UNIT – I

SIMPLE STRESSES AND STRAINS

Elasticity and plasticity:

Elasticity is defined as the property which enables a material to get back to (or recover) its original shape, after the removal of applied force.

Plasticity is defined as the property which enables a material to be deformed continuously and permanently without rupture during the application of force.

Types of stresses and strains:

Stress: Stress is proportional to strain within its elastic limit. This law is known as Hookes law. The material will not return to original shape if the applied stress is more than E.

$$\zeta = \frac{P}{A}$$

P – Load

A- Area of the section where the load is applied.

Stresses are three types tensile, compressive, and shear stress. Moment and torsion will produced any of these stresses.

Strain: Strain is nothing but deformation (change in length, breadth, height, diameter, therefore area or volume) of the body or material due to load. Therefore strain is change in dimension to the original dimension.

$$\varepsilon = \frac{\delta L}{L}$$

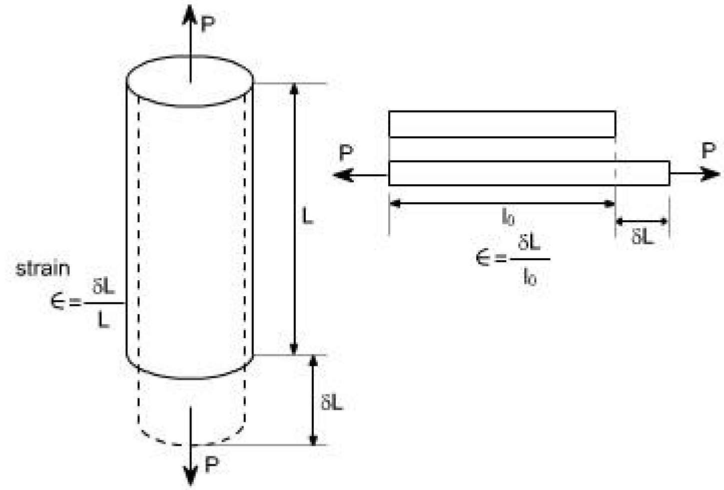
δL– Change in length

L – Original length

Concept of Strain: if a bar is subjected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount dL, the strain produce is defined as follows:

$$\text{strain}(\varepsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a non dimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit.



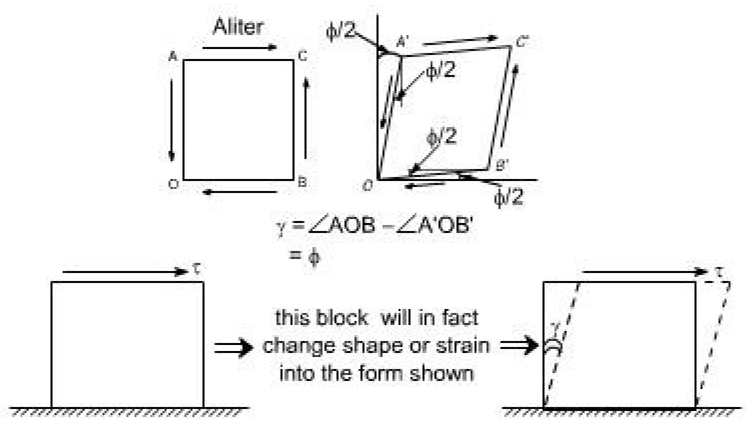
Since in practice, the extensions of materials under load are very very small, it is often convenient to measure the strain in the form of strain x 10^{-6} i.e. micro strain, when the symbol used becomes m Î.

Sign convention for strain:

Tensile strains are positive whereas compressive strains are negative. The strain defined earlier was known as linear strain or normal strain or the longitudinal strain now let us define the shear strain.

Definition: An element which is subjected to a shear stress experiences a deformation as shown in the figure below. The tangent of the angle through which two adjacent sides rotate relative to their initial position is termed shear strain. In many cases the angle is very small and the angle it self is used, (in radians), instead of tangent, so that g = Ð AOB - Ð A'OB' = f

Shear strain: As we know that the shear stresses acts along the surface. The action of the stresses is to produce or being about the deformation in the body consider the distortion produced b shear sheer stress on an element or rectangular block



This shear strain or slide is f and can be defined as the change in right angle. or The angle of deformation g is then termed as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.So we have two types of strain i.e. normal stress & shear stresses.

Hook's Law :

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

Hook's law therefore states that

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

Modulus of elasticity : Within the elastic limits of materials i.e. within the limits in which Hook's law applies, it has been shown that

Stress / strain = constant

This constant is given by the symbol E and is termed as the modulus of elasticity or Young's modulus of elasticity

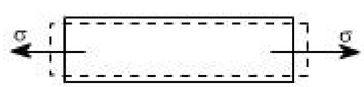
$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$

$$= \frac{P/A}{\delta L/L}$$

$$\boxed{E = \frac{PL}{A\delta L}}$$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most engineering material has high, numerical value of the order of 200 GPa

Poisson's ratio: If a bar is subjected to a longitudinal stress there will be a strain in this direction equal to s / E . There will also be a strain in all directions at right angles to s . The final shape being shown by the dotted lines.

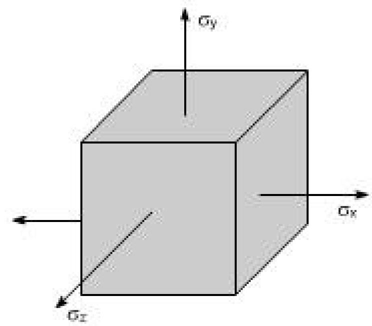


It has been observed that for an elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio .

Poison's ratio (m) = - lateral strain / longitudinal strain

For most engineering materials the value of m his between 0.25 and 0.33.

Three – dimensional state of strain : Consider an element subjected to three mutually perpendicular tensile stresses s_x , s_y and s_z as shown in the figure below.



If s_y and s_z were not present the strain in the x direction from the basic definition of Young's modulus of Elasticity E would be equal to

$$\hat{I}_x = s_x / E$$

The effects of s_y and s_z in x direction are given by the definition of Poisson's ratio ' m ' to be equal as -m s_y/ E and -m s_z/ E

The negative sign indicating that if s_y and s_z are positive i.e. tensile, these they tend to reduce the strain in x direction thus the total linear strain is x direction is given by

$$\epsilon_x = \frac{\sigma_x}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E} - \mu\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \mu\frac{\sigma_y}{E} - \mu\frac{\sigma_x}{E}$$

Principal strains in terms of stress:

In the absence of shear stresses on the faces of the elements let us say that s_x , s_y , s_z are in fact the principal stress. The resulting strain in the three directions would be the principal strains.

i.e. We will have the following relation.

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu\sigma_2 - \mu\sigma_3\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu\sigma_1 - \mu\sigma_3\right]$$

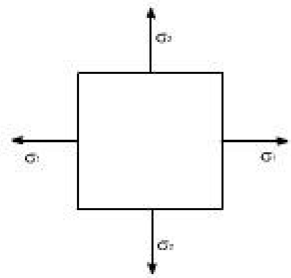
$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu\sigma_1 - \mu\sigma_2\right]$$

For Two dimensional strain: system, the stress in the third direction becomes zero i.e $s_z = 0$ or $s_3 = 0$

Although we will have a strain in this direction owing to stresses s_1 & s_2 .

Hence the set of equation as described earlier reduces to

$$\epsilon_1 = \frac{1}{E}[\sigma_1 - \mu\sigma_2]$$
$$\epsilon_2 = \frac{1}{E}[\sigma_2 - \mu\sigma_1]$$
$$\epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$$



Hence a strain can exist without a stress in that direction

i.e if $\sigma_3 = 0$; $\epsilon_3 = \frac{1}{E}[-\mu\sigma_1 - \mu\sigma_2]$

Also

$\epsilon_1 .E = \sigma_1 - \mu\sigma_2$

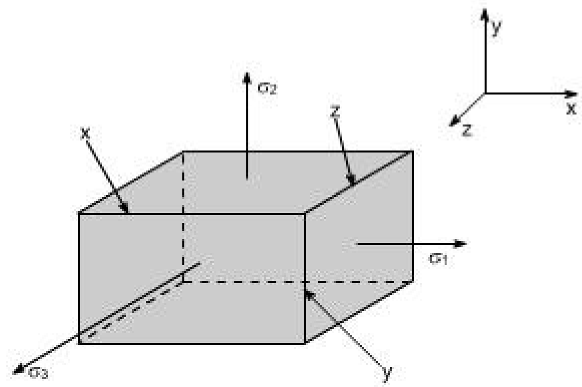
$\epsilon_2 .E = \sigma_2 - \mu\sigma_1$

so the solution of above two equations yields

$$\boxed{\begin{aligned}\sigma_1 &= \frac{E}{(1-\mu^2)}[\epsilon_1 + \mu\epsilon_2] \\ \sigma_2 &= \frac{E}{(1-\mu^2)}[\epsilon_2 + \mu\epsilon_1]\end{aligned}}$$

Hydrostatic stress : The term Hydrostatic stress is used to describe a state of tensile or compressive stress equal in all directions within or external to a body. Hydrostatic stress causes a change in volume of a material, which if expressed per unit of original volume gives a volumetric strain denoted by $\hat{I}_v$. So let us determine the expression for the volumetric strain.

Volumetric Strain:



Consider a rectangle solid of sides x, y and z under the action of principal stresses s_1 , s_2 , s_3 respectively.

Then $\hat{I}_1$, $\hat{I}_2$, and $\hat{I}_3$ are the corresponding linear strains, than the dimensions of the rectangle becomes

(x + $\hat{I}_1$. x); (y + $\hat{I}_2$. y); (z + $\hat{I}_3$. z)

$$\text{Volumetric strain} = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{x(1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3)z - xyz}{xyz}$$

$$= (1+\epsilon_1)y(1+\epsilon_2)(1+\epsilon_3) - 1 \cong \epsilon_1 + \epsilon_2 + \epsilon_3 \left[\text{Neglecting the products of } \epsilon\text{'s}\right]$$

<u>**ALITER :**</u> Let a cuboid of material having initial sides of Length x, y and z. If under some load system, the sides changes in length by dx, dy, and dz then the new volume (x + dx) (y + dy) (z +dz)

New volume = xyz + yzdx + xzdy + xydz

Original volume = xyz

Change in volume = yzdx +xzdy + xydz

Volumetric strain = (yzdx +xzdy + xydz) / xyz = $\hat{I}_x$+ $\hat{I}_y$+ $\hat{I}_z$

Neglecting the products of epsilon's since the strains are sufficiently small.

Volumetric strains in terms of principal stresses:

As we know that

$$\epsilon_1 = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_2 = \frac{\sigma_2}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_3}{E}$$

$$\epsilon_3 = \frac{\sigma_3}{E} - \mu\frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$

Futher Volumetric strain $= \epsilon_1 + \epsilon_2 + \epsilon_3$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)}{E} - \frac{2\mu(\sigma_1 + \sigma_2 + \sigma_3)}{E}$$

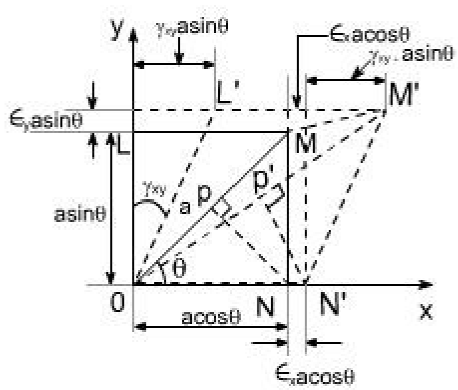
$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}$$

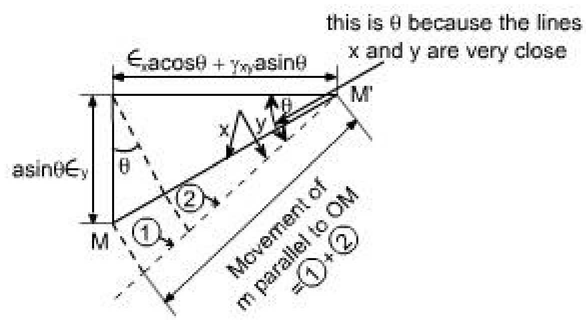
hence the

$$\boxed{\text{Volumetric strain} = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\mu)}{E}}$$

Strains on an oblique plane

(a) Linear strain





Consider a rectangular block of material OLMN as shown in the xy plane. The strains along ox and oy are Î$_x$ and Î$_y$, and g$_{xy}$ is the shearing strain.

Then it is required to find an expression for Î$_q$, i.e the linear strain in a direction inclined at q to OX, in terms of Î$_x$,Î$_y$, g$_{xy}$ and q.

Let the diagonal OM be of length 'a' then ON = a cos q and OL = a sin q , and the increase in length of those under strains are Î$_x$acos q and Î$_y$a sin q (i.e. strain x original length) respectively.

If M moves to M', then the movement of M parallel to x axis is Î$_x$acos q + g$_{xy}$ sin q and the movement parallel to the y axis is Î$_y$asin q

Thus the movement of M parallel to OM , which since the strains are small is practically coincident with MM'. and this would be the summation of portions (1) and (2) respectively and is equal to

$= (\epsilon_y \; a\sin\theta)\sin\theta + (\epsilon_x \; a\cos\theta + \gamma_{xy} a\sin\theta)\cos\theta$

$= a\left[\epsilon_y \; \sin\theta.\sin\theta + \epsilon_x \; \cos\theta.\cos\theta + \gamma_{xy} \sin\theta.\cos\theta\right]$

hence the strain along OM

$$= \frac{\text{extension}}{\text{originallength}}$$

$\epsilon_\theta = \epsilon_x \; \cos^2\theta + \gamma_{xy} \sin\theta.\cos\theta + \epsilon_y \; \sin^2\theta$

$\epsilon_\theta = \epsilon_x \; \cos^2\theta + \epsilon_y \; \sin^2\theta + \gamma_{xy} \sin\theta.\cos\theta$

Recalling $\cos^2\theta - \sin^2\theta = \cos 2\theta$

or $2\cos^2\theta - 1 = \cos 2\theta$

$$\cos^2\theta = \left[\frac{1+\cos 2\theta}{2}\right]$$

$$\sin^2\theta = \left[\frac{1-\sin 2\theta}{2}\right]$$

hence

$$\epsilon_\theta = \epsilon_x \left[\frac{1+\cos 2\theta}{2}\right] + \epsilon_y \left[\frac{1-\sin 2\theta}{2}\right] + \gamma_{xy} a\sin\theta.\cos\theta$$

$$= \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

$$\epsilon_\theta = \left\{\frac{\epsilon_x + \epsilon_y}{2}\right\} + \left\{\frac{\epsilon_x - \epsilon_y}{2}\right\}\cos 2\theta + \frac{1}{2}\gamma_{xy}\sin 2\theta$$

This expression is identical in form with the equation defining the direct stress on any inclined plane q with Îx and Îy replacing sx and sy and ½ gxy replacing txy i.e. the shear stress is replaced by half the shear strain

Shear strain: To determine the shear stain in the direction OM consider the displacement of point P at the foot of the perpendicular from N to OM and the following expression can be

$$\frac{1}{2}\gamma_\theta = -\left[\frac{1}{2}(\epsilon_x - \epsilon_y)\sin 2\theta - \frac{1}{2}\gamma_{xy}\cos 2\theta\right]$$

In the above expression ½ is there so as to keep the consistency with the stress relations.

Futher -ve sign in the expression occurs so as to keep the consistency of sign convention, because OM' moves clockwise with respect to OM it is considered to be negative strain.

The other relevant expressions are the following :

Principalplanes :

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Principalstrains :

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

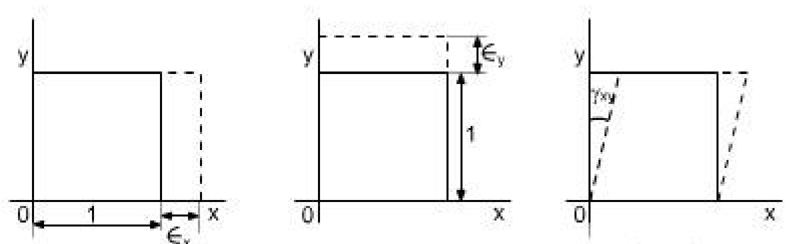
Maximumshearstrains :

$$\frac{\gamma_{max}}{2} = \pm\sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

Let us now define the plane strain condition

Plane Strain :

In xy plane three strain components may exist as can be seen from the following figures:



(Fig.1) (Fig.2) (Fig.3)

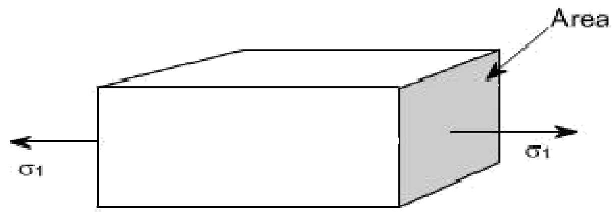
Therefore, a strain at any point in body can be characterized by two axial strains i.e Î$_x$ in x direction, Î$_y$ in y - direction and g$_{xy}$ the shear strain.

In the case of normal strains subscripts have been used to indicate the direction of the strain, and Î$_x$, Î$_y$ are defined as the relative changes in length in the co-ordinate directions.

With shear strains, the single subscript notation is not practical, because such strains involves displacements and length which are not in same direction.The symbol and subscript g$_{xy}$ used for the shear strain referred to the x and y planes. The order of the subscript is unimportant. g$_{xy}$ and g$_{yx}$ refer to the same physical quantity. However, the sign convention is important.The shear strain g$_{xy}$ is considered to be positive if it represents a decrease the angle between the sides of an element of material lying parallel the positive x and y axes. Alternatively we can think of positive shear strains produced by the positive shear stresses and viceversa.

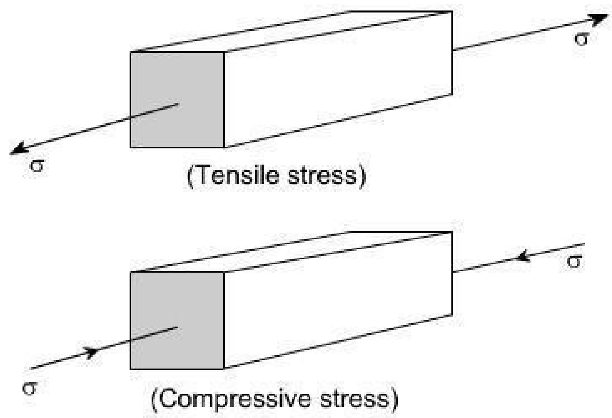
TYPES OF STRESSES : Only two basic stresses exists : (1) normal stress and (2) shear stress. Other stresses either are similar to these basic stresses or are a combination of this e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress. Let us define the normal stresses and shear stresses in the following sections.

Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses.



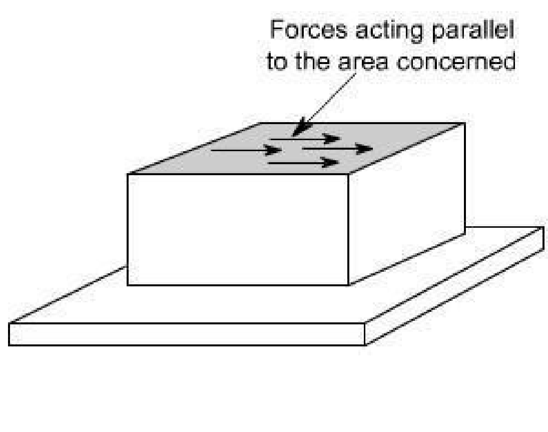
Tensile or compressive Stresses:

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area



Shear Stresses:

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting stress is known as shear stress.



Hooke's law:

Hooke’s law states that whenever a material is loaded within the elastic limit, the stress is proportional to the strain.

Stress – strain diagram for mild steel:

In the course of operation or use, all the articles and structures are subjected to the action of external forces, which create stresses that inevitably cause deformation. To keep these stresses, and, consequently deformation within permissible limits it is necessary to select suitable materials for the Components of various designs and to apply the most effective heat treatment. i.e. a Comprehensive knowledge of the chief character tics of the semi-finished metal products & finished metal articles (such as strength, ductility, toughness etc) are essential for the purpose.

For this reason the specification of metals, used in the manufacture of various products and structure, are based on the results of mechanical tests or we say that the mechanical tests conducted on the specially prepared specimens (test pieces) of standard form and size on special machines to obtained the strength, ductility and toughness characteristics of the metal.

The conditions under which the mechanical test are conducted are of three types

(1) **Static:** When the load is increased slowly and gradually and the metal is loaded by tension, compression, torsion or bending.

(2) **Dynamic:** when the load increases rapidly as in impact

(3) **Repeated or Fatigue:** (both static and impact type) . i.e. when the load repeatedly varies in the course of test either in value or both in value and direction Now let us consider the uniaxial tension test.

[For application where a force comes on and off the structure a number of times, the material cannot withstand the ultimate stress of a static tool. In such cases the ultimate strength depends on no. of times the force is applied as the material works at a particular stress level. Experiments one conducted to compute the number of cycles requires to break to specimen at a particular stress when fatigue or fluctuating load is acting. Such tests are known as fatque tests]

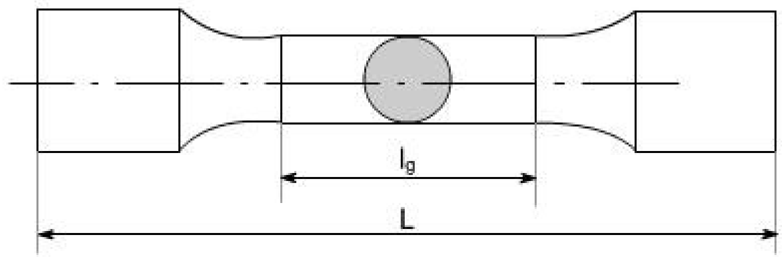
Uniaxial Tension Test: This test is of static type i.e. the load is increased comparatively slowly from zero to a certain value.

Standard specimen's are used for the tension test.

There are two types of standard specimen's which are generally used for this purpose, which have been shown below:

Specimen I:

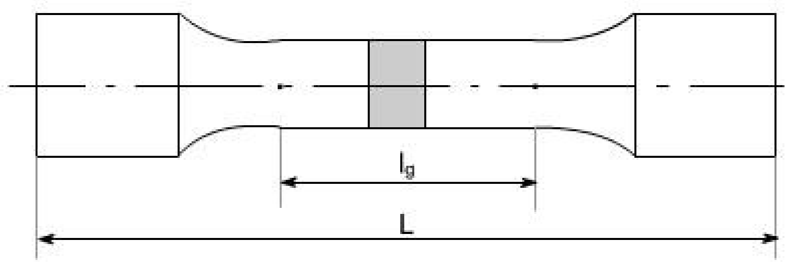
This specimen utilizes a circular X-section.



[specimen with circular X-section]

Specimen II:

This specimen utilizes a rectangular X-section.



[specimen with rectangular X-section]

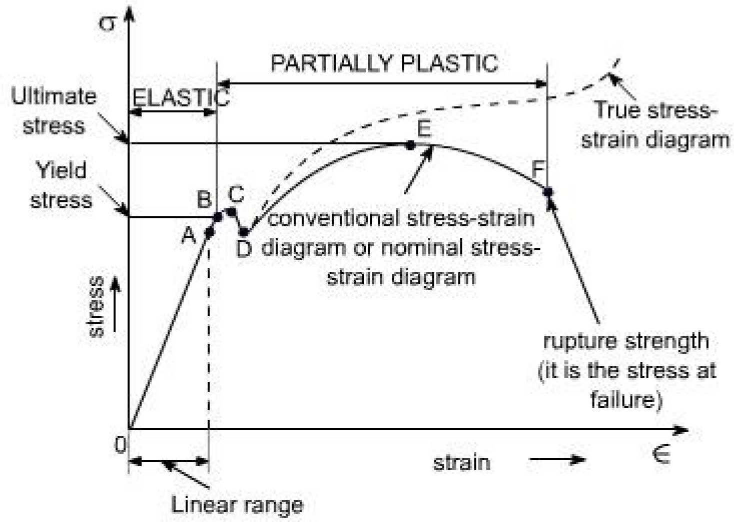
l_g = gauge length i.e. length of the specimen on which we want to determine the mechanical properties.The uniaxial tension test is carried out on tensile testing machine and the following steps are performed to conduct this test.

(i) The ends of the specimen's are secured in the grips of the testing machine.

(ii) There is a unit for applying a load to the specimen with a hydraulic or mechanical drive.

(iii) There must be a some recording device by which you should be able to measure the final output in the form of Load or stress. So the testing machines are often equipped with the pendulum type lever, pressure gauge and hydraulic capsule and the stress Vs strain diagram is plotted which has the following shape.

A typical tensile test curve for the mild steel has been shown below



Nominal stress – Strain OR Conventional Stress – Strain diagrams:

Stresses are usually computed on the basis of the original area of the specimen; such stresses are often referred to as conventional or nominal stresses.

True stress – Strain Diagram:

Since when a material is subjected to a uniaxial load, some contraction or expansion always takes place. Thus, dividing the applied force by the corresponding actual area of the specimen at the same instant gives the so called true stress.

SALIENT POINTS OF THE GRAPH:

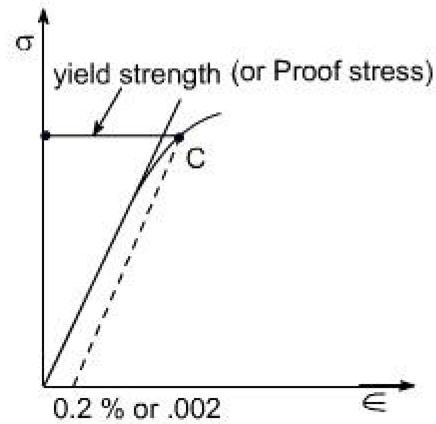
(A) So it is evident form the graph that the strain is proportional to strain or elongation is proportional to the load giving a st.line relationship. This law of proportionality is valid upto a point A. or we can say that point A is some ultimate point when the linear nature of the graph ceases or there is a deviation from the linear nature. This point is known as **the limit of proportionality or the proportionality limit**.

(B) For a short period beyond the point A, the material may still be elastic in the sense that the deformations are completely recovered when the load is removed. The limiting point B is termed as **Elastic Limit** .

(C) and (D) - Beyond the elastic limit plastic deformation occurs and strains are not totally recoverable. There will be thus permanent deformation or permanent set when load is removed. These two points are termed as upper and lower yield points respectively. The stress at the yield point is called the yield strength.

A study a stress – strain diagrams shows that the yield point is so near the proportional limit that for most purpose the two may be taken as one. However, it is much easier to locate the former. For material which do not posses a well define yield points, In order to find the yield point or yield strength, an offset method is applied.

In this method a line is drawn parallel to the straight line portion of initial stress diagram by off setting this by an amount equal to 0.2% of the strain as shown as below and this happens especially for the low carbon steel.



(E) A further increase in the load will cause marked deformation in the whole volume of the metal. The maximum load which the specimen can with stand without failure is called the load at the ultimate strength.

The highest point 'E' of the diagram corresponds to the ultimate strength of a material.

s_u = Stress which the specimen can with stand without failure & is known as Ultimate Strength or Tensile Strength.

s_u is equal to load at E divided by the original cross-sectional area of the bar.

(F) Beyond point E, the bar begins to forms neck. The load falling from the maximum until fracture occurs at F.

[Beyond point E, the cross-sectional area of the specimen begins to reduce rapidly over a relatively small length of bar and the bar is said to form a neck. This necking takes place whilst the load reduces, and fracture of the bar finally occurs at point F]

Note: Owing to large reduction in area produced by the necking process the actual stress at fracture is often greater than the above value. Since the designers are interested in maximum loads which can be carried by the complete cross section, hence the stress at fracture is seldom of any practical value.

Factor of safety:

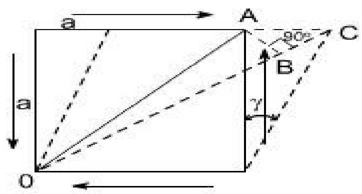
Factor of safety can be defined as the ratio of ultimate strength to the design strength. It is a constant factor that is considered for designing of machine components or structure beyond its working strength. F.O.S. is taken generally around 1.5 to 3

RELATION AMONG ELASTIC CONSTANTS

Relation between E, G and u :

Let us establish a relation among the elastic constants E,G and u. Consider a cube of material of side ‘a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below.

Assuming that the strains are small and the angle A C B may be taken as 45^0.



Therefore strain on the diagonal OA

= Change in length / original length

Since angle between OA and OB is very small hence OA @ OB therefore BC, is the change in the length of the diagonal OA

$$\text{Thus, strain on diagonal OA} = \frac{BC}{OA} = \frac{AC\cos 45^0}{OA}$$

$$OA = \frac{a}{\sin 45^0} = a.\sqrt{2}$$

$$\text{hence} \quad \text{strain} = \frac{AC}{a\sqrt{2}}.\frac{1}{\sqrt{2}} = \frac{AC}{2a}$$

but $AC = a\gamma$

where γ = shear strain

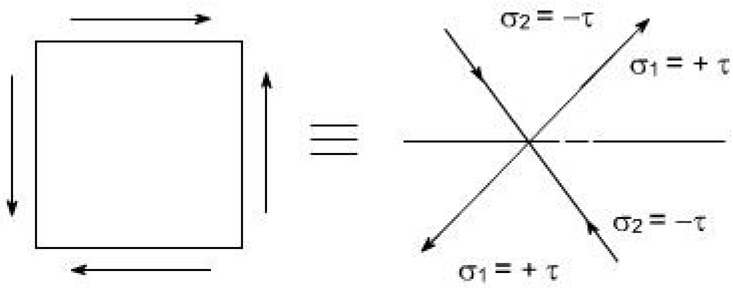
$$\text{Thus, the strain on diagonal} = \frac{a\gamma}{2a} = \frac{\gamma}{2}$$

From the definition

$$G = \frac{\tau}{\gamma} \text{ or } \gamma = \frac{\tau}{G}$$

$$\text{thus, the strain on diagonal} = \frac{\gamma}{2} = \frac{\tau}{2G}$$

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain.



Thus, for the direct state of stress system which applies along the diagonals:

$$\text{strain on diagonal} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$
$$= \frac{\tau}{E} - \mu\frac{(-\tau)}{E}$$
$$= \frac{\tau}{E}(1+\mu)$$

equating the two strains one may get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\mu)$$

or $$\boxed{E = 2G(1+\mu)}$$

We have introduced a total of four elastic constants, i.e E, G, K and g. It turns out that not all of these are independent of the others. Infact given any two of then, the other two can be found.

Again $E = 3K(1-2\gamma)$

$$\Rightarrow \frac{E}{3(1-2\gamma)} = K$$

if $\gamma = 0.5$ $K = \infty$

$$\epsilon_v = \frac{(1-2\gamma)}{E}(\epsilon_x + \epsilon_y + \epsilon_z) = 3\frac{\sigma}{E}(1-2\gamma)$$

(for $\epsilon_x = \epsilon_y = \epsilon_z$ hydrostatic state of stress)

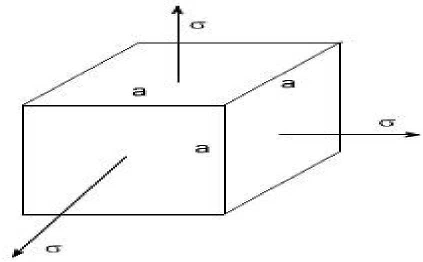
$\epsilon_v = 0$ if $\gamma = 0.5$

irrespective of the stresses i.e, the material is incompressible.

When g = 0.5 Value of k is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

<u>Relation between E, K and u :</u>

Consider a cube subjected to three equal stresses s as shown in the figure below



The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress s is given as

$$= \frac{\sigma}{E} - \gamma\frac{\sigma}{E} - \gamma\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1-2\gamma)$$

volumetre strain = 3.linear strain

volumetre strain = $\epsilon_x + \epsilon_y + \epsilon_z$

or thus, $\epsilon_x = \epsilon_y = \epsilon_z$

volumetric strain = $3\frac{\sigma}{E}(1-2\gamma)$

By definition

Bulk Modulus of Elasticity (K) = $\frac{\text{Volumetric stress}(\sigma)}{\text{Volumetric strain}}$

or

Volumetric strain = $\frac{\sigma}{k}$

Equating the two strains we get

$$\frac{\sigma}{k} = 3.\frac{\sigma}{E}(1-2\gamma)$$

$$\boxed{E = 3K(1-2\gamma)}$$

Relation between E, G and K :

The relationship between E, G and K can be easily determained by eliminating u from the already derived relations

E = 2 G (1 + u) and E = 3 K (1 - u)

Thus, the following relationship may be obtained

$$\boxed{E = \frac{9\,GK}{(3K+G)}}$$

Relation between E, K and g :

From the already derived relations, E can be eliminated

$$E = 2G(1+\gamma)$$
$$E = 3K(1-2\gamma)$$

Thus, we get

$$3k(1-2\gamma) = 2G(1+\gamma)$$

therefore

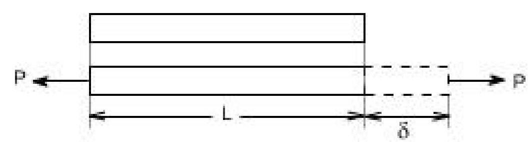
$$\gamma = \frac{(3K-2G)}{2(G+3K)}$$

or

$$\boxed{\gamma = 0.5(3K-2G)(G+3K)}$$

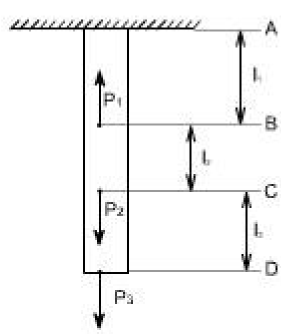
Bars of varying section:

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \quad \text{———} \quad (1)$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

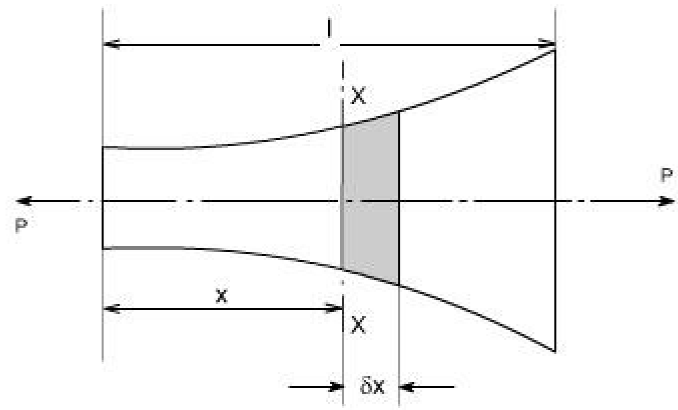
$$d\delta = \frac{P_x\,dx}{E.A_x}$$

$$\delta = \int_0^l \frac{P_x\,dx}{E.A_x}$$

i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

stresses in Non – Uniform bars

Consider a bar of varying cross section subjected to a tensile force P as shown below.



Let

a = cross sectional area of the bar at a chosen section XX

then

Stress, s = p / a

If E = Young's modulus of bar then the strain at the section XX can be calculated

Î = s / E

Then the extension of the short element d x. = Î .original length = s / E. d^x

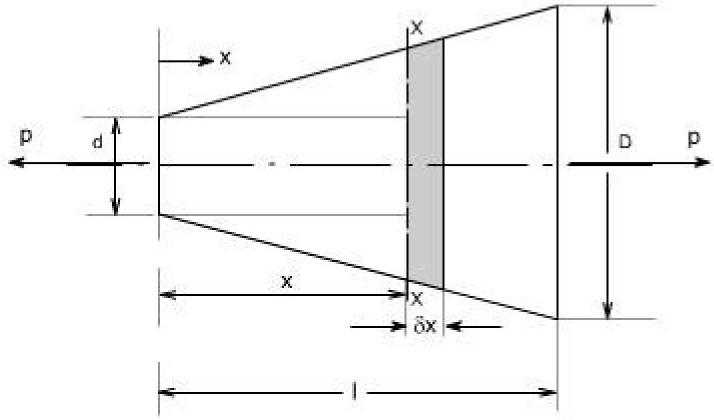
$$= \frac{P}{E.}\frac{\delta x}{a}$$

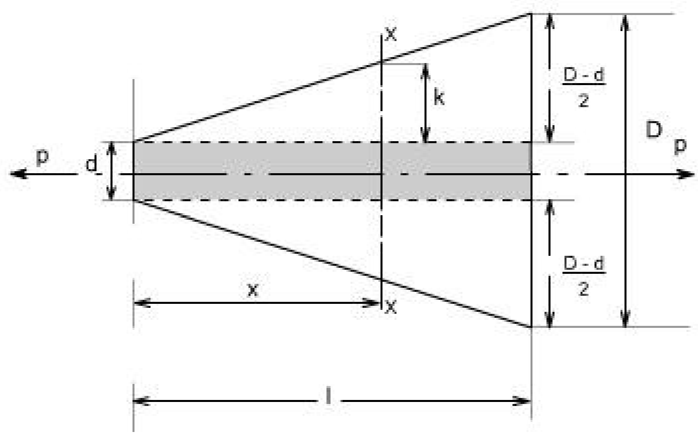
Thus, the extension for the entire bar is

$$\delta = \int_0^l \frac{P}{E}\frac{\delta x}{a}$$

$$\text{or total extension} = \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

Now let us for example take a case when the bar tapers uniformly from d at x = 0 to D at x = l





In order to compute the value of diameter of a bar at a chosen location let us determine the value of dimension k, from similar triangles

$$\frac{(D-d)/2}{l} = \frac{k}{x}$$

$$\text{Thus}, k = \frac{(D-d)x}{2l}$$

therefore, the diameter 'y' at the X-section is

or = d + 2k

$$y = d + \frac{(D-d)x}{l}$$

Hence the cross –section area at section X- X will be

$$A_x \text{ or } a = \frac{\pi}{4} y^2$$

$$= \frac{\pi}{4}\left[d + (D-d)\frac{x}{l}\right]^2$$

hence the total extension of the bar will be given by expression

$$= \frac{P}{E}\int_0^l \frac{\delta x}{a}$$

subsitituting the value of 'a' to get the total extention of the bar

$$= \frac{\pi P}{4E}\int_0^l \frac{\delta x}{\left[d + (D - d)\frac{x}{l}\right]^2}$$

after carrying out the intergration we get

$$= -\frac{4.P.l}{\pi E}\left[\frac{1}{D} - \frac{1}{d}\right]$$

$$= \frac{4.P.l}{\pi E D.d}$$

hence the total strain in the bar = $\mathbf{\dfrac{4.P.l}{\pi E\, D.d}}$

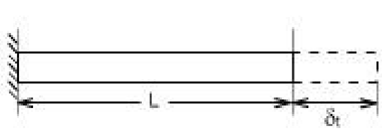
Composite Bars and Temperature Stresses

- A **composite bar** made of two **bars** of different materials rigidly fixed together so that both **bars** strain together under external load.
- Since strains in the two **bars** are same, the stresses in the two **bars** depend on their Young's modulus of elasticity.

Compound bars subjected to Temp. Change : Ordinary materials expand when heated and contract when cooled, hence , an increase in temperature produce a positive thermal strain. Thermal strains usually are reversible in a sense that the member returns to its original shape when the temperature return to its original value. However, there here are some materials which do not behave in this manner. These metals differs from ordinary materials in a sence that the strains are related non linearly to temperature and some times are irreversible .when a material is subjected to a change in temp. is a length will change by an amount.

d_t = a .L.t

or Î$_t$= a .L.t or s $_t$= E .a.t



a = coefficient of linear expansoin for the material

L = original Length

t = temp. change

Thus an increase in temperature produces an increase in length and a decrease in temperature results in a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not to be considered here.

If however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. They stress is equal in magnitude to that which would be produced in the bar by initially allowing the bar to its free length and then applying sufficient force to return the bar to its original length.

Change in Length = a L t

Therefore, strain = a L t / L

= a t

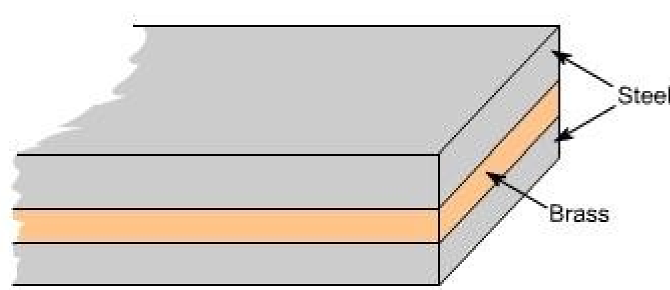
Therefore ,the stress generated in the material by the application of sufficient force to remove this strain

= strain x E

or Stress = E a t

Consider now a compound bar constructed from two different materials rigidly joined together, for simplicity.

Let us consider that the materials in this case are steel and brass.



If we have both applied stresses and a temp. change, thermal strains may be added to those given by generalized hook's law equation –e.g.

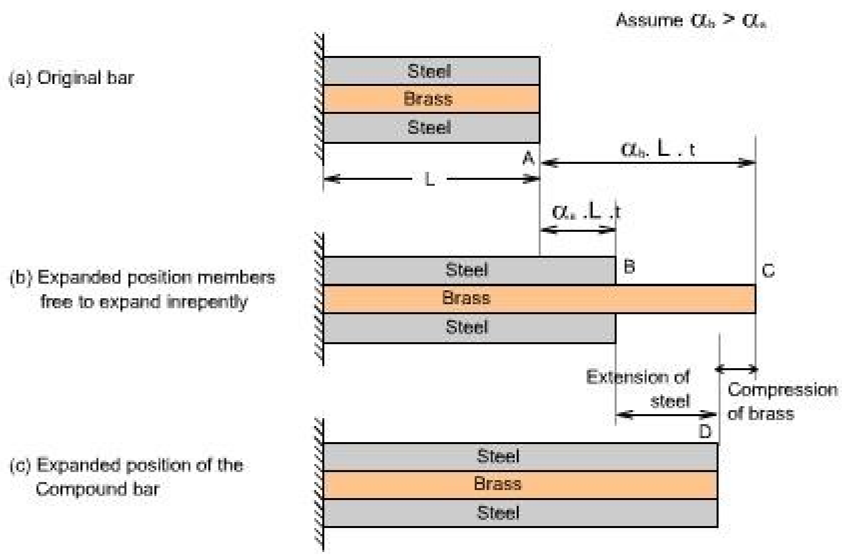
$$\in_x = \frac{1}{E}\left[\sigma_x - \gamma(\sigma_y + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_y - \gamma(\sigma_x + \sigma_z)\right] + \alpha\Delta t$$

$$\in_x = \frac{1}{E}\left[\sigma_z - \gamma(\sigma_x + \sigma_y)\right] + \alpha\Delta t$$

While the normal strains a body are affected by changes in temperatures, shear strains are not. Because if the temp. of any block or element changes, then its size changes not its shape therefore shear strains do not change.

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temp. rises each material will attempt to expand by different amounts. Figure below shows the positions to which the individual materials will expand if they are completely free to expand (i.e not joined rigidly together as a compound bar). The extension of any Length L is given by a L t



In general, changes in lengths due to thermal strains may be calculated form equation d_t = a Lt, provided that the members are able to expand or contract freely, a situation that exists in statically determinates structures. As a consequence no stresses are generated in a statically determinate structure when one or more members undergo a uniform temperature change. If in a structure (or a compound bar), the free expansion or contraction is not allowed then the member becomes s statically indeterminate, which is just being discussed as an example of the compound bar and thermal stresses would be generated.

Thus the difference of free expansion lengths or so called free lengths

= a_B.L. t - a_s .L .t

= (a_B - a_s).L .t

Since in this case the coefficient of expansion of the brass a_B is greater then that for the steel a_s. the initial lengths L of the two materials are assumed equal.

If the two materials are now rigidly joined as a compound bar and subjected to the same temp. rise, each materials will attempt to expand to its free length position but each will be affected by the movement of the other. The higher coefficient of expansion material (brass) will therefore, seek to pull the steel up to its free length position and conversely, the lower coefficient of expansion martial (steel) will try to hold the brass back. In practice a compromised is reached, the compound bar extending to the position shown in fig (c), resulting in an effective compression of the brass from its free length position and an effective extension of steel from its free length position.

Elastic constants:

There are three types of elastic constants (moduli) are:

- Modulus of **elasticity** or Young's modulus (E),
- Bulk modulus (K) and.
- Modulus of rigidity or shear modulus (M, C or G).

Strain Energy – Resilience – Gradual, sudden, impact and shock loadings – simple applications.

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

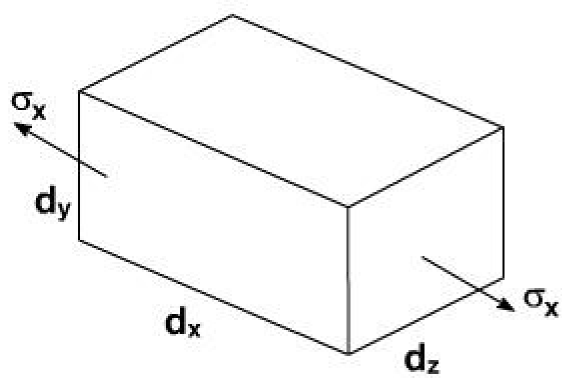


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress s_x.

The forces acting on the face of this element is s_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = $Î_x$ dx

$Î_x$ = strain in the material in x – direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

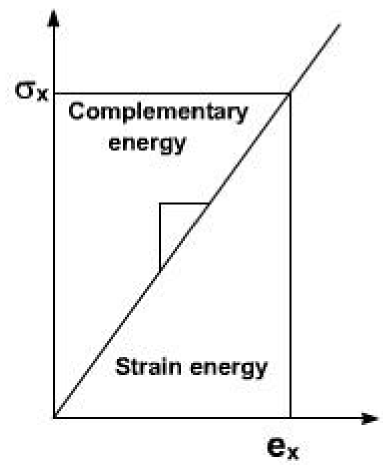


Fig .2

From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ s_x . dy. dz.

Therefore the workdone by the above force

Force = average force x deformed length

= ½ s_x. dydz . $\hat{I}_x$. dx

For a perfectly elastic body the above work done is the internal strain energy "du".

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \qquad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x\, \epsilon_x\, dv} \qquad ....(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x \epsilon_x} \qquad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy – density 'u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^2}{2E} = \frac{E\epsilon_x^2}{2}} \qquad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^2}{2E} dv} \qquad ....(6)$$

In the case of a rod of uniform cross – section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

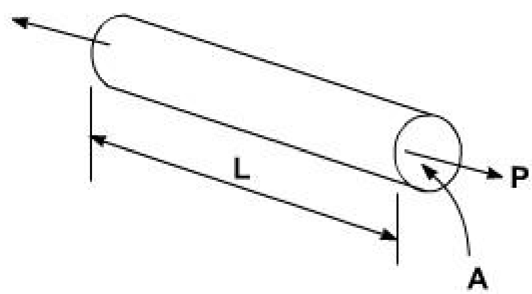


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^2}{2E} dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_0^L \frac{P^2}{2EA^2} Adx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.

L = Length of the bar

$$\boxed{U = \frac{P^2 L}{2AE}} \qquad .....(7)$$

Modulus of resilience :

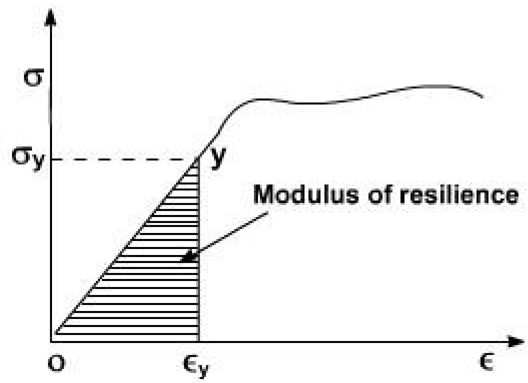


Fig .4

Suppose ‘ s_x‘ in strain energy equation is put equal to s_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $\boxed{U_y = \frac{\sigma_y^2}{2E}}$ .....(8)

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion ‘OY' of the stress – strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

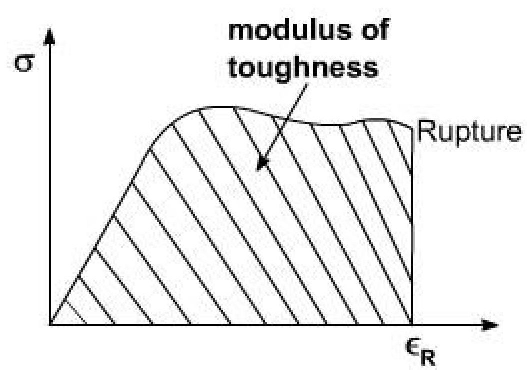


Fig .5

Suppose ‘Î' [strain] in strain energy expression is replaced by $Î_R$ strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x \,dx = \frac{E\,\epsilon_R^2}{2}\,dv$$

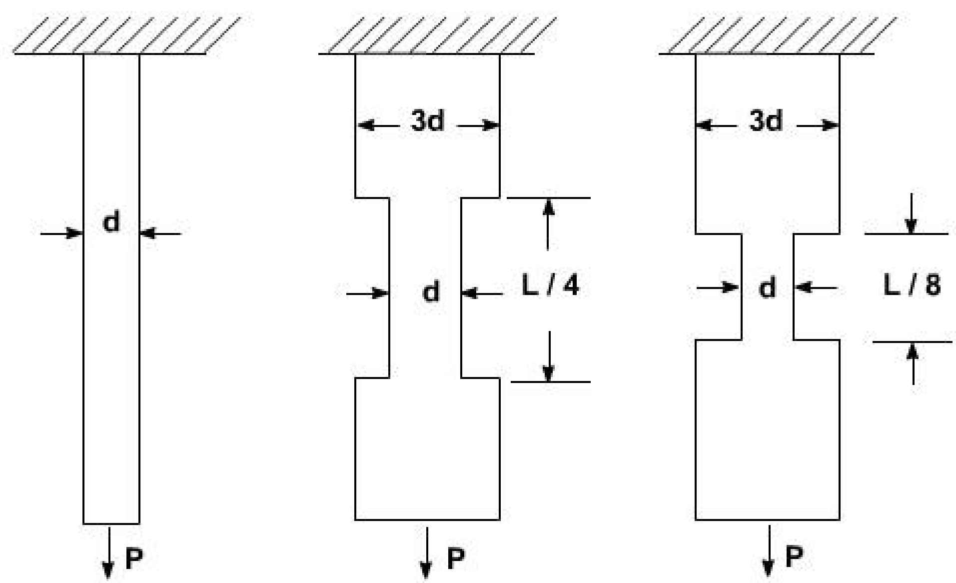
$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress – strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter'd' over its entire length, the second had this diameter over one – fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2(L/4)}{2EA} + \frac{P^2(3L/4)}{2E9A} = \frac{P^2L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

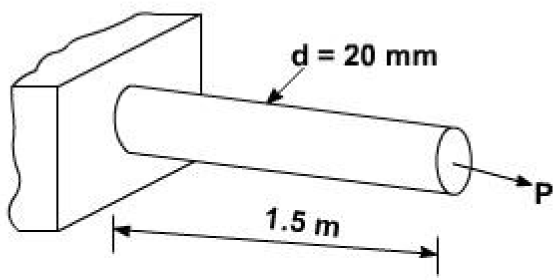
$$U_3 = \frac{P^2(L/8)}{2EA} + \frac{P^2(7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4}d^2L$$

$$= \frac{\pi}{4}\, 20 \times 1.5 \times 10^3$$

$$= 471 \times 10^3 \ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to s_x .

$$U = \frac{\sigma_y^2}{2E}$$

$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$

$$\boxed{\sigma_y = 200\ \text{Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

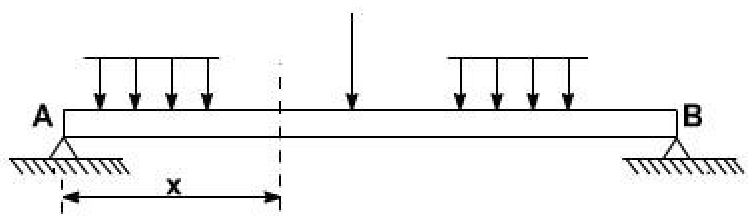


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2.y^2}{2EI^2} dv \qquad .....(10)$$

Substituting $dv = dxdA$

Where dA = elemental cross-sectional area

$\frac{M^2.y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.