Thin Linear wire Antennas

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Introduction:- The electric charges are the sources of electro-magnetic fields. when these sources are time varying, the EM waves Propagate away from sources and radiation takes place.

Radiation can be considered as the process of transmitting electric energy. The radiation of EM waves into space is effectively achieved with the help of conducting or dielectric structure called as antennas or radiators.

Note:- Antenna can also be defined as a transmission or matching unit between the sources & waves in space.

In the design of antenna system we must consider important requirements such as the antenna pattern, the total power radiated the i/p impedance of radiator, the radiation efficiency etc.

The direct solution for these requirements can be obtained by solving maxwell's equations with appropriate boundary conditions of the radiator and at infinity.

Retarded potentials:-

In case of electrostatic fields or steady magnetic fields, the electric field or magnetic field can be obtained easily by first setting the potentials interms of the charges or currents.

For obtaining the potentials for the electromagnetic field, there are different approches.

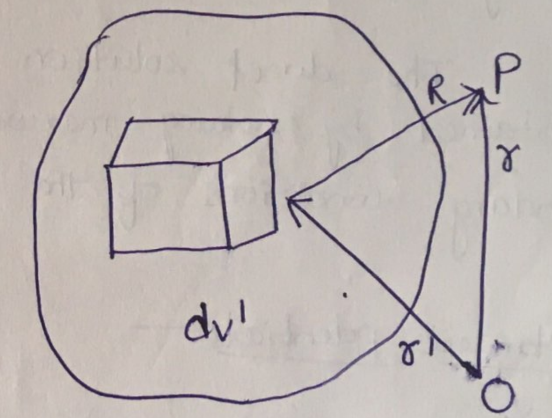
In the first approach, using trail and error, the potentials for the electric and magnetic fields are generated (generalized). Then it is shown that these potentials satisfy the maxwell equations. and this approach is called heuristic approach.

The second approach is to start with the maxwell's equations and then derive the differential equations that the potentials satisfy

The third approach is to obtain directly the solutions of derived differential equations for the potentials.

Heuristic approach:-

For this first of all we have to obtain the potentials for electric and magnetic fields. Consider a uniform volume charge density ρ_v, over the given volume shown as fig below



Consider the differential volume dv' at a point distance "r'" from the orign where charge density $\rho_v(r')$.

Then the scalar electric potential 'V' at a point 'p' can be expressed interms of static charge distribution as

Potential due to volume charge density

$$v = \frac{q}{4\pi\epsilon R} = \frac{1}{4\pi\epsilon}\frac{q}{R}$$

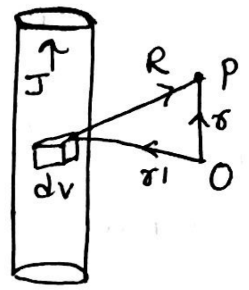
$$V(r) = \int_v \frac{\rho_v(r')\,dv'}{4\pi\epsilon R} \quad -(1)$$

Then the fundamental electric field can be obtained ② by finding the gradient of scalar potential 'V'.

$$E = -\nabla V \quad -②$$ (−ve of gradient of potential is equal to electric field).

Similarly, for steady magnetic field, in a homogeneous medium, the vector magnetic potential $\bar{A}$ can be expressed interms of current distribution which is constant with time as

$$A(r) = \int_v \frac{J(r)\mu\, dv'}{4\pi R} \quad -③$$



Then the fundamental magnetic field can be obtained by finding the curl of vector magnetic field.

$$\therefore \quad \bar{B} = \nabla \times \bar{A} \quad -④$$

The potentials given by equations ① & ③ represent the potentials for the static electric & magnetic fields respectively.

ie; Charge and current distributions are not varying with time 't'.

But the charge and current distributions producing the electromagnetic field vary w.r.t time. So, the above eq's as a function of 't' can be given as

$$V(r,t) = \frac{1}{4\pi\epsilon} \int_v \frac{\rho_v(r,t)}{R} \, dv' \quad — (5)$$

$$A(r,t) = \frac{\mu}{4\pi} \int_v \frac{J(r,t)}{R} \, dv \quad — (6)$$

where $R = |r - r'|$

From the above equations it is clear that the potentials vary instantaneously as the charge and current distribution varies

In general, the Em ~~wo~~ fields propagate with finite velocity where $v = \frac{1}{\sqrt{\frac{\mu}{\epsilon}}} = \sqrt{\frac{\epsilon}{\mu}}$.

The equations expressing the potentials are modified by introducing time delay $\frac{R}{v}$ as follows.

$$V(r,t) = \frac{1}{4\pi\epsilon} \int_v \frac{\rho_v\left(r, t - \frac{R}{v}\right)}{R} \, dv' \quad — (7)$$

$$A(r,t) = \frac{1}{4\pi} \int_v \frac{J\left(r, t - \frac{R}{v}\right)}{R} \, dv' \quad — (8)$$

From eq's (7) & (8) it is clear that the potentials are delayed or retarded by time $\frac{R}{v}$. Hence these potentials are called as retarded potentials.

Antenna theorms:-

③

The Properties of transmitting antenna and receiving antenna are related to each other through varias antenna theorms.

ie; Antenna theorms are applied to analyse the properties of transmitting antenna.

For such antennas a sinusoidal signal is given at Tx antenna, the same sinusoidal current distribution will not be collected at receiving antenna.

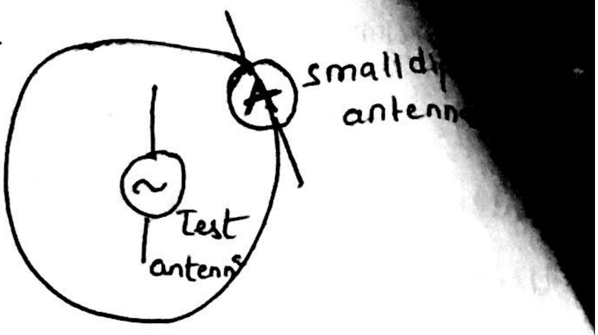
This indicates that the basic antenna property is such as directional property, impedance property. The antenna theorms are

→ 1. Equality of directional patterns:-

Statement:— The directional patterns of an antenna as a receiving antenna is idential to that when used as a Tx antenna.

Proof:— This antenna theorm follows the applications of reciprocity theorm used in linear bilaterlal network. Basically a directional pattern of a Tx antenna is represented as a polar characteristics, because it indicates the strength (amplitude) of the radiated field at a fixed distance in several directions

To measure the directional pattern of antenna as a transmitting antenna, the test antenna is kept at center of very large sphere and small dipole antenna is moved along the surface of sphere as shown in fig.



Directional pattern measurement for a directional pattern antenna.

A voltage 'V' is connected to test antenna placed at the center of sphere and current "I" following in short dipole antenna is measured by using ammeter at different positions.

By the applications of reciprocity theorm, we interchange test antenna and dipole antenna positions and a voltage of 'V' is applied to the dipole antenna and current 'I' at different positions has been calculated for test antenna.

At every position the ratio of $\frac{V}{I}$ is same as before obtained for test antenna as Tx antenna. ie; The directional pattern of Rx antenna is identical to that of Tx antenna.

→ 2. Equivalence of transmitting & receiving antenna impedances:—

Statement:— The impedance of an isolated antenna used for transmitting as well as receiving purposes is identical.

Proof.— Consider two antennas A_1 and A_2 seperated widely, then the self impedance of an antenna A_1 can be given as

$$Z_{11} = \frac{V_1}{I_1}$$

When two antennas are seperated widely, in that case (4) the mutual impedance Z_{12} of antenna "A_1" can be neglected, if antenna 'A_1' is used as Tx antenna.

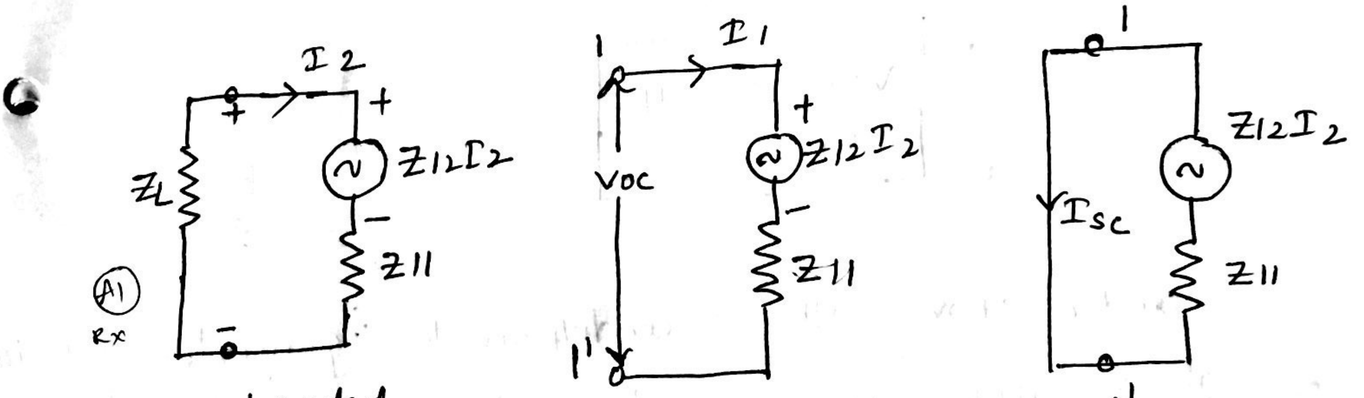
<u>Note</u>:- If the same antenna A_1 is used as receiving antenna, then the mutual impedance Z_{12} cannot be neglected as it is the only parameter indicating coupling b/w two antennas.

→ Let us consider the load Z_L is connected to antenna A_1 used as receiving antenna.

→ The coupling b/w antennas A_1 & A_2 can be represented with the help of $Z_{12}I_2$.

where Z_{12} - Mutual impedance

I_2 - current in antenna A_2. and its equivalent circuit can be given as



(a) Loaded condition (b) Open ckt condition (c) short ckt cond

<u>Equivalent ckts of receiving antennas</u>

Since two antennas are seperated with along a distance the variation in load impedance Z_L connected to antenna A_1 will not change current I_2 in antenna A_2.

ie; The generator value of $Z_{12} I_2$ can be treated as ideal generator with zero internal impedance providing const voltage at its o/p terminal.

Under open circuit condition the voltage across terminals 1 1' can be given as $V_{oc} = Z_{12} I_2$, under short circuit condition, The short circuit current I_{sc} from terminals 1 1' can be given as

$$I_{sc} = \frac{Z_{12} I_2}{Z_{11}}$$

Hence the ratio of V_{oc} and I_{sc} is called transfer impedance and is given by.

$$\frac{V_{oc}}{I_{sc}} = \frac{\cancel{Z_{12} I_2}}{\frac{\cancel{Z_{12} I_2}}{Z_{11}}} = Z_{11}$$

$$\therefore \boxed{\frac{V_{oc}}{I_{sc}} = Z_{11}}$$

Under above all 3 conditions the generator ~~of~~ value of $Z_{12} I_2$ acts as generator with internal impedance Z_{11}. Hence the receiving antenna impedance is equal to transmitting antenna impedance.

Equality of effective lengths:- (5)

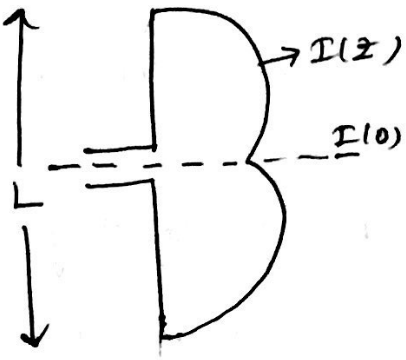
Basically the effective length (Leff) of an antenna represents the effectiveness of antenna as a radiator or collector.

The effective length is defined as the length of equivalent linear antenna which has current Io along its length at all points radiating field strength in direction ⊥er to the length of actual antenna.

ie; For a transmitting antenna

$$I(0)\, L_{eff} = \int_{-L/2}^{L/2} I(z)\, dz$$

$$L_{eff} = \frac{1}{I(0)} \int_{-L/2}^{L/2} I(z)\, dz \quad \text{---} (1)$$



Representation of effective length

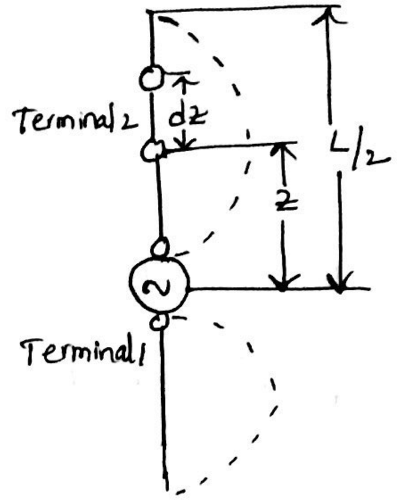
For receiving antenna, the effective length can be given as

$$L_{eff(Rec)} = \frac{-V_{oc}}{E} \quad \text{---} (2)$$ where

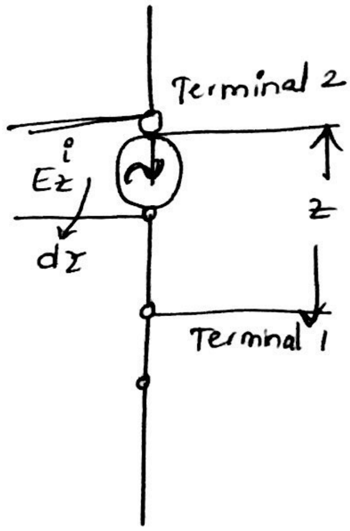
V_{oc} – Open circuit voltage developed at the antenna terminals

E – Received field strength

To show the equality of transmitting and receiving effective lengths, Let us apply reciprocity theorm as shown in fig below



(a) Tx antenna



(b) Rx antenna

For transmitting antenna:—

The current produced at antenna terminals can be given as

$$I(0) = \frac{V}{Z_a}$$ where V – applied voltage at terminals
Z_a – Antenna impedance

Similarly the current at any point 'z' along the antenna is $I(z)$ and is called as prime situtation.

For receiving antenna:—

Let E_z^i will be the electromagnetic field shown in fig (b). This induces voltage $E_z^i dz$ in the element dz.

As the induced voltage is independent of current through the antenna, it can be indicated as ideal generator of voltage $E_z^i dz$ in series, producing a current I_{sc} in the antenna. This is called double prime situtation.

According to reciprocity theorm

$$\frac{V}{I(z)} = \frac{E_z^i dz}{I_{sc} d}$$

$$I_{sc}\, d = \frac{E_z^i dz\, I(z)}{V}$$

$$I_{sc} = \int \frac{E_z^i dz\, I(z)}{V} = \frac{1}{V}\int E_z^i dz\, I(z) \quad — (3)$$

But according to thevenin's theorm, the open cut voltage at antenna terminals is given by

$$V_{oc} = -I_{sc} Z_a.$$

Substitute in the above equation in the value I_{sc}

$$V_{oc} = -\frac{1}{V}\int I(z)\, dz\, E_z^i\, Z_a.$$

$$= \frac{-Z_a}{V}\int I(z)\, E_z^i\, dz$$

$$V_{oc} = -\frac{1}{I(0)}\int I(z)\, dz\, E_z^i$$

$$\frac{-V_{oc}}{E(z)} = \frac{1}{I(0)}\int I(z)\, dz.$$

$$L_{eff} \text{ of receiver} = \frac{-V_{oc}}{E(z)}$$

For constant electric fields. $E_z^i = E(z)$

Since by above relation it is proved that effective length of Tx antenna is equal to receiving antenna.

Loop antennas:—

Introduction:— In general radiowaves propagate from transmitter follow circular route to reach receiver. The method of finding the direction of unknown transmitter with respect to receiver uses the findingout of direction of radiation of the radiowave from it. This method is known " Direction finder" (DF).

Because this method uses a radiowave to find directic this method is called as "Radio Direction Finder" (RDF).

This method is useful in navigating the ships and aeroplanes. The direction finders use loop antennas as they are basic directional antennas.

Basics of loop antennas:— In general a loop antenna is nothing but a radiating coil of any shape with one or more turns carring RF current. The loop antenna may assume one of the following shapes and generally a loop is formed on a ferrite foil (pole) or air.

Rectangle Square circular Triangular Rhombic

Generally a loop may consists of one or more turns on a ferrite or air pole. If a loop consists more than one turn, then it is called as frame.

Based on the dimensions of loop antennas, these are classified as two Categories as

a → The dimensions of loop are very small as compared with one wavelength

b → The dimensions of loop are compared with one wavelength

Consider a Rectangular loop which inturn sides 2314 as vertical arms and 1234 as horizontal arms as shown in fig below

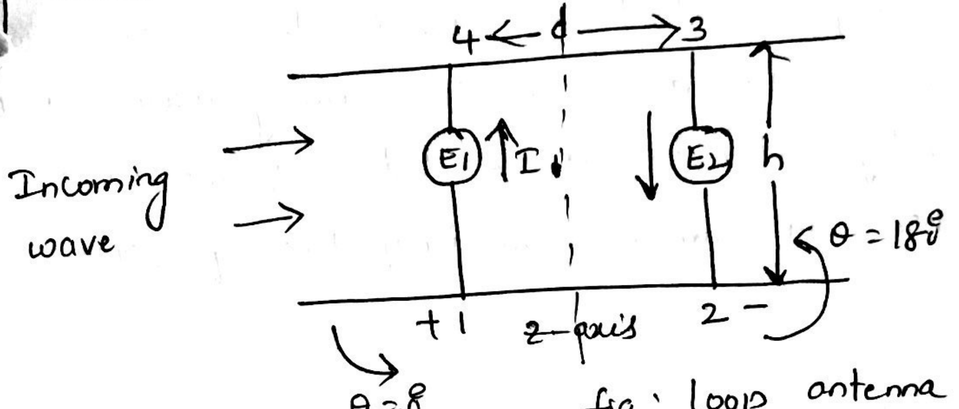


fig : Loop antenna

The horizontal arms and vertical arms of loop antenna acts as horizontal antenna & vertical antenna respectively

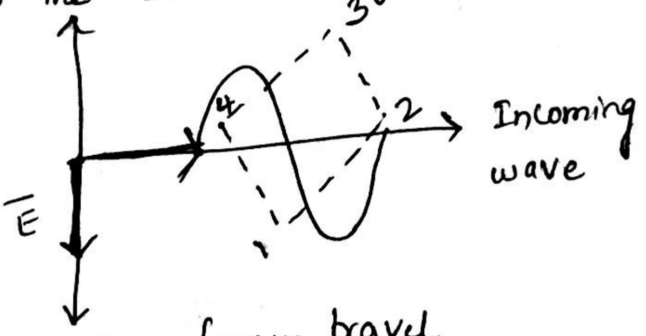
Consider that the loop is such that its plane is ~~at~~ right angles to the wave travells.

If the incoming waves are vertically polarized. then the voltage will be induced in two vertical arms of loop & these voltages are in same magnitude & phase.

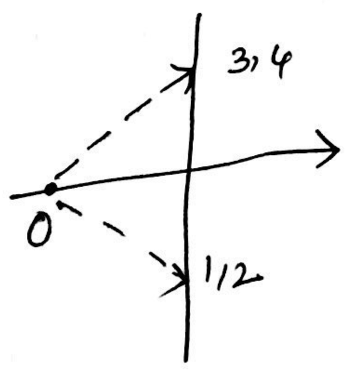
But these two vertical arm are set due to voltages in opposite direction, the two voltages will get Cancelled.

direction right

fig: Loop antenna with its plane in the direction right to the direction of wave.



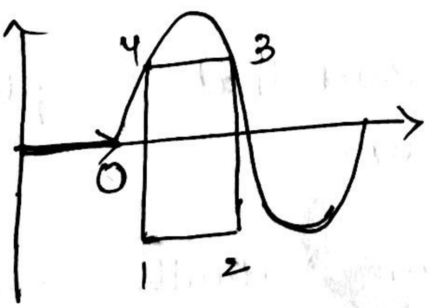
(a) perspective view



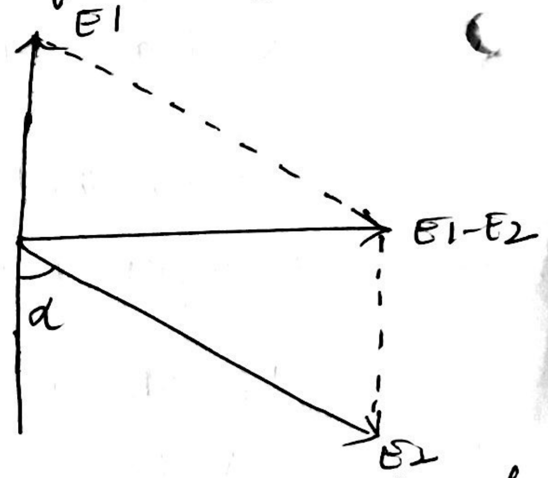
(b) plane or top view

The emf induced in the horizontal arms due to vertical polarized wave always equal to zero while the emf induced in the horizontal arms if the incoming wave would have been horizontally polarized is neglected. ie; The emf induced in the horizontal arm is zero irrespective of polarization of incoming wave.

If the loop is rotated by 90° such that the plane of loop aligns with the direction of incoming wave.



(a) perspective view



(b) Phase representation of EM wave

Now the distance b/w transmitter and vertical antenna is no more same the two emf's induced respectively in the vertical arms will also be of same magnitude (amplitude) and phase

Hence the resultant emf induced along the vertical arms be E1 - E2. Thus we can conclude that the induced emf is max only the plane of loop is in the direction of incoming waves. This condition is successfully used in the direction finding of unknown transmitter.

Field pattern of small loop antenna:-

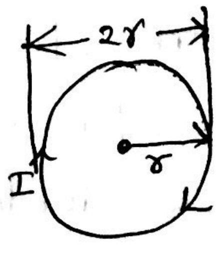
For the transmission purposes the radiation efficiency of closed loop antenna is very low if the dimensions of loop are very small as compared to wavelength

ie) For smaller dimensions the current throughout the loop is same.

→ whenever a loop antenna is used for transmission purposes its dimensions are made comparable to wavelength.

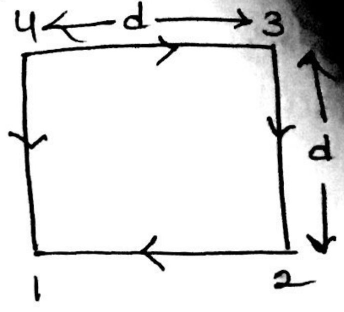
→ Consider a circular loop with radius 'r' whose dimensions are very less comparable to wavelength.

ie; The current throughout the loop is same and can be represented as



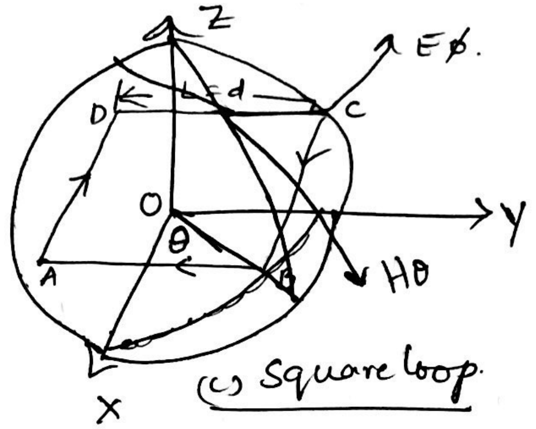
(a) circular loop $r << \lambda$

→ The representation of circular loop in the form of square loop with dimensions d/d such that area of both loops



(b) square loop

→ Consider a square loop is located at the center of Cartesian coordinate system as shown in fig.



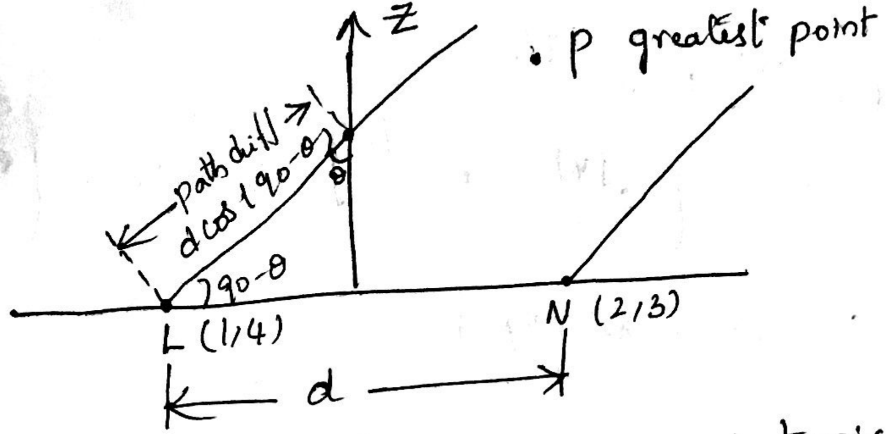
(c) square loop.

To find a far field pattern instead of considering all four short dipoles, it is sufficient to consider two short dipoles as sides 1-4, 2-3.

From fig xy - Horizontal plane

yz - Vertical plane

The far field radiation due to two point sources with reference to 'O' can be represented as Eϕ = field component due to dipole 14 + field component due to dipole 23



d) dipoles in yz planes as isotropic point sources

By observing above fig(d), the waves radiated from dipoles 1,4 will take more time compared to waves radiated from dipoles 2,3.

ie; The extra distance travelled by dipole 1,4 is L to M is called path difference and can be given as $d\cos(90-\theta)$.

$$\text{path difference in wavelengths} = \frac{d\cos(90-\theta)}{\lambda}$$

The phase diff ψ can be given as $= 2\pi(\text{path diff})$

$$= 2\pi \frac{d\cos(90-\theta)}{\lambda}$$

$$= \frac{2\pi}{\lambda} d\cos(90-\theta)$$

$$= \beta d\cos(90-\theta)$$

$$\therefore \psi = \beta d\cos(90-\theta)$$

$$\boxed{\psi = \beta d\sin\theta}$$

To calculate electric field $E\phi$

$$E\phi = E\phi_1 + E\phi_2$$

$$E\phi_1 = -E_0 e^{j\psi/2}$$

$$E\phi_2 = E_0 e^{-j\psi/2}$$

$$E\phi = -E_0 e^{j\psi/2} + E_0 e^{-j\psi/2}$$

$$= -E_0\left[e^{j\psi/2} - e^{-j\psi/2}\right] \Rightarrow -2jE_0\left[\frac{e^{j\psi/2} - e^{-j\psi/2}}{2j}\right]$$

$$E\phi \Rightarrow -2jE_0 \sin(\psi/2)$$

$$= -2jE_0 \sin\left(\frac{\beta d \sin\theta}{2}\right) \text{ V/m.}$$

we know that relation b/w electric & Magnetic field components as related as

$$\frac{E\phi}{H\theta} = \eta_0 \quad \text{where} \quad \boxed{\eta_0 = 120\pi}$$

$$H\theta = \frac{E\phi}{\eta_0} = \frac{-2jE_0 \sin\left(\frac{\beta d \sin\theta}{2}\right)}{120\pi}$$

$$H\theta = \frac{-jE_\phi}{60\pi} \sin\left(\frac{\beta d \sin\theta}{2}\right)$$

The magnitude of individual component can be given by

$$E\theta = j60\pi(I)L\frac{\sin 90}{r\lambda} \qquad \sin 90 = 1$$

$$E\theta = j60\pi[I]\frac{L}{r\lambda}.$$

for smaller values $E\theta = E_0$ and $\sin\theta \approx 0$

$$E\phi = -\cancel{2}jE_0\frac{\beta d \sin\theta}{\cancel{2}} = jE_0\beta d \sin\theta$$

$$E\phi \Rightarrow -2jE_0\,(j60\pi)\frac{IL}{r\lambda}\left[\frac{\beta d \sin\theta}{\cancel{2}}\right]$$

$$= \frac{60\pi IL}{r\lambda}\left[\frac{2\pi}{\lambda} d\sin\theta\right] \qquad [L = d]$$

$$\therefore E\phi = \frac{120\pi^2 I d^2}{r\lambda^2} \sin\theta \qquad \left[\beta = \frac{2\pi}{\lambda}\right]$$

$$E\phi = \frac{120\pi^2 I A \sin\theta}{r\lambda^2}$$

where A - Area of square loop of dimension d/d

[I] - Retarded current

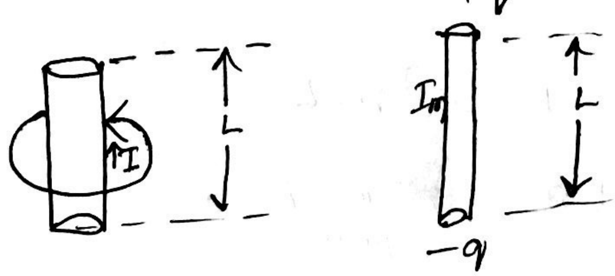
WKT $\frac{E\phi}{H\phi} = \eta_0$

$$H\phi = \frac{E\phi}{\eta_0} = \frac{\frac{120\pi^2 I A \sin\theta}{r\lambda^2}}{120\pi}$$

$$\therefore \boxed{H\phi = \frac{\pi I A \sin\theta}{r\lambda^2}}$$

Comparision of small loop with short dipole:—

A small loop can be considered as equivalent to a short magnetic dipole. Thus a small loop of area 'A' carrying current 'I' can be replaced by short magnetic dipole of length 'L' and carrying magnetic current shown as fig



(a) Small loop (b) short dipole

Equivalence of small loop of area 'A' with short magnetic dipole of length 'L'

Let q_m be the pole strength at each end of magnetic dipole. the magnetic dipole moment is $q_m L$.

Then magnetic current can be expressed interms of pole strength as

$$I_m = -\mu \frac{d}{dt} q_m \quad \text{—(1)} \qquad \text{where } I_m = I_o e^{j\omega t}$$

$$I_{mo} e^{j\omega t} = -\mu \frac{d}{dt} q_m$$

Applying integration on b.s

$$I_{mo} \frac{e^{j\omega t}}{j\omega} = -\mu q_m$$

$$q_m = \frac{-I_m}{j\omega\mu}$$

$$\therefore q_m = \frac{-I_m}{j\omega\mu} \quad \text{— (2)}$$

The magnetic moment of loop is $I \cdot A$, where A is the small loop area and I is the uniform phase current through loop.

ie; equating magnetic moment of loop and magnetic moment of dipole, we can write

$$I \cdot A = q_m \cdot L$$

$$I \cdot A = \frac{-I_m}{j\omega\mu} \cdot L$$

$$-I_m L = IA\,(j\omega\mu) = IA\,(j2\pi f\mu)$$

$$-I_m \cdot L = IA\left(j2\pi f\mu \frac{\lambda}{\lambda}\right) = IA\,[j]\,2\pi\,[\mu]\,\frac{f\lambda}{\lambda}$$

$$-I_m L = jIA \frac{2\pi\mu}{\lambda}[c]$$

$$= jIA \frac{2\pi}{\lambda}[4\pi \times 10^{-7}][3 \times 10^{8}]$$

$$= \frac{jIA\,\pi^2\, 24 \times 10}{\lambda} = \frac{j240\pi^2 IA}{\lambda}$$

$$\boxed{I_m L = \frac{-j\,2\pi\mu A I}{\lambda}} \quad \because \mu = 120\pi$$

General case of loop antenna:—

Consider a loop antenna with uniform inphase current and size is not small, but comparable to one wavelength is placed at the center of cartesian coordinate system with radius 'a'.

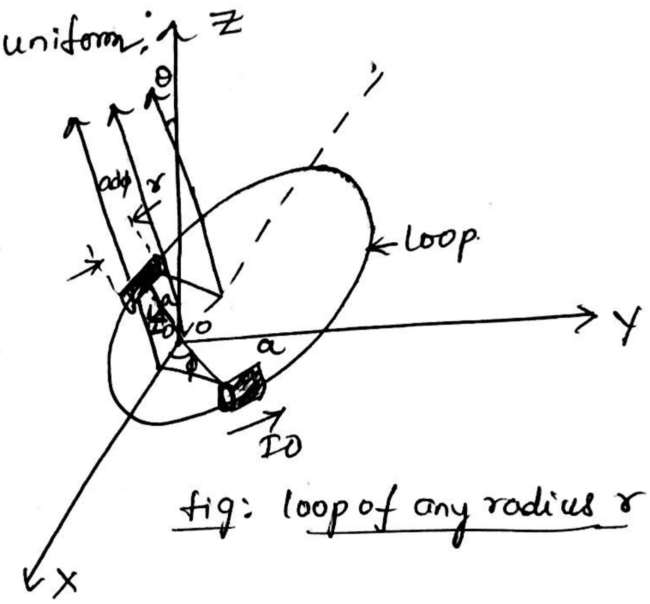


fig: loop of any radius r.

By the diagramatic representation those two magnetic short dipoles are placed in opposite direction with length of "a dφ".

As the current is confined to the loop, the vector potential will have only φ component while the components θ, r directions are zero i.e: a_r, a_θ are zero.

The infinitesimal value of ϕ component of vector potential 'A' at point 'p' can be given as

$$dA_\phi = \frac{\mu\, dm}{4\pi r}$$, where dm - current moment due to pair of infinitesimal dipoles

$$I = I_0 e^{j\omega(t-\frac{r}{c})}$$, I_0 is peak current

Now consider the cross section through the loop in xz plane shown as

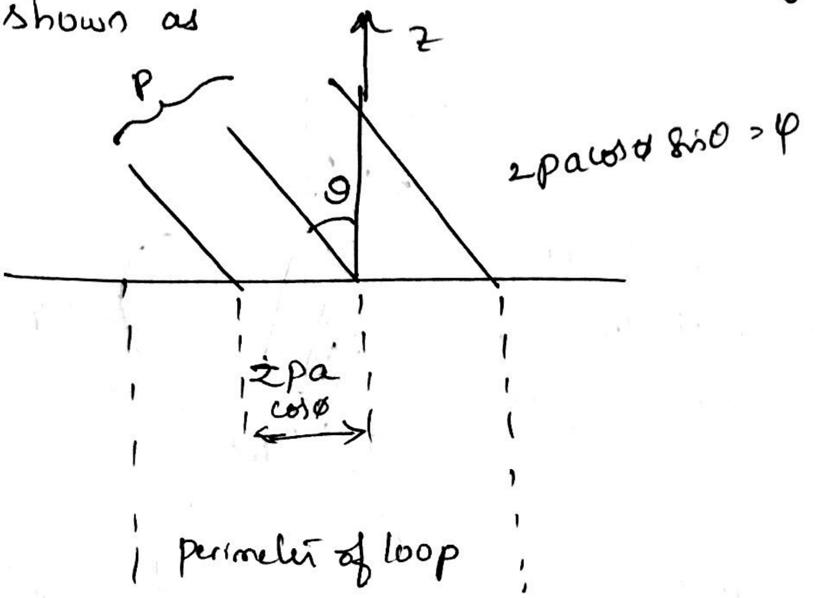


fig: Cross section of loop with radius 'a' in xz plane

The resultant moment can be given as

$$dm = 2j\,[I] \cos\phi \,[a\, d\phi]\, \sin\frac{\psi}{2} \quad \text{—①}$$

$$dm = 2j[I]\,[a\, d\phi] \cos\phi \,\sin\left[\frac{\cancel{2}\beta a \sin\theta \cos\phi}{\cancel{2}}\right]$$

$$dm = 2j\,[I]\,[a\, d\phi] \cos\phi\, \sin[\beta a \sin\theta \cos\phi] \quad \text{—②}$$

Substitute the value of dm in the equation

$$dA\phi = \frac{\mu\, dm}{4\pi r}$$

$$= \frac{\mu}{\cancel{4}\pi r}\, \cancel{2} j [I][a\, d\phi] \cos\phi \sin[\beta a \sin\theta \cos\phi]$$

$$= \frac{j\mu[I][a\, d\phi]}{2\pi r} \cos\phi \sin[\beta a \sin\theta \cos\phi]$$

Integrate both sides with $\phi = 0$ to π

$$A\phi = \int_{\phi=0}^{\pi} \frac{j\mu[I][a\, d\phi]}{2\pi r} \cos\phi \sin[\beta a \sin\theta \cos\phi]$$

$$= \frac{j\mu[I]}{2\pi r} \int_{\phi=0}^{\pi} a\, d\phi \cos\phi \sin(\beta a \sin\theta \cos\phi).$$

$$A\phi = \frac{j\mu[I]a}{2r} J_1 \beta a \sin\theta$$ where

$J_1 \beta a \sin\theta$ is Bessel function of the order $\beta a \sin\theta$

The far field of loop consists only ϕ component and is given by

$$E\phi = -j\omega A\phi$$

$$= -j\omega\left[\frac{j\mu I a}{2r} J_1 \beta a \sin\theta\right]$$

$$= \frac{\mu\omega I a}{2r} J_1 \beta a \sin\theta$$

$$= \frac{\mu\, 2\pi f I a}{2r} J_1 \beta a \sin\theta$$

$$= \frac{\mu \left(\frac{2\pi}{\lambda}\right) f \lambda I a}{2r} J_1 \beta a \sin\theta$$

$$= \frac{\mu c I a^2 \beta^2}{2r} J_1 \sin\theta$$

$$= \frac{4\pi \times 10^{-7} \times 3 \times 10^{8} \times I a^2 J_1 \beta^2 \sin\theta}{2r}$$

$$\therefore E\phi = \frac{60\pi\beta [I][a]}{r} J_1 \beta a \sin\theta$$

$$\eta = \frac{E\phi}{H\theta}$$

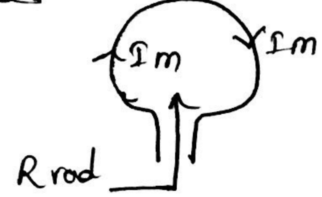
$$\frac{E\phi}{H\theta} = 120\pi \Rightarrow H\theta = \frac{\frac{60\pi \beta [I][a]}{r}}{\frac{120\pi}{2}} J_1 \beta a \sin\theta$$

$$\boxed{\therefore H\theta = \frac{\beta [I][a]}{2r} J_1 \beta a \sin\theta}$$

Radiation resistance of loop antenna:—

To find radiation resistance of loop antenna the poynting vector is integrated over the sphere and its total power radiated is equated to square of rms current and is multiplied with radiation resistance.

$$\therefore P_{rad} = \frac{I_m^2}{2} R_{rad} \quad \text{— (1)}$$



Rrad of loop antenna

The average poynting vector of a far field is given (13)

$$\vec{P} = \frac{1}{2} \left| E\phi \times H\theta^* \right| \quad \text{— (2)}$$

$$= \frac{1}{2} \left| 120\pi \times H\theta \times H\theta^* \right| \qquad \because \eta_0 = \frac{E\theta}{H\phi} = 120\pi$$

$$= \frac{1}{2}\eta_0 \left| H\theta \times H\theta^* \right|$$

$$\therefore \vec{P} = \frac{1}{2}\eta_0 \left| H\theta \right|^2$$

where $H\theta^*$ is conjugate of $H\theta$

$$\boxed{P_{rad} = \frac{1}{2}\eta_0 \left| H\theta \right|^2} \longrightarrow (3)$$

we know that $H\theta = \frac{\beta[I][a]}{2r} J_1 \beta a \sin\theta$

$$P = \frac{1}{2}\eta_0 \left[\frac{\beta[I][a]}{2r} J_1 \beta a \sin\theta \right]^2$$

$$= \frac{1}{2} \, 120\pi \, \frac{\beta^2 [I]^2 [a]^2}{4r^2} J_1 \beta a \sin\theta$$

$$= \frac{60\pi \beta^2 a^2 [I]^2}{4r^2} J_1^2 \beta a \sin\theta$$

$$\therefore P = \frac{15\pi \beta^2 a^2 [I]^2 J_1^2 \beta a \sin\theta}{r^2}$$

$$\boxed{\therefore P = 15\pi \left(\frac{\beta a I}{r^2} \right)^2 J_1^2 \beta a \sin\theta} \longrightarrow (4)$$

To find total power radiated, integrating the above over a large sphere we can write

$$P = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \vec{P}\, ds.$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} 15\pi \frac{(\beta a I)^2}{\cancel{r^2}} J_1^2 \beta a \sin\theta \, \cancel{r^2} \sin\theta \, d\theta\, d\phi$$

$$\Rightarrow 2\pi(15\pi) \int_{\theta=0}^{\pi/2} (\beta a I)^2 J_1^2 \beta a \sin^2\theta \, d\theta$$

$$\Rightarrow 30\pi^2 (\beta a I)^2 \int_{\theta=0}^{\pi/2} \beta a J_1^2 \sin^2\theta \, d\theta$$

$$\boxed{P = 30\pi^2 (\beta a I)^2 \int_{\theta=0}^{\pi/2} J_1^2(\beta a \sin\theta) \sin\theta \, d\theta} \quad \text{——} \; (5)$$

But for loop which is smaller interms of wavelength 'λ', we can write $J_1^2(\beta a \sin\theta) = \left(\frac{\beta a \sin\theta}{2}\right)^2$,

substitute in the above eq's.

$$P = 30\pi^2 (\beta a I)^2 \int_{\theta=0}^{\pi/2} \left(\frac{\beta a \sin\theta}{2}\right)^2 \sin\theta \, d\theta$$

$$= 30\pi^2 (\beta a I)^2 \left(\frac{\beta a}{2}\right)^2 \int_{\theta=0}^{\pi/2} \sin^3\theta \, d\theta$$

(14)

$$\Rightarrow \frac{30\pi^2}{4} (\beta a)^4 I^2 \int_{\theta=0}^{\pi/2} \sin^3\theta \, d\theta$$

$I = I_m$

$$P \Rightarrow \frac{30\pi^2}{4} \left(\beta a\right)^4 I_m^2 \int_{\theta=0}^{\pi/2} \sin^3\theta \, d\theta$$

Solving the above integral, finally we get

$$\boxed{P = 10\pi^2 (\beta a)^4 I_m^2} \quad \text{— (6)}$$

Assuming ideal condition equate eq (1) & (6), we get

$$\therefore R_{rad} = 20\pi^2 (\beta a)^4$$

$$= 20\pi^2 \left(\frac{2\pi}{\lambda} a\right)^4$$

$$R_{rad} = 20\beta^4 (\pi a^2)^2 = 20\beta^4 A^2$$

$$\boxed{\therefore R_{rad} = 20\beta^4 A^2} \ \Omega \quad \text{— (7)}$$

we know that $\beta = \frac{2\pi}{\lambda}$

$$\therefore R_{rad} = 20 \left(\frac{2\pi}{\lambda}\right)^4 A^2 = 20 \frac{(2\pi)^4}{\lambda^4} A^2$$

$$\boxed{\therefore R_{rad} = 31,170 \frac{A^2}{\lambda^4} \ \Omega}$$

$$\boxed{\therefore R_{rad} = 31200 \left(\frac{A}{\lambda^2}\right)^2 \Omega} \longrightarrow (8)$$

$\therefore \quad R_{rad} = \pi^4 320 \left(\frac{A}{\lambda^2}\right)^2 \Omega$ — (9)

All the above equations are for the loop in single term.

If the loop consists N no of turns, then Radiation resistance is N^2 times the radiation resistance of loop

$$\boxed{\therefore \quad R_{rad} = 320\pi^2\left(\frac{NA}{\lambda^2}\right)^2 \Omega = 31200\left(\frac{A}{\lambda^2}\right)^2 N^2 \Omega} \quad — (10)$$

Considering eq (7), again

$$R_{rad} = 20\beta^4 A^2, \text{where } \beta = \frac{2\pi}{\lambda}$$

$$R_{rad} = 20\left(\frac{2\pi}{\lambda}\right)^4 A^2$$

$$= 20\left(\frac{2\pi}{\lambda}\right)^4 (\pi a^2)^2$$

$$= 20\frac{\pi^2}{\lambda^4}(2\pi a)^4$$

$$= 20\left(\frac{\pi}{\lambda^2}\right)^2 c^4$$

$$\boxed{R_{rad} = 20\pi^2\left(\frac{c}{\lambda}\right)^4} \quad — (11)$$

$$R_{rad} = 20(\pi^2)(c/\lambda)^4 = 197(c/\lambda)^4$$

$$\boxed{\therefore \quad R_{rad} = 197(c/\lambda)^4} \quad — (12)$$