UNIT - IV

DEFLECTION OF BEAMS

Introduction:

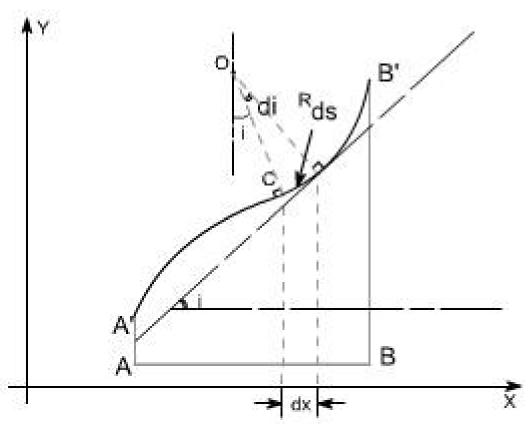
In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.
2. The curvature is always small.
3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$$\tan i = \frac{dy}{dx} \quad ......(1) \quad \text{or} \quad i = \frac{dy}{dx} \quad \text{Assuming } \tan i = i$$

Futher

$$ds = R\,di$$

however,

$$ds = dx \text{ [usually for small curvature]}$$

Hence

$$ds = dx = R\,di$$

$$\text{or } \boxed{\frac{di}{dx} = \frac{1}{R}}$$

substituting the value of i, one get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R} \text{ or } \frac{d^2y}{dx^2} = \frac{1}{R}$$

From the simple bending theory

$$\frac{M}{I} = \frac{E}{R} \text{ or } M = \frac{EI}{R}$$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as

Differentiating the equation as derived

$$\frac{dM}{dx}=EI\frac{d^3y}{dx^3} \quad \text{Recalling } \frac{dM}{dx}=F$$

Thus,

$$F=EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{slope = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{Shear\ force = EI\frac{d^3y}{dx^3}}$$

$$\boxed{load\ distribution = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int\frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

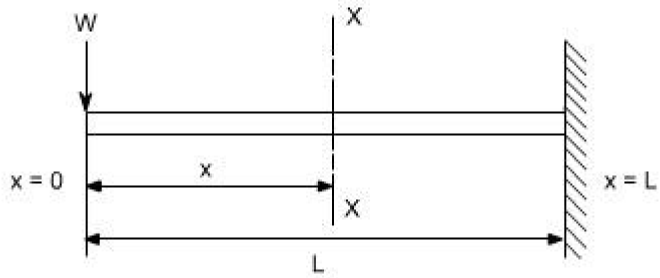
Integrate once again to get the deflection.

$$y = \int\int\frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarty with the direct integration method

<u>Case 1:</u> <u>Cantilever Beam with Concentrated Load at the end:-</u> A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 -----------------------------(1)

at x = L ; dy/dx = 0 --------------------------(2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{WL^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

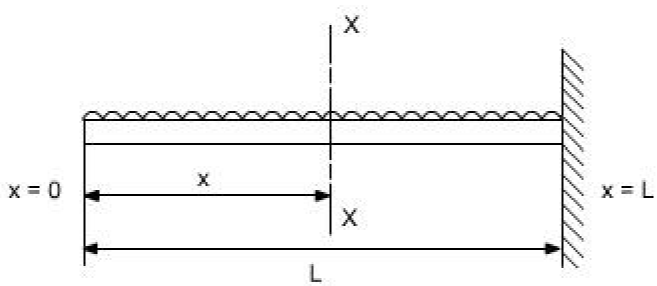
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(Slope)_{maxm} = +\frac{WL^2}{2EI}}$$

<u>**Case 2:** A Cantilever with Uniformly distributed Loads:-</u> In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int\frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int\frac{dy}{dx} = \int -\frac{wx^3}{6EI}dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

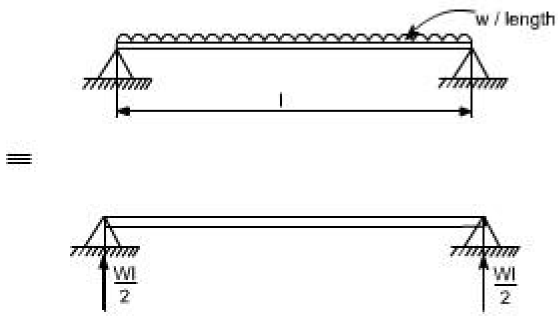
Thus, $y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$

So y_{maxm} will be at $x = 0$

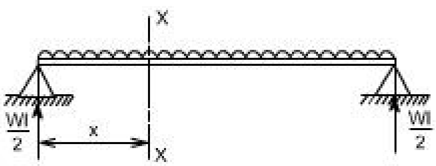
$$\boxed{y_{maxm} = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{maxm} = \frac{wL^3}{6EI}}$$

Case 3: <u>Simply Supported beam with uniformly distributed Loads:-</u> In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI} dx - \int \frac{wx^2}{2EI} dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----(1)}$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

$$\text{Then} \quad y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

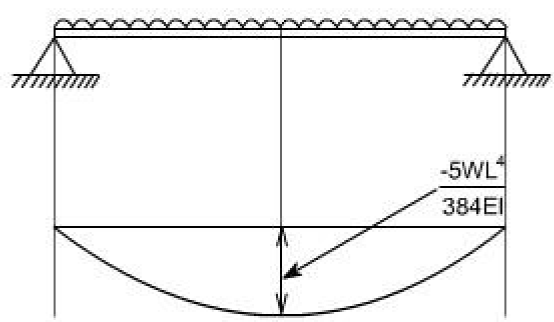
(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

The final equation which is governs the deflection of the loaded beam in this case is

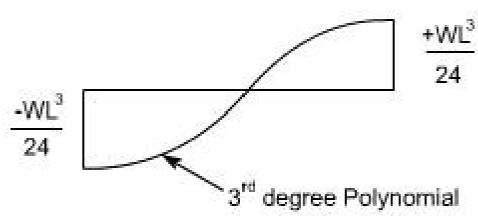
$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.



Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

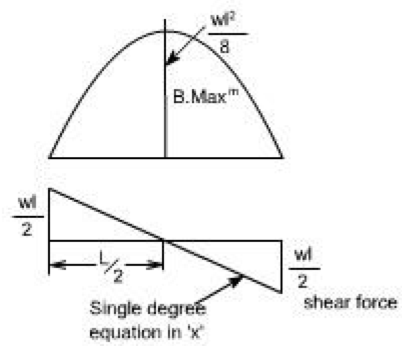


Slope (dy/dx)

$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$

Bending Moment So the bending moment diagram would be

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear Force

Shear force is obtained by taking

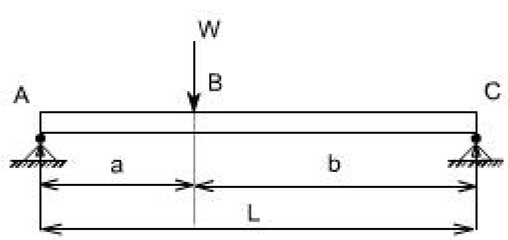
third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

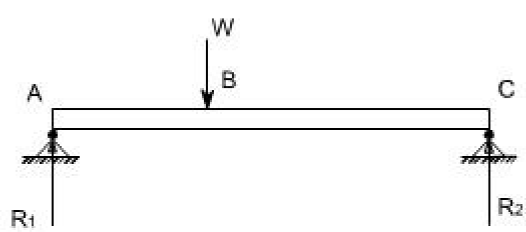
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \le x \le a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \le x \le l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1 x$$

$$EI\frac{d^2y}{dx^2} = R_1 x - W(x - a)$$

These two equations can be integrated in the usual way to find 'y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$K_1 = K_2 = K$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{------}(3)$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{-------}(4)$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{------}(5)$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{------}(6)$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0$;

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{----} \mathbf{for\ 0 \le x \le a} \text{-----(7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \quad \text{For } \mathbf{for\ a \le x \le l} \text{-----(8)}$$

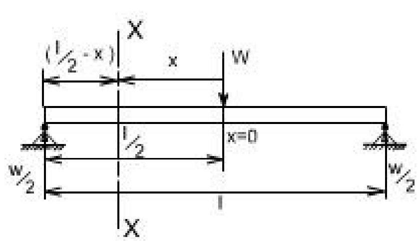
so either of the equation (7) or (8) may be used to find the deflection at x = a
hence substituting x = a in either of the equation we get

$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{\max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a more simpler way. Let us considering the origin at the point of application of the load,



$$S.F\big|_{xx} = \frac{W}{2}$$

$$B.M\big|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

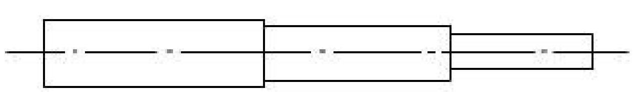
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$Y_{max^m}\big|_{at\,x=0}$	$= -\dfrac{WL^3}{48EI}$	At the centre
$\left(\dfrac{dy}{dx}\right)_{max^m}\Big|_{at\,x=\pm\frac{L}{2}}$	$= \pm\dfrac{WL^2}{16EI}$	At the ends

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.

2. Area moment methods
3. Energy principle methods