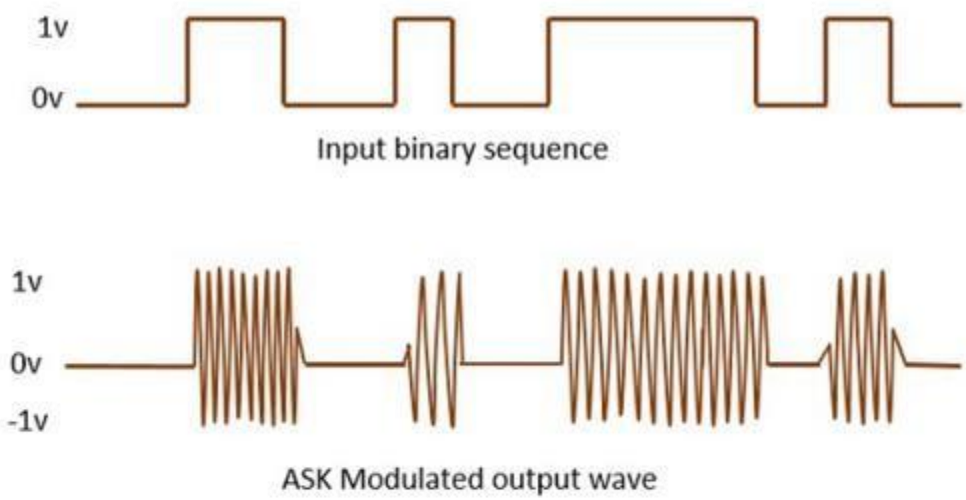
a simplified block diagram for a digital modulation system.

Amplitude Shift Keying

The amplitude of the resultant output depends upon the input data whether it should be a zero level or a variation of positive and negative, depending upon the carrier frequency.

Amplitude Shift Keying (ASK) is a type of Amplitude Modulation which represents the binary data in the form of variations in the amplitude of a signal.

Following is the diagram for ASK modulated waveform along with its input.



Any modulated signal has a high frequency carrier. The binary signal when ASK is modulated, gives a zero value for LOW input and gives the carrier output for HIGH input.
Mathematically, amplitude-shift keying is

$$v_{(ask)}(t) = [1 + v_m(t)]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

where vask(t) = amplitude-shift keying wave

vm(t) = digital information (modulating) signal (volts)

A/2 = unmodulated carrier amplitude (volts)

ωc= analog carrier radian frequency (radians per second, 2πfct)

In above Equation, the modulating signal [vm(t)] is a normalized binary waveform, where + 1 V = logic 1 and -1 V = logic 0. Therefore, for a logic 1 input, vm(t) = + 1 V, Equation 2.12 reduces to

$$v_{(ask)}(t) = [1 + 1]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

$$= A\cos(\omega_c t)$$

Mathematically, amplitude-shift keying is (2.12) where vask(t) = amplitude-shift keying wave vm(t) = digital information (modulating) signal (volts) A/2 = unmodulated carrier amplitude (volts) ωc= analog carrier radian frequency (radians per second, 2πfct) In Equation 2.12, the modulating signal [vm(t)] is a normalized binary waveform, where + 1 V = logic 1 and -1 V = logic 0. Therefore, for a logic 1 input, vm(t) = + 1 V, Equation 2.12 reduces to and for a logic 0 input, vm(t) = -1 V,Equation reduces to

$$v_{(ask)}(t) = [1 - 1]\left[\frac{A}{2}\cos(\omega_c t)\right]$$

Thus, the modulated wave vask(t),is either A cos(ωct) or 0. Hence, the carrier is either "on "or "off," which is why amplitude-shift keying is sometimes referred to as on-off keying (OOK).
it can be seen that for every change in the input binary data stream, there is one change in the ASK waveform, and the time of one bit (tb) equals the time of one analog signaling element (t,).
B = fb/1 = fb baud = fb/1 = fb

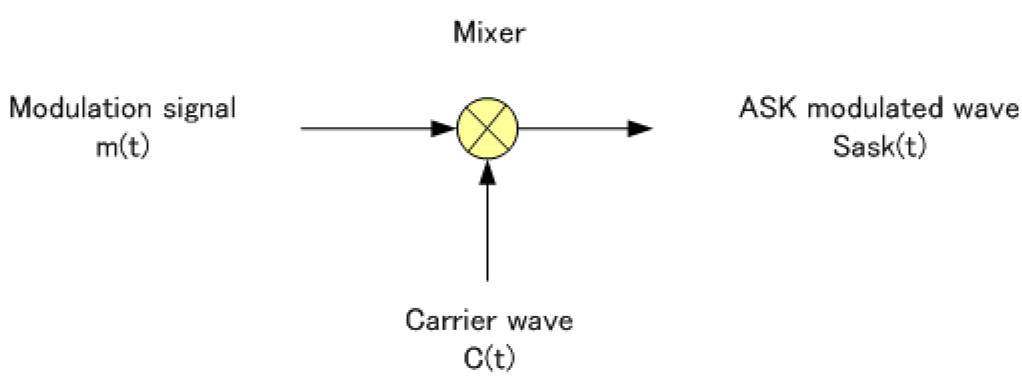
Example :
Determine the baud and minimum bandwidth necessary to pass a 10 kbps binary signal using amplitude shift keying. 10Solution For ASK, N = 1, and the baud and minimum bandwidth are determined from Equations 2.11 and 2.10, respectively:

B = 10,000 / 1 = 10,000

baud = 10, 000 /1 = 10,000

The use of amplitude-modulated analog carriers to transport digital information is a relatively low-quality, low-cost type of digital modulation and, therefore, is seldom used except for very low-speed telemetry circuits.

ASK TRANSMITTER:



The input binary sequence is applied to the product modulator. The product modulator amplitude modulates the sinusoidal carrier .it passes the carrier when input bit is ‘1’ .it blocks the carrier when input bit is ‘0.’

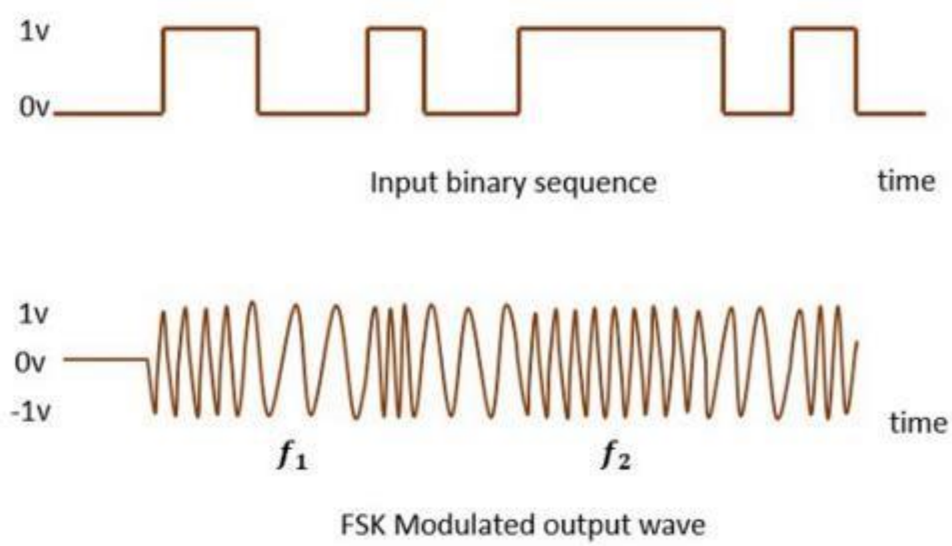
Coherent ASK DETECTOR:

FREQUENCYSHIFT KEYING

The frequency of the output signal will be either high or low, depending upon the input data applied.

Frequency Shift Keying (FSK) is the digital modulation technique in which the frequency of the carrier signal varies according to the discrete digital changes. FSK is a scheme of frequency modulation.

Following is the diagram for FSK modulated waveform along with its input.



The output of a FSK modulated wave is high in frequency for a binary HIGH input and is low in frequency for a binary LOW input. The binary 1s and 0s are called **Mark** and **Space frequencies**.

FSK is a form of constant-amplitude angle modulation similar to standard frequency modulation (FM) except the modulating signal is a binary signal that varies between two discrete voltage levels rather than a continuously changing analog waveform.Consequently, FSK is sometimes called *binary FSK* (BFSK). The general expression for FSK is where $v_{fsk}(t) = V_c \cos\{2\pi[f_c + v_m(t)\,\Delta f]t\}$

$v_{fsk}(t)$ = binary FSK waveform

V_c = peak analog carrier amplitude (volts)

f_c = analog carrier center frequency (hertz)

f=peak change (shift)in the analog carrier frequency(hertz)

$v_m(t)$ = binary input (modulating) signal (volts)

From Equation 2.13, it can be seen that the peak shift in the carrier frequency (f) is proportional to the amplitude of the binary input signal (v_m[t]), and the direction of the shift is determined by the polarity.

The modulating signal is a normalized binary waveform where a logic 1 = + 1 V and a logic 0 = -1 V. Thus, for a logic l input, $v_m(t)$ = + 1, Equation 2.13 can be rewritten as

$$v_{fsk}(t) = V_c \cos[2\pi(f_c + \Delta f)t]$$

For a logic 0 input, $v_m(t)$ = -1, Equation becomes

$$v_{fsk}(t) = V_c \cos[2\pi(f_c - \Delta f)t]$$

With binary FSK, the carrier center frequency (f_c) is shifted (deviated) up and down in the frequency domain by the binary input signal as shown in Figure 2-3.

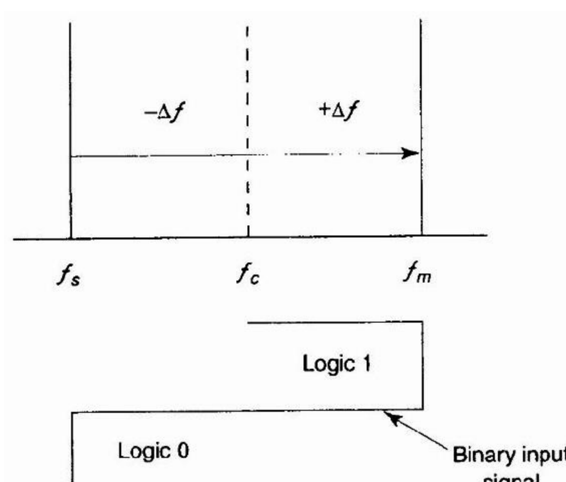


FIGURE: FSK in the frequency domain

As the binary input signal changes from a logic 0 to a logic 1 and vice versa, the output frequency shifts between two frequencies: a mark, or logic 1 frequency (f_m), and a space, or logic 0 frequency (f_s). The mark and space frequencies are separated from the carrier frequency by the peak frequency deviation (f) and from each other by 2 f.

Frequency deviation is illustrated in Figure 2-3 and expressed mathematically as

$$f = |f_m - f_s| / 2 \qquad (2.14)$$

where f = frequency deviation (hertz)

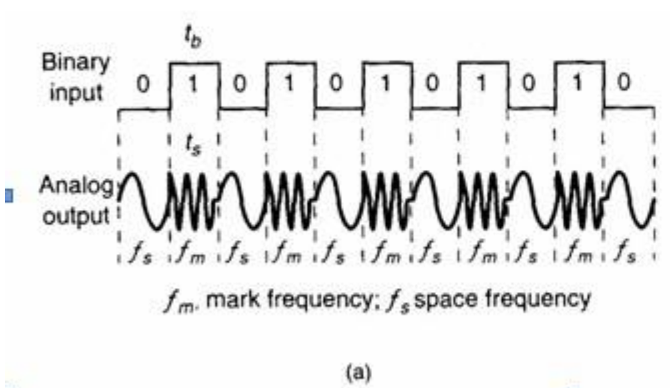
$|f_m - f_s|$ = absolute difference between the mark and space frequencies (hertz)

Figure 2-4a shows in the time domain the binary input to an FSK modulator and the corresponding FSK output.

When the binary input *(f_b)* changes from a logic 1 to a logic 0 and vice versa, the FSK output frequency shifts from a mark (f_m) to a space (f_s) frequency and vice versa.

In Figure 2-4a, the mark frequency is the higher frequency *(f_c* + f) and the space frequency is the lower frequency (f_c - f), although this relationship could be just the opposite.

Figure 2-4b shows the truth table for a binary FSK modulator. The truth table shows the input and output possibilities for a given digital modulation scheme.



binary input	frequency output
0	space (f_s)
1	mark (f_m)

(b)

FIGURE 2-4 FSK in the time domain: (a) waveform: (b) truth table

FSK Bit Rate, Baud, and Bandwidth

In Figure 2-4a, it can be seen that the time of one bit *(t_b)* is the same as the time the FSK output is a mark of space frequency *(t_s)*. Thus, the bit time equals the time of an FSK signaling element, and the bit rate equals the baud.

The baud for binary FSK can also be determined by substituting N = 1 in Equation 2.11:

baud = $f_b / 1 = f_b$

The minimum bandwidth for FSK is given as

$$B= |(f_s - f_b) - (f_m - f_b)|$$

$$=|(f_s - f_m)| + 2f_b$$

and since $|(f_s - f_m)|$ equals 2 f, the minimum bandwidth can be approximated as

$$B= 2(f + f_b) \qquad (2.15)$$

where

B= minimum Nyquist bandwidth (hertz)

f= frequency deviation $|(f_m - f_s)|$ (hertz)

f_b = input bit rate (bps)

Example 2-2

Determine (a) the peak frequency deviation, (b) minimum bandwidth, and (c) baud for a binary FSK signal with a mark frequency of 49 kHz, a space frequency of 51 kHz, and an input bit rate of 2 kbps.

Solution

a. The peak frequency deviation is determined from Equation 2.14:

$$f= |149kHz - 51 kHz| / 2 =1 kHz$$

b. The minimum bandwidth is determined from Equation 2.15:

$$B = 2(100+ 2000)$$

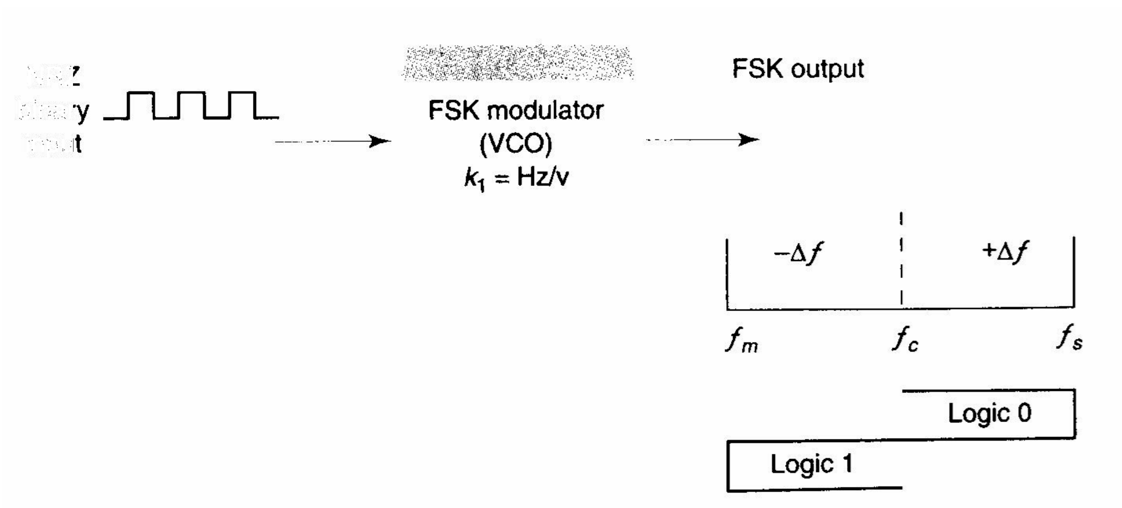
$$=6 kHz$$

c. For FSK, N = 1, and the baud is determined from Equation 2.11 as

$$\text{baud} = 2000 / 1 = 2000$$

FSK TRANSMITTER:

Figure 2-6 shows a simplified binary FSK modulator, which is very similar to a conventional FM modulator and is very often a voltage-controlled oscillator (VCO).The center frequency (f_c) is chosen such that it falls halfway between the mark and space frequencies.



A logic 1 input shifts the VCO output to the mark frequency, and a logic 0 input shifts the VCO output to the space frequency. Consequently, as the binary input signal changes back and forth between logic 1 and logic 0 conditions, the VCO output shifts or deviates back and forth between the mark and space frequencies.

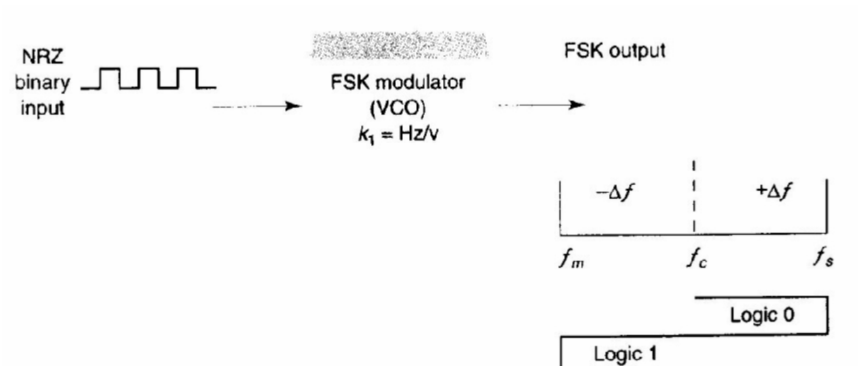


FIGURE 2-6 FSK modulator

A VCO-FSK modulator can be operated in the sweep mode where the peak frequency deviation is simply the product of the binary input voltage and the deviation sensitivity of the VCO.

With the sweep mode of modulation, the frequency deviation is expressed mathematically as

$$f = v_m(t)k_l \qquad (2\text{-}19)$$

$v_m(t)$ = peak binary modulating-signal voltage (volts)

k_l = deviation sensitivity (hertz per volt).

FSK Receiver

FSK demodulation is quite simple with a circuit such as the one shown in Figure 2-7.

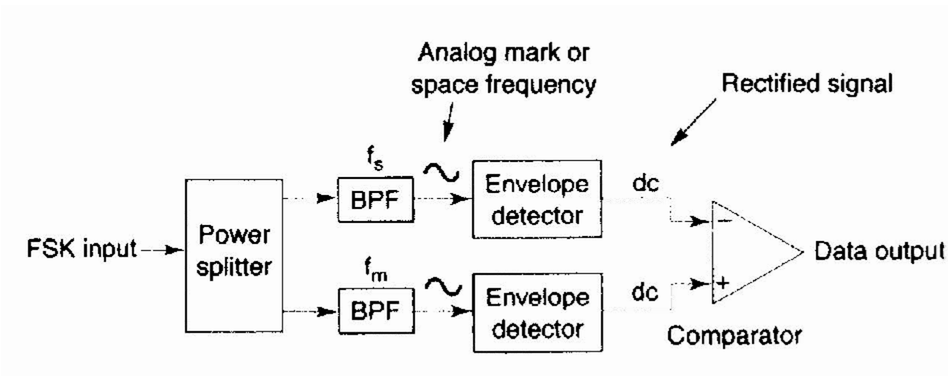


FIGURE 2-7 Noncoherent FSK demodulator

The FSK input signal is simultaneously applied to the inputs of both bandpass filters (BPFs) through a power splitter.The respective filter passes only the mark or only the space frequency on to its respective envelope detector.The envelope detectors, in turn, indicate the total power in each passband, and the comparator responds to the largest of the two powers.This type of FSK detection is referred to as noncoherent detection.

Figure 2-8 shows the block diagram for a coherent FSK receiver.The incoming FSK signal is multiplied by a recovered carrier signal that has the exact same frequency and phase as the transmitter reference.

However, the two transmitted frequencies (the mark and space frequencies) are not generally continuous; it is not practical to reproduce a local reference that is coherent with both of them. Consequently, coherent FSK detection is seldom used.

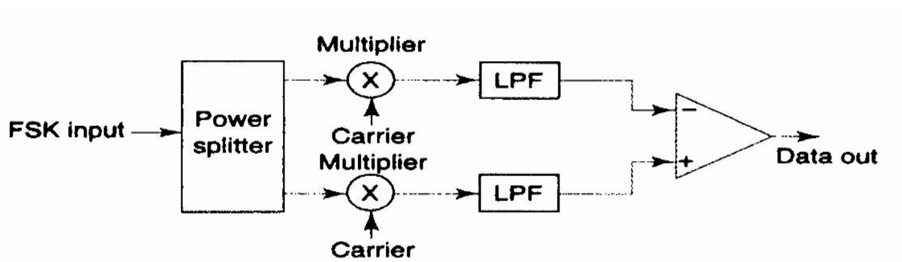
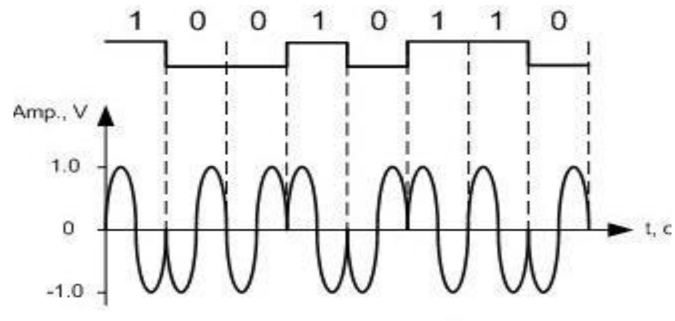


FIGURE 2-8 Coherent FSK demodulator

PHASESHIFT KEYING:

The phase of the output signal gets shifted depending upon the input. These are mainly of two types, namely BPSK and QPSK, according to the number of phase shifts. The other one is DPSK which changes the phase according to the previous value.



Phase shift keying (PSK)

Phase Shift Keying (PSK) is the digital modulation technique in which the phase of the carrier signal is changed by varying the sine and cosine inputs at a particular time. PSK technique is widely used for wireless LANs, bio-metric, contactless operations, along with RFID and Bluetooth communications.

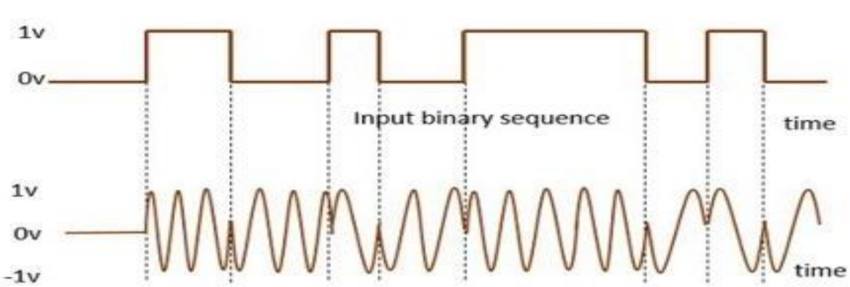
PSK is of two types, depending upon the phases the signal gets shifted. They are −

Binary Phase Shift Keying (BPSK)

This is also called as **2-phase PSK** (or) **Phase Reversal Keying**. In this technique, the sine wave carrier takes two phase reversals such as 0° and 180°.

BPSK is basically a DSB-SC (Double Sideband Suppressed Carrier) modulation scheme, for message being the digital information.

Following is the image of BPSK Modulated output wave along with its input.



BPSK Modulated output wave

Binary Phase-Shift Keying

The simplest form of PSK is *binary phase-shift keying* (BPSK), where N = 1 and *M* = *2.*Therefore, with BPSK, two phases ($2^1 = 2$) are possible for the carrier.One phase represents a logic 1, and the other phase represents a logic 0. As the input digital signal changes state (i.e., from a 1 to a 0 or from a 0 to a 1), the phase of the output carrier shifts between two angles that are separated by 180°.

Hence, other names for BPSK are *phase reversal keying* (PRK) and *biphase modulation.* BPSK is a form of square-wave modulation of a *continuous wave (CW)* signal.

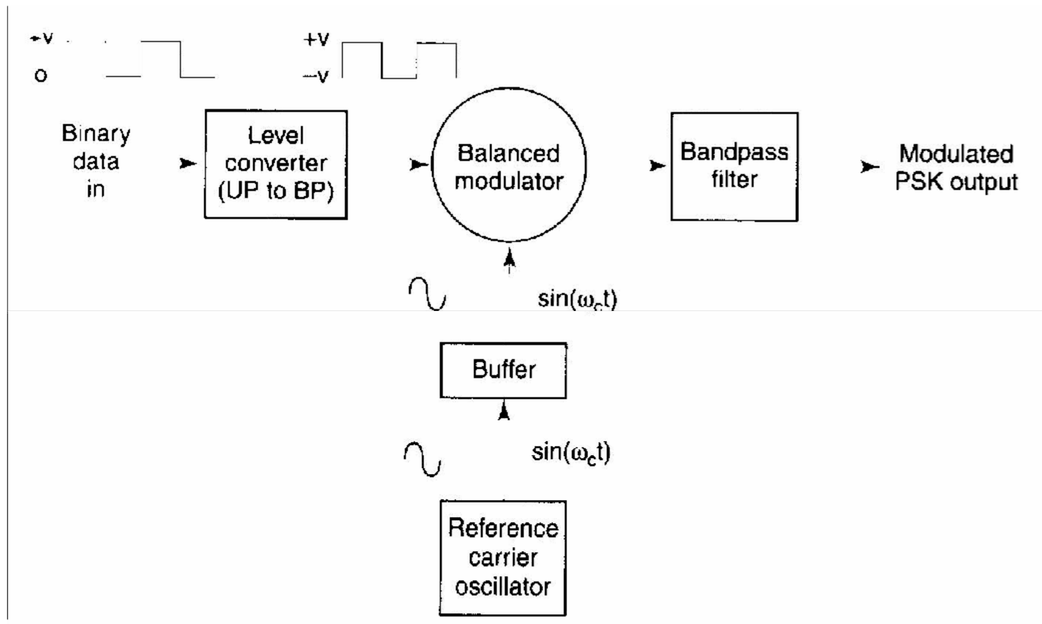


FIGURE 2-12 BPSK transmitter

BPSK TRANSMITTER:

Figure 2-12 shows a simplified block diagram of a BPSK transmitter. The balanced modulator acts as a phase reversing switch. Depending on the logic condition of the digital input, the carrier is transferred to the output either in phase or 180° out of phase with the reference carrier oscillator.

Figure 2-13 shows the schematic diagram of a balanced ring modulator. The balanced modulator has two inputs: a carrier that is in phase with the reference oscillator and the binary digital data. For the balanced modulator to operate properly, the digital input voltage must be much greater than the peak carrier voltage.

This ensures that the digital input controls the on/off state of diodes D1 to D4. If the binary input is a logic 1(positive voltage), diodes D 1 and D2 are forward biased and on, while diodes D3 and D4 are reverse biased and off (Figure 2-13b). With the polarities shown, the carrier voltage is developed across transformer T2 in phase with the carrier voltage across T

1. Consequently, the output signal is in phase with the reference oscillator.
If the binary input is a logic 0 (negative voltage), diodes Dl and D2 are reverse biased and off, while diodes D3 and D4 are forward biased and on (Figure 9-13c). As a result, the carrier voltage is developed across transformer T2 180° out of phase with the carrier voltage across T 1.

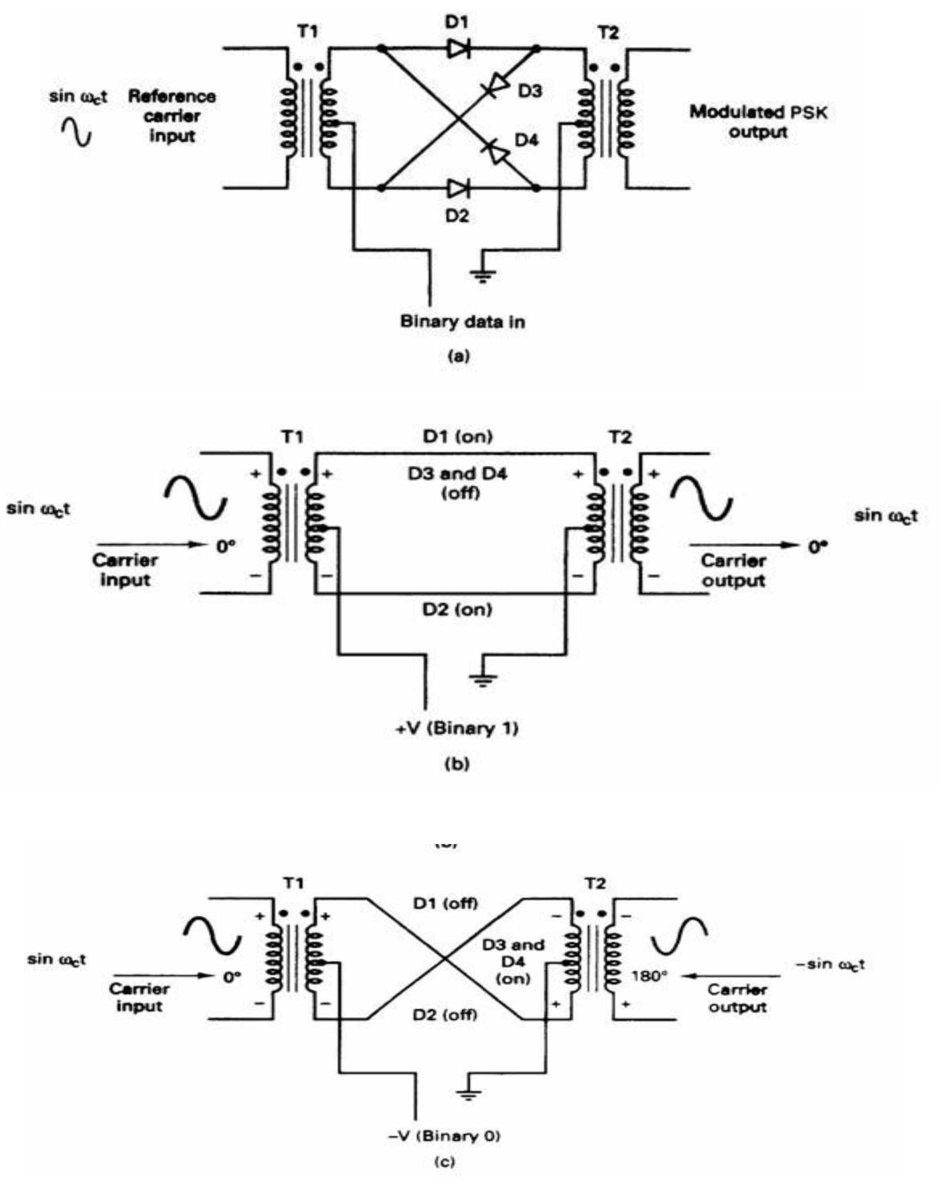


FIGURE 9-13 (a) Balanced ring modulator; (b) logic 1 input; (c) logic 0 input

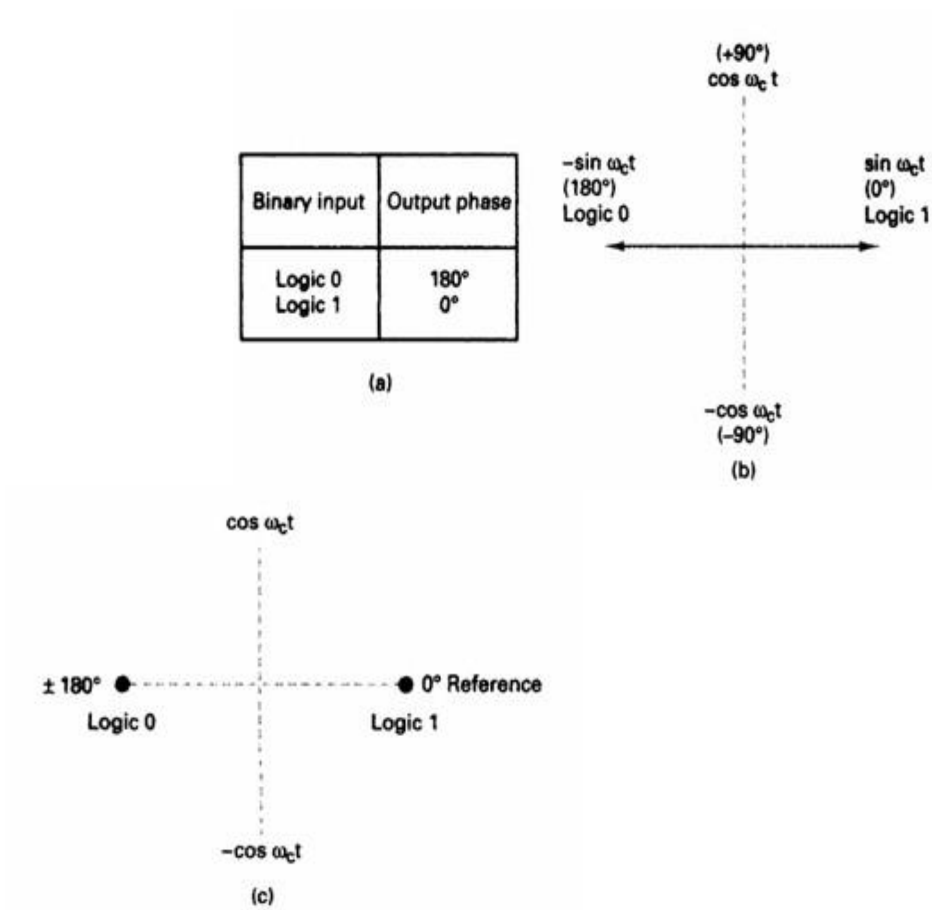


FIGURE 2-14 BPSK modulator: (a) truth table; (b) phasor diagram; (c) constellation diagram

BANDWIDTH CONSIDERATIONS OF BPSK:

In a BPSK modulator. the carrier input signal is multiplied by the binary data.

If + 1 V is assigned to a logic 1 and -1 V is assigned to a logic 0, the input carrier (sin $\omega_c t$) is multiplied by either a + or - 1 .

The output signal is either + 1 sin $\omega_c t$ or -1 sin $\omega_c t$ the first represents a signal that is *in phase* with the reference oscillator, the latter a signal that is 180° out of phase with the reference oscillator.Each time the input logic condition changes, the output phase changes.

Mathematically, the output of a BPSK modulator is proportional to

$$\text{BPSK output} = [\sin (2\pi f_a t)] \times [\sin (2\pi f_c t)] \qquad (2.20)$$

where

f_a = maximum fundamental frequency of binary input (hertz)

f_c = reference carrier frequency (hertz)

Solving for the trig identity for the product of two sine functions,

$0.5\cos[2\pi(f_c - f_a)t] - 0.5\cos[2\pi(f_c + f_a)t]$

Thus, the minimum double-sided Nyquist bandwidth *(B) is*

$$f_c + f_a \qquad\qquad f_c + f_a$$

$$-(f_c + f_a) \quad \text{or} \quad \underline{-f_c + f_a}$$

$$2f_a$$

and because $f_a = f_b / 2$, where f_b = input bit rate,

where *B is* the minimum double-sided Nyquist bandwidth.

Figure 2-15 shows the output phase-versus-time relationship for a BPSK waveform. Logic 1 input produces an analog output signal with a 0° phase angle, and a logic 0 input produces an analog output signal with a 180° phase angle.

As the binary input shifts between a logic 1 and a logic 0 condition and vice versa, the phase of the BPSK waveform shifts between 0° and 180°, respectively.

BPSK signaling element *(t_s)* is equal to the time of one information bit *(t_b)*, which indicates that the bit rate equals the baud.

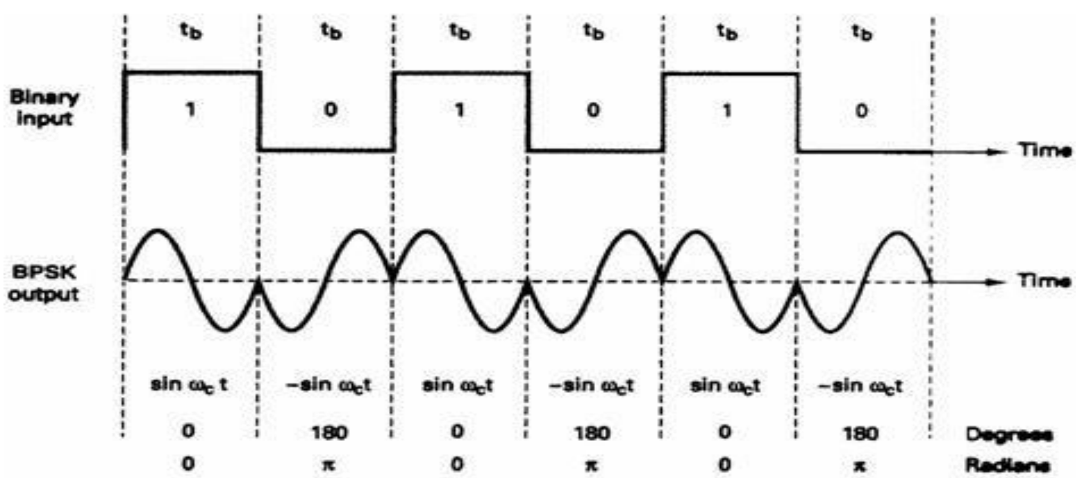


FIGURE 2-15 Output phase-versus-time relationship for a BPSK modulator

Example:

For a BPSK modulator with a carrier frequency of 70 MHz and an input bit rate of 10 Mbps, determine the maximum and minimum upper and lower side frequencies, draw the output spectrum, de-termine the minimum Nyquist bandwidth, and calculate the baud..

Solution

Substituting into Equation 2-20 yields

$$\text{output} = [\sin(2\pi f_a t)] \times [\sin(2\pi f_c t)];\ f_a = f_b / 2 = 5\ \text{MHz}$$

$$= [\sin 2\pi(5\text{MHz})t)] \times [\sin 2\pi(70\text{MHz})t)]$$

$$= 0.5\cos[2\pi(70\text{MHz} - 5\text{MHz})t] - 0.5\cos[2\pi(70\text{MHz} + 5\text{MHz})t]$$

lower side frequency upper side frequency

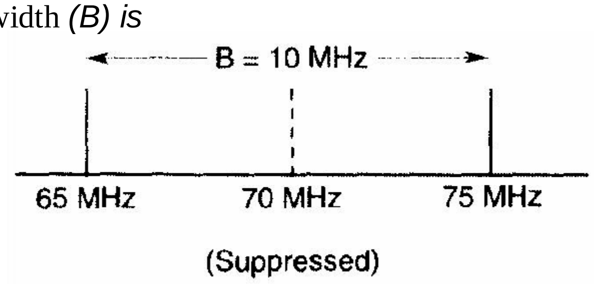
Minimum lower side frequency (LSF):

LSF=70MHz - 5MHz = 65MHz

Maximum upper side frequency (USF):

USF = 70 MHz + 5 MHz = 75 MHz

Therefore, the output spectrum for the worst-case binary input conditions *is* as follows: The minimum Nyquist bandwidth *(B) is*



B = 75 MHz - 65 MHz = 10 MHz

and the baud = f_b or 10 megabaud.

BPSK receiver:.

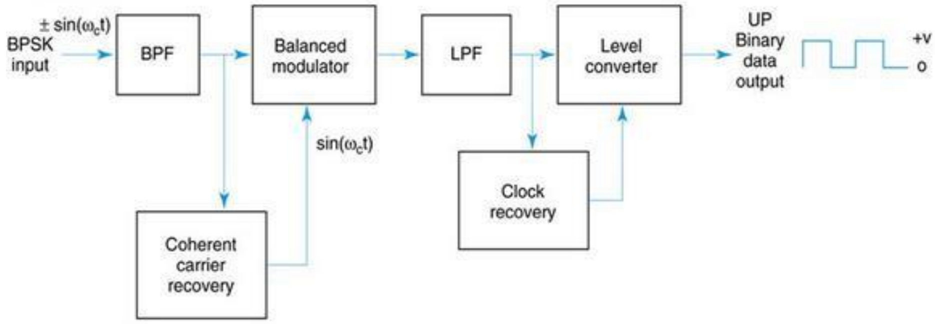
Figure 2-16 shows the block diagram of a BPSK receiver.

The input signal maybe+ sin $\omega_c t$ or - sin $\omega_c t$ .The coherent carrier recovery circuit detects and regenerates a carrier signal that is both frequency and phase coherent with the original transmit carrier.

The balanced modulator is a product detector; the output is the product d the two inputs (the BPSK signal and the recovered carrier).

The low-pass filter (LPF) operates the recovered binary data from the complex demodulated signal.

FIGURE 2-16 Block diagram of a BPSK receiver



Mathematically, the demodulation process is as follows.

For a BPSK input signal of + sin $\omega_c t$ (logic 1), the output of the balanced modulator is

$$\text{output} = (\sin \omega_c t)(\sin \omega_c t) = \sin^2\omega_c t \qquad (2.21)$$

or

$$\sin^2\omega_c t = 0.5(1 - \cos 2\omega_c t) = 0.5 - 0.5\cos 2\omega_c t$$

filtered out

leaving output = + 0.5 V = logic 1

It can be seen that the output of the balanced modulator contains a positive voltage (+[1/2]V) and a cosine wave at twice the carrier frequency (2 $\omega_c t$).

The LPF has a cutoff frequency much lower than 2 $\omega_c t$, and, thus, blocks the second harmonic of the carrier and passes only the positive constant component. A positive voltage represents a demodulated logic 1.

For a BPSK input signal of -sin $\omega_c t$ (logic 0), the output of the balanced modulator is

$$\text{output} = (-\sin \omega_c t)(\sin \omega_c t) = \sin^2\omega_c t$$

or

$$\sin^2\omega_c t = -0.5(1 - \cos 2\omega_c t) = 0.5 + 0.5\cos 2\omega_c t$$

filtered out

leaving

output = - 0.5 V = logic 0

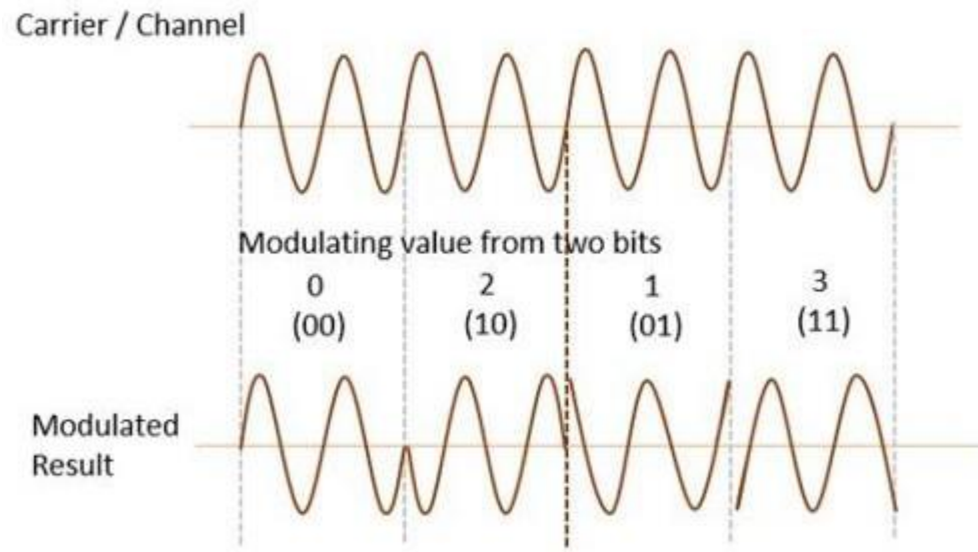
The output of the balanced modulator contains a negative voltage (-[l/2]V) and a cosine wave at twice the carrier frequency ($2\omega_c t$).

Again, the LPF blocks the second harmonic of the carrier and passes only the negative constant component. A negative voltage represents a demodulated logic 0.

QUADRATURE PHASE SHIFT KEYING (QPSK):

This is the phase shift keying technique, in which the sine wave carrier takes four phase reversals such as 0°, 90°, 180°, and 270°.

If this kind of techniques are further extended, PSK can be done by eight or sixteen values also, depending upon the requirement. The following figure represents the QPSK waveform for two bits input, which shows the modulated result for different instances of binary inputs.



QPSK is a variation of BPSK, and it is also a DSB-SC (Double Sideband Suppressed Carrier) modulation scheme, which sends two bits of digital information at a time, called as **bigits**.

Instead of the conversion of digital bits into a series of digital stream, it converts them into bit-pairs. This decreases the data bit rate to half, which allows space for the other users.

QPSK transmitter.

A block diagram of a QPSK modulator is shown in Figure 2-17Two bits (a dibit) are clocked into the bit splitter. After both bits have been serially inputted, they are simultaneously parallel outputted.

The I bit modulates a carrier that is in phase with the reference oscillator (hence the name "I" for "in phase" channel), and theQ bit modulate, a carrier that is 90° out of phase.

For a logic 1 = + 1 V and a logic 0= - 1 V, two phases are possible at the output of the I balanced modulator (+sin ω_ct and - sin ω_ct), and two phases are possible at the output of the Q balanced modulator (+cos ω_ct), and (-cos ω_ct).

When the linear summer combines the two quadrature (90° out of phase) signals, there are four possible resultant phasors given by these expressions: + sin ω_ct + cos ω_ct, + sin ω_ct - cos ω_ct, -sin ω_ct + cos ω_ct, and -sin ω_ct - cos ω_ct.

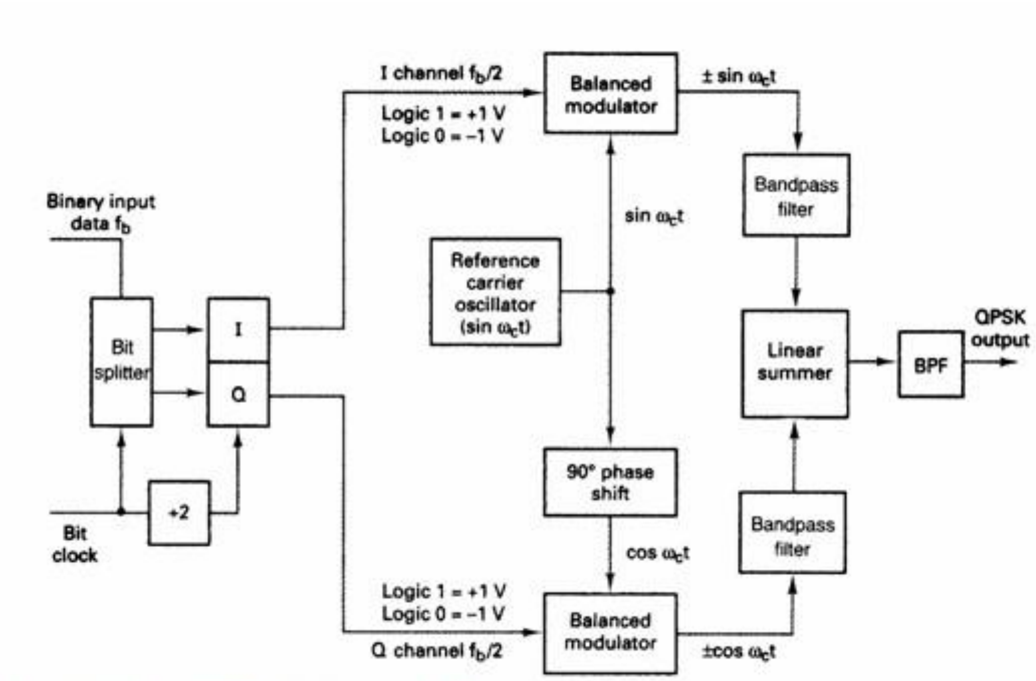


FIGURE 2-17 QPSK modulator

Example:

For the QPSK modulator shown in Figure 2-17, construct the truthtable, phasor diagram, and constellation diagram.

Solution

For a binary data input of Q = O and I= 0, the two inputs to the Ibalanced modulator are -1 and sin ω_ct, and the two inputs to the Q balanced modulator are -1 and cos ω_ct.

Consequently, the outputs are

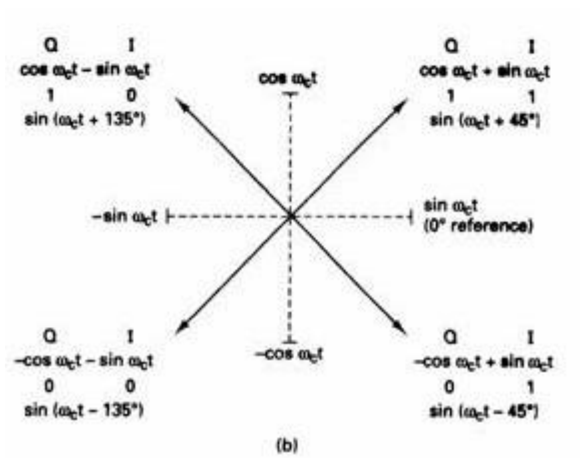
I balanced modulator =(-1)(sin ω_ct) = -1 sin ω_ct

Q balanced modulator =(-1)(cos ω_ct) = -1 cos ω_ct and the output of the linear summer is
-1 cos ω_ct - 1 sin ω_ct = 1.414 sin(ω_ct - 135°)

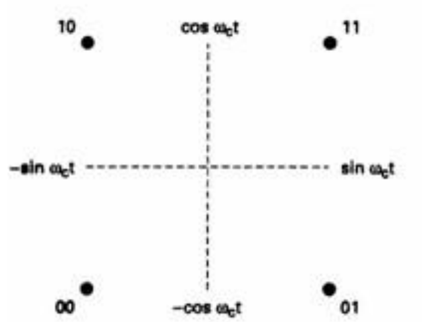
For the remaining dibit codes (01, 10, and 11), the procedure is the same. The results are shown in Figure 2-18a.

Binary input Q	Binary input I	QPSK output phase
0	0	−135°
0	1	−45°
1	0	+135°
1	1	+45°

(a)



(b)



(c)

FIGURE 2-18 QPSK modulator: (a) truth table; (b) phasor diagram; (c) constellation diagram

In Figures 2-18b and c, it can be seen that with QPSK each of the four possible output phasors has exactly the same amplitude. Therefore, the binary information must be encoded entirely in the phase of the output signal

Figure 2-18b, it can be seen that the angular separation between any two adjacent phasors in QPSK is 90°.Therefore, a QPSK signal can undergo almost a+45° or -45° shift in phase during transmission and still retain the correct encoded information when demodulated at the receiver.

Figure 2-19 shows the output phase-versus-time relationship for a QPSK modulator.

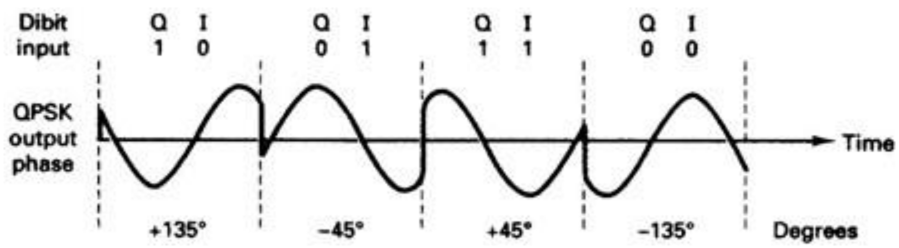


FIGURE 2-19 Output phase-versus-time relationship for a PSK modulator

Bandwidth considerations of QPSK

With QPSK, because the input data are divided into two channels, the bit rate in either the I or the Q channel is equal to one-half of the input data rate ($f_b/2$) (one-half of $f_b/2 = f_b/4$).

QPSK RECEIVER:

The block diagram of a QPSK receiver is shown in Figure 2-21

The power splitter directs the input QPSK signal to the I and Q product detectors and the carrier recovery circuit. The carrier recovery circuit reproduces the original transmit carrier oscillator signal. The recovered carrier must be frequency and phase coherent with the transmit reference carrier. The QPSK signal is demodulated in the I and Q product detectors, which generate the original I and Q data bits. The outputs of the product detectors are fed to the bit combining circuit, where they are converted from parallel I and Q data channels to a single binary output data stream. The incoming QPSK signal may be any one of the four possible output phases shown in Figure 2-18. To illustrate the demodulation process, let the incoming QPSK signal be $-\sin \omega_c t + \cos \omega_c t$. Mathematically, the demodulation process is as follows.

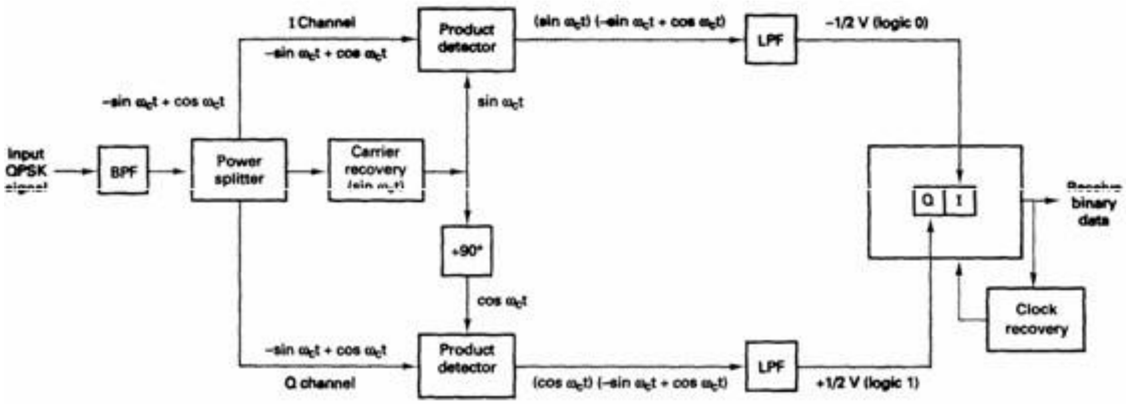


FIGURE 2-21 QPSK receiver

The receive QPSK signal (-sin $\omega_c t$ + cos $\omega_c t$) is one of the inputs to the I product detector. The other input is the recovered carrier (sin $\omega_c t$). The output of the I product detector is

$$I = \underbrace{(-\sin \omega_c t + \cos \omega_c t)}_{\text{QPSK input signal}} \underbrace{(\sin \omega_c t)}_{\text{carrier}}$$

$$= (-\sin \omega_c t)(\sin \omega_c t) + (\cos \omega_c t)(\sin \omega_c t)$$

$$= -\sin^2 \omega_c t + (\cos \omega_c t)(\sin \omega_c t)$$

$$= -\frac{1}{2}(1 - \cos 2\omega_c t) + \frac{1}{2}\sin(\omega_c + \omega_c)t + \frac{1}{2}\sin(\omega_c - \omega_c)t$$

(filtered out) (equals 0)

$$I = -\frac{1}{2} + \frac{1}{2}\cos 2\omega_c t + \frac{1}{2}\sin 2\omega_c t + \frac{1}{2}\sin 0$$

$$= -\frac{1}{2}\text{V (logic 0)} \qquad (2.23)$$

Again, the receive QPSK signal (-sin $\omega_c t$ + cos $\omega_c t$) is one of the inputs to the Q product detector. The other input is the recovered carrier shifted 90° in phase (cos $\omega_c t$). The output of the Q product detector is

$$Q = \underbrace{(-\sin \omega_c t + \cos \omega_c t)}_{\text{QPSK input signal}} \underbrace{(\cos \omega_c t)}_{\text{carrier}}$$

$$= \cos^2 \omega_c t - (\sin \omega_c t)(\cos \omega_c t)$$

$$= \frac{1}{2}(1 + \cos 2\omega_c t) - \frac{1}{2}\sin(\omega_c + \omega_c)t - \frac{1}{2}\sin(\omega_c - \omega_c)t$$

$$Q = \frac{1}{2} + \frac{1}{2}\underbrace{\cos 2\omega_c t}_{\text{(filtered out)}} - \frac{1}{2}\sin 2\omega_c t - \underbrace{\frac{1}{2}\sin 0}_{\text{(equals 0)}}$$

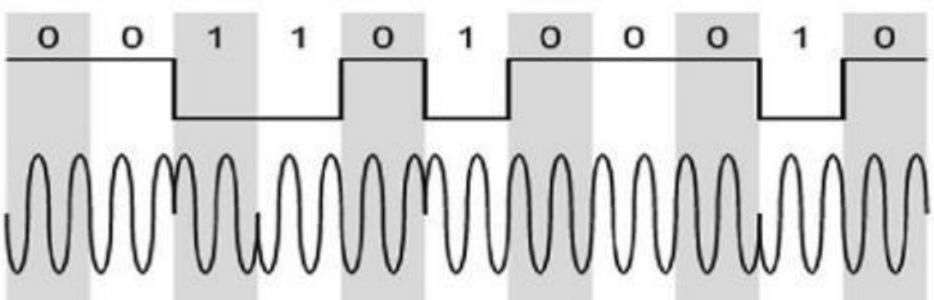
$$= \frac{1}{2}\text{V(logic 1)} \qquad (2.24)$$

The demodulated I and Q bits (0 and 1, respectively) correspond to the constellation diagram and truth table for the QPSK modulator shown in Figure 2-18.

DIFFERENTIAL PHASE SHIFT KEYING (DPSK):

In DPSK (Differential Phase Shift Keying) the phase of the modulated signal is shifted relative to the previous signal element. No reference signal is considered here. The signal phase follows the high or low state of the previous element. This DPSK technique doesn’t need a reference oscillator.

The following figure represents the model waveform of DPSK.



It is seen from the above figure that, if the data bit is LOW i.e., 0, then the phase of the signal is not reversed, but is continued as it was. If the data is HIGH i.e., 1, then the phase of the signal is reversed, as with NRZI, invert on 1 (a form of differential encoding).

If we observe the above waveform, we can say that the HIGH state represents an **M** in the modulating signal and the LOW state represents a **W** in the modulating signal.

The word binary represents two-bits. **M** simply represents a digit that corresponds to the number of conditions, levels, or combinations possible for a given number of binary variables.

This is the type of digital modulation technique used for data transmission in which instead of one-bit, two or **more bits are transmitted at a time**. As a single signal is used for multiple bit transmission, the channel bandwidth is reduced.

DBPSK TRANSMITTER.:

Figure 2-37a shows a simplified block diagram of a *differential binary phase-shift keying* (DBPSK) transmitter. An incoming information bit is XNORed with the preceding bit prior to entering the BPSK modulator (balanced modulator).

For the first data bit, there is no preceding bit with which to compare it. Therefore, an initial reference bit is assumed. Figure 2-37b shows the relationship between the input data, the XNOR output data, and the phase at the output of the balanced modulator. If the initial reference bit is assumed a logic 1, the output from the XNOR circuit is simply the complement of that shown.

In Figure 2-37b, the first data bit is XNORed with the reference bit. If they are the same, the XNOR output is a logic 1; if they are different, the XNOR output is a logic 0. The balanced modulator operates the same as a conventional BPSK modulator; a logic I produces $+\sin \omega_c t$ at the output, and A logic 0 produces $-\sin \omega_c t$ at the output.

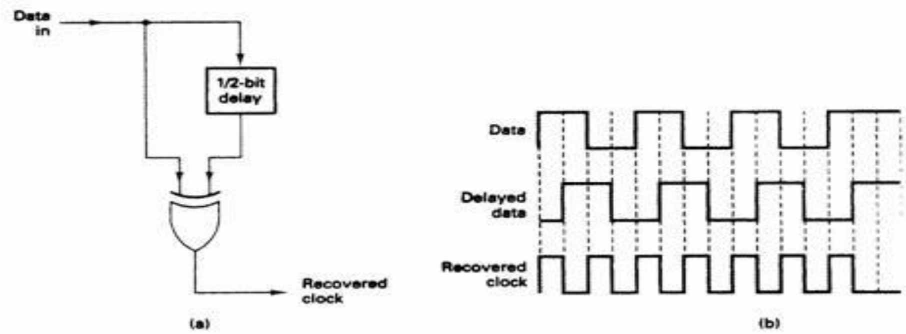


FIGURE 9-40 (a) Clock recovery circuit; (b) timing diagram

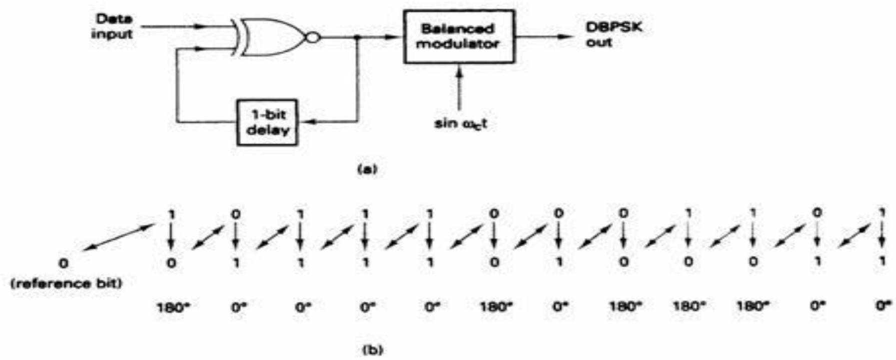


FIGURE 2-37 DBPSK modulator (a) block diagram (b) timing diagram

BPSK RECEIVER:

Figure 9-38 shows the block diagram and timing sequence for a DBPSK receiver. The received signal is delayed by one bit time, then compared with the next signaling element in the balanced modulator. If they are the same. J logic 1(+ voltage) is generated. If they are different, a logic 0 (-voltage) is generated. [f the reference phase is incorrectly assumed, only the first demodulated bit is in error. Differential encoding can be implemented with higher-than-binary digital modulation schemes, although the differential algorithms are much more complicated than for DBPS K.

The primary advantage of DBPSK is the simplicity with which it can be implemented. With DBPSK, no carrier recovery circuit is needed. A disadvantage of DBPSK is, that it requires between 1 dB and 3 dB more signal-to-noise ratio to achieve the same bit error rate as that of absolute PSK.

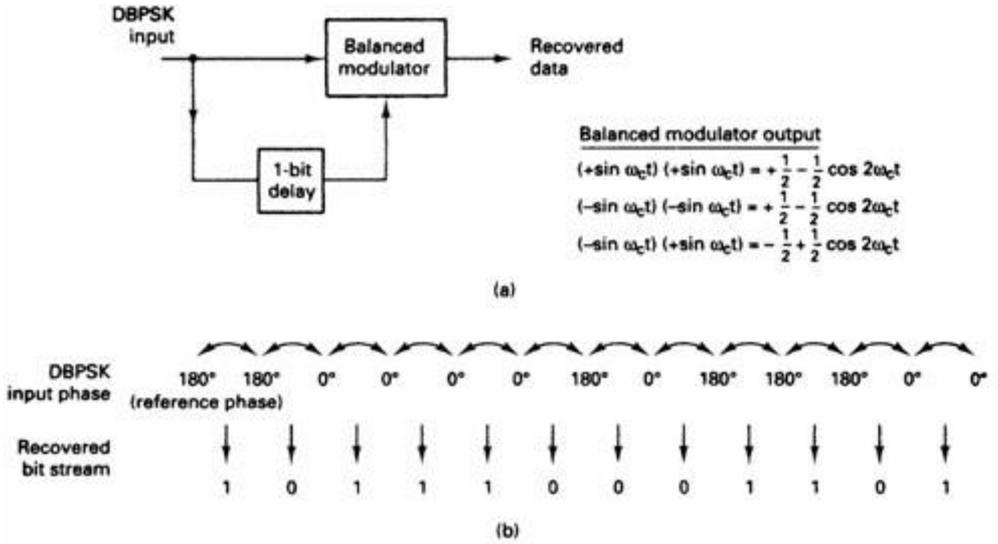


FIGURE 2-38 DBPSK demodulator: (a) block diagram; (b) timing sequence

COHERENT RECEPTION OF FSK:

The coherent demodulator for the coherent FSK signal falls in the general form of coherent demodulators described in Appendix B. The demodulator can be implemented with two correlators as shown in Figure 3.5, where the two reference signals are cos(27r f t) and cos(27r fit). They must be synchronized with the received signal. The receiver is optimum in the sense that it minimizes the error probability for equally likely binary signals. Even though the receiver is rigorously derived in Appendix B, some heuristic explanation here may help understand its operation. When s 1 (t) is transmitted, the upper correlator yields a signal 1 with a positive signal component and a noise component. However, the lower correlator output 12, due to the signals' orthogonality, has only a noise component. Thus the output of the summer is most likely above zero, and the threshold detector will most likely produce a 1. When s2(t) is transmitted, opposite things happen to the two correlators and the threshold detector will most likely produce a 0. However, due to the noise nature that its values range from -00 to m, occasionally the noise amplitude might overpower the signal amplitude, and then detection errors will happen. An alternative to Figure 3.5 is to use just one correlator with the reference signal cos (27r f t) - cos(2s f2t) (Figure 3.6). The correlator in Figure can be replaced by a matched filter that matches cos(27r fit) - cos(27r f2t) (Figure 3.7). All implementations are equivalent in terms of error performance (see Appendix B). Assuming an AWGN channel, the received signal is

$$r(t) = s_i(t) + n(t), \quad i = 1.2$$

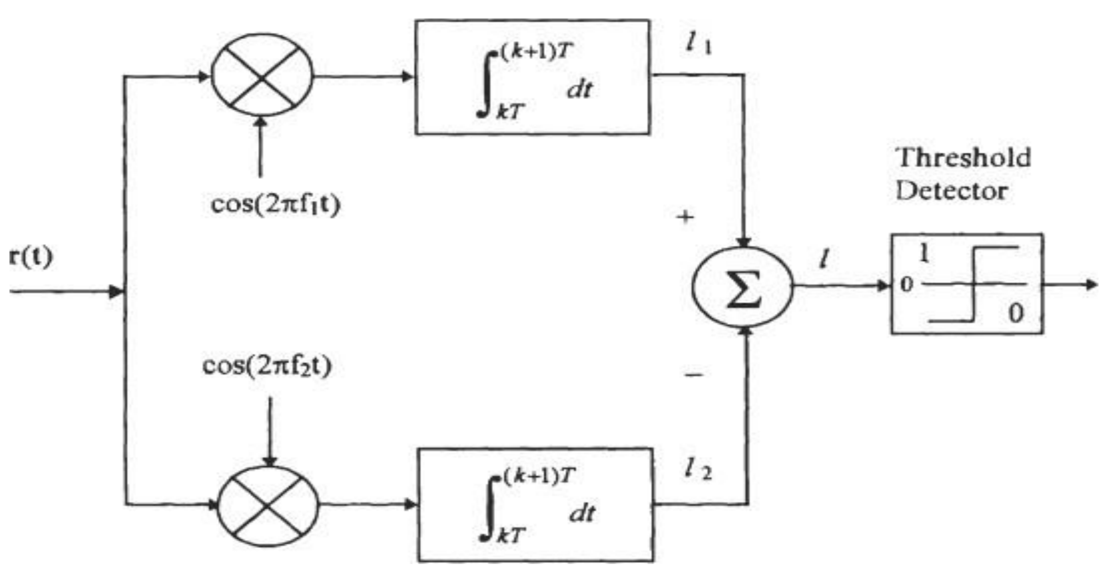
where n(t) is the additive white Gaussian noise with zero mean and a two-sided power spectral density A',/2. From (B.33) the bit error probability for any equally likely binary signals is

$$P_b = Q\left(\sqrt{\frac{E_1 + E_2 - 2\rho_{12}\sqrt{E_1 E_2}}{2N_o}}\right)$$

where No/2 is the two-sided power spectral density of the additive white Gaussian noise. For Sunde's FSK signals El = Ez = Eb, pI2 = 0 (orthogonal). thus the error probability is

$$P_b = Q\left(\sqrt{\frac{E_b}{N_o}}\right)$$

where Eb = A2T/2 is the average bit energy of the FSK signal. The above Pb is plotted in Figure 3.8 where Pb of noncoherently demodulated FSK, whose expression will be given shortly, is also plotted for comparison.



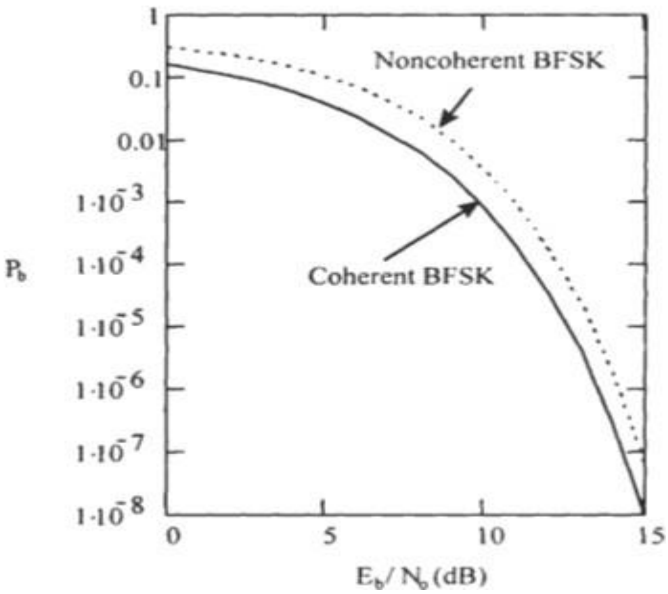


Figure: Pb of coherently and non-coherently demodulated FSK signal.

NONCOHERENT DEMODULATION AND ERROR PERFORMANCE:

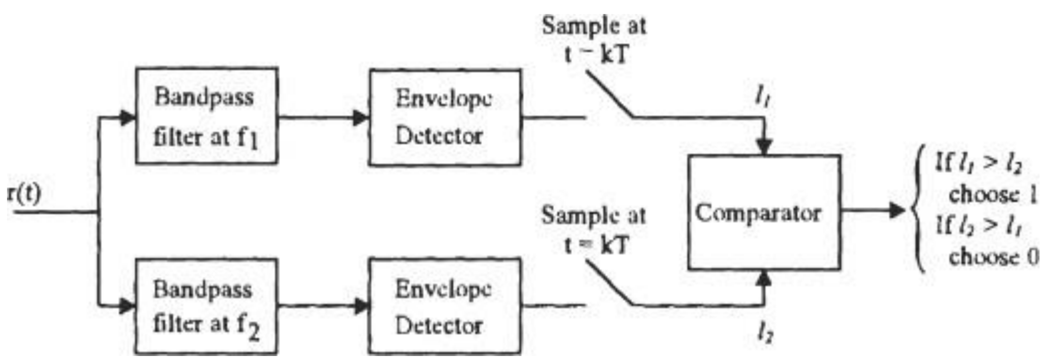
Coherently FSK signals can be noncoherently demodulated to avoid the carrier recovery. Noncoherently generated FSK can only be noncoherently demodulated. We refer to both cases as noncoherent FSK. In both cases the demodulation problem becomes a problem of detecting signals with unknown phases. In Appendix B we have shown that the optimum receiver is a quadrature receiver. It can be implemented using correlators or equivalently, matched filters. Here we assume that the binary noncoherent FSK signals are equally likely and with equal energies. Under these assumptions, the demodulator using correlators is shown in Figure 3.9. Again, like in the coherent case, the optimality of the receiver has been rigorously proved (Appendix B). However, we can easily understand its operation by some heuristic argument as follows. The received signal (ignoring noise for the moment) with an unknown phase can be written as

$$\begin{aligned} s_i(t, \theta) &= A\cos(2\pi f_i t + \theta), \qquad i = 1, 2 \\ &= A\cos\theta\cos 2\pi f_i t - A\sin\theta\sin 2\pi f_i t \end{aligned}$$

The signal consists of an in phase component A cos 8 cos 27r f t and a quadrature component A sin 8 sin 2x f,t sin 0. Thus the signal is partially correlated with cos 2s fit and partiah'y correlated with sin 27r fit. Therefore we use two correlators to collect the signal energy in these two parts. The outputs of the in phase and quadrature correlators will be cos 19 and sin 8, respectively. Depending on the value of the unknown phase 8, these two outputs could be anything in (- 5, y). Fortunately the squared sum of these two signals is not dependent on the unknown phase. That is

$$(\frac{AT}{2}\cos\theta)^2 + (\frac{AT}{2}\sin\theta)^2 = \frac{A^2T^2}{2}$$

This quantity is actually the mean value of the statistics I? when signal si (t) is transmitted and noise is taken into consideration. When si (t) is not transmitted the mean value of 1: is 0. The comparator decides which signal is sent by checking these I?. The matched filter equivalence to Figure 3.9 is shown in Figure 3.10 which has the same error performance. For implementation simplicity we can replace the matched filters by bandpass filters centered at f and fi, respectively (Figure 3.1 1). However, if the bandpass filters are not matched to the FSK signals, degradation to



various extents will result. The bit error probability can be derived using the correlator demodulator (Appendix B). Here we further assume that the FSK signals are orthogonal, then from Appendix B the error probability is

$$P_b = \frac{1}{2}e^{-E_b/2N_o}$$

PART-2

DATATRANSMISSION

BASE BAND SIGNAL RECEIVER:

Consider that a binary encoded signal consists of a time sequence of voltage levels +V or –V. if there is a guard interval between the bits, the signal forms a sequence of positive and negative pulses. in either case there is no particular interest in preserving the waveform of the signal after reception .we are interested only in knowing within each bit interval whether the transmitted voltage was +V or –V. With noise present, the receives signal and noise together will yield sample values generally different from ±V. In this case, what deduction shall we make from the sample value concerning the transmitted bit?

Suppose that the noise is gaussian and therefore the noise voltage has a probability density which is entirely symmetrical with respect to zero volts. Then the probability that the noise has increased the sample value is the same as the probability that the noise has decreased the sample value. It then seems entirely reasonable that we can do no better than to assume that if the sample value is positive the transmitted level was $+V$, and if the sample value is negative the transmitted level was $-V$. It is, of course, possible that at the sampling time the noise voltage may be of magnitude larger than V and of a polarity opposite to the polarity assigned to the transmitted bit. In this case an error will be made as indicated in Fig. 11.1-1. Here the transmitted bit is represented by the voltage $+V$ which is sustained over an interval T from t_1 to t_2. Noise has been superimposed on the level $+V$ so that the voltage v represents the received signal and noise. If now the sampling should happen to take place at a time $t = t_1 + \Delta t$, an error will have been made.

We can reduce the probability of error by processing the received signal plus noise in such a manner that we are then able to find a sample time where the sample voltage due to the signal is emphasized relative to the sample voltage due to the noise. Such a processer (receiver) is shown in Fig. 11.1-2. The signal input during a bit interval is indicated. As a matter of convenience we have set $t = 0$ at the beginning of the interval. The waveform of the signal $s(t)$ before $t = 0$ and after $t = T$ has not been indicated since, as will appear, the operation of the receiver during each bit interval is independent of the waveform during past and future bit intervals.

The signal $s(t)$ with added white gaussian noise $n(t)$ of power spectral density $\eta/2$ is presented to an integrator. At time $t = 0+$ we require that capacitor C be uncharged. Such a discharged condition may be ensured by a brief closing of switch SW_1 at time $t = 0-$, thus relieving C of any charge it may have acquired during the previous interval. The sample is taken at the output of the integrator by closing this sampling switch SW_2. This sample is taken at the end of the bit interval, at $t = T$. The signal processing indicated in Fig. 11.1-2 is described by the phrase *integrate and dump*, the term *dump* referring to the abrupt discharge of the capacitor after each sampling.

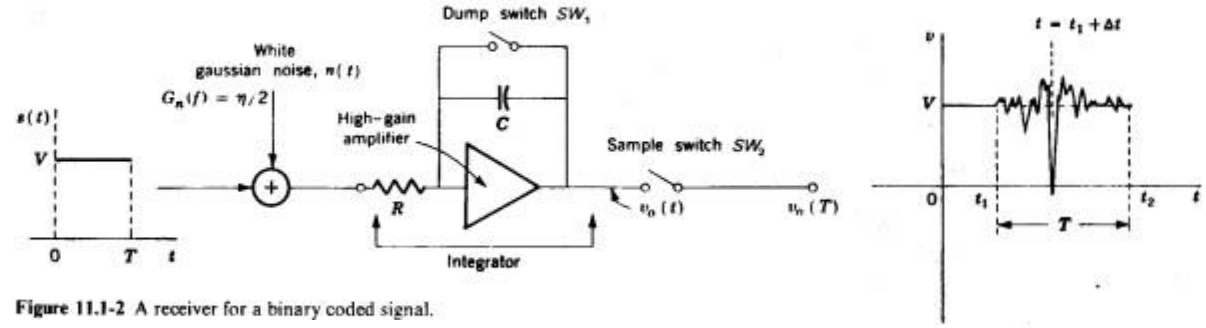


Figure 11.1-2 A receiver for a binary coded signal.

Peak Signal to RMS Noise Output Voltage Ratio

The integrator yields an output which is the integral of its input multiplied by $1/RC$. Using $\tau = RC$, we have

$$v_o(T) = \frac{1}{\tau}\int_0^T [s(t) + n(t)]\,dt = \frac{1}{\tau}\int_0^T s(t)\,dt + \frac{1}{\tau}\int_0^T n(t)\,dt \qquad (11.1\text{-}1)$$

The sample voltage due to the signal is

$$s_o(T) = \frac{1}{\tau}\int_0^T V\,dt = \frac{VT}{\tau} \qquad (11.1\text{-}2)$$

The sample voltage due to the noise is

$$n_o(T) = \frac{1}{\tau}\int_0^T n(t)\,dt \qquad (11.1\text{-}3)$$

This noise-sampling voltage $n_o(T)$ is a gaussian random variable in contrast with $n(t)$, which is a gaussian random process.

The variance of $n_o(T)$ was found in Sec. 7.9 [see **Eq.** (7.9-17)] to be

$$\sigma_o^2 = \overline{n_o^2(T)} = \frac{\eta T}{2\tau^2} \qquad (11.1\text{-}4)$$

and, as noted in Sec. 7.3, $n_o(T)$ has a gaussian probability density.

The output of the integrator, before the sampling switch, is $v_o(t) = s_o(t) + n_o(t)$. As shown in Fig. 11.1-3*a*, the signal output $s_o(t)$ is a ramp, in each bit interval, of duration T. At the end of the interval the ramp attains the voltage $s_o(T)$ which is $+VT/\tau$ or $-VT/\tau$, depending on whether the bit is a 1 or a 0. At the end of each interval the switch SW_1 in Fig. 11.1-2 closes momentarily to discharge the capacitor so that $s_o(t)$ drops to zero. The noise $n_o(t)$, shown in Fig. 11.1-3*b*, also starts each interval with $n_o(0) = 0$ and has the random value $n_o(T)$ at the end of each interval. The sampling switch SW_2 closes briefly just before the closing of SW_1 and hence reads the voltage

$$v_o(T) = s_o(T) + n_o(T) \qquad (11.1\text{-}5)$$

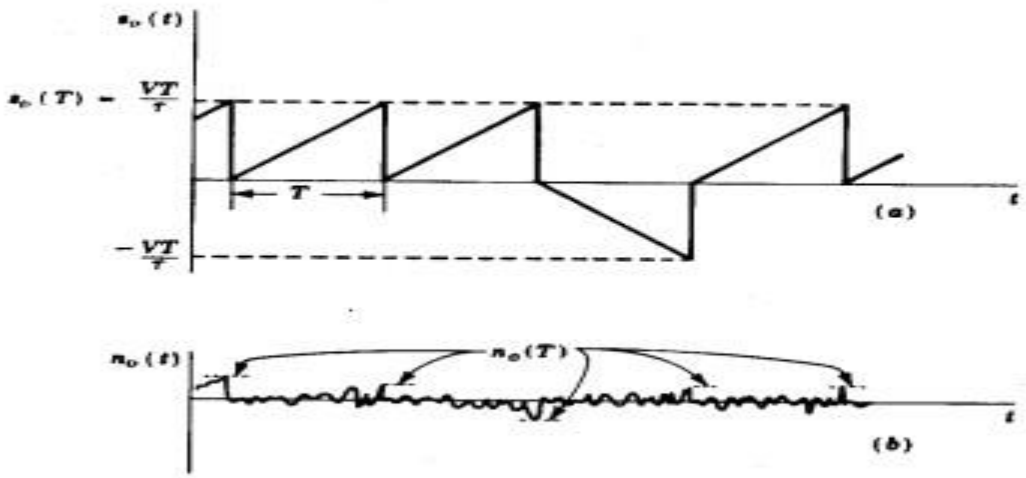


Figure 11.1-3 (*a*) The signal output and (*b*) the noise output of the integrator of Fig. 11.1-2.

We would naturally like the output signal voltage to be as large as possible in comparison with the noise voltage. Hence a figure of merit of interest is the signal-to-noise ratio

$$\frac{[s_o(T)]^2}{[n_o(T)]^2} = \frac{2}{\eta} V^2 T \qquad (11.1\text{-}6)$$

This result is calculated from Eqs. (11.1-2) and (11.1-4). Note that the signal-to-noise ratio increases with increasing bit duration T and that it depends on $V^2 T$ which is the normalized energy of the bit signal. Therefore, a bit represented by a narrow, high amplitude signal and one by a wide, low amplitude signal are equally effective, provided $V^2 T$ is kept constant.

It is instructive to note that the integrator filters the signal and the noise such that the signal voltage increases linearly with time, while the standard deviation (rms value) of the noise increases more slowly, as $\sqrt{T}$. Thus, the integrator enhances the signal relative to the noise, and this enhancement increases with time as shown in Eq. (11.1-6).

PROBABILITY OF ERROR

Since the function of a receiver of a data transmission is to ditinguish the bit 1 from the bit 0 in the presence of noise, a most important charcteristic is the probability that an error will be made in such a determination.we now calculate this error probabilty P_e for the integrate and dump receiver of fig 11.1-2

We have seen that the probability density of the noise sample $n_o(T)$ is gaussian and hence appears as in Fig. 11.2-1. The density is therefore given by

$$f[n_o(T)] = \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}} \tag{11.2-1}$$

where σ_o^2, the variance, is $\sigma_o^2 \equiv \overline{n_o^2(T)}$ given by Eq. (11.1-4). Suppose, then, that during some bit interval the input-signal voltage is held at, say, $-V$. Then, at the sample time, the signal sample voltage is $s_o(T) = -VT/\tau$, while the noise sample is $n_o(T)$. If $n_o(T)$ is positive and larger in magnitude than VT/τ, the total sample voltage $v_o(T) = s_o(T) + n_o(T)$ will be positive. Such a positive sample voltage will result in an error, since as noted earlier, we have instructed the receiver to interpret such a positive sample voltage to mean that the signal voltage was $+V$ during the bit interval. The probability of such a misinterpretation, that is, the probability that $n_o(T) > VT/\tau$, is given by the area of the shaded region in Fig. 11.2-1. The probability of error is, using Eq. (11.2-1),

$$P_e = \int_{VT/\tau}^{\infty} f[n_o(T)]\, dn_o(T) = \int_{VT/\tau}^{\infty} \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}}\, dn_o(T) \tag{11.2-2}$$

Defining $x \equiv n_o(T)/\sqrt{2}\sigma_o$, and using Eq. (11.1-4), Eq. (11.2-2) may be rewritten as

$$P_e = \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{x=V\sqrt{T/\eta}}^{\infty} e^{-x^2}\, dx$$

$$= \frac{1}{2}\operatorname{erfc}\left(V\sqrt{\frac{T}{\eta}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{V^2T}{\eta}\right)^{1/2} = \frac{1}{2}\operatorname{erfc}\left(\frac{E_s}{\eta}\right)^{1/2} \tag{11.2-3}$$

in which $E_s = V^2T$ is the signal energy of a bit.

If the signal voltage were held instead at $+V$ during some bit interval, then it is clear from the symmetry of the situation that the probability of error would again be given by P_e in Eq. (11.2-3). Hence Eq. (11.2-3) gives P_e quite generally.

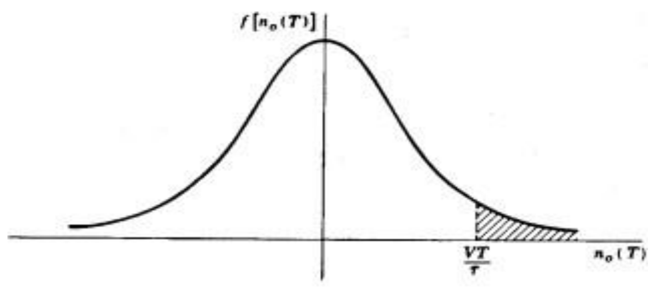


Figure 11.2-1 The gaussian probability density of the noise sample $n_o(T)$.

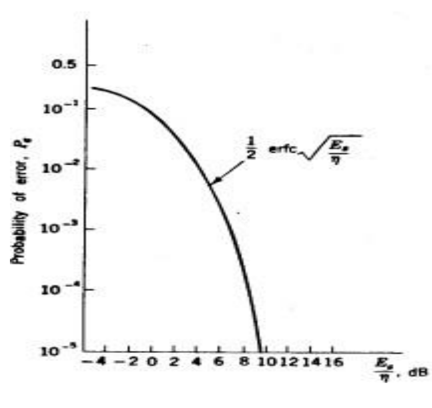


Figure 11.2-2 Variation of P_e versus E_s/η.

The probability of error p_e, as given in eq.(11.2-3),is plotted in fig.11.2-2.note that p_e decreases rapidly as $E_{s.}/\eta$ increases. The maximum value of p_e is ½.thus ,even if the signal is entirely lost in the noise so that any determination of the receiver is a sheer guess, the receiver cannot bi wrong more than half the time on the average.

THE OPTIMUM FILTER:

In the receiver system of Fig 11.1-2, the signal was passed through a filter(integrator),so that at the sampling time the signal voltage might be emphasized in comparison with the noise voltage. We are naturally led to risk whether the integrator is the optimum filter for the purpose of minimizing the probability of error. We shall find that the received signal contemplated in system of fig 11.1-2 the integrator is indeed the optimum filter. However, before returning specifically to the integrator receiver.

We assume that the received signal is a binary waveform. One binary digit is represented by a signal waveformS_1 (t) which persists for time T, while the4 other bit is represented by the waveform S_2(t) which also lasts for an interval T. For example, in the transmission at baseband, as shown in fig 11.1-2 S_1(t)=+V; for other modulation systems, different waveforms are transmitted. for example for PSK signaling , $S_1(t)=A\cos w_0t$ and $S_2(t)=-A\cos w_0t$;while for FSK, $S_1(t)=A\cos(w_{0+\Omega})t$.

As shown in Fig. 11.3-1 the input, which is $s_1(t)$ or $s_2(t)$, is corrupted by the addition of noise $n(t)$. The noise is gaussian and has a spectral density $G(f)$. [In most cases of interest the noise is white, so that $G(f) = \eta/2$. However, we shall assume the more general possibility, since it introduces no complication to do so.] The signal and noise are filtered and then sampled at the end of each bit interval. The output sample is either $v_o(T) = s_{o1}(T) + n_o(T)$ or $v_o(T) = s_{o2}(T) + n_o(T)$. We assume that immediately after each sample, every energy-storing element in the filter has been discharged.

We have already considered in Sec. 2.22, the matter of signal determination in the presence of noise. Thus, we note that in the absence of noise the output sample would be $v_o(T) = s_{o1}(T)$ or $s_{o2}(T)$. When noise is present we have shown that to minimize the probability of error one should assume that $s_1(t)$ has been transmitted if $v_o(T)$ is *closer* to $s_{o1}(T)$ than to $s_{o2}(T)$. Similarly, we assume $s_2(t)$ has been transmitted if $v_o(T)$ is *closer* to $s_{o2}(T)$. The decision boundary is therefore midway between $s_{o1}(T)$ and $s_{o2}(T)$. For example, in the baseband system of Fig. 11.1-2, where $s_{o1}(T) = VT/\tau$ and $s_{o2}(T) = -VT/\tau$, the decision boundary is $v_o(T) = 0$. In general, we shall take the decision boundary to be

$$v_o(T) = \frac{s_{o1}(T) + s_{o2}(T)}{2} \qquad (11.3\text{-}1)$$

The probability of error for this general case may be deduced as an extension of the considerations used in the baseband case. Suppose that $s_{o1}(T) > s_{o2}(T)$ and that $s_2(t)$ was transmitted. If, at the sampling time, the noise $n_o(T)$ is positive and larger in magnitude than the voltage difference $\frac{1}{2}[s_{o1}(T) + s_{o2}(T)] - s_{o2}(T)$, an error will have been made. That is, an error [we decide that $s_1(t)$ is transmitted rather than $s_2(t)$] will result if

$$n_o(T) \geq \frac{s_{o1}(T) - s_{o2}(T)}{2} \qquad (11.3\text{-}2)$$

Hence probability of error is

$$P_e = \int_{[s_{o1}(T)-s_{o2}(T)]/2}^{\infty} \frac{e^{-n_o^2(T)/2\sigma_o^2}}{\sqrt{2\pi\sigma_o^2}}\, dn_o(T) \qquad (11.3\text{-}3)$$

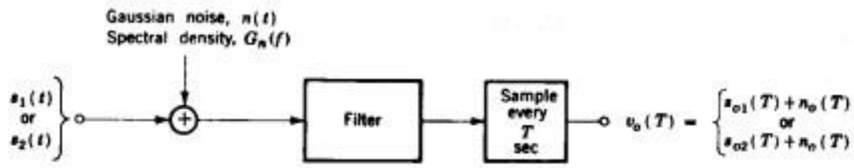


Figure 11.3-1 A receiver for binary coded signalling.

If we make the substitution $x \equiv n_o(T)/\sqrt{2}\sigma_o$, Eq. (11.3-3) becomes

$$P_e = \frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{[s_{o1}(T)-s_{o2}(T)]/2\sqrt{2}\sigma_o}^{\infty} e^{-x^2}\, dx \qquad (11.3\text{-}4a)$$

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{s_{o1}(T)-s_{o2}(T)}{2\sqrt{2}\sigma_o}\right] \qquad (11.3\text{-}4b)$$

Note that for the case $s_{o1}(T) = VT/\tau$ and $s_{o2}(T) = -VT/\tau$, and, using Eq. (11.1-4), Eq. (11.3-4b) reduces to Eq. (11.2-3) as expected.

The complementary error function is a monotonically decreasing function of its argument. (See Fig. 11.2-2.) Hence, as is to be anticipated, P_e decreases as the difference $s_{o1}(T) - s_{o2}(T)$ becomes larger and as the rms noise voltage σ_o becomes smaller. The optimum filter, then, is the filter which maximizes the ratio

$$\gamma = \frac{s_{o1}(T) - s_{o2}(T)}{\sigma_o} \qquad (11.3\text{-}5)$$

We now calculate the transfer function $H(f)$ of this optimum filter. As a matter of mathematical convenience we shall actually maximize γ^2 rather than γ.

Calculation of the Optimum-Filter Transfer Function $H(f)$

The fundamental requirement we make of a binary encoded data receiver is that it distinguishes the voltages $s_1(t) + n(t)$ and $s_2(t) + n(t)$. We have seen that the ability of the receiver to do so depends on how large a particular receiver can make γ. It is important to note that γ is proportional not to $s_1(t)$ nor to $s_2(t)$, but rather to the *difference* between them. For example, in the baseband system we represented the signals by voltage levels $+V$ and $-V$. But clearly, if our only interest was in distinguishing levels, we would do just as well to use $+2$ volts and 0 volt, or $+8$ volts and $+6$ volts, etc. (The $+V$ and $-V$ levels, however, have the advantage of requiring the least average power to be transmitted.) Hence, while $s_1(t)$ or $s_2(t)$ is the received signal, the signal which is to be compared with the noise, i.e., the signal which is relevant in all our error-probability calculations, is the difference signal

$$p(t) \equiv s_1(t) - s_2(t) \qquad (11.3\text{-}6)$$

Thus, for the purpose of calculating the minimum error probability, we shall assume that the input signal to the optimum filter is $p(t)$. The corresponding *output signal* of the filter is then

$$p_o(t) \equiv s_{o1}(t) - s_{o2}(t) \qquad (11.3\text{-}7)$$

We shall let $P(f)$ and $P_o(f)$ be the Fourier transforms, respectively, of $p(t)$ and $p_o(t)$.

If $H(f)$ is the transfer function of the filter,

$$P_o(f) = H(f)P(f) \tag{11.3-8}$$

and

$$p_o(T) = \int_{-\infty}^{\infty} P_o(f)e^{j2\pi fT}\, df = \int_{-\infty}^{\infty} H(f)P(f)e^{j2\pi fT}\, df \tag{11.3-9}$$

The input noise to the optimum filter is $n(t)$. The output noise is $n_o(t)$ which has a power spectral density $G_{n_o}(f)$ and is related to the power spectral density of the input noise $G_n(f)$ by

$$G_{n_o}(f) = |H(f)|^2 G_n(f) \tag{11.3-10}$$

Using Parseval's theorem (Eq. 1.13-5), we find that the normalized output noise power, i.e., the noise variance σ_o^2, is

$$\sigma_o^2 = \int_{-\infty}^{\infty} G_{n_o}(f)\, df = \int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df \tag{11.3-11}$$

From Eqs. (11.3-9) and (11.3-11) we now find that

$$\gamma^2 = \frac{p_o^2(T)}{\sigma_o^2} = \frac{|\int_{-\infty}^{\infty} H(f)P(f)e^{j2\pi Tf}\, df|^2}{\int_{-\infty}^{\infty} |H(f)|^2 G_n(f)\, df} \tag{11.3-12}$$

Equation (11.3-12) is unaltered by the inclusion or deletion of the absolute value sign in the numerator since the quantity within the magnitude sign $p_o(T)$ is a positive real number. The sign has been included, however, in order to allow further development of the equation through the use of the *Schwarz inequality*.

The *Schwarz inequality* states that given arbitrary complex functions $X(f)$ and $Y(f)$ of a common variable f, then

$$\left|\int_{-\infty}^{\infty} X(f)Y(f)\, df\right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^2\, df \int_{-\infty}^{\infty} |Y(f)|^2\, df \tag{11.3-13}$$

The equal sign applies when

$$X(f) = KY^*(f) \tag{11.3-14}$$

where K is an arbitrary constant and $Y^*(f)$ is the complex conjugate of $Y(f)$.

We now apply the Schwarz inequality to Eq. (11.3-12) by making the identification

$$X(f) \equiv \sqrt{G_n(f)}\, H(f) \tag{11.3-15}$$

and

$$Y(f) \equiv \frac{1}{\sqrt{G_n(f)}} P(f)e^{j2\pi Tf} \tag{11.3-16}$$

Using Eqs. (11.3-15) and (11.3-16) and using the Schwarz inequality, Eq. (11.3-13), we may rewrite Eq. (11.3-12) as

$$\frac{p_o^2(T)}{\sigma_o^2} = \frac{|\int_{-\infty}^{\infty} X(f)Y(f)\, df|^2}{\int_{-\infty}^{\infty} |X(f)|^2\, df} \leq \int_{-\infty}^{\infty} |Y(f)|^2\, df \tag{11.3-17}$$

or, using Eq. (11.3-16),

$$\frac{p_o^2(T)}{\sigma_o^2} \leq \int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} \, df \qquad (11.3\text{-}18)$$

The ratio $p_o^2(T)/\sigma_o^2$ will attain its maximum value when the equal sign in Eq. (11.3-18) may be employed as is the case when $X(f) = KY^*(f)$. We then find from Eqs. (11.3-15) and (11.3-16) that the optimum filter which yields such a maximum ratio $p_o^2(T)/\sigma_o^2$ has a transfer function

$$H(f) = K \frac{P^*(f)}{G_n(f)} e^{-j2\pi fT} \qquad (11.3\text{-}19)$$

Correspondingly, the maximum ratio is, from Eq. (11.3-18),

$$\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max} = \int_{-\infty}^{\infty} \frac{|P(f)|^2}{G_n(f)} \, df \qquad (11.3\text{-}20)$$

In succeeding sections we shall have occasion to apply Eqs. (11.13-19) and (11.13-20) to a number of cases of interest.

11.4 WHITE NOISE: THE MATCHED FILTER

An optimum filter which yields a maximum ratio $p_o^2(T)/\sigma_o^2$ is called a *matched filter* when the input noise is *white*. In this case $G_n(f) = \eta/2$, and Eq. (11.3-19) becomes

$$H(f) = K \frac{P^*(f)}{\eta/2} e^{-j2\pi fT} \qquad (11.4\text{-}1)$$

The impulsive response of this filter, i.e., the response of the filter to a unit strength impulse applied at $t = 0$, is

$$h(t) = \mathscr{F}^{-1}[H(f)] = \frac{2K}{\eta} \int_{-\infty}^{\infty} P^*(f) e^{-j2\pi fT} e^{j2\pi ft} \, df \qquad (11.4\text{-}2a)$$

$$= \frac{2K}{\eta} \int_{-\infty}^{\infty} P^*(f) e^{j2\pi f(t-T)} \, df \qquad (11.4\text{-}2b)$$

A physically realizable filter will have an impulse response which is real, i.e., not complex. Therefore $h(t) = h^*(t)$. Replacing the right-hand member of Eq. (11.4-2*b*) by its complex conjugate, an operation which leaves the equation unaltered, we have

$$h(t) = \frac{2K}{\eta} \int_{-\infty}^{\infty} P(f) e^{j2\pi f(T-t)} \, df \qquad (11.4\text{-}3a)$$

$$= \frac{2K}{\eta} p(T-t) \qquad (11.4\text{-}3b)$$

Finally, since $p(t) \equiv s_1(t) - s_2(t)$ [see Eq. (11.3-6)], we have

$$h(t) = \frac{2K}{\eta} [s_1(T-t) - s_2(T-t)] \qquad (11.4\text{-}4)$$

The significance of these results for the matched filter may be more readily appreciated by applying them to a specific example. Consider then, as in Fig. 11.4-1*a*, that $s_1(t)$ is a triangular waveform of duration T, while $s_2(t)$, as shown in Fig. 11.4-1*b*, is of identical form except of reversed polarity. Then $p(t)$ is as shown in Fig. 11.4-1*c*, and $p(-t)$ appears in Fig. 11.4-1*d*. The waveform $p(-t)$ is the waveform $p(t)$ rotated around the axis $t = 0$. Finally, the waveform $p(T - t)$ called for as the impulse response of the filter in Eq. (11.4-3*b*) is this rotated waveform $p(-t)$ translated in the positive t direction by amount T. This last translation ensures that $h(t) = 0$ for $t < 0$ as is required for a *causal* filter.

In general, the impulsive response of the matched filter consists of p(t) rotated about t=0 and then delayed long enough(i.e., a time T) to make the filter realizable. We may note in passing, that any additional delay that a filter might introduce would in no way interfere with the performance of the filter ,for both signal and noise would be delayed by the same amount, and at the sampling time (which would need similarity to be delayed)the ratio of signal to noise would remain unaltered.

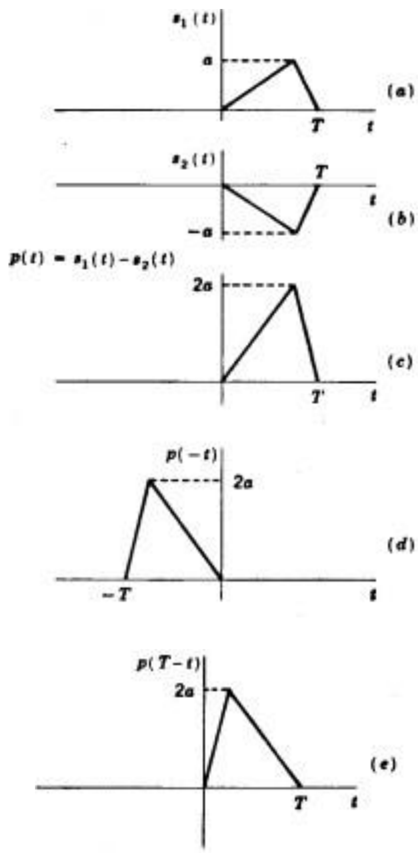


Figure 11.4-1 The signals (*a*) $s_1(t)$, (*b*) $s_2(t)$, and (*c*) $p(t) = s_1(t) - s_2(t)$. (*d*) $p(t)$ rotated about the axis $t = 0$. (*e*) The waveform in (*d*) translated to the right by amount T.

11.5 PROBABILITY OF ERROR OF THE MATCHED FILTER

The probability of error which results when employing a matched filter, may be found by evaluating the maximum signal-to-noise ratio $[p_o^2(T)/\sigma_o^2]_{max}$ given by Eq. (11.3-20). With $G_n(f) = \eta/2$, Eq. (11.3-20) becomes

$$\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{2}{\eta}\int_{-\infty}^{\infty} |P(f)|^2 \, df \qquad (11.5\text{-}1)$$

From parseval's theorem we have

$$\int_{-\infty}^{\infty} |P(f)|^2\, df = \int_{-\infty}^{\infty} p^2(t)\, dt = \int_0^T p^2(t)\, dt \qquad (11.5\text{-}2)$$

In the last integral in Eq. (11.5-2), the limits take account of the fact that $p(t)$ persists for only a time T. With $p(t) = s_1(t) - s_2(t)$, and using Eq. (11.5-2), we may write Eq. (11.5-1) as

$$\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{2}{\eta}\int_0^T [s_1(t) - s_2(t)]^2\, dt \qquad (11.5\text{-}3a)$$

$$= \frac{2}{\eta}\left[\int_0^T s_1^2(t)\, dt + \int_0^T s_2^2(t)\, dt - 2\int_0^T s_1(t)s_2(t)\, dt\right. \qquad (11.5\text{-}3b)$$

$$= \frac{2}{\eta}(E_{s1} + E_{s2} - 2E_{s12}) \qquad (11.5\text{-}3c)$$

Here E_{s1} and E_{s2} are the energies, respectively, in $s_1(t)$ and $s_2(t)$, while E_{s12} is the energy due to the correlation between $s_1(t)$ and $s_2(t)$.

Suppose that we have selected $s_1(t)$, and let $s_1(t)$ have an energy E_{s1}. Then it can be shown that if $s_2(t)$ is to have the *same energy*, the optimum choice of $s_2(t)$ is

$$s_2(t) = -s_1(t) \qquad (11.5\text{-}4)$$

The choice is optimum in that it yields a maximum output signal $p_o^2(T)$ for a given signal energy. Letting $s_2(t) = -s_1(t)$, we find

$$E_{s1} = E_{s2} = -E_{s12} \equiv E_s$$

and Eq. (11.5-3c) becomes

$$\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max} = \frac{8E_s}{\eta} \qquad (11.5\text{-}5)$$

Rewriting Eq. (11.3-4b) using $p_o(T) = s_{o1}(T) - s_{o2}(T)$, we have

$$P_e = \frac{1}{2}\,\text{erfc}\left[\frac{p_o(T)}{2\sqrt{2}\,\sigma_o}\right] = \frac{1}{2}\,\text{erfc}\left[\frac{p_o^2(T)}{8\sigma_o^2}\right]^{1/2} \qquad (11.5\text{-}6)$$

Combining Eq. (11.5-6) with (11.5-5), we find that the minimum error probability $(P_e)_{min}$ corresponding to a maximum value of $p_o^2(T)/\sigma_o^2$ is

$$(P_e)_{min} = \frac{1}{2}\,\text{erfc}\left\{\frac{1}{8}\left[\frac{p_o^2(T)}{\sigma_o^2}\right]_{max}\right\}^{1/2} \qquad (11.5\text{-}7)$$

$$= \frac{1}{2}\,\text{erfc}\left(\frac{E_s}{\eta}\right)^{1/2} \qquad (11.5\text{-}8)$$

We note that Eq. (11.5-8) establishes more generally the idea that the error probability depends only on the signal energy and not on the signal waveshape. Previously we had established this point only for signals which had constant voltage levels.

We note also that Eq. (11.5-8) gives $(P_e)_{min}$ for the case of the matched filter and when $s_1(t) = -s_2(t)$. In Sec. 11.2 we considered the case when $s_1(t) = +V$ and $s_2(t) = -V$ and the filter employed was an integrator. There we found [Eq. (11.2-3)] that the result for P_e was identical with $(P_e)_{min}$ given in Eq. (11.5-8). This agreement leads us to suspect that for an input signal where $s_1(t) = +V$ and $s_2(t) = -V$, the integrator is the matched filter. Such is indeed the case. For when we have

$$s_1(t) = V \qquad 0 \le t \le T \tag{11.5-9a}$$

$$s_2(t) = -V \qquad 0 \le t \le T \tag{11.5-9b}$$

the impulse response of the matched filter is, from Eq. (11.4-4),

$$h(t) = \frac{2K}{\eta}[s_1(T-t) - s_2(T-t)] \tag{11.5-10}$$

The quantity $s_1(T-t) - s_2(T-t)$ is a pulse of amplitude $2V$ extending from $t = 0$ to $t = T$ and may be rewritten, with $u(t)$ the unit step,

$$h(t) = \frac{2K}{\eta}(2V)[u(t) - u(t-T)] \tag{11.5-11}$$

The constant factor of proportionality $4KV/\eta$ in the expression for $h(t)$ (that is, the gain of the filter) has no effect on the probability of error since the gain affects signal and noise alike. We may therefore select the coefficient K in Eq. (11.5-11) so that $4KV/\eta = 1$. Then the inverse transform of $h(t)$, that is, the transfer function of the filter, becomes, with s the Laplace transform variable,

$$H(s) = \frac{1}{s} - \frac{e^{-sT}}{s} \tag{11.5-12}$$

The first term in Eq. (11.5-12) represents an integration beginning at $t = 0$, while the second term represents an integration with reversed polarity beginning at $t = T$. The overall response of the matched filter is an integration from $t = 0$ to $t = T$ and a zero response thereafter. In a physical system, as already described, we achieve the effect of a zero response after $t = T$ by sampling at $t = T$, so that so far as the determination of one bit is concerned we ignore the response after $t = T$.

COHERENT RECEPTION: CORRELATION:

We discuss now an alternative type of receiving system which, as we shall see, is identical in performance with the matched filter receiver. Again, as shown in Fig. 11.6-1, the input is a binary data waveform $s_1(t)$ or $s_2(t)$ corrupted by noise $n(t)$. The bit length is T. The received signal plus noise $v_i(t)$ is multiplied by a locally generated waveform $s_1(t) - s_2(t)$. The output of the multiplier is passed through an integrator whose output is sampled at $t = T$. As before, immediately after each sampling, at the beginning of each new bit interval, all energy-storing elements in the integrator are discharged. This type of receiver is called a *correlator*, since we are *correlating* the received signal and noise with the waveform $s_1(t) - s_2(t)$.

The output signal and noise of the correlator shown in Fig. 11.6-1 are

$$s_o(T) = \frac{1}{\tau}\int_0^T s_i(t)[s_1(t) - s_2(t)]\, dt \tag{11.6-1}$$

$$n_o(T) = \frac{1}{\tau}\int_0^T n(t)[s_1(t) - s_2(t)]\, dt$$

(11.6-2)

Where $s_1(t)$ is either $s_1(t)$ or $s_2(t)$,and wthere π is the constant of the integrator(i.e.,the integrator output is 1/π times the integral of its input).we now compare these outputs with the matched filter outputs.

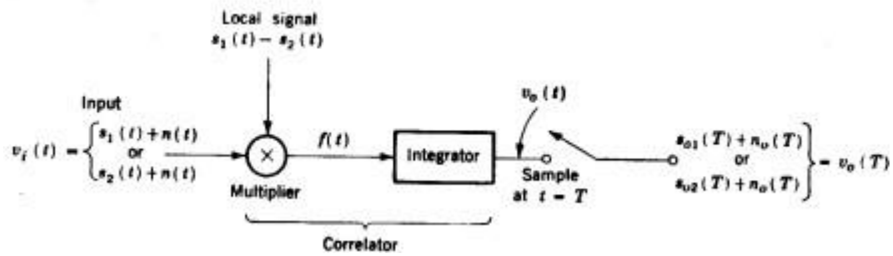


Fig:11.6-1 Coherent system of signal reception

If h(t) is the impulsive response of the matched filter ,then the output of the matched filter $v_0(t)$ can be found using the convolution integral. we have

$$v_o(t) = \int_{-\infty}^{\infty} v_i(\lambda)h(t-\lambda)\, d\lambda = \int_0^T v_i(\lambda)h(t-\lambda)\, d\lambda \qquad (11.6\text{-}3)$$

The limits on the integral have been charged to 0 and T since we are interested in the filter response to a bit which extends only over that interval. Using Eq.(11.4-4) which gives h(t) for the matched filter, we have

$$h(t) = \frac{2K}{\eta}[s_1(T-t) - s_2(T-t)] \qquad (11.6\text{-}4)$$

so that

$$h(t-\lambda) = \frac{2K}{\eta}[s_1(T-t+\lambda) - s_2(T-t+\lambda)] \qquad (11.6\text{-}5)$$

sub 11.6-5 in 11.6-3

$$v_o(t) = \frac{2K}{\eta}\int_0^T v_i(\lambda)[s_1(T-t+\lambda) - s_2(T-t+\lambda)]\, d\lambda \qquad (11.6\text{-}6)$$

Since $v_i(\lambda) = s_i(\lambda) + n(\lambda)$, and $v_o(t) = s_o(t) + n_o(t)$, setting $t = T$ yields

$$s_o(T) = \frac{2K}{\eta}\int_0^T s_i(\lambda)[s_1(\lambda) - s_2(\lambda)]\, d\lambda \qquad (11.6\text{-}7)$$

where $s_i(\lambda)$ is equal to $s_1(\lambda)$ or $s_2(\lambda)$. Similarly we find that

$$n_o(T) = \frac{2K}{\eta}\int_0^T n(\lambda)[s_1(\lambda) - s_2(\lambda)]\, d\lambda \qquad (11.6\text{-}8)$$

Thus $s_0(T)$ and $n_0(T)$, as calculated from eqs.(11.6-1) and (11.6-2) for the correlation receiver, and as calculated from eqs.(11.6-7) and (11.6-8) for the matched filter receiver, are identical .hence the performances of the two systems are identical. The matched filter and the correlator are not simply two distinct, independent techniques which happens to yield the same result. In fact they are two techniques of synthesizing the optimum filter h(t)

5.13.1 Error Probability of ASK

In Amplitude Shift Keying (ASK), some number of carrier cycles are transmitted to send '1' and no signal is transmitted for binary '0'. Thus,

$$\text{Binary '1'} \Rightarrow \quad x_1(t) = \sqrt{2P_s}\cos(2\pi f_0 t) \text{ and}$$

$$\text{Binary '0'} \Rightarrow \quad x_2(t) = 0 \quad \text{(i.e. no signal)} \qquad \text{... (5.13.1)}$$

Here P_s is the normalized power of the signal in 1Ω load. i.e. power $P_s = \frac{A^2}{2}$. Hence $A = \sqrt{2P_s}$. Therefore in above equation for $x_1(t)$ amplitude 'A' is replaced by $\sqrt{2P_s}$.

We know that the probability of error of the optimum filter is given as,

$$P_e = \frac{1}{2} erfc\left\{\frac{x_{01}(T) - x_{02}(T)}{2\sqrt{2}\,\sigma}\right\} \qquad \text{... (5.13.2)}$$

Here $\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{S_{ni}(f)}\, df$

The above equations can be applied to matched filter when we consider white Gaussian noise. The power spectral density of white Gaussian noise is given as,

$$S_{ni}(f) = \frac{N_0}{2}$$

Putting this value of $S_{ni}(f)$ in above equations we get,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \int_{-\infty}^{\infty} \frac{|X(f)|^2}{\frac{N_0}{2}}\, df$$

$$= \frac{2}{N_0}\int_{-\infty}^{\infty} |X(f)|^2\, df \qquad \text{... (5.13.3)}$$

Parseval's power theorem states that,

$$\int_{-\infty}^{\infty} |X(f)|^2\, df = \int_{-\infty}^{\infty} x^2(t)\, dt$$

Hence equation 5.13.3 becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_{-\infty}^{\infty} x^2(t)\, dt$$

We know that $x(t)$ is present from 0 to T. Hence limits in above equation can be changed as follows :

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x^2(t)\,dt \qquad \dots (5.13.4)$$

We know that $x(t) = x_1(t) - x_2(t)$. For ASK $x_2(t)$ is zero, hence $x(t) = x_1(t)$. Hence above equation becomes,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T x_1^2(t)\,dt$$

Putting equation of $x_1(t)$ from equation 5.13.1 in above equation we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2}{N_0}\int_0^T \left[\sqrt{2P_s}\cos(2\pi f_0 t)\right]^2 dt$$

$$= \frac{4P_s}{N_0}\int_0^T \cos^2(2\pi f_0 t)\,dt$$

We know that $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. Here applying this formula to $\cos^2(2\pi f_0 t)$ we get,

$$\left[\frac{x_{01}(T)-x_{02}(T)}{\sigma}\right]^2_{max} = \frac{4P_s}{N_0}\int_0^T \frac{1+\cos 4\pi f_0 t}{2}\,dt$$

$$= \frac{4P_s}{N_0}\cdot\frac{1}{2}\left\{\int_0^T dt + \int_0^T \cos 4\pi f_0 t\,dt\right\}$$

$$= \frac{2P_s}{N_0}\left\{[t]_0^T + \left[\frac{\sin 4\pi f_0 t}{4\pi f_0}\right]_0^T\right\}$$

$$= \frac{2P_s}{N_0}\left\{T + \frac{\sin 4\pi f_0 T}{4\pi f_0}\right\} \qquad \dots (5.13.5)$$

We know that T is the bit period and in this one bit period, the carrier has integer number of cycles. Thus the product $f_0 T$ is an integer. This is illustrated in Fig. 5.13.1

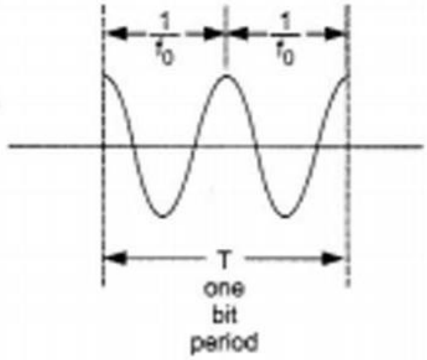


Fig. 5.13.1 In one bit period T, the carrier completes its two cycles. The carrier has frequency f_0. From figure we can write,

$$T = \frac{1}{f_0} + \frac{1}{f_0}$$

i.e. $$T = \frac{2}{f_0}$$

$\therefore$ $$f_0 T = 2 \quad \text{(integer no. of cycles)}$$

As shown in above figure, the carrier completes two cycles in one bit duration. Hence

$$f_0 T = 2$$

Therefore, in general if carrier completes 'k' number of cycles, then,

$$f_0 T = k \qquad \text{(Here } k \text{ is an integer)}$$

Therefore the sine term in equation 5.13.5 becomes, $\sin 4\pi k$ and k is integer.

For all integer values of k, $\sin 4\pi k = 0$. Hence equation 5.13.5 becomes,

$$\left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]^2_{max} = \frac{2P_s T}{N_0} \qquad \text{... (5.13.6)}$$

$$\therefore \quad \left[\frac{x_{01}(T) - x_{02}(T)}{\sigma}\right]_{max} = \sqrt{\frac{2P_s T}{N_0}} \qquad \text{... (5.13.7)}$$

Putting this value in equation 5.13.2 we get error probability of ASK using matched filter detection as,

$$P_e = \frac{1}{2} erfc\left\{\frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{2P_s T}{N_0}}\right\} = \frac{1}{2} erfc\sqrt{\frac{P_s T}{4N_0}}$$

Here $P_s T = E$, i.e. energy of one bit hence above equation becomes,

$$\textbf{Error probability of ASK : } P_e = \frac{1}{2} erfc\sqrt{\frac{E}{4N_0}} \qquad \text{... (5.13.8)}$$

This is the expression for error probability of ASK using matched filter detection.

Error Probability of Binary FSK

The observation vector **x** has two elements x_1 and x_2 that are defined by, respectively,

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\, dt \tag{6.92}$$

$$x_2 = \int_0^{T_b} x(t)\phi_2(t)\, dt \tag{6.93}$$

where $x(t)$ is the received signal, the form of which depends on which symbol was transmitted. Given that symbol 1 was transmitted, $x(t)$ equals $s_1(t) + w(t)$, where $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. If, on the other hand, symbol 0 was transmitted, $x(t)$ equals $s_2(t) + w(t)$.

Now, applying the decision rule of Equation (5.59), we find that the observation space is partitioned into two decision regions, labeled Z_1 and Z_2 in Figure 6.25. The decision boundary, separating region Z_1 from region Z_2 is the perpendicular bisector of the line joining the two message points. The receiver decides in favor of symbol 1 if the received signal point represented by the observation vector **x** falls inside region Z_1. This occurs when $x_1 > x_2$. If, on the other hand, we have $x_1 < x_2$, the received signal point falls inside region Z_2, and the receiver decides in favor of symbol 0. On the decision boundary, we have $x_1 = x_2$, in which case the receiver makes a random guess in favor of symbol 1 or 0.

Define a new Gaussian random variable Y whose sample value y is equal to the difference between x_1 and x_2; that is,

$$y = x_1 - x_2 \tag{6.94}$$

The mean value of the random variable Y depends on which binary symbol was transmitted. Given that symbol 1 was transmitted, the Gaussian random variables X_1 and X_2, whose sample values are denoted by x_1 and x_2, have mean values equal to $\sqrt{E_b}$ and zero, respectively. Correspondingly, the conditional mean of the random variable Y, given that symbol 1 was transmitted, is

$$\begin{aligned} E[Y|1] &= E[X_1|1] - E[X_2|1] \\ &= +\sqrt{E_b} \end{aligned} \tag{6.95}$$

On the other hand, given that symbol 0 was transmitted, the random variables X_1 and X_2 have mean values equal to zero and $\sqrt{E_b}$, respectively. Correspondingly, the conditional mean of the random variable Y, given that symbol 0 was transmitted, is

$$E[Y|0] = E[X_1|0] - E[X_2|0] \\ = -\sqrt{E_b} \tag{6.96}$$

The variance of the random variable Y is independent of which binary symbol was transmitted. Since the random variables X_1 and X_2 are statistically independent, each with a variance equal to $N_0/2$, it follows that

$$\text{var}[Y] = \text{var}[X_1] + \text{var}[X_2] \\ = N_0 \tag{6.97}$$

Suppose we know that symbol 0 was transmitted. The conditional probability density function of the random variable Y is then given by

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(y + \sqrt{E_b})^2}{2N_0}\right] \tag{6.98}$$

Since the condition $x_1 > x_2$, or equivalently, $y > 0$, corresponds to the receiver making a decision in favor of symbol 1, we deduce that the conditional probability of error, given that symbol 0 was transmitted, is

$$\begin{aligned} p_{10} &= P(y > 0 | \text{symbol 0 was sent}) \\ &= \int_0^\infty f_Y(y|0)\, dy \\ &= \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty \exp\left[-\frac{(y + \sqrt{E_b})^2}{2N_0}\right] dy \end{aligned} \tag{6.99}$$

$$\frac{y + \sqrt{E_b}}{\sqrt{2N_0}} = z \tag{6.100}$$

Then, changing the variable of integration from y to z, we may rewrite Equation (6.99) as follows:

$$\begin{aligned} p_{10} &= \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/2N_0}}^\infty \exp(-z^2)\, dz \\ &= \frac{1}{2} \text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \end{aligned} \tag{6.101}$$

Similarly, we may show the p_{01}, the conditional probability of error given that symbol 1 was transmitted, has the same value as in Equation (6.101). Accordingly, averaging p_{10} and p_{01}, we find that the *average probability of bit error* or, equivalently, the *bit error rate for coherent binary FSK* is (assuming equiprobable symbols)

$$P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) \qquad (6.102)$$

Comparing Equations (6.20) and (6.102), we see that, in a coherent binary FSK system, we have to double the *bit energy-to-noise density ratio*, E_b/N_0, to maintain the same bit error rate as in a coherent binary PSK system. This result is in perfect accord with the signal-space diagrams of Figures 6.3 and 6.25, where we see that in a binary PSK system the Euclidean distance between the two message points is equal to $2\sqrt{E_b}$, whereas in a binary FSK system the corresponding distance is $\sqrt{2E_b}$. For a prescribed E_b, the minimum distance $d_{\min}$ in binary PSK is therefore $\sqrt{2}$ times that in binary FSK. Recall from Chapter 5 that the probability of error decreases exponentially as $d^2_{\min}$, hence the difference between the formulas of Equations (6.20) and (6.102).

Error Probability of QPSK

In a coherent QPSK system, the received signal $x(t)$ is defined by

$$x(t) = s_i(t) + w(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, 3, 4 \end{cases} \qquad (6.28)$$

where $w(t)$ is the sample function of a white Gaussian noise process of zero mean and power spectral density $N_0/2$. Correspondingly, the observation vector $\mathbf{x}$ has two elements, x_1 and x_2, defined by

$$\begin{aligned} x_1 &= \int_0^T x(t)\phi_1(t)\,dt \\ &= \sqrt{E}\cos\left[(2i-1)\frac{\pi}{4}\right] + w_1 \qquad (6.29) \\ &= \pm\sqrt{\frac{E}{2}} + w_1 \end{aligned}$$

$$\begin{aligned} x_2 &= \int_0^T x(t)\phi_2(t)\, dt \\ &= -\sqrt{E}\sin\left[(2i-1)\frac{\pi}{4}\right] + w_2 \\ &= \mp\sqrt{\frac{E}{2}} + w_2 \end{aligned} \tag{6.30}$$

Thus the observable elements x_1 and x_2 are sample values of independent Gaussian random variables with mean values equal to $\pm\sqrt{E/2}$ and $\mp\sqrt{E/2}$, respectively, and with a common variance equal to $N_0/2$.

The decision rule is now simply to decide that $s_1(t)$ was transmitted if the received signal point associated with the observation vector **x** falls inside region Z_1, decide that $s_2(t)$ was transmitted if the received signal point falls inside region Z_2, and so on. An erroneous decision will be made if, for example, signal $s_4(t)$ is transmitted but the noise $w(t)$ is such that the received signal point falls outside region Z_4.

To calculate the average probability of symbol error, we note from Equation (6.24) that a coherent QPSK system is in fact equivalent to two coherent binary PSK systems working in parallel and using two carriers that are in phase quadrature; this is merely a statement of the quadrature-carrier multiplexing property of coherent QPSK. The in-phase channel output x_1 and the quadrature channel output x_2 (i.e., the two elements of the observation vector **x**) may be viewed as the individual outputs of the two coherent binary PSK systems. Thus, according to Equations (6.29) and (6.30), these two binary PSK systems may be characterized as follows:

- The signal energy per bit is $E/2$.
- The noise spectral density is $N_0/2$.

Hence, using Equation (6.20) for the average probability of bit error of a coherent binary PSK system, we may now state that the average probability of bit error in *each* channel of the coherent QPSK system is

$$\begin{aligned} P' &= \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E/2}{N_0}}\right) \\ &= \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned} \tag{6.31}$$

Another important point to note is that the bit errors in the in-phase and quadrature channels of the coherent QPSK system are statistically independent. The in-phase channel makes a decision on one of the two bits constituting a symbol (dibit) of the QPSK signal, and the quadrature channel takes care of the other bit. Accordingly, the *average probability of a correct decision* resulting from the combined action of the two channels working together is

$$\begin{aligned} P_c &= (1 - P')^2 \\ &= \left[1 - \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)\right]^2 \\ &= 1 - \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) + \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned} \tag{6.32}$$

The average probability of symbol error for coherent QPSK is therefore

$$\begin{aligned} P_e &= 1 - P_c \\ &= \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) \end{aligned} \tag{6.33}$$

In the region where $(E/2N_0) \gg 1$, we may ignore the quadratic term on the right-hand side of Equation (6.33), so we approximate the formula for the average probability of symbol error for coherent QPSK as

$$P_e \simeq \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \tag{6.34}$$

The formula of Equation (6.34) may also be derived in another insightful way, using the signal-space diagram of Figure 6.6. Since the four message points of this diagram are circularly symmetric with respect to the origin, we may apply Equation (5.92), reproduced here in the form

$$P_e \le \frac{1}{2}\sum_{\substack{k=1 \\ k \ne i}}^{4} \operatorname{erfc}\left(\frac{d_{ik}}{2\sqrt{N_0}}\right) \qquad \text{for all } i \tag{6.35}$$

Consider, for example, message point m_1 (corresponding to dibit 10) chosen as the transmitted message point. The message points m_2 and m_4 (corresponding to dibits 00 and 11) are the *closest* to m_1. From Figure 6.6 we readily find that m_1 is equidistant from m_2 and m_4 in a Euclidean sense, as shown by

$$d_{12} = d_{14} = \sqrt{2E}$$

Assuming that E/N_0 is large enough to ignore the contribution of the most distant message point m_3 (corresponding to dibit 01) relative to m_1, we find that the use of Equation (6.35) yields an approximate expression for P_e that is the same as Equation (6.34). Note that in mistaking either m_2 or m_4 for m_1, a single bit error is made; on the other hand, in mistaking m_3 for m_1, two bit errors are made. For a high enough E/N_0, the likelihood of both bits of a symbol being in error is much less than a single bit, which is a further justification for ignoring m_3 in calculating P_e when m_1 is sent.

In a QPSK system, we note that since there are two bits per symbol, the transmitted signal energy per symbol is twice the signal energy per bit, as shown by

$$E = 2E_b \tag{6.36}$$

Thus expressing the average probability of symbol error in terms of the ratio E_b/N_0, we may write

$$P_e \simeq \text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{6.37}$$

With Gray encoding used for the incoming symbols, we find from Equations (6.31) and (6.36) that the *bit error rate* of QPSK is exactly

$$\text{BER} = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{6.38}$$

We may therefore state that a coherent QPSK system achieves the same average probability of bit error as a coherent binary PSK system for the same bit rate and the same E_b/N_0, but uses only half the channel bandwidth. Stated in a different way, for the same E_b/N_0 and therefore the same average probability of bit error, a coherent QPSK system transmits information at twice the bit rate of a coherent binary PSK system for the same channel

bandwidth. For a prescribed performance, QPSK uses channel bandwidth better than binary PSK, which explains the preferred use of QPSK over binary PSK in practice.

ERROR PROBABILITY OF BINARY PSK:

To realize a rule for making a decision in favor of symbol 1 or symbol 0,we partition the signal space into two regions:

- The set of points closest to message point 1 at $+\sqrt{E_b}$.
- The set of points closest to message point 2 at $-\sqrt{E_b}$.

This is accomplished by constructing the midpoint of the line joining these two message points, and then marking off the appropriate decision regions. In Figure 6.3 these decision regions are marked Z_1 and Z_2, according to the message point around which they are constructed.

The decision rule is now simply to decide that signal $s_1(t)$ (i.e., binary symbol 1) was transmitted if the received signal point falls in region Z_1, and decide that signal $s_2(t)$ (i.e., binary symbol 0) was transmitted if the received signal point falls in region Z_2. Two kinds of erroneous decisions may, however, be made. Signal $s_2(t)$ is transmitted, but the noise is such that the received signal point falls inside region Z_1 and so the receiver decides in favor of signal $s_1(t)$. Alternatively, signal $s_1(t)$ is transmitted, but the noise is such that the received signal point falls inside region Z_2 and so the receiver decides in favor of signal $s_2(t)$.

To calculate the probability of making an error of the first kind, we note from Figure 6.3 that the decision region associated with symbol 1 or signal $s_1(t)$ is described by

$$Z_1: 0 < x_1 < \infty$$

where the observable element x_1 is related to the received signal $x(t)$ by

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)\,dt \tag{6.15}$$

The conditional probability density function of random variable X_1, given that symbol 0 [i.e., signal $s_2(t)$] was transmitted, is defined by

$$\begin{aligned} f_{X_1}(x_1|0) &= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{21})^2\right] \\ &= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] \end{aligned} \tag{6.16}$$

The conditional probability of the receiver deciding in favor of symbol 1, given that symbol 0 was transmitted, is therefore

$$\begin{aligned} p_{10} &= \int_0^{\infty} f_{X_1}(x_1|0)\,dx_1 \\ &= \frac{1}{\sqrt{\pi N_0}} \int_0^{\infty} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2\right] dx_1 \end{aligned} \tag{6.17}$$

Putting

$$z = \frac{1}{\sqrt{N_0}}(x_1 + \sqrt{E_b}) \tag{6.18}$$

and changing the variable of integration from x_1 to z, we may rewrite Equation (6.17) in the compact form

$$\begin{aligned} p_{10} &= \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp(-z^2)\, dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \end{aligned} \tag{6.19}$$

where erfc(·) is the complementary error function.

Thus, averaging the conditional error probabilities p_{10} and p_{01}, we find that the *average probability of symbol error* or, equivalently, the *bit error rate for coherent binary PSK* is (assuming equiprobable symbols)

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) \tag{6.20}$$

As we increase the transmitted signal energy per bit, E_b, for a specified noise spectral density N_0, the message points corresponding to symbols 1 and 0 move further apart, and the average probability of error P_e is correspondingly reduced in accordance with Equation (6.20), which is intuitively satisfying.