FUNCTION OF A COMPLEX VARIABLE

Day : Thursday.

Surya

Introduction.

A complex number (z) is of the form $z = x + iy$ where, $x, y \in R$ and 'i' is an imaginary ~~property~~ unit with property $i^2 = -1$ ($i = \sqrt{-1}$).

The numbers x, y are called Real and Imaginary part of a complex number z. And we write $x = Re(z)$; $y = Img(z)$.

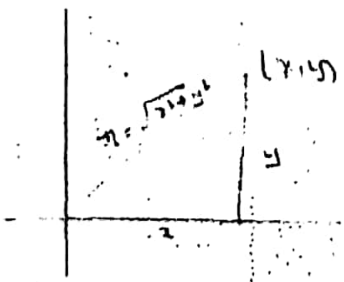
We can represent any complex number by a point in (x, y) in $(R \times R)$.

The plane representing the complex no's in this way is called complex plane or Argand plane.

The x-axis referred as real axis. And

The Y-axis Imaginary axis.

$|z|$ represents the distance b/w the origin and the point $z(x, y)$.



Polar form of a complex number.

Consider any non zero complex number $z = x + iy$.

Let (r, θ) be a polar co-ordinates.

Then $x = r\cos\theta$; $y = r\sin\theta$; $r = \sqrt{x^2 + y^2}$; $\theta = \tan^{-1}(y/x)$.

Let a positive number r is the length of the vector representing $z = r$. The number θ is called an argument of z and can be written as $\theta = \arg z$. Find the argument of $z = 1 + i$.

Example:- Find the $\arg z = 1 + i$.

$z = 1 + i$.

$x = 1$; $y = 1$ $\Rightarrow r = \sqrt{2}$. $\theta = \pi/4$

$\theta = \arg z = \pi/4$ and $z = re^{i\theta} = re^{i\pi/4} = \sqrt{2}e^{i\pi/4}$.

Complex conjugate

The complex conjugate of a complex number $z = x+iy$ is defined as $\bar{z} = x-iy$.

Note. z is a real number $\Leftrightarrow$

Complex function or (Function of a com Var)

Let 's' be the set of the complex no's. For every 'z' in a set 's', a unique value 'w' is associated then 'w' is said to be function of z and it is denoted by $f(z) = w$.

'S' is known as domain of F.

Range of $f = \{ w / w = f(z) \}$

In general we write $w = f(z) = u(x,y) + iv(x,y)$.

where, $u(x,y)$ and $v(x,y)$ are real and Imaginary parts of $f(z)$.

Example.

* Find $u(x,y)$ and $v(x,y)$ for $f(z) = z^2$

Sol: We know that $z = x+iy$.

$f(z) = z^2 \Rightarrow f(x+iy) = (x+iy)^2 = x^2 - y^2 + (2xy)i$.

$\therefore u = x^2 - y^2$; $v = 2xy$.

Day: saturday

Limit of a complex: A complex function $f(z)$ tends to 'l'. if $\lim_{z \to z_0} f(z) = l$

Here. z may approach z_0 in any direction.

Continuity of a complex function. A complex function $f(z)$ is said to be continuous at 'z_0' if $f(z_0)$ exist and $\lim_{z \to z_0} f(z) = f(z_0)$.

A function $f(z)$ is said to be continuous in domain 'D', if it is continuous at every point in the domain D.

If it is not continuous at z_0, then $f(z)$ is
A function $f(z)$ is not continuous at z_0. said to be discontinuous at z_0.

Differentiability or Derivate of a complex function.

A complex function $f(z)$ is said to be differentiable at a point z_0. If the limit $\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - f(z)}{\Delta z}$ or $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exist. And it is denoted by $f'(z)$.

* The above limit should be same along any path. *

Note.

* $\frac{d}{dz}(c) = 0$. * $\frac{d}{dz}[f \pm g] = \frac{df}{dz} \pm \frac{dg}{dz}$ * $\frac{d}{dz}[cf] = c \cdot \frac{df}{dz}$

* $\frac{d}{dz}[f \cdot g] = fg' + gf'$.

* $\frac{d}{dz}\left[f/g\right] = \frac{g \cdot f' - f g'}{g^2}$.

* $\frac{d}{dz}\left[(f(z))^n\right] = n \cdot \left[f(z)\right]^{n-1} \cdot \frac{df}{dz}$.

* **Analytic function.** A function $f(z)$ is said to be analytic at the point z_0, if it is differentiable some neighbourhood of z_0.

A function $f(z)$ is analytic in a domain D. If it is analytic at every point in the domain D.

Analytic function is also known called as Regular function.

* **Entire function.** If $f(z)$ is analytic at every point on the complex plane. then $f(z)$ is said to be entire function.

Cauchy-Riemann Equation (C-R. Eq'n).

* **Statement.** The neccessary and sufficient condition for $f(z)$ analytic in a domain R. are

* u_x, u_y, v_x, v_y are continuous functions in R

* $u_x = v_y$; $u_y = -v_x$.

The above equations are called C-R equation.

C-R Equations in polar form.

Let $z = x+iy$.

Polar co-ordinates are.

$x = r\cos\theta$.

$y = r\sin\theta$.

$z = r\cos\theta + ir\sin\theta$.

$z = re^{i\theta}$.

Let $f(z) = u+iv$.

$\therefore f(re^{i\theta}) = u+iv \longrightarrow (1)$

Diff eq'n (1) w.r.t 'θ' on Both sides.

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ri\, e^{i\theta} \longrightarrow (2)$$

Diff eq'n (1) w.r.t 'r' on Both sides.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta} \longrightarrow (3)$$

from (2) & (3).

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = \frac{1}{e^{i\theta}}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] ri\, e^{i\theta}$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$\Rightarrow \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and Imaginary parts.

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}\ ;\ \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}.$$

$v_r = -\frac{1}{r}u_\theta$.

$\therefore \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \Rightarrow \boxed{u_\theta = -r v_r}$

$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \Rightarrow \boxed{u_r = \frac{1}{r}v_\theta}$

These two Eq'ns are called C-R Eq'ns in

Date: 7-1-19
Day: wednesday.

Properties of Analytic functions.

* If F, g are two analytic functions +, −, ×, ÷. $f \pm g$, $f \cdot g$, $\frac{f}{g}$ $(g \neq 0)$ are also analytic functions.
* Analytic function of an analytic function is also an analytic.
* An entire function is also an entire function.

Note.
f is analytic ⇒ C.R eq'n satisfies f is analytic ⇐ u_x, u_y, v_x, v_y are continuous.

→ S.T $f(z) = z^2$ is analytic for all z.

Sol Given $f(z) = z^2$.
Let $z = x+iy$.
therefore $f(z) = (x+iy)^2 \Rightarrow f(z) = x^2 - y^2 + i2xy$
compare it with $u(x,y) + i\,v(x,y)$.
Here, $u = x^2 - y^2$; $v(x,y) = 2xy$.
∴ $u_x = 2x$; $v_x = 2y$.
$u_y = -2y$; $v_y = 2x$.
There fore $\boxed{u_x = v_y}$; $\boxed{u_y = -v_x}$.
∴ C.R Eq'ns are satisfied.
∴ 'f' is analytic for all 'z'.

S.T $f(z) = e^z$ is analytic every where in the complex plane and find $f'(z)$.

Sol Given $f(z) = e^z$.
Let $z = x+iy$.
∴ $f(z) = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$.
compare it with $u(x,y) + i\,v(x,y)$
Here, $u = e^x \cos y$; $v = e^x \sin y$.
$u_x = e^x \cos y$; $v_x = e^x \sin y$
$u_y = e^x(-\sin y)$; $v_y = e^x \cos y$.
$u_y = -e^x \sin y$.
∴ $\boxed{u_x = v_y}$ and $\boxed{u_y = -v_x}$

∴ C.R. Eq'ns are satisfied.

∴ 'f' is analytic for all 'z'.

∴ $f'(z) = u_x + i v_x$.

$\Rightarrow f'(z) = e^x \cos y + i e^x \sin y$.

$= e^x (\cos y + i \sin y)$.

$= e^z$.

∴ $\boxed{f'(z) = e^z}$.

P.T the function $f(z) = \bar{z}$ is not analytic at any pt.

sol Given that $f(z) = \bar{z}$.

Let. $z = x + iy$.

Therefore $f(z) = x - iy$.

compare it with $u(x,y) + i v(x,y)$.

Here, $u(x,y) = x$; $v(x,y) = -y$.

$\Rightarrow u_x = 1$; $v_x = 0$

$u_y = 0$; $v_y = -1$

$\boxed{u_x \neq v_y}$

∴ C.R. Eq'ns are not satisfied.

So f is not analytic at any point.

Determine whether the function $2xy + i(x^2 - y^2)$ is analytic.

sol Let given that $f(z) = 2xy + i(x^2 - y^2)$.

compare it with $u(x,y) + i v(x,y)$.

Here $u(x,y) = 2xy$; $v(x,y) = x^2 - y^2$.

$\Rightarrow u_x = 2y$; $v_x = 2x$.

$\Rightarrow u_y = 2x$; $v_y = -2y$.

$\boxed{u_x \neq v_y}$; $u_y \neq -v_x$

∴ C.R. Eq'ns are not satisfied.

∴ f is not analytic.

S.T $f(z) = z + 2\bar{z}$ is not analytic any where in the complex plane

sol Given that $f(z) = z + 2\bar{z}$

Let $z = x + iy$.

Therefore. $f(z) = x + iy + 2(x - iy) = x + iy + 2x - i2y = 3x - iy$.

compare it with $u(x,y) + v(x,y)$

Here $u(x,y) = 3x$; $v(x,y) = -y$.

$\Rightarrow u_x = 3$; $u_y = 0$; $v_x = 0$; $v_y = -1$

$u_x \neq v_y$ and $u_y \neq -v_x$.

$\therefore$ C.R. Eq'ns are not satisfied.

$\therefore$ f is not analytic for all z.

→ Find all values of k such that $f(z) = e^x(\cos ky + i \sin ky)$ is analytic.

Sol: Let $f(z) = e^x(\cos ky + i \sin ky)$.

Compare it with $u(x,y) + i v(x,y)$

Here,

$u(x,y) = e^x \cos ky$; $v(x,y) = \sin ky \, e^x$.

$u_x = e^x \cos ky$; $v_x = e^x \sin ky$.

$u_y = e^x - \sin ky (k)$; $v_y = e^x \cos ky \, (k)$.

If $f(z)$ is analytic.

$u_x = v_y \Rightarrow e^x \cos ky = e^x \cos ky \cdot k \Rightarrow k = 1$.

$u_y = -v_x \Rightarrow -e^x \sin ky \, k = -e^x \sin ky \Rightarrow k = 1$.

If $k = 1$, then $f(z)$ is analytic.

→ Determine 'p' such that the function $f(z) = \frac{1}{2}\log(x^2+y^2) + i \tan^{-1}\left(\frac{px}{y}\right)$ be an analytic function.

Sol: Given that $f(z) = \frac{1}{2}\log(x^2+y^2) + i \tan^{-1}(px/y)$.

compare it with $u(x,y) + i v(x,y)$

Here, $u(x,y) = \frac{1}{2}\log(x^2+y^2)$; $v(x,y) = \tan^{-1}(px/y)$.

$\Rightarrow u_x = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot (2x)$; $v_x = \frac{1}{1+\frac{p^2x^2}{y^2}} \cdot p/y \, (1)$

$= x/x^2+y^2$.

$= \frac{y^2}{y^2+p^2x^2} \cdot \frac{p}{y}$

$\Rightarrow u_y = \frac{1}{2} \frac{1}{x^2+y^2} 2y$

$= py / p^2x^2+y^2$.

$u_y = \frac{y}{x^2+y^2}$

$v_y = \frac{y^2}{y^2+p^2x^2} \cdot px \cdot (-1/y^2)$

$v_y = \frac{-px}{y^2+p^2x^2}$.

If $f(z)$ is analytic then $u_x = v_y$

$\Rightarrow \frac{x}{x^2+y^2} = \frac{-px}{p^2x^2+y^2}$

$\Rightarrow \frac{-p}{p^2x^2+y^2} = x^2+y^2.$

$\Rightarrow$ ~~$p = (x^2+y^2)$~~

$\Rightarrow \frac{-px}{p^2x^2+y^2} = \frac{x}{x^2+y^2} \Rightarrow -px(x^2+y^2) = x(p^2x^2+y^2)$

$\Rightarrow -px^2 - py^2 = p^2x^2 + y^2$

$\Rightarrow -px^2 - p^2x^2 = y^2 + py^2.$

$\Rightarrow$ If $\boxed{p=-1}$ then $\boxed{u_x = v_y}$

If $f(z)$ is analytic $u_y = -v_x$.

$\Rightarrow \frac{y}{x^2+y^2} = -\frac{py}{p^2x^2+y^2} \Rightarrow \boxed{p=-1}$

$\therefore f$ is analytic If $\boxed{p=-1}$

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7. If $w = \log z$ find $\frac{dw}{dz}$ and determine where w is non-analytic.

Sol: Given $w = \log z$

Let $z = x+iy$ and $f(z) = w$.

$\therefore f(z) = \log z = \log(x+iy)$.

$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$

$x+iy = re^{i\theta}$
$\log(x+iy) = \log re^{i\theta}$

$\Rightarrow \log(x+iy) = \log re^{i\theta}$
$= \log r + \log e^{i\theta}$
$= \log r + i\theta$
$= \log\sqrt{x^2+y^2} + i\theta.$
$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

$\therefore f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x).$

compare it with $u(x,y) + iv(x,y)$.

Here $u(x,y) = \frac{1}{2}\log(x^2+y^2)$. ; $v(x,y) = \tan^{-1}(y/x)$

$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2x$; $v(x,y) = \frac{1}{1+\frac{y^2}{x^2}}\cdot y(-1/x^2)$

$u_y = \frac{1}{2}\cdot\frac{1}{x^2+y^2}\cdot 2y$ $\quad = \frac{x^2}{x^2+y^2}\cdot -y/x^2 = \frac{-y}{x^2+y^2}$

$\therefore v_x = -y/x^2+y^2$

$\Rightarrow v_y = \frac{x^2}{x^2+y^2}\cdot\frac{1}{x} = x/x^2+y^2.$

$\therefore u_x = \frac{x}{x^2+y^2}$; $v_x = \frac{-y}{x^2+y^2}$

$u_y = \frac{y}{x^2+y^2}$ $\quad v_y = x/x^2+y^2.$

$\therefore u_x = v_y$
$u_y = -v_x$

There fore. f satisfies C.R. Equations. and partial derivatives are not continuous at (0,0).

∴ f is. analytic for all z.

$f'(z) = u_x + iv_x$.

$= \frac{x}{x^2+y^2} + i(-y/x^2+y^2)$.

$= \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{z.\bar{z}}$ ∴ $f'(z) = 1/z \Rightarrow \boxed{\frac{d\omega}{dz} = 1/z}$.

f is not analytic.

There fore. 'ω' is not analytic at z = 0. Hence "ω" is. analytic except z = 0.

→ P.T z^n. 'n' is a (+ve) integer. Is analytic.

Sol: Let given that $f(z) = z^n$. and $z = x+iy$.

$x = r\cos\theta$; $y = r\sin\theta$. ∴ $f(z) = (r\cos\theta + ir\sin\theta)^n = r^n e^{in\theta}$

∴ $f(z) = r^n e^{in\theta}$.

$f(z) = r^n [\cos n\theta + i\sin n\theta]$.

compare it with. $u(r,\theta) + iv(r,\theta)$.

Here $u(r,\theta) = r^n\cos n\theta$; $v(r,\theta) = r^n\sin n\theta$.

⇒ $u_r = \cos n\theta.\, n r^{n-1}$; $v_r = \sin n\theta.\, n r^{n-1}$

⇒ $u_\theta = r^n(-\sin n\theta).\, n$; $v_\theta = r^n\cos n\theta.\, n$

$u_r = \frac{1}{r} v_\theta$; $u_\theta = -r v_r$.

⇒ $\cos n\theta\, n r^n.\frac{1}{r} = \frac{1}{r} r^n \cos n\theta.\, n$.

∴ $\boxed{u_r = 1/r\, v_\theta}$ $\boxed{u_\theta = -\frac{1}{r} v_\theta}$

∴ f is analytic for all z

Date : 8-7-2019
Day : Thursday

Harmonic function.

A function $u(x,y)$ satisfies. Laplace eq'n i.e $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ ($\nabla^2 u=0$) is called. Harmonic function.

Theorem.

If $f(z)=u+iv$ is analytic then real and imaginary. Parts are Harmonic function.

Proof. Let $f(z)=u+iv$ is analytic.

$\therefore$ 'f' satisfies C.R. Eq'ns. $u_x=v_y$; $u_y=-v_x$ $\longrightarrow$ (1)

claim.

To P.T u,v are harmonic.

Differentiating eq'n ① w.r.t x and y.

Therefore, $u_{xx}=v_{yx}$, $u_{yy}=-v_{xy}$.

$\therefore u_{xx}+u_{yy}=v_{yx}-v_{xy}=0$. [$\because v_{xy}=v_{yx}$]

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

$\therefore$ u is harmonic.

Differentiating eq'n① w.r.t y and 'x'.

Therefore, $u_{xy}=v_{yy}$, $u_{yx}=-v_{xx}$.

$\therefore v_{xx}+v_{yy}=-u_{yx}+u_{xy}=0$

$\therefore \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$

$\therefore$ v is harmonic.

Hence $f(z)=u+iv$ is analytic then u,v are harmonic functions.

Conjugate harmonic function.

If $f(z)=u+iv$ is analytic and if u,v satisfies Laplace equation then u (or v) is called conjugate harmonic of v (or u)

Construction of conjugate Harmonic function.

Let $f(z)=u+iv$ be analytic function whose real part u is known. we can find imaginary part v in the following procedure.

We know that $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$. $\left[\begin{matrix} u_x=v_y \\ u_y=-v_x\end{matrix}\right]$

$=-\frac{\partial u}{\partial y}dx+\frac{\partial u}{\partial x}dy$

Let $M=-\frac{\partial u}{\partial y}$, $N=\frac{\partial u}{\partial x}$

$\therefore$ $\boxed{dv = M\,dx + N\,dy}$ $\longrightarrow$ (1)

$\therefore$ Equation is exact.

Integration equation (1), we get Imaginary part B.

→ S.T. $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

sol given. $u = e^{-x}(x\sin y - y\cos y)$

Let $u(x,y) = e^{-x}(x\sin y - y\cos y)$.

$\Rightarrow u_x = e^{-x} x\sin y - e^{-x} y\cos y$

$= \sin y[-xe^{-x} + e^{-x}] + y\cos y\, e^{-x}$.

$= e^{-x}\sin y - xe^{-x}\sin y + e^{-x} y\cos y$.

$\Rightarrow u_{xx} = -e^{-x}\sin y - \sin y[-xe^{-x} + e^{-x}(1)] + y\cos y\,(e^{-x})(-1)$

$= -e^{-x}\sin y + xe^{-x}\sin y - e^{-x}\sin y - e^{-x} y\cos y$

$u_{xx} = xe^{-x}\sin y - 2e^{-x}\sin y - e^{-x} y\cos y$.

$u_y = e^{-x} x\cdot\cos y - e^{-x}[-y\sin y + \cos y(1)]$

$= xe^{-x}\cos y + e^{-x} y\sin y - e^{-x}\cos y$.

$u_{yy} = -xe^{-x}\sin y + e^{-x}[y\cos y + \sin y(1)] - e^{-x}(-\sin y)$

$= -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + e^{-x}\sin y$

$= -xe^{-x}\sin y + ye^{-x}\cos y + 2e^{-x}\sin y$

consider $u_{xx} + u_{yy}$

$xe^{-x}\sin y - 2e^{-x}\sin y - e^{-x}y\cos y - xe^{-x}\sin y + 2e^{-x}\sin y + ye^{-x}\cos y$

$= 0$

$\therefore u_{xx} + u_{yy} = 0$.

$\therefore$ "u" is harmonic function.

→ S.T the function $u = 2\log(x^2+y^2)$ is harmonic. And find its conjugate Harmonic.

Sol Let $u(x,y) = 2\log(x^2+y^2)$.

$\Rightarrow u_x = 2\,\frac{1}{(x^2+y^2)}\cdot 2x$

$u_x = \frac{4x}{x^2+y^2}$

$\Rightarrow u_{xx} = 4\left[\frac{(x^2+y^2)(1) - x(2x+0)}{(x^2+y^2)^2}\right] = 4\left[\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right] = 4\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$

$\Rightarrow u_y = 2\cdot\left[\frac{1}{x^2+y^2}\right]2y = 4\left[\frac{y}{x^2+y^2}\right]$; $u_{yy} = 4\left[\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right] = 4\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$

consider $u_{xx} + u_{yy} = 4\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right] + 4\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right] = \frac{4(y^2) - 4x^2 + 4x^2 - 4y^2}{(x^2+y^2)^2} = 0$

$\therefore u_{xx} + u_{yy} = 0$

$\therefore$ "u" is harmonic.

Let V be the conjugate Harmonic of u

Let $dv = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = -u_y\,dx + u_x\,dy$

$\therefore dv = \frac{-4y}{x^2+y^2}dx + \frac{4x}{x^2+y^2}dy \Rightarrow dv = 4\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right]$

$\therefore \int dv = 4\int \frac{x\,dy - y\,dx}{x^2+y^2}$ $\quad\therefore V = 4\,Tan^{-1}(y/x) + c.$

Construction of analytic function whose real part and Imaginary part are known.

Milne - Thomson's Method

Let $f(z) = u + iv$, whose real part u or imaginary v is known.

(i) * If u is given, taken $f' = u_x - iu_y$

* If V is given, taken $f' = V_y + iV_x$.

II. Replace x by z and y by "0".

III. Integrate w.r.t "z".

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1. Find Most general analytic function whose real part is $u = x^2 - y^2 - x$.

Sol Let $u(x,y) = x^2 - y^2 - x$.

$\Rightarrow u_x = 2x - 1$; $u_{xx} = 2$.

$\Rightarrow u_y = -2y$; $u_{yy} = -2$.

Let $f(z) = u + iv$ be analytic.

$f'(z) = u_x + iV_x$.

$\therefore f'(z) = u_x - iu_y$ [$\because$ from ①].

$= 2x - 1 - i(-2y)$.

$f'(z) = 2x - 1 + i2y$.

Replace x by z and y by "0".

$\therefore f'(z) = 2z - 1 + 0 \Rightarrow f'(z) = 2z - 1$.

$\therefore f(z) = 2\cdot\frac{z^2}{2} - z$.

$f(z) = z^2 - z + ic$.

→ Find an analytic function whose real part is $e^{-x}(x\sin y - y\cos y)$.

Sol: Let $u(x,y) = e^{-x}(x\sin y - y\cos y)$.

$\Rightarrow u_x = e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y$.

$\Rightarrow u_y = xe^{-x}\cos y + e^{-x}y\sin y - e^{-x}\cos y$

Let $f(z) = u + iv$ be analytic.

$\therefore f'(z) = u_x + iv_x$

$\therefore f'(z) = u_x - iu_y$.

$\therefore f'(z) = [e^{-x}\sin y - xe^{-x}\sin y + e^{-x}y\cos y] - i[xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y]$

Replace x by z and y by "0"

$\therefore f'(z) = (0 - 0 + 0) - i[ze^{-z}(1) + 0 - e^{-z}(1)]$

$\therefore f'(z) = -i[ze^{-z} - e^{-z}]$; $\Rightarrow f'(z) = ie^{-z}(1-z)$

$\therefore f(z) = i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)(-1)}\right]\right] = i[(z-1)e^{-z} + e^{-z}]$

$= i[e^{-z}(z - 1 + 1)]$

$= ize^{-z}$.

$\therefore$ $f(z) = ize^{-z} + iC$.

→ Find an analytic function whose real part is $\frac{x}{x^2+y^2}$.

Sol: Let $u(x,y) = \frac{x}{x^2+y^2}$

$\Rightarrow u_x = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\Rightarrow u_y = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

Let $f(z) = u + iv$ be analytic.

$\therefore f'(z) = u_x + iv_x$.

$f'(z) = u_x - iu_y$.

$\therefore f'(z) = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2}$

Replace x by z and y by "0".

$\therefore f'(z) = \frac{-z^2}{z^4} + 0$.

$= -1/z^2$.

$f(z) = \frac{1}{z} + iC$.

→ Find an analytic function whose real part is $u = e^x \cos y$.

Sol: Let $u(x,y) = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$

$\Rightarrow u_y = -e^x \sin y$.

Let $f(z) = u + iv$ be analytic

$\therefore f'(z) = u_x + iv_x \Rightarrow f'(z) = u_x - iu_y \Rightarrow f'(z) = e^x \cos y + i e^x \sin y$

Replace x by z and y by "0" $\therefore f'(z) = e^z (1) + i e^z (0)$

$\therefore f'(z) = e^z$ $\therefore$ $\boxed{f(z) = e^z + ic}$

→ Find an analytic function whose real part is $\frac{\sin 2x}{\cosh 2y - \cos 2x}$.

Sol: Let $u(x,y) = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)\cos 2x (2) - \sin 2x (0 + \sin 2x \cdot 2)}{(\cosh 2y - \cos 2x)^2}$$

$$u_x = \frac{2\cos 2x \cdot \cosh 2y - 2 \cdot \cos^2 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{0 - \sin 2x (\sinh 2y \cdot 2 - 0)}{(\cosh 2y - \cos 2x)^2} = \frac{-2 \sinh 2y \sin 2x}{(\cosh 2y - \cos 2x)^2}$$

Let $f(z) = u + iv$ be analytic.

$$f'(z) = u_x - iu_y \Rightarrow f'(z) = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} + i \frac{2\sinh 2y \sin 2x}{(\cosh 2y - \cos 2x)^2}$$

~~Let~~ Replace x by z and y by "0"

$$\therefore f'(z) = \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} + 0$$

$$= \frac{-2 \cancel{(1 - \cos 2z)}}{(1 - \cos 2z)^{\cancel{2}}}$$

$$= \frac{-2}{1 - \cos 2z}$$

$\therefore f'(z) = -2/1 - \cos 2z$.

$$f(z) = -2 \int \frac{1}{1 - \cos 2z} dz$$

$$= -\cancel{2} \int \frac{1}{\cancel{2}\sin^2 z} dz$$

$$= -\int \csc^2 z \, dz$$

$$= \cot z + ic$$

$\therefore$ $\boxed{f(z) = \cot z + ic}$

→ Find a function "w" such that $w=u+iv$ is an analytic. Given $u=\sin x\cosh y + 2\cos x \sinh y + x^2-y^2+4xy$.

Sol: Let $u(x,y) = \sin x\cosh y + 2\cos x\sinh y + x^2-y^2+4xy$.

$\Rightarrow u_x = \cosh y\cdot\cos x + 2\sinh y(-\sin x) + 2x - 0 + 4y$

$= \cosh y\cos x - 2\sinh y\sin x + 2x + 4y$.

$\Rightarrow u_y = \sin x\sinh y + 2\cos x\cdot\cosh y + 0 - 2y + 4x$

$= \sin x\sinh y + 2\cos x\cosh y - 2y + 4x$

Let $w=u+iv$ (is analytic) $\Rightarrow w'(z) = u_x + iv_x \Rightarrow w'(z) = u_x - iu_y$

$\Rightarrow w'(z) = \cosh y\cos x - 2\sinh y\sin x + 2x + 4y - i[\sin x\sinh y + 2\cos x\cosh y - 2y + 4x]$

Replace x by z and y by "0"

$= (1)\cos z - 0 + 2z + 0 - i[0 + 2\cos z(1) + 4z]$

$w'(z) = \cos z + 2z - i[2\cos z + 4z]$.

$w(z) = \int(\cos z + 2z)dz - i\int(2\cos z + 4z)dz$

$= \sin z + 2\cdot\frac{z^2}{2} - 2i\cdot\sin z - i\cdot 4\frac{z^2}{2}$

$= \sin z + z^2 - i[2\sin z + 2z^2]$

$= \sin z + z^2 - 2i[\sin z + z^2] + c_0$

Date: 12-7-19
Day: Monday

→ Find the analytic function whose imaginary part is $v = e^x\sin y$

Sol: Given that $v(x,y) = e^x\sin y$.

$\Rightarrow v_x = e^x\sin y$; $v_y = e^x\cos y$.

Let $f=u+iv$ be analytic then $u_x=v_y$; $u_y=-v_x$.

Let $f' = u_x + iv_x$ $\Rightarrow f' = v_y + iv_x$

$\Rightarrow f'(z) = e^x\cos y + i\, e^x\sin y$. By Milne's thomson.

Replace x by z and y by "0"

$\Rightarrow f'(z) = e^z(1) -$

$f = \int e^z = e^z$ ∴ ~~$f=e^z+ic_0$~~ $f=e^z+c_0$

point.

⇒ ...

real part $= e^x\cos y$..

→ Find the regular function (analytic function) whose imaginary part is $\dfrac{x-y}{x^2+y^2}$.

Sol: Given that $v(x,y) = \dfrac{x-y}{x^2+y^2}$

$$\Rightarrow v_x = \frac{(x^2+y^2)(1) - (x-y)2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-(x-y)(2x)}{(x^2+y^2)^2} \Rightarrow v_x = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2} \Rightarrow v_x = \frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$

$$\Rightarrow v_y = \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2} \Rightarrow v_y = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2} \Rightarrow v_y = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

Let $f = u+iv$ be analytic then $u_x = v_y \Rightarrow u_y = -v_x$

$\therefore f' = v_y + i v_x$.

$$\Rightarrow f' = \frac{y^2-x^2-2xy}{(x^2+y^2)^2} + i\,\frac{y^2-x^2+2xy}{(x^2+y^2)^2}$$

Replace x by z and y by '0'

$$\Rightarrow f'(z) = \frac{0-z^2-0}{(z^2+0)^2} + i\,\frac{0-z^2+0}{(z^2+0)^2} = -\frac{z^2}{z^4} - i\,\frac{z^2}{z^4}$$

$$= -\frac{z^2}{z^4}[1+i] \Rightarrow -\frac{1}{z^2} - i\,\frac{1}{z^2}$$

Integrating.

$$f = \int -\frac{1}{z^2}\,dz - i\int \frac{1}{z^2}\,dz = \frac{1}{z} + \frac{i}{z} + c = \frac{1+i}{z} + c$$

→ Find the analytic function whose Imaginary part is $\log(x^2+y^2)+x-2y$.

Sol: Let $v(x,y) = \log(x^2+y^2) + x - 2y$.

$$\Rightarrow v_x = \frac{1}{x^2+y^2}(2x) + 1 \Rightarrow v_x = \frac{2x}{x^2+y^2} + 1$$

$$\Rightarrow v_y = \frac{1}{x^2+y^2}(2y) - 2 \Rightarrow v_y = \frac{2y}{x^2+y^2} - 2$$

Let $f = u+iv \Rightarrow f' = v_y + i v_x$.

$$\Rightarrow f' = \frac{2y}{x^2+y^2} - 2 + i\left[\frac{2x}{x^2+y^2} + 1\right]$$

Replace x by z and y by "0"

$$\Rightarrow f' = 0 - 2 + i\left[\frac{2z}{z^2} + 1\right] \Rightarrow f' = -2 + i\left[\frac{2}{z} + 1\right]$$

Integrating $f = \int -2\,dz + i\int \frac{2}{z}\,dz + i\int dz = -2z + i\log z + iz$

$= -2z + i(2\log z + z) + c$.

→ If $f(z) = u+iv$ is analytic and $v = \frac{2\sin x \sin y}{\cos 2x + \cosh 2y}$ find u.

Sol Let $V(x,y) = \frac{2\sin x \sin y}{\cos 2x + \cosh 2y}$

$$\Rightarrow V_x = \frac{(\cos 2x + \cosh 2y)(2\sin y)\cos x - 2\sin x \sin y(-2\sin 2x)}{(\cos 2x + \cosh 2y)^2}$$

$$\Rightarrow V_x = \frac{(\cos 2x + \cosh 2y)(2\sin y \cos x) + 4\sin x \sin y \sin 2x}{(\cos 2x + \cosh 2y)^2}$$

$$V_y = \frac{(\cos 2x + \cosh 2y)(2\sin x)\cos y - 2\sin x \sin y(2\sinh 2y)}{(\cos 2x + \cosh 2y)^2}$$

Let $f(z) = u+iv$ is analytic then $u_x = v_y$; $v_x = -u_y$.

$\Rightarrow f'(z) = v_y + iv_x$

$$\Rightarrow f'(z) = \frac{(\cos 2x + \cosh 2y)(2\sin x \cos y) - 2\sin x \sin y(2\sinh 2y)}{(\cos 2x + \cosh 2y)^2} + i\left[\frac{(\cos 2x + \cosh 2y)(2\sin y \cos x) + 4\sin x \sin y \sin 2x}{(\cos 2x + \cosh 2y)^2}\right]$$

Replace x by z and y by "0"

$$\Rightarrow f'(z) = \frac{(\cos 2z + 1)(2\sin z) - 0}{(\cos 2z + 1)^2} + i[0]$$

$$f'(z) = \frac{2\sin z}{1+\cos 2z}$$

$$\Rightarrow f(z) = \int \frac{2\sin z}{2\cos^2 z}dz = \int \frac{\sin z}{\cos^2 z}dz$$ put $\cos z = t$, $-\sin z\, dz = dt$

$$= -\int \frac{dt}{t^2}$$

$$= 1/t$$

$$= 1/\cos z + c_1$$

$\therefore f(z) = 1/\cos z + c$.

$\Rightarrow f(z) = \sec z + c$

$f(z) = \sec(x+iy) + c = \frac{1}{\cos(x+iy)} + c$

$$\Rightarrow \frac{\cos(x-iy)}{\cos(x-iy)\cos(x+iy)} + c = \frac{\cos x \cos iy + \sin x \sin iy}{\frac{1}{2}[\cos 2x + \cos 2iy]} + c$$

$$= \frac{2\cos x \cos iy}{\cos 2x + \cos 2iy} + \frac{\sin x \sin iy}{\cos 2x + \cos 2iy} + c$$

→ (Formula).

[∵ $\sin iy = i\sinh y$

∴ $\cos iy = \cosh y$].

$$\therefore f(z) = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y} + \frac{\sin x\,(i\sinh y)}{\cos 2x + \cosh 2y}.$$

$$\therefore u = \frac{2\cos x \cosh y}{\cos 2x + \cosh 2y}$$

→ $V = (x^2-y^2) + \frac{x}{x^2+y^2}$.

→ $v = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$. Find analytic function.

→ $V = (x^2-y^2) + \frac{x}{x^2+y^2}$

Let $V(x,y) = x^2 - y^2 + \frac{x}{x^2+y^2}$.

$$\Rightarrow V_x = 2x - 0 + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$V_x = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\Rightarrow V_y = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2}.$$

$f(z)$ is analytic function.

$f(z) = u_x + iv_x$.

$f'(z) = V_y + iV_x$

$$\Rightarrow f'(z) = \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right) + i\left(2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\right)$$

Replace x by 'z' & y by '0'

$$\Rightarrow f'(z) = -2(0) - 2\frac{(z)0}{(z^2+0)^2} + i\left[2z + \frac{z^2+0-2z^2}{(z^2)^2}\right]$$

$$= 0 + i\left[2z - \frac{z^2}{z^4}\right].$$

$$= i\left[2z - \frac{1}{z^2}\right].$$

$$\Rightarrow f(z) = i\int \left(2z - \frac{1}{z^2}\right) dz = i\left[2\cdot\frac{z^2}{2} + \frac{1}{z}\right] = i\left[z^2 + \frac{1}{z}\right] + c.$$

$u = -2xy + \frac{y}{x^2+y^2}$

$$= i\left[(x+iy)^2 + \frac{1}{x+iy}\right]$$

$$= i\left(x^2 - y^2 + 2ixy\right) + i\,\frac{(x-iy)}{x^2+y^2}$$

$$= i(x^2-y^2) - 2xy + \frac{ix}{x^2+y^2} + \frac{y}{x^2+y}$$

2) $v(x,y) = \sin x \cdot \cosh y + 2\cos x \cdot \sinh y + x^2 - y^2 + 4xy$.

Sol: $v_x = \cos x \cdot \cosh y + 2(-\sin x) \cdot \sinh y + 2x - 0 + 4y$

$v_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$v_y = \sin x \sinh y + 2\cos x \cdot \cosh y + 0 - 2y + 4x$.

$v_y = \sin x \sinh y + 2\cos x \cdot \cosh y - 2y + 4x$

$f(z)$ is analytic function. $f(z) = U_x + iU_y$.

$f'(z) = v_y + i v_x$

$= \sin x \sinh y + 2\cos x \cdot \cosh y - 2y + 4x$.

$+ i[\cos x \cosh y - 2\sin x \sinh y + 2x + 4y]$

Replace x by z & y by "0"

$\Rightarrow f'(z) = \sin z(0) + 2\cos z \cdot (1) - 0 + 4z + i[\cos z(1) - 0 + 2z + 0]$

$= 0 + 2\cos z + 4z + i[\cos z + 2z]$

$= 2\cos z + 4z + i[\cos z + 2z)$

$\Rightarrow f(z) = 2 \cdot \sin z + 4\frac{z^2}{2} + i \cdot \sin z + 2 \cdot z^2/2$

$= 2\sin z + 2z^2 + i\sin z + z^2 + c$.

$\therefore f(z) = 2\sin z + 2z^2 + i\sin z + z^2 + c$,

where "c" is a complex constant.

→ S.T $u = x^2 - y^2$ is Harmonic and find its conjugate Harmonic.

Sol: Given that $u = x^2 - y^2$.

$\Rightarrow u_x = 2x$; $u_y = -2y$. $\Rightarrow u_{xx} = 2$; $u_{yy} = -2$.

$u_{xx} + u_{yy} = 2 - 2$
$= 0$

∴ "u" is Harmonic.

Let $f(z) = u + iv$ be analytic.

∴ $f'(z) = u_x + iv_x$. $\Rightarrow f'(z) = u_x - iu_y = 2x - i(-2y)$.

Replace x by z and y by "0".

$f'(z) = 2z + i(0) = 2z \Rightarrow f(z) = z^2 + c$,

$f(z) = z \cdot z + c$

$= (x + iy)^2 + ic$.

$= x^2 + i^2y^2 + 2xiy + ic$

$= x^2 - y^2 + i(2xy + c)$

∴ Conjugate Harmonic of u is $v = (2xy + c)$.

→ S.T $u = e^{2x}(x\cos 2y - y\sin 2y)$ is harmonic and find its conjugate Harmonic.

Sol. Given that $u = e^{2x} x\cos 2y - e^{2x} y\sin 2y$.

$\Rightarrow u_x = \cos 2y(2xe^{2x} + e^{2x}) - y\sin 2y\, 2e^{2x}$

$= 2xe^{2x}\cos 2y + e^{2x}\cos 2y - 2ye^{2x}\sin 2y$

$\Rightarrow u_{xx} = 2\cos 2y\,[xe^{2x}(2) + e^{2x}] + 2e^{2x}\cos 2y - 4ye^{2x}\sin 2y$.

$\Rightarrow u_y = -2xe^{2x}\sin 2y - e^{2x}[2y\cos 2y + \sin 2y]$.

$= -2xe^{2x}\sin 2y - 2ye^{2x}\cos 2y - e^{2x}\sin 2y$.

$\Rightarrow u_{yy} = -2xe^{2x}\cos 2y(2) - 2e^{2x}[2y(-\sin 2y) + \cos 2y(1)]$
$- 2e^{2x}\cos 2y$

$= -4xe^{2x}\cos 2y + 4ye^{2x}\sin 2y - 2e^{2x}\cos 2y - 2e^{2x}\cos 2y$

$u_{xx} + u_{yy} = 4xe^{2x}\cos 2y + 2e^{2x}\cos 2y + 2e^{2x}\cos 2y - 4ye^{2x}\sin 2y$
$- 4xe^{2x}\cos 2y + 4ye^{2x}\sin 2y - 2e^{2x}\cos 2y - 2e^{2x}\cos 2y$

$\therefore u_{xx}+u_{yy}=0$

$\therefore$ "u" is harmonic

Let $f(z)=u+iv$ be analytic function.

$f'(z)=u_x-iu_y$

$=2xe^{2x}\cos 2y+e^{2x}\cos 2y-2ye^{2x}\sin 2y-i\left[-2xe^{2x}\sin 2y-\right.$

$\left.2ye^{2x}\cos 2y-e^{2x}\sin 2y\right]$

Replace x by 'z' and "y" by "0"

$\Rightarrow f'(z)=2ze^{2z}(1)+e^{2z}(1)-0-i[0-0-0]$

$=2ze^{2z}+e^{2z}$

$=e^{2z}[2z+1)$

$\therefore f(z)=(2z+1)\frac{e^{2z}}{2}-(2)\frac{e^{2z}}{4}+c$

$=ze^{2z}+c.$

$f(z)=ze^{2z}+c.$

$=2e^{2(x+iy)}+ic$

$=2e^{2x}e^{2iy}+c$

$(x+iy)\,e^{2(x+iy)}+c$

$=(x+iy)e^{2x}.e^{2iy}+ic$

$=e^{2x}(x+iy)[\cos 2y+i\sin 2y)+ic.$ $=e^{2x}[x\cos 2y-y\sin 2y]+i[y\cos 2y+x\sin 2y]+c$

~~the real part is $-e^{2x}[x\cos 2y-y\sin 2y]$~~

$=e^{2x}(x\cos 2y-y\sin 2y)+ie^{2x}[y\cos 2y+x\sin 2y+c]$

$\therefore$ Harmonic conjugate of u is.

$V=e^{2x}[y\cos 2y+x\sin 2y+c]$

→ Find "k" such that $u(x,y) = x^3 + 3kxy^2$ may be Harmonic and find its conjugate Harmonic

Sol: Given that $u(x,y) = x^3 + 3kxy^2$

$\Rightarrow u_x = 3x^2 + 3ky^2 \Rightarrow u_{xx} = 6x + 0.$

$\Rightarrow u_y = 3kx(2y) \Rightarrow u_{yy} = 6kx.$

$\therefore u_{xx} + u_{yy} = 0 \Rightarrow 6x + 6kx = 0 \Rightarrow \boxed{k = -1}.$

Let $f(z) = u + iv$ be analytic function. $\Rightarrow f'(z) = u_x - iu_y.$

$\Rightarrow f'(z) = 3x^2 + 3(-1)y^2 - i[3(-1)x\,2y]$

$= 3x^2 - 3y^2 + i[6xy]$

Replace x by z and y by "0".

$\Rightarrow f'(z) = 3z^2 \quad \therefore f(z) = z^3 + ic$

$\Rightarrow f(z) = (x+iy)^2(x+iy) + ic$

$= (x^2 - y^2 + 2xyi)(x+iy) + ic$

$= x^3 + ix^2y - xy^2 - iy^3 + 2x^2yi + 2xy^2i^2 + ic$

$= x^3 - xy^2 - 2xy^2 + ix^2y - iy^3 + i2x^2y + ic$

$= (x^3 - 3xy^2) + i[3x^2y - y^3 + c]$

$\therefore$ conjugate Harmonic of "u" is $(3x^2y - y^3 + c)$

→ S.T the function $u(x,y) = 4xy - 3x + 2$ is real part of an analytic function and find its conjugate Harmonic

→ Find the analytic eq'n $f(z) = u + iv$. If $u(r,\theta) = -r^3 \sin 3\theta$.

Sol: Let $f(z) = u + iv$ be analytic.

The C.R eq'ns in polar form is: $u_r = \frac{1}{r} v_\theta$; $v_r = -\frac{1}{r} u_\theta$.

Given that $u(r,\theta) = -r^3 \sin 3\theta$.

$\Rightarrow u_r = -\sin 3\theta (3r^2)$; $\Rightarrow u_\theta = -3r^3 \cos 3\theta$.

But $v_r = -\frac{1}{r} u_\theta \Rightarrow v_r = -\frac{1}{r}(-3r^3 \cos 3\theta) = 3r^2 \cos 3\theta$.

$\therefore v_r = 3r^2 \cos 3\theta$.

Integrating w.r.t "r" $\Rightarrow v = 3\cos 3\theta \cdot \frac{r^3}{3} \Rightarrow$ $\boxed{v = r^3 \cos 3\theta + c(\theta)}$ ↓θ

Differentiating w.r.t "θ" $\Rightarrow v_\theta = -3r^3 \sin 3\theta + c'(\theta)$.

But $u_r = \frac{1}{r} v_\theta \Rightarrow u_r r = -3r^3 \sin 3\theta + c'(\theta)$

$\Rightarrow r(-3r^2 \sin 3\theta) = -3r^3 \sin 3\theta + c'(\theta) \Rightarrow$ $\boxed{c'(\theta) = 0}$

Integrating w.r.t θ.

$c(\theta) = \text{constant}$

From eq'n ① $\Rightarrow$ $\boxed{v = r^3 \cos 3\theta + c}$

$\therefore f(z) = u + iv$.

$= -r^3 \sin 3\theta + i(r^3 \cos 3\theta + c)$.

$= -r^3 (\sin 3\theta - i\cos 3\theta) + c$.

→ Find the analytic eq'n $f(z) = u + iv$. If $u(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$

Sol: Let $f(z) = u + iv$ be an analytic function.

The C.R eq'ns in polar form is: $u_r = \frac{1}{r} v_\theta$; $v_r = -\frac{1}{r} u_\theta$.

Given that $u(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$.

$\Rightarrow u_r = 2r \cos 2\theta - \cos\theta$; $u_\theta = -2r^2 \sin 2\theta + r\sin\theta$.

But $v_r = -\frac{1}{r} u_\theta \Rightarrow v_r = -\frac{1}{r}(-2r^2 \sin 2\theta + r\sin\theta) = -\frac{(-r)}{r}[2r \sin 2\theta - \sin\theta]$

$v_r = 2r \sin 2\theta - \sin\theta$.

Integrating w.r.t "r" $\Rightarrow v = r^2 \sin 2\theta - r\sin\theta + c(\theta)$ ——→ (1)

Differentiating w.r.t "θ" $\Rightarrow v_\theta = 2r^2 \cos 2\theta - r\cos\theta + c'(\theta)$.

But $u_r = \frac{1}{r} v_\theta$.

$\Rightarrow r(u_r) = 2r^2 \cos 2\theta - r\cos\theta + c'(\theta)$

$\Rightarrow r(2r\cos 2\theta - \cos\theta) = 2r^2 \cos 2\theta - r\cos\theta + c'(\theta)$.

$\therefore c'(\theta) = 0$ Integrating w.r.t θ $\Rightarrow$ $\boxed{c(\theta) = c}$

From (1)

$V = r^2\sin 2\theta - r\sin\theta + c$

$f(z) = u + iv$

$= r^2\cos 2\theta - r\cos\theta + 2 + i(r^2\sin 2\theta - r\sin\theta) + c.$

$f(z) = r^2[\cos 2\theta + i\sin 2\theta] - r[\cos\theta + i\sin\theta] + c + 2$ //

→ Find the analytic fn $f(z) = u + iv$. If $V(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$.

Sol: Let $f(z) = u + iv$ be an analytic fn.

The C.R eqn in polar form is $u_r = \frac{1}{r}v_\theta$, $v_r = -\frac{1}{r}u_\theta$.

Given that $v(r,\theta) = r^2\cos 2\theta - r\cos\theta + 2$.

$v_r = 2r\cos 2\theta - \cos\theta$; $v_\theta = -2r^2\sin 2\theta + r\sin\theta$.

But $u_r = \frac{1}{r}v_\theta \Rightarrow ru_r = v_\theta \Rightarrow ru_r = -2r^2\sin 2\theta + r\sin\theta$.

$\Rightarrow u_r = -2r\sin 2\theta + \sin\theta$.

Integrating w.r.t "r" $\Rightarrow u = -r^2\sin 2\theta + r\sin\theta + c(\theta)$.

Differentiating w.r.t "θ" $\Rightarrow u_\theta = -2r^2\cos 2\theta + r\cos\theta + c'(\theta)$.

But $v_r = -\frac{1}{r}u_\theta$

$\Rightarrow rv_r = 2r^2\cos 2\theta - r\cos\theta - c'(\theta)$.

$\Rightarrow r[2r\cos 2\theta - \cos\theta] = 2r^2\cos 2\theta - r\cos\theta - c'(\theta)$

$\therefore c'(\theta) = 0$.

Integrating w.r.t "θ" $\Rightarrow c(\theta) = c$.

$\therefore u = -r^2\sin 2\theta + r\sin\theta + c$.

$\therefore f(z) = u + iv \Rightarrow f(z) = -r^2\sin 2\theta + r\sin\theta + c + i[r^2\cos 2\theta - r\cos\theta + 2]$

$= -r^2[\sin 2\theta - i\cos 2\theta] + r[\sin\theta - i\cos\theta] + c + 2$ //

• Find the analytic function $f(z) = u + iv$. If $v(r,\theta) = (r - \frac{1}{r})\sin\theta$, $r \neq 0$.

$= \left(\frac{r^2-1}{r}\right)\sin\theta$

$= r\sin\theta - \frac{\sin\theta}{r}$

Sol: Let $f(z) = u + iv$ be an analytic function.

The C.R eqns in polar form is $u_r = \frac{1}{r}v_\theta$,

$v_r = -\frac{1}{r}u_\theta$.

Given that $V(r,\theta) = r\sin\theta - \frac{\sin\theta}{r}$

$\Rightarrow V_r = \sin\theta(1) + \sin\theta\left(\frac{1}{r^2}\right)$; $V_\theta = r\cos\theta - \frac{1}{r}\cos\theta$.

But $u_r = \frac{1}{r} V_\theta \Rightarrow r u_r = r\cos\theta - \frac{1}{r}\cos\theta \Rightarrow r u_r = r\left[\cos\theta - \frac{1}{r^2}\cos\theta\right]$

$\therefore u_r = \cos\theta - \frac{1}{r^2}\cos\theta$

Integrating w.r.t "r" $\Rightarrow U = r\cos\theta - \cos\theta\left(-\frac{1}{r}\right)$

$U = r\cos\theta + \frac{\cos\theta}{r} + c(\theta)$

Differentiating w.r.t "θ" $\Rightarrow u_\theta = -r\sin\theta - \frac{1}{r}\sin\theta + c'(\theta)$

But $V_r = -\frac{1}{r} u_\theta \Rightarrow r V_r = r\sin\theta + \frac{1}{r}\sin\theta - c'(\theta)$

$\Rightarrow r\sin\theta + \frac{1}{r}\sin\theta = r\sin\theta + \frac{1}{r}\sin\theta - c'(\theta)$

$\Rightarrow c'(\theta) = 0$

Integrating w.r.t "θ" $\Rightarrow c(\theta) = c$.

$\therefore U = r\cos\theta + \frac{\cos\theta}{r} + c$.

$\therefore f(z) = u + iv$.

$= r\cos\theta + \frac{1}{r}\cos\theta + c + i\left[r\sin\theta - \frac{\sin\theta}{r}\right]$

$= r\left[\cos\theta + i\sin\theta\right] + \frac{1}{r}\left[\cos\theta - i\sin\theta\right] + c$

→ If $f(z)$ is regular function of z. P.T $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Sol: Let $f(z) = u + iv$ be analytic function. $|f(z)|^2 = u^2 + v^2$. $\therefore |f(z)| = \sqrt{u^2 + v^2}$

Consider $\frac{\partial^2}{\partial x^2}|f(z)|^2 = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(u^2 + v^2)\right) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]$.

$\therefore \frac{\partial^2}{\partial x^2}|f(z)|^2 = 2\left[u\cdot\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\cdot\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$

Similarly, $\frac{\partial^2}{\partial y^2}|f(z)|^2 = 2\left[u\cdot\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$ "v" Harmonic

Consider, $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2\left[u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right.$

"u" harmonic $\left. + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$.

$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2\left[0 + 0 + u_x^2 + v_x^2 + u_y^2 + v_y^2\right]$.

[$\because f(z) = u + iv$, is analytic function, then u, v are harmonic functions].

$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2\left[u_x^2 + v_x^2 + (-v_x)^2 + (u_x)^2\right] = 2\left[2u_x^2 + 2v_x^2\right] = 4\left[u_x^2 + v_x^2\right]$

$\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$

[$\because$ $f(z) = u + iv$, $f'(z) = u_x + iv_x$, $|f'(z)|^2 = u_x^2 + v_x^2$] f satisfies C.R. eqns

→ If $f(z)$ is analytic function (or) (regular function) of z. P.T.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(z)|^2 = 2|f'(z)|^2$$

Sol: Let $u+iv$ be analytic function.

$|\text{Real } f(z)|^2 = u^2$

consider. $\frac{\partial^2}{\partial x^2} |\text{Real } f(z)|^2 = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} u^2\right] = \frac{\partial}{\partial x}\left[2u \frac{\partial u}{\partial x}\right] = 2\left[u \cdot \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$

$\Rightarrow \frac{\partial^2}{\partial x^2} |\text{Real } f(z)|^2 = 2\left[u \cdot \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right]$.

$\|^{ly}$

$\Rightarrow \frac{\partial^2}{\partial y^2} |\text{Real } f(z)|^2 = 2\left[u \cdot \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right]$.

consider $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(z)|^2 = 2\left[u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

[∵ $f(z) = u+iv$ be an analytic function then. "u" is Harmonic]

$\Rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(z)|^2 = 2\left[u_x^2 + u_y^2\right] = 2\left[u_x^2 + (-v_x)^2\right] = 2\left[u_x^2 + v_x^2\right]$

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(z)|^2 = 2\left[u_x^2 + v_x^2\right]$

[∵ Since "f" satisfied C.R eqns.
∴ $f(z) = u+iv \Rightarrow f'(z) = u_x + iv_x$
$|f'(z)|^2 = u_x^2 + v_x^2$]

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\text{Real } f(z)|^2 = 2|f'(z)|^2$.

Date :- 14-7-19
Day :- Wednesday

Orthogonal Trajectories

Two family of curves $u(x,y)=c_1$, $v(x,y)=c_2$ are said be an orthogonal system. If the intersect at right angles at each point of their intersection.

Properties

* every analytic function of $f(z)=u+iv$ defines two family of curves $u(x,y)=c_1$, $v(x,y)=c_2$ forming an orthogonal system. (or) If $w=f(z)$ is an analytic function then prove that the family of curves defined by $u(x,y)=c_1$ cuts orthogonally the family of curves $u(x,y)=c_1$.

Proof

Let $f(z)=u+iv$ is an analytic function.

So, $f(z)$ satisfies C.R. Eq'ns.

$u_x=v_y$; $u_y=-v_x$. $\rightarrow$ (1)

Let $u(x,y)=c_1$, $v(x,y)=c_2$.

$u(x,y)=c_1$

differentiating w.r.t 'x', y ;

$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dx}=0$; $\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}\frac{dy}{dx}=0$.

(m_1)

$m_1m_2=-\frac{u_x}{u_y}\times\frac{-v_x}{v_y}$ $\therefore$ $m_1m_2=-1$

$=\frac{u_x}{u_y}\times\frac{-u_y}{u_x}=-1$

Hence, $f(z)$ is analytic, then $u(x,y)=c_1$, $v(x,y)=c_2$ forming an orthogonal System.

* For $w=u+iv=z^3$, P.T $u(x,y)=c_1$, $v(x,y)=c_2$ are each other orthogonally

Sol: Let $w=f(z)=z^3=(x+iy)^3$

$=(x^3-3xy^2)+i(3x^2y-y^3)$.

$u(x,y)=x^3-3xy^2$, $v(x,y)=3x^2y-y^3$.

$\Rightarrow u_x=3x^2-3y^2$ $\quad v_y=3x^2-3y^2$

$\Rightarrow u_y=0-6xy$ $\quad v_x=6xy$

$\therefore u_x=v_y$; $u_y=-v_x$.

$m_1m_2=-1$; $-\frac{u_x}{u_y}\times-\frac{v_x}{v_y}$; $-\frac{u_x}{u_y}\times\frac{u_y}{u_x}$; $-\frac{u_x}{u_y}\times\frac{u_y}{u_x}$; -1

$\therefore f(z)=u+iv$ satisfies C.R equations

W.K.T If $f(z)=u+iv$ be an analytic function. then $u(x,y)=c_1$, $v(x,y)=c_2$ form an orthogonal System. $\therefore u(x,y)=x^3-3xy^2$; $v(x,y)=3x^2y-y^3$ are orthogonal

⇒ Find the orthogonal trajectories of the family of curves $x^3y - xy^3$ = constant

Sol: Let $u(x,y) = x^3y - xy^3$ = constant (c_1).

Let $f(z) = u + iv$ be analytic.

We know that, $f(z) = u + iv$ is analytic then $u(x,y) = c_1$, $v(x,y) = c_2$ forming an orthogonal system.

Here, to find out $v(x,y) = c_2$ by Milne-Thomson method.

$\Rightarrow f' = u_x + iv_x \Rightarrow f' = u_x - iu_y$

Let $u(x,y) = x^3y - xy^3 = c_1$

$\Rightarrow u_x = 3x^2y - y^3$; $u_y = x^3 - 3y^2x$.

$f'(z) = u_x + iv_x \Rightarrow f'(z) = u_x - iu_y$; $u_x = v_y$; $u_y = -v_x \Rightarrow v_x = -u_y$

$\Rightarrow f'(z) = 3x^2y - y^3 - i(x^3 - 3y^2x)$

$f'(z) = 3x^2y - y^3 - i(x^3 - 3y^2x)$

Replace x by z and y by 0 $\Rightarrow f'(z) = 0 - 0 - i[z^3 - 0]$.

$= -iz^3$.

$\therefore f(z) = -i\frac{z^4}{4} + c = \frac{z^4 i^2}{4} + ic$,

$(x+iy)^2 = x^2 + i^2y^2 + 2xiy$.

$= x^2 - y^2 + 2xiy$

$(x+iy)^2 = ''$

$f(z) = -\frac{i}{4}(x+iy)^4 + ic$

$(x+iy)^4 = (x^2 - y^2 + 2xiy)(x^2 - y^2 + 2xiy)$

$= -\frac{i}{4}\left[x^4 + y^4 - 2x^2y^2 + 4x^3iy + 4x^3iy^3 - 4x^2y^2\right] + ic$ $\quad x^4 - x^2y^2 + 2x^3iy - x^2y^2 + y^4 - 2xiy^3 + 2x^3iy - 2xiy^3 + 4x^2i^2y^2$

$= -\frac{i}{4}\left[x^4 + y^4 - 6x^2y^2 + i[4x^3y - 4xy^3]\right] + ic$.

$= -\frac{i}{4}(x^4 + y^4 - 6x^2y^2) + \frac{1}{4}(4x^3y - 4xy^3) + ic$.

$= x^3y - xy^3 - \frac{i}{4}[x^4 + y^4 - 6x^2y^2] + ic$.

$\therefore v(x,y) = -\frac{1}{4}(x^4 + y^4 - 6x^2y^2 + c)$.

Orthogonal trajectories of $u(x,y) = c_1$ is $v(x,y) = -\frac{1}{4}(x^4 + y^4 - 6x^2y^2 + c) = c_2$

⇒ Find the orthogonal trajectories of the family of curves $r^2\cos 2\theta = c_1$

Sol: Let $u(r,\theta) = r^2\cos 2\theta = c_1$.

Let $f(z) = u + iv$ be an analytic.

We know that $f(z) = u + iv$ is analytic then $u(x,y) = c_1$, $v(x,y) = c_2$ forming an orthogonal system.

$u(r,\theta) = r^2\cos 2\theta$.

$u_r = 2r\cos 2\theta$; $u_\theta = -2r^2\sin 2\theta$.

But $v_r = -\frac{1}{r}u_\theta = -\frac{1}{r}(-2r^2\sin 2\theta) = 2r\sin 2\theta$.

$\therefore v_r = 2r\sin 2\theta$.

Integrating w.r.t $r \Rightarrow V = 2\sin 2\theta \cdot \frac{r^2}{2} + c(\theta)$

$V = r^2\sin 2\theta + c(\theta)$

Complex potential function.

Let $\omega = \phi(x,y) + i\psi(x,y)$. If "$\omega$" is analytic then it is called complex potential function. Its real part $\phi(x,y)$ is called velocity potential function, and imaginary part $\psi(x,y)$ is called stream function in the context of fluid Mechanics.

If both ϕ and ψ satisfy Laplace eq'n with one function is given, find from another function.

→ If $\omega = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine the function ϕ.

Sol: Let $\omega = \phi + i\psi$.

$$\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

At $f'(z) = \phi_x + i\psi_x$ — ①

$$= \psi_y + i\psi_x.$$

$$\psi_x = 2x + \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} = 2x + \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\psi_y = -2y + x\left[\frac{-1}{(x^2+y^2)^2}\right](2y) = -2y - \frac{2xy}{(x^2+y^2)^2}.$$

$$f'(z) = \psi_y + i\psi_x$$

$$= -2y - \frac{2xy}{(x^2+y^2)^2} + i\left(2x + \frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

By Milne Thomson Method x by z, y by 0

$$\Rightarrow 0 - 0 + i\left(2z - \frac{z^2}{z^4}\right) = i\left(2z - \frac{1}{z^2}\right)$$

$$f(z) = i\left(2\frac{z^2}{2} + \frac{1}{z}\right) + ic = i\left(z^2 + \frac{1}{z}\right) + ic$$

$$\therefore f(z) = i\left[x^2 - y^2 + 2ixy + \frac{x - iy}{x^2+y^2}\right] + ic.$$

$$= i\left[\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + i\left(2xy - \frac{y}{x^2+y^2} + c\right)\right]$$

$$\therefore \phi = -2xy + \frac{y}{x^2+y^2} - c$$

→ If the potential function is $\log(x^2+y^2)$, find the complex potential function.

Sol: Let $\omega = \phi + i\psi$.

Let $\phi(x,y) = \log(x^2+y^2)$.

~~At $f'(z) = \phi_x + i\psi_x$~~ $\Rightarrow \phi_x - i\phi_y = f'(z)$

$\Rightarrow \phi_x = \frac{1}{x^2+y^2}(2x)$; $\phi_y = \frac{1}{x^2+y^2}(2y)$

$\Rightarrow f'(z) = \frac{2x}{x^2+y^2} - i\frac{2y}{x^2+y^2}$

Replace y by "0" and x by "z"

$\Rightarrow f'(z) = \frac{2z}{z^2} - 0 \Rightarrow \frac{2}{z}$ $\therefore f(z) = -2/z^2 + ic$. & $2\log z + ic$

$\Rightarrow f(z) = 2\log z + ic = 2\log(x+iy) + ic$,

$= 2\log(r\cos\theta + ir\sin\theta) + i$

$= 2(\log re^{i\theta}) + ic$

$= 2(\log r + \log e^{i\theta}) + ic$

$= \log r^2 + 2i\theta + ic$

$= \log(x^2+y^2) + 2i\tan^{-1}(y/x) + ic$

$\therefore f(z) = \log(x^2+y^2) + i[2\tan^{-1}(y/x) + c]$.

$\rightarrow$ If $w = \phi + i\psi$ represents the complex potential of an electrical field and $\psi = 3x^2y - y^3$, find ϕ.

Sol: Let $w = \phi + i\psi$.

$\psi = 3x^2y - y^3$.

At $f'(z) = \phi_x + i\psi_x$.

$= \psi_y + i\psi_x$.

$\psi_x = 6xy$; $\psi_y = 3x^2 - 3y^2$.

$\therefore f'(z) = 3x^2 - 3y^2 + i(6xy)$.

Replace x by "z" and y by "0" $\Rightarrow f'(z) = 3z^2 - 0 + 0 = 3z^2$.

$\therefore f(z) = z^3 + ic$

$= (x+iy)^3 + ic$

$= (x^2 - y^2 + 2xiy)(x+iy) + ic$

$= x^3 + ix^2y - xy^2 - iy^3 + 2x^2iy + 2xi^2y^2 + ic$

$= x^3 + ix^2y - xy^2 - iy^3 + 2x^2iy - 2xy^2 + ic$

$= x^3 - 3xy^2 + i[x^2y - y^3 + 2x^2y + c]$

S.T $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, if $z \neq 0$ is not analytic at $z=0$ although C.R. eq'ns are satisfied at the origin.

$f(z) = 0$ if $z = 0$.

Sol:

$$f(z) = \frac{xy^2(x+iy)}{x^2+y^4}, \quad \text{if } z \neq 0$$
$$= 0 \quad \text{if } z = 0 \quad \Bigg\} \longrightarrow (1)$$

To S.T C-R eq'ns are satisfied.

From eq'n (1) $u(x,y) = \frac{x^2y^2}{x^2+y^4}$, $v(x,y) = \frac{xy^3}{x^2+y^4}$

$u(0,0) = 0$, $v(0,0) = 0$.

Now $\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0)-u(0,0)}{x-0} = \underset{x\to 0}{Lt} \frac{0-0}{x} = 0.$

$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y)-u(0,0)}{y-0} = \underset{y\to 0}{Lt} \frac{0-0}{y} = 0.$

$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0)-v(0,0)}{x-0} = \underset{x\to 0}{Lt} \frac{0-0}{x} = 0.$

$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y-0} = \underset{y\to 0}{Lt} \frac{0-0}{y} = 0.$

$\therefore u_x = v_y$, $u_y = -v_x$ at $(x,y) = (0,0)$.

$\therefore$ f satisfies C-R eq'n at origin

To S.T $f'(z)$ does not exist at origin.

Case 1

Along x-axis $(y \to 0)$.

consider $\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{x+iy} = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{xy^2}{x^2+y^4}$

$= \underset{x\to 0}{Lt} (0) = 0.$

Case 2 Along y axis $(x \to 0)$.

consider $\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{\frac{xy^2(x+iy)}{x^2+y^4} - 0}{x+iy} = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{xy^2}{x^2+y^4}$

$= \underset{y\to 0}{Lt} (0) = 0.$

Case 3

Along the st. line.

consider $\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{\substack{y\to mx\\ x\to 0}}{Lt} \frac{xy^2}{x^2+y^4} = \underset{x\to 0}{Lt} \frac{x(mx)^2}{x^2+m^4x^4}$

$= \underset{x\to 0}{Lt} \frac{m^2x}{(1+m^4x^2)}$

$= \underset{x\to 0}{Lt} \frac{m^2x}{1+m^4x^2}$

$= 0$

Case IV.

Along the path. $x = my^2$.

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to my^2\\ y\to 0}} \frac{xy^2}{x^2+y^4}$

$= \lim_{y\to 0} \frac{my^4}{y^4(1+my)}$ ~~$= \lim_{y\to 0} \frac{my^2}{1+my^2} = 0$~~

$= \lim_{y\to 0} \frac{my^4}{m^2y^4+y^4} = \lim_{y\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$

Here $f'(z)$ does not same value if value of m is change.

Therefore $f'(z)$ does not exist at origin.

Hence $f(z)$ is not analytic at $z=0$ even though the C.R eq'ns are satisfied.

→ If $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$ if $z\neq 0$. is not analytic at $z=0$ although the C.R eq'ns are satisfied at the origin.

$= 0$ if $z=0$

Sol Given that $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$ if $z\neq 0$ } → ①

$= 0$ if $z=0$.

To S.T C.R eq'ns are satisfied

from eq'n ① $u(x,y) = \frac{x^3y^2}{x^6+y^2}$ $v(x,y) = -\frac{x^4y}{x^6+y^2}$

$u(0,0)=0$; $v(0,0)=0$.

Now,

$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$.

$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$.

$\therefore u_x = v_y$, $u_y = -v_x$ at $(x,y) = (0,0)$.

$\therefore f$ satisfies C.R eq'n at origin.

To S.T $f'(z)$ does not exist at origin

Case 1:- Along x-axis ($y\to 0$)

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\frac{x^3y(y-ix)}{x^6+y^2}-0}{x+iy}$ ~~$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3y}{x^6+y^2}$~~

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\frac{x^3y(y-ix)}{x^6+y^2}}{x+iy} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3y(y-ix)}{x^6+y^2} \times \frac{1}{x+iy} = 0$

case 2

Along Y-axis. $(x \to 0)$

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3y\,(y-ix)}{x^6+y^2} = 0 \times \frac{1}{x+iy} = 0$

case 3

Along the st. line. $y = mx$

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$

$\lim_{\substack{y\to mx\\ x\to 0}} \frac{x^3y\,(y-ix)}{x^6+y^2} \times \frac{1}{x+iy} = \frac{mx^4(mx-ix)}{x^6+m^2x^2} \times \frac{1}{x+i(mx)}$

case 4

Along the path $y = mx^2$.

$= \lim_{x\to 0} \frac{mx^2(mx-ix)}{x^4+m^2} \times \frac{1}{x+imx} = 0$

con

Here $f'(z)$ does not same value. if value of m is change.

Therefore $f'(z)$ does not exist at origin.

Here, $f(z)$ is not analytic at $z=0$. even though the C.R eqns are satisfied.

Case 4

Along the path $y = mx^3$

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{y\to mx^3\\ x\to 0}} \frac{x^3y\,(y-ix)}{x^6+y^2} \times \frac{1}{x+iy}$

$= \lim_{x\to 0} \frac{x^3(mx^3)(mx^3-ix)}{x^6+(mx^3)^2} \times \frac{1}{x+i(mx^3)}$

$= \lim_{x\to 0} \frac{x^6m\,(mx^3-ix)}{x^6+m^2x^6} \times \frac{1}{x+imx^3}$

$= \lim_{x\to 0} \frac{x^6 m\,(mx^3-ix)}{x^6(1+m^2)} \times \frac{1}{x+imx^3}$

$= \lim_{x\to 0} \frac{m\,(mx^3-ix)}{1+m^2} \times \frac{1}{ix}$

$= \lim_{x\to 0} \frac{m}{i(1+m^2)}$

$= m/i(1+m^2)$

S.T $f(z) = \frac{xy^2(x+iy)}{x^2+y^2}$, if $z \neq 0$ is not analytic at $z=0$ although C.R. eq'ns are satisfied at the origin.

$f(z)=0$ if $z=0$.

Q) $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, if $z \neq 0$
$= 0$ if $z=0$ $\Big\} \to ①$

To S.T C-R eq'ns are satisfied.

From eq'n ①. $u(x,y) = \frac{x^2y^2}{x^2+y^4}$, $v(x,y) = \frac{xy^3}{x^2+y^4}$

$u(0,0)=0$, $v(0,0)=0$.

Now $\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$.

$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$.

$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x} = 0$

$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$

$\therefore u_x = v_y$, $u_y = -v_x$ at $(x,y)=(0,0)$.

$\therefore$ f satisfies C.R eq'n at origin.

To S.T $f'(z)$ does not exist at origin.

case 1

Along x-axis ($y \to 0$).

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{x\to 0\\y\to 0}} \frac{\frac{xy^2(x+iy)}{x^2+y^4}-0}{x+iy} = \lim_{\substack{x\to 0\\y\to 0}} \frac{xy^2}{x^2+y^4}$

$= \lim_{x\to 0} (0) = 0.$

case 2 Along y-axis ($x \to 0$).

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{y\to 0\\x\to 0}} \frac{\frac{xy^2(x+iy)}{x^2+y^4}-0}{x+iy} = \lim_{\substack{y\to 0\\x\to 0}} \frac{xy^2}{x^2+y^4}$

$= \lim_{y\to 0} (0) = 0$.

case 3 Along the st. line.

consider $\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{\substack{y\to mx\\x\to 0}} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x(mx)^2}{x^2+m^4x^4}$

$= \lim_{x\to 0} \frac{m^2x^3}{(1+m^4x^2)x^2}$

$= \lim_{x\to 0} \frac{m^2x}{1+m^4x^2}$

$= 0$

→ P.T the function f(z) defined by $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ If $z \neq 0$
$= 0$ If $z = 0$
is continuous and C-R equations are satisfied at the origin but $f'(0)$ does not exist.

Sol Given that,

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \frac{x^3 + ix^3 - y^3 + y^3 i}{x^2+y^2} = \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}$$ if ...
$= 0$ if ...

Here, $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$; $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$.

$u(0,0) = 0$, $v(0,0) = 0$

* To s.t f(z) is continuous at origin. *

case 1. Along x-axis,

consider $\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} = \lim_{x\to 0} \frac{x^3 + i(x^3)}{x^2} = \lim_{x\to 0} (1+i)x = 0$.

case 2

Along y-axis,

consider $\lim_{z\to 0} f(z) = \lim_{\substack{y\to 0\\ x\to 0}} \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} = \lim_{y\to 0} \frac{-y^3 + i(y^3)}{y^2} = \lim_{y\to 0} \frac{y^3(i-1)}{y^2} = 0$.

case 3.

Along the st. line. $y = mx$.

consider $\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0\\ y=mx}} \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} = \lim_{x\to 0} \frac{(x^3 - m^3x^3) + i(x^3 + m^3x^3)}{x^2 + m^2x^2}$

$= \lim_{x\to 0} \frac{x^3(1-m^3) + x^3 \cdot i(1+m^3)}{x^2(1+m^2)} = \lim_{x\to 0} \frac{x[1-m^3 + i(1+m^3)]}{(1+m^2)} = 0$

∴ f(z) is continuous at the origin.

* To show, f(z) satisfies C.R eq'ns. *

$$\left(\frac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{u(x,0) - u(0)}{x-0} = \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x} = \frac{x}{x} = 1.$$

$$\left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{u(0,y) - u(0)}{y-0} = \lim_{y\to 0} \frac{\frac{0-y^3}{y^2} - 0}{y} = \frac{-y}{y} = -1.$$

$$\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \frac{v(x,0) - v(0)}{x-0} = \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x} = \frac{x}{x} = 1.$$

$$\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \frac{v(0,y) - v(0)}{y-0} = \lim_{y\to 0} \frac{y^3}{y^2}/y = y/y = 1.$$

∴ $u_x = v_y$; $u_y = -v_x$.

∴ f(z) satisfies C.R equations.

* To show that $f'(0)$ does not exist *

case 1 Along x-axis.

consider $\lim_{z\to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2} \times \frac{1}{x+iy} = \lim_{x\to 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{x^3(1+i)}{x^3}$

$= \lim_{x\to 0} (1+i)$

$= 1+i$

Case II

Along Y-axis.

$$\text{consider } \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy} = \lim_{y\to 0}\frac{-y^3+iy^3}{y^2\cdot(iy)}$$

$$= \lim_{y\to 0}\frac{(i-1)}{i} = \lim_{y\to 0}(i-1)(-i) = \lim_{y\to 0} -i^2+i = \lim_{y\to 0} 1+i$$

$$= 1+i.$$

case 3.

Along the st. line $y=mx$.

$$\text{consider } \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{y\to mx\\ x\to 0}} \frac{(x^3-y^3)+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}$$

$$= \lim_{x\to 0}\frac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{x^2+m^2x^2}\cdot\frac{1}{x+i(mx)}.$$

$$= \lim_{x\to 0}\frac{x^3(1-m^3+i(1+m^3))}{x^3(1+m^2)(1+im)}$$

$$= \lim_{x\to 0}\frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)} = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}.$$

$\therefore$ Hence $f'(z)$ does not exist zero.

. S.T the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin although C.R eqns are satisfied at that point.

Sol Given that $f(z)=\sqrt{|xy|}$

Here $u(x,y)=\sqrt{|xy|}$; $v(x,y)=0$.

To S.T C.R eq'ns are satisfied

$$\left(\frac{\partial u}{\partial x}\right)_{0,0} = \lim_{x\to 0}\frac{u(x,0)-u(0)}{x-0} = \lim_{x\to 0}\frac{0-0}{x}=0 \;;\; \left(\frac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0}\frac{u(0,y)-u(0)}{y-0} = \lim_{y\to 0}\frac{0-0}{0}=0.$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = 0 \;;\; \left(\frac{\partial v}{\partial y}\right)_{(0,0)}=0$$

$\therefore u_x = v_y$; $u_y = -v_x$.

Therefore $f(z)$ satisfies C.R equations.

To S.T $f'(z)$ does not exist at the origin.

CASE 1.

Along x-axis.

$$\text{consider, } \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{|xy|}-0}{x+iy} = \lim_{x\to 0}\frac{\sqrt{0}-0}{x} = \frac{0-0}{0}=0$$

CASE 2:

Along Y-axis

$$\text{consider } \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{y\to 0\\ x\to 0}}\frac{\sqrt{|xy|}-0}{x+iy} = \frac{0-0}{iy} = 0$$

CASE 3.

Along the st. line.

$$\text{consider } \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{\substack{y\to mx\\ x\to 0}}\frac{\sqrt{|xy|}-0}{x+iy} = \lim_{x\to 0}\frac{\sqrt{|x\,mx|}}{x+mx\,i} = \lim_{x\to 0}\frac{\sqrt{|m|}\cdot x}{(1+im)\,x}$$

$$= \lim_{x\to 0}\frac{\sqrt{|m|}}{1+im}$$

$$= \lim \frac{\sqrt{|m|}}{1+im}.$$

Here, $f'(z)$ depends on m

Therefore $f'(z)$ does not exist at origin. Hence $f(z)$ is not an[illegible] at $z=0$ eventhough the C.R eqns are satisfied.

COMPLEX - INTEGRATION

In this chapter we shall deal with integration of a (com) function of a complex variable. the concept of a indefinite integral to the function of a complex variable is same as to the function of a real variable. In case of definite integration of a function of a complex variable. $\int_a^b f(z)\,dz$. The path of the definite integral may be along any curve from $z=a$ to $z=b$.

Definition.

Let $F(t) = u(t) + iv(t)$, $a \le t \le b$, where u & v are continuous function of a real variable "t".

We define $\int_a^b F(t)\,dt = \int_a^b u(t) + i\,v(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$.

The real part of $\int_a^b F(t)\,dt = \int_a^b u(t)\,dt$.

The imaginary part of $\int_a^b F(t)\,dt = \int_a^b v(t)\,dt$.

Definition.

Let 'c' be a curve in the plane.

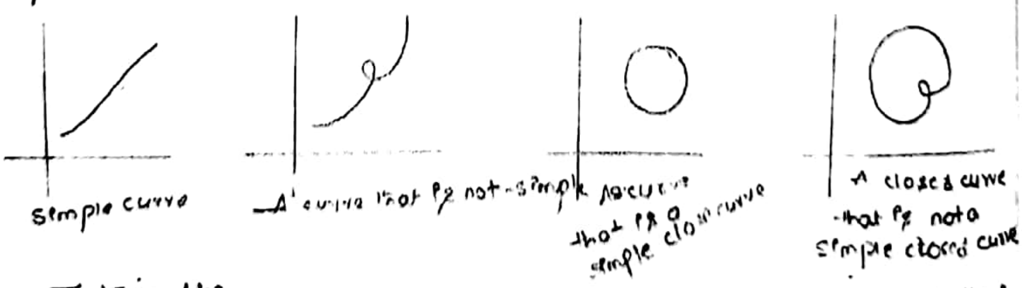
Let $c: z(t) = x(t) + i\,y(t)$ for $a \le t \le b$. where $x(t)$ and $y(t)$ are continuous functions of a variable "t" is called "continuous curve" or "continuous arc".

Suppose, the curve "c" is said to be. does not cross itself, then 'c' is called "simple arc" or "Jordan arc".

If $z(a) = z(b)$, then the curve "c" is said to a "closed curve" or "closed arc."

If $z(a) = z(b)$ is the only point of intersection then "c" is called "simple closed curve".

Example



Line Integration The complex line integration along the path "c" is denoted by integration along 'z'.

The complex integration $\int_c f(z)\,dz$ can be expressed as a sum of line integrated.

Let $z = x+iy$, $f(z) = u+iv$ $\therefore \int f(z)\,dz = \int (u+iv)\cdot(dx+idy)$

$\Rightarrow dz = dx + i\,dy$. $\Rightarrow \boxed{\int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i(v\,dx + u\,dy)}$

$\therefore \int_c f(z)\,dz = \int_c (u\,dx - v\,dy) + i\int_c u\,dy + v\,dx$

the real part of $f(z)$ is $= \int_c (u\,dx - v\,dy)$

the Imaginary part of $f(z)$ is $= \int u\,dy + v\,dx$.

→ Evaluate $\int_0^{1+i} z^2\,dz$ along the curve $y = x^2$

Sol: Given that $\int_0^{1+i} z^2\,dz$.

$0 \to 1+i$

$(0,0) \to (1,1)$

$x \to 0$ to 1

$y \to 0$ to 1

Let $f(z) = z^2$

Let $z = x+iy \Rightarrow dz = dx + i\,dy$.

Given curve is $y = x^2 \Rightarrow dy = 2x\,dx$.

consider $f(z)\,dz = z^2\,dz = (x+iy)^2\,(dx + i\,dy)$

$= (x^2 - y^2 + 2ixy)(dx + i\,dy)$

$= (x^2 + 2ix(x^2) - (x^2)^2)\,(dx + i\,2x\,dx)$

$\Rightarrow f(z)\,dz = (x^2 + 2ix^3 - x^4)(dx + 2xi\,dx)$.

$= (x^2 + 2ix^3 - x^4)\,dx + (x^2 + 2ix^3 - x^4)\,2xi\,dx$.

$= x^2\,dx + 2ix^3\,dx - x^4\,dx + i2x^3\,dx - 4x^4\,dx - i2x^5\,dx$.

$z^2\,dz = -i2x^5\,dx - 5x^4\,dx + 4ix^3\,dx + x^2\,dx$.

Therefore the line integral is.

$\Rightarrow \int_0^{1+i} z^2\,dz = \int_0^1 -i2x^5\,dx - \int_0^1 5x^4\,dx + \int_0^1 4ix^3\,dx + \int_0^1 x^2\,dx$

$= 2i\frac{x^6}{6}\Big]_0^1 - 5\frac{x^5}{5}\Big]_0^1 + 4i\frac{x^4}{4}\Big]_0^1 + \frac{x^3}{3}\Big]_0^1$

$= -\frac{i}{3}[x^6]_0^1 - x^5]_0^1 + i[x^4]_0^1 + \frac{1}{3}[x^3]_0^1$

$= -\frac{i}{3}[1-0] - [1-0] + i[1-0] + \frac{1}{3}[1-0]$

$= -i/3 - 1 + i + \frac{1}{3}$.

$= i\left[1-\frac{1}{3}\right] + \frac{1}{3} - 1$

$= i\left[\frac{2}{3}\right] + \frac{-2}{3}$.

$= 2/3\,[i-1]$.

→ Evaluate $\int_{(0,0)}^{(1,1)} 3x^2 + 5y + i(x^2 - y^2)\,dz$ along $y^2 = x$.

Sol: Given that. $\Rightarrow \int_{(0,0)}^{(1,1)} 3x^2 + 5y + i(x^2 - y^2)\,dz$

Let $f(z) = 3x^2 + 5y + i(x^2 - y^2)$.

Let $z = x+iy \Rightarrow dz = dx + i\,dy$.

Given curve is $y^2 = x \Rightarrow 2y\,dy = dx \Rightarrow dy = \frac{dx}{2y}$

consider $f(z)\,dz = 3x^2 + 5y + i(x^2-y^2)\cdot dz$

$= [3x^2+5y+i(x^2-y^2)]\,[dx+i\,dy]$

$= 3(y^2)^2+5y+i[(y^2)^2-y^2]\,[2y\,dy+i\,dy]$

$= (3y^4+5y+i[y^4-y^2])\,(2y\,dy+i\,dy)$

$= (3y^4+5y+iy^4-iy^2)\,(2y\,dy) + (3y^4+5y+i(y^4-y^2))\,(i\,dy)$

$= 6y^5dy+10y^2dy+2iy^5dy-2iy^3dy+3iy^4dy+i5y\,dy + i^2y^4dy - i^2y^2dy$

$= 6y^5dy+10y^2dy+2iy^5dy-2iy^3dy+3iy^4dy+i5y\,dy-y^4dy+y^2dy$

$= 6y^5dy+2iy^5dy+3iy^4dy-y^4dy-2iy^3dy+11y^2dy+i5y\,dy$

$\int_0^1 [y^6]_0^1 + 2i\frac{y^6}{6}\Big]_0^1 + 3i\frac{y^5}{5}\Big]_0^1 - \frac{y^5}{5}\Big]_0^1 - 2i\frac{y^4}{4}\Big]_0^1 + 11\frac{y^3}{3}\Big]_0^1 + i5\frac{y^2}{2}\Big]_0^1$

$= 1 + \frac{2i}{6} + \frac{3i}{5} - \frac{1}{5} - \frac{2i}{4} + \frac{11}{3} + \frac{5i}{2}$

$= 1-\frac{1}{5}+\frac{11}{3}+i\left[\frac{2}{6}+\frac{3}{5}-\frac{2}{4}+\frac{5}{2}\right]$

$\frac{1}{3}+\frac{3}{5}+2$

$= \frac{15-3+55}{15} + i\left[\frac{1}{3}+\frac{3}{5}-\frac{1}{2}+\frac{5}{2}\right]$

$= \frac{67}{15} + i\left[\frac{5+9+30}{15}\right] = \frac{67}{15}+i\left[\frac{44}{15}\right]$

$\therefore \int_{(0,0)}^{(1,1)} 3x^2+5y+i(x^2-y^2)\,dz = \frac{67}{15}+i\left(\frac{44}{15}\right)$

Date: 2-8-19.
Day: Monday.

→ Evaluate $\int_0^{1+i}(x^2-iy)\,dz$ along the paths (i) $y=x$ (ii) $y=x^2$.

Sol Let $f(z)=x^2-iy$

Let $z=x+iy \Rightarrow dz=dx+i\,dy$.

Along the path $y=x$

$\therefore y=x \Rightarrow dy=dx$ & $x: 0\to 1$.

consider $f(z)\,dz = (x^2-iy)\,(dx+i\,dy)$.

$= (x^2-ix)\,(dx+i\,dx)$

$= (x^2-ix)\,dx + (x^2-ix)\,i\,dx$

$= x^2dx - ix\,dx + ix^2dx + x\,dx$.

$\therefore \int_0^{1+i}(x^2-iy)\,dz = \int_0^1 x^2dx + x\,dx + ix^2dx - ix\,dx$.

$= \frac{x^3}{3} + \frac{x^2}{2} + i\frac{x^3}{3} - i\frac{x^2}{2} \Big]_0^1 = \frac{1}{3} + \frac{1}{2} + i\frac{1}{3} - i\frac{1}{2} = \frac{1}{3}[1+i] + \frac{1}{2}[1 \dots$

$= \frac{5}{6} + i\left(\frac{1}{3} - \frac{1}{2}\right) = 5/6 + i\left(\frac{2-3}{6}\right) = \frac{5}{6} + i\left(\frac{1}{6}\right) = \frac{1}{6}[5+i]$

$\therefore \int_0^{1+i} (x^2 - iy)\, dz = \frac{1}{6}[5-i]$

Along the path $y = x^2$.

$\because y = x^2 \Rightarrow dy = 2x\,dx. \Rightarrow$ x : 0 to 1, y : 0 to 1.

consider $f(z)\, dz = (x^2 - iy)(dx + i\,dy)$.

$= (x^2 - ix^2)(dx + i\,2x\,dx)$

$= (x^2 - ix^2)dx + (x^2 - ix^2)\, i\,2x\, dx$

$= x^2 dx - ix^2 dx + 2ix^3 dx + 2x^3 dx.$

$\therefore \int_0^{1+i} (x^2 - iy)\, dz = \int_0^1 (x^2 dx - ix^2 dx + 2ix^3 dx + 2x^3 dx)$

$= \left[\frac{x^3}{3} - i\frac{x^3}{3} + 2i\frac{x^4}{4} + 2\frac{x^4}{4}\right]_0^1 = \frac{1}{3} - i\frac{1}{3} + 2i\frac{1}{4} + 2\frac{1}{4}$

$= \left(\frac{1}{3} + \frac{2}{4}\right) + i\left(\frac{2}{4} - \frac{1}{3}\right) = \left(\frac{4+6}{12}\right) + i\left(\frac{6-4}{12}\right)$

$= \frac{10}{12} + i\left(\frac{2}{12}\right) = \frac{5}{6} + i\frac{1}{6} = \frac{1}{6}(5+i).$

$\therefore \int_0^{1+i} (x^2 - iy)\, dz = \frac{1}{6}(5+i)$

→ Evaluate $\int_{(0,0)}^{(1,3)} 3x^2y\,dx + (x^3 - 3y^2)\,dy$ along the curve i, $y = 3x$ (ii) $y = 3x^2$

Sol Given that $\int_{0,0}^{1,3} 3x^2 y\, dx + (x^3 - 3y^2)\, dy$

$y = 3x$

$\Rightarrow y = 3x \Rightarrow dy = 3\,dx.$

$\therefore f(z)\, dz = 3x^2 y\, dx + (x^3 - 3y^2)\, dy.$

$\int_{0,0}^{1,3} 3x^2 y\, dx + (x^3 - 3y^2)\, dy = \int_0^1 3x^2(3x)\, dx + (x^3 - 3(3x)^2)\, 3dx.$

$= \int_0^1 9x^3 dx + (x^3 - 27x^2)\, 3dx$

$= \int_0^1 9x^3 dx + 3x^3 dx - 81x^2 dx.$

$= 9\frac{x^4}{4} + 3\cdot\frac{x^4}{4} - 81\frac{x^3}{3}\Big]_0^1 = \frac{9}{4} + \frac{3}{4} - \frac{81}{3}27 = \frac{9}{4} + \frac{3}{4} - 27 = \frac{9+3}{4} - 27 = \frac{12}{4} - 27 = 3 - 27 = -24$

$Y = 3x^2$

$y = 3x^2 \Rightarrow dy = 6x\,dx.$

$\therefore f(z)\,dz = 3x^2y\,dx + (x^3 - 3y^2)\,dy = 3x^2(3x^2)\,dx + (x^3 - 3(3x^2)^2)\,6x\,dx$

$= 9x^4dx + [x^3 - 3(9x^4)]\,6x\,dx = 9x^4dx + 6x^4dx - 162x^5dx.$

$\therefore \int_{0,0}^{1,3} f(z)\,dz = \int_0^1 (15x^4dx - 162x^5dx) = 15\frac{x^5}{5} - 162\frac{x^6}{6}\Big]_0^1$

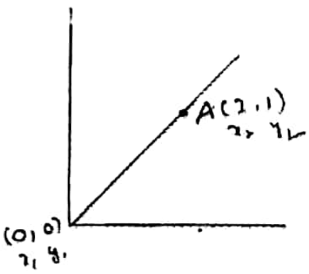
$= (3x^5 - 27x^6)\Big|_0^1$

$= 3 - 27 = -24$

$\therefore \int_0^1 f(z)\,dz = -24$

→ Evaluate $\int_0^{2+i} z^2 dz$ where c is st. line segment from $O(z=0)$ to $A(z=2+i)$

Sol



Let $f(z) = z^2$.

$\Rightarrow z = (x+iy).$

$\Rightarrow dz = dx + i\,dy.$

The eq'n of st. line joining $O(0,0)$ to $A(2,1)$ is

$\Rightarrow y - 0 = \frac{1-0}{2-0}(x-0). \Rightarrow y = x/2 \Rightarrow dy = \frac{dx}{2}$

$\Rightarrow x = 2y.$

consider $f(z)\,dz = z^2 dz$

$= (x+iy)^2(dx + i\,dy).$

$= (2y + iy)^2(2dy + i\,dy)$

$= [(2y)^2 + (iy)^2 + 2(2y)(iy)]\,(2dy + i\,dy)$

$= (4y^2 - y^2 + 4y^2 i)(2dy + i\,dy)$

$= (3y^2 + 4iy^2)(2dy + i\,dy)$

$= (3y^2 + 4iy^2)(2dy) + (3y^2 + 4iy^2)(i\,dy)$

$= 6y^2dy + 8iy^2dy + 3iy^2dy - 4y^2dy.$

$= 2y^2dy + 11iy^2dy$

$\therefore \int_0^1 z^2dz = \int_0^1 2y^2dy + 11iy^2dy = \left[2\frac{y^3}{3} + 11i\frac{y^3}{3}\right]_0^1 = \frac{2}{3} + \frac{11i}{3}$

$= \frac{1}{3}(2 + 11i)$

$\therefore \int_0^1 z^2 dz = \frac{1}{3}(2 + 11i)$

→ Integrate $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,8)$ along the curve.

$C: x = t;\ y = t^3$

Sol. Let $f(z) = x^2 + ixy$

$x = t$, $dx = dt$

$y = t^3$, $dy = 3t^2\,dt$

At $A = (1,1)$: $x = t \Rightarrow t = 1$; $y = t^3 \Rightarrow t = 1$ $\therefore t = 1$

At $B = (2,8)$: $x = t \Rightarrow t = 2$; $2^3 = t^3 \Rightarrow t = 2$ $\therefore t = 2$.

Consider $f(z)\,dz = (x^2 + ixy)(dx + i\,dy)$.

$$= (t^2 + i(t)(t^3))\,(dt + i(3t^2\,dt))$$

$$= [t^2 + it^4][dt + 3it^2\,dt]$$

$$= t^2\,dt + it^4\,dt + 3it^4\,dt - 3t^6\,dt$$

$$\therefore \int_{AB} f(z)\,dz = \int_1^2 t^2\,dt + 4it^4\,dt - 3t^6\,dt$$

$$= \left[\frac{t^3}{3} + 4i\frac{t^5}{5} - 3\frac{t^7}{7}\right]_1^2$$

$$= \left(\frac{8}{3} + 4i\frac{32}{5} - 3\cdot\frac{128}{7}\right) - \left[\frac{1}{3} + \frac{4i}{5} - \frac{3}{7}\right].$$

$$= \frac{8}{3} - \frac{1}{3} - \frac{384}{7} + \frac{3}{7} + \frac{128i}{5} - \frac{4i}{5}$$

$$= \frac{7}{3} - \frac{381}{7} + \frac{124i}{5}.$$

* Evaluate $\int_C (x-2y)dx + (x^2-y^2)dy$ where C is the boundary of the first-quadrant of the circle $x^2+y^2=4$.

Sol Given that $f(z)dz = (x-2y)dx + (x^2-y^2)dy$ the parametric eq'n of the circle $x^2+y^2=4$ are

$x = 2\cos\theta, \; y = 2\sin\theta$.

$dx = -2\sin\theta d\theta \; ; \; dy = 2\cos\theta d\theta$.

consider. $(x-2y)dx + (x^2-y^2)dy = (2\cos\theta - 4\sin\theta)(-2\sin\theta d\theta)$
$+ (4\cos^2\theta - 4\sin^2\theta)(2\cos\theta d\theta)$.

$= -4\sin\theta\cos\theta d\theta + 8\sin^2\theta d\theta + 4(\cos 2\theta)(2\cos\theta d\theta)$

$= -2(\sin 2\theta)d\theta + 8\left(\frac{1-\cos 2\theta}{2}\right) + 8\cos 2\theta \cos\theta d\theta$

$= -2\sin 2\theta d\theta + 4 - 4\cos 2\theta + 8\cos 2\theta \cos\theta d\theta$

$= -2\sin 2\theta d\theta + 4d\theta - 4\cos 2\theta d\theta + 4\cos 3\theta + 4\cos\theta$.

$\therefore \int_0^{\pi/2} f(z)dz = \int_0^{\pi/2} -2\sin 2\theta d\theta + 4d\theta - 4\cos 2\theta d\theta + 4\cos 3\theta d\theta + 4\cos\theta d\theta$

$= -2\left(\frac{-\cos 2\theta}{2}\right) + 4\theta - 4\frac{\sin 2\theta}{2} + 4\frac{\sin 3\theta}{3} + 4\sin\theta\Big]_0^{\pi/2}$

$= \cos 2(\pi/2) + 4(\pi/2) - 2\sin 2(\pi/2) + \frac{4}{3}\sin\left(\frac{3\pi}{2}\right) + 4\sin\pi/2$
$- \left[\cos 0 + 0 - 2\sin 0 + \frac{4}{3}\sin 0 + 4\sin 0\right]$

$= -1 + 2\pi - 0 + 0 + 4 - (1)$

$= -1 + 2\pi + 4 - 1$

$= 2\pi + 2$

$= 2(\pi+1)$ //

Day :- Thursday.

Cauchy's Theorem.

Cauchy's Theorem

Statement. Let $f(z)=u(x,y)+iv(x,y)$ be analytic on and within a simple closed contour "c" and let $f'(z)$ be continuous there, then $\int_c f(z)dz=0$.

Proof. Let $f(z)=u+iv$ be analytic function on and within a "c".

i.e. $u_x=v_y$ and $u_y=-v_x \rightarrow (1)$.

consider $f(z)\,dz=(u+iv)(dx+idy)$

$$=(udx-vdy)+i(vdx+udy).$$

Since $f'(z)$ is continuous, Then u_x, v_y, u_y, v_x are also continuous functions.

By Green's theorem.

$$\int vdx+udy=\iint \left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)dx\,dy \longrightarrow (2)$$

$$\Rightarrow \int (udx-vdy)=\iint \left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dx\,dy \longrightarrow (3).$$

consider $\int_c f(z)dz=\int_c \left[(udx-vdy)+i(vdx+udy)\right]dx\,dy$

$$=\iint\left[\left(-\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+i\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)\right]dx\,dy.$$

$$=\iint\left[\left(\frac{\partial u}{\partial y}-\frac{\partial u}{\partial y}\right)+i\left(\frac{\partial u}{\partial x}-\frac{\partial u}{\partial x}\right)\right]dx\,dy.$$

[$\because$ from (1)]

$$\Rightarrow \int_c f(z)dz=\iint (0+0)\,dx\,dy$$

$$\therefore \boxed{\int_c f(z)dz=0.}$$

Note:

If 'c' is a simple closed contour and $c_1, c_2, c_3, \ldots c_n$ are contours with in "c". and outside $f(z)$ is analytic with in "c" and outside $c_1, c_2, c_3, \ldots c_n$.

Thus $\int_C f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \cdots + \int_{c_n} f(z)\,dz$.

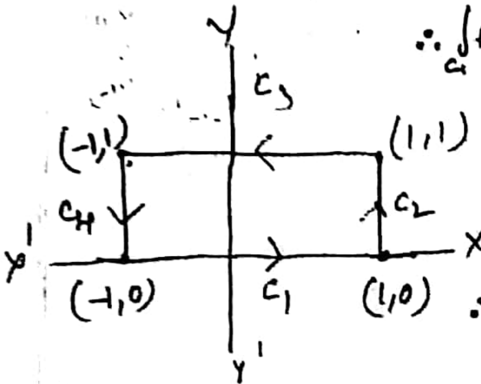
→ Verify cauchy integral theorem for the integral of z^3 taken the boundary of the rectangle with the vertices $-1, 1, 1+i, -1+i$.

Sol: Let $f(z) = z^3$.

$\therefore f(z)$ is analytic and $f'(z)$ is continuous.

Along c_1

$y=0 \Rightarrow dy=0$; $x: -1$ to 1.



$\therefore \int_{c_1} f(z)\,dz = \int_{-1}^{1} (x+iy)^3 (dx+idy)$

$= \int_{-1}^{1} (x+0)^3 (dx+0) = \int_{-1}^{1} x^3\,dx = \left.\frac{x^4}{4}\right]_{-1}^{1}$

$= \left[\frac{1}{4} - \left(\frac{1}{4}\right)\right] = 0$

$\therefore \boxed{\int_{c_1} f(z)\,dz = 0}$

$(1+iy)\ (1+iy)$

$(1+iy+iy-y^2)(1+iy)$

$2iy$

$(1+iy+2iy-2y^2-y^2-iy^3)$

$1+3iy-3y^2-iy^3$

Along c_2

$x=1 \Rightarrow dx=0$; $y: 0$ to 1

$\therefore \int_{c_2} f(z)\,dz = \int (x+iy)^3 (dx+idy). = \int_0^1 (1+iy)^3 (i\,dy).$

$= \int_0^1 (1+3iy-3y^2-iy^3)\, i\,dy$

$= \int_0^1 i\,dy + \int_0^1 3(-1)y\,dy - \int_0^1 3y^2 i\,dy - \int_0^1 (-1)\,y^3\,dy$

$= i(y)]_0^1 - 3\left(\frac{y^2}{2}\right)_0^1 - 3i\left(\frac{y^3}{3}\right]_0^1 + \left.\frac{y^4}{4}\right]_0^1$

$= i - \frac{3}{2} - i + \frac{1}{4}$

$= \frac{1}{4} - \frac{3}{2} = \frac{1-6}{4} = -5/4$. $\therefore \boxed{\int_{c_2} f(z)\,dz = -5/4}$

Along c_3

$y=1 \Rightarrow dy=0$; $x: 1$ to -1

$(x+i)^2$ $(x+i)(x+i)$

$(x^2-1+2xi)(x+i)$

$(x^3+x^2i-x-i+2x^2i...)$

$(x^3+3x^2i-3x-i)$

$\therefore \int_{c_3} f(z)\,dz = \int_{c_3} (x+iy)^3 (dx+idy). = \int_1^{-1} (x+i)^3 (dx).$

$= \int_1^{-1} (x^3+3x^2i-3x-i)\,dx$

$= \left[\frac{x^4}{4} + 3i\frac{x^3}{3} - 3\frac{x^2}{2} - ix\right]_1^{-1}$

$= \left[\frac{1}{4} - i - \frac{3}{2} + i\right] - \left[\frac{1}{4} + i - \frac{3}{2} i\right]$

$= \frac{1}{4} - \frac{3}{2} - \frac{1}{4} + 3/2$

$= 0$

$\therefore \int_{C_3} f(z)\,dz = 0$

$(x+iy)^2$
$(x^2 - y^2 + 2xiy)(x+iy)$
$= x^3 + ix^2y - xy^2 - iy^3 + 2x^2iy - 2xy^2$
$= -1 + iy + y^2 - iy^3 + 2iy + 2y^2$
$= -1 + 3iy + 3y^2 - iy^3$

Along C_4

$x = -1 \Rightarrow dx = 0$; $y : 1$ to 0.

$$\therefore \int_{C_4} f(z)\,dz = \int_{C_4} (x+iy)^3 (dx + i\,dy)$$

$$= \int_1^0 (-1 + 3iy + 3y^2 - iy^3)\, i\,dy$$

$$= \int_1^0 -i\,dy + \int_1^0 3(-1)y\,dy + \int_1^0 3i y^2\,dy - \int_1^0 (-1) y^3\,dy$$

$$= -i[y]_1^0 - 3\left[\frac{y^2}{2}\right]_1^0 + 3i\left[\frac{y^3}{3}\right]_1^0 + \left[\frac{y^4}{4}\right]_1^0$$

$$= (0 - 0 + 0 + 0) - \left[-i(1) - 3\left(\frac{1}{2}\right) + 3i\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)\right]$$

$$= i + \frac{3}{2} - i - \frac{1}{4} = \frac{3}{2} - \frac{1}{4} = \frac{6-1}{4} = 5/4.$$

$$\therefore \int_{C_4} f(z)\,dz = \frac{5}{4}.$$

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz$$

$$= 0 + \frac{5}{4} + 0 + \frac{5}{4}$$

$$= 0$$

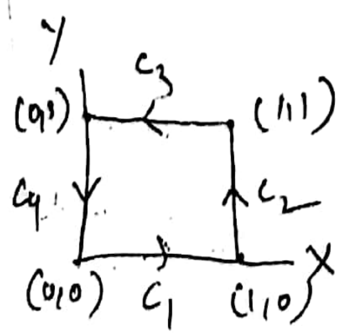
$$\therefore \int_C f(z)\,dz = 0$$

→ S.T. $\int_C (z+1)\,dz = 0$ where "c" is the boundary of the sphere whose vertices are $z = 0,\ z = 1,\ z = 1+i,\ z = i$.

$x+iy$: $0+0$: $x=0, y=0$; $x=1, y=0$; $x=1, y=1$; $x=0, y=1$

Sol: Let $f(z) = z + 1$.

$z = x + iy \Rightarrow dz = dx + i\,dy$.



Along C_1

$y = 0 \Rightarrow dy = 0$; $x : 0$ to 1

Consider $\int_C f(z)\,dz = \int_C (x + iy + 1)(dx + i\,dy)$

$$= \int_0^1 (x + 0 + 1)(dx + 0)$$

$$\therefore \int_{C_1} f(z)dz = \int_0^1 (x+1)\,dx = \frac{x^2}{2} + x \Big]_0^1 = \frac{1}{2} + 1 = 3/2$$

Along C_2

$x = 1$; $dx = 0$; y : 0 to 1

consider $\int_{C_2} f(z)dz = \int_{C_2} (x+iy+1)\,(dx + idy)$

$$= \int_0^1 (2+iy)(idy) = \int_0^1 2i\,dy - \int_0^1 y\,dy$$

$$= 2iy - \frac{y^2}{2}\Big]_0^1 = 2i - \frac{1}{2} \quad \therefore \int_{C_2} f(z)\,dz = 2i - 1/2$$

Along C_3

$y = 1 \Rightarrow dy = 0$; x : 1 to 0

consider $\int_{C_3} f(z)dz = \int_{C_3} (x+iy+1)\,(dx + idy)$

$$= \int_1^0 (x+i+1)(dx)$$

$$= \frac{x^2}{2} + ix + x\Big]_1^0$$

$$= -\left[\frac{1}{2} + i + 1\right] = -\left[3/2 + i\right]$$

$\therefore \int_{C_3} f(z)dz = -\left(\frac{3}{2} + i\right)$

Along C_4

$x = 0 \Rightarrow dx = 0$; y : 1 to 0.

$$\therefore \int_{C_4} f(z)dz = \int_1^0 (x+iy+1)(dx+idy) = \int_1^0 (iy+1)(idy)$$

$$= -\int_1^0 y\,dy + \int_1^0 i\,dy$$

$$= \left[-1 \cdot \frac{y^2}{2} + iy\right]_1^0 = -\left[-\frac{1}{2} + i\right]$$

$\therefore \int_{C_4} f(z)dz = \frac{1}{2} - i$

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)dz + \int_{C_4} f(z)\,dz.$$

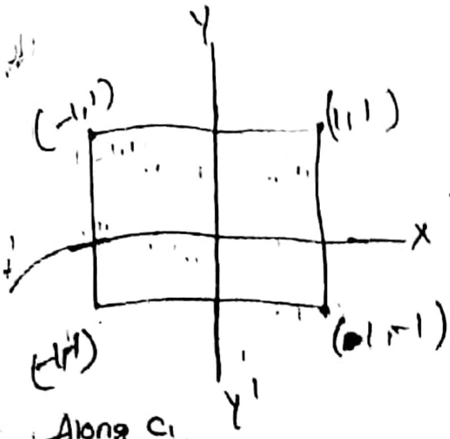
$$\therefore \frac{3}{2} + 2i - \frac{1}{2} - \frac{3}{2} - i + \frac{1}{2} - i = 0.$$

$$\therefore \int_C f(z)dz = 0$$

Verify cauchy theorem for the function $f(z) = 3z^2 + iz - 4$; If C is the Sphere with vertices $1 \pm i$, $-1 \pm i$.

$1+i$, $1-i$, $-1+i$, $-1-i$

$x=1$, $y=1$; $x=1$, $y=-1$; $x=-1$, $y=1$; $x=-1$, $y=-1$.



Let $f(z)$ is an analytic fd every point.

Along C_1.

The eq'n of st. line.

$y = -1$; $x = -1$ to 1

$dy = 0$.

$f(z)\,dz = (3z^2 + iz - 4)\,dz = (3(x+iy)^2 + i(x+iy) - 4)(dx + i\,dy)$

$= [3(x^2 - y^2 + 2xiy) + i(x+iy) - 4](dx + i\,dy)$

$= (3x^2 - 3y^2 + 6xiy + ix - y - 4)(dx + i\,dy)$

$= (3x^2 - 3y^2 + 6xiy + ix - y - 4)(dx + i\,dy)$

Along C_1 $\quad y = -1 \Rightarrow dy = 0$; $x : -1$ to 1.

$$\int_{C_1} f(z)\,dz = \int_{-1}^{1} (3x^2 - 3y^2 + 6xiy + ix - y - 4)(dx + i\,dy)$$

$$= \int_{-1}^{1} (3x^2 - 3 - 6xi + ix + 1 - 4)\,(dx).$$

$$= 3 \cdot \frac{x^3}{3} - 3x - 6i\frac{x^2}{2} + i\frac{x^2}{2} - 3x \Big]_{-1}^{1}$$

$$= x^3 - 6x - 3ix^2 + \frac{ix^2}{2} \Big]_{-1}^{1}$$

$$= \left(1 - 6 - 3i + \frac{i}{2}\right) - \left(-1 + 6 - 3i + \frac{i}{2}\right)$$

$$= \left(-5 - 3i + \frac{i}{2}\right) + 1 - 6 + 3i - \frac{i}{2}.$$

$$= -10$$

Along c_2

$x=1 \Rightarrow dx=0$; y : -1 to 1

$$\therefore \int_{c_2} f(z)\,dz = \int_{-1}^{1} (3x^2-3y^2+6xiy+ix-y-4)\,(dx+i\,dy)$$

$$= \int_{-1}^{1} (3-3y^2+6iy+i-y-4)\,(i\,dy)$$

$$= \int_{-1}^{1} 3i\,dy - \int_{-1}^{1} 3iy^2\,dy - 6\int_{-1}^{1} y\,dy - \int_{-1}^{1} dy - i\int_{-1}^{1} y\,dy - 4i\int_{-1}^{1} dy$$

$$= [3iy]_{-1}^{1} - [iy^3]_{-1}^{1} - [3y^2]_{-1}^{1} - [y]_{-1}^{1} - i\left[\frac{y^2}{2}\right]_{-1}^{1} - [4iy]_{-1}^{1}$$

$$= (3i+3i) - [i+i] - [3-3] - [1+1] - \left[\frac{i}{2}-\frac{i}{2}\right] - (4i+4i)$$

$i(1)^3 - (i(-1)^3)$

$i+1\ldots$

$$= 6i - i - i - 2 - 8i$$

$$= -3-3i \qquad 6i-2i-2-8i$$

$$= -3(1+i) \qquad -4i-2$$

$$\therefore \int_{c_2} f(z)\,dz = -4i-2$$

Along c_3

$y=1 \Rightarrow dy=0$; x : 1 to -1

$$\therefore \int_{c_3} f(z)\,dz = \int_{1}^{-1} (3x^2-3+6xi+ix-1-4)\,(dx)$$

$$= \int_{1}^{-1} (3x^2+7xi-8)\,dx$$

$$= \left[x^3 + 7i\frac{x^2}{2} - 8x\right]_{1}^{-1}$$

$$= \left[-1+\frac{7i}{2}+8\right] - \left[1+\frac{7i}{2}-8\right]$$

$$= -1+\frac{7i}{2}+8-1-\frac{7i}{2}+8$$

$$= 14.$$

$$\therefore \int_{c_3} f(z)\,dz = 14.$$

Along c_4

$x=-1 \Rightarrow dx=0$; y : 1 to -1

$$\therefore \int_{c_4} f(z)\,dz = \int_{c_4} (3x^2-3y^2+6xiy+ix-y-4)\,(dx+i\,dy)$$

$$\Rightarrow \int_{C_4} f(z)\,dz = \int_1^{-1} (3-3y^2-6iy-i-y-4)\,(i\,dy)$$

$$= \int_1^{-1} (-3y^2-6iy-i-y-1)\,i\,dy$$

$$= -3i\int_1^{-1} y^2\,dy + 6\int_1^{-1} y\,dy + 1\int_1^{-1} dy - i\int_1^{-1} y\,dy - i\int_1^{-1} dy$$

$$= \left[-i\,y^3\right]_1^{-1} + \left[3y^2\right]_1^{-1} + \left[y\right]_1^{-1} - i\left[\frac{y^2}{2}\right]_1^{-1} - i\left[y\right]_1^{-1}$$

$$= (+i+i) + [3-3] + [-1-1] - i\left[\frac{1}{2}-\frac{1}{2}\right] - i[-1-1]$$

$$= 2i - 2 + 2i$$

$$= 4i-2.$$

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \int_{C_4} f(z)\,dz$$

$$= -16 - 4i - 2 + 16 + 4i - 2$$

$$= 0$$

$$\therefore \int_C f(z)\,dz = 0$$

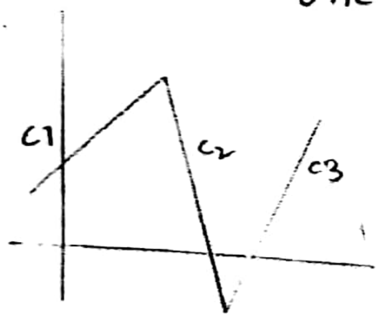
Day ÷ Friday.

Smooth arc. — A curve $c: z(t) = x(t) + i y(t)$, $a \le t \le b$ is said to be smooth arc', if $z'(t)$ is continuous and non-zero on that interval.

contour: — A curve 'c' that is ~~contin~~ constructed by joining finitely smooth arcs end to end is called a contour.

Example

The contour $c = c_1 + c_2 + c_3$ //



Def: - A point $z = a$ is called singular point of $f(z)$, if $f(z)$ is not analytic at $z = a$.

Cauchy Integral formula.

Statement. Let $f(z)$ be a an analytic function. everywhere on and with in a closed curve 'c'. If $z = a$ is a point with in 'c'. then $f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{f(z-a)} dz$. Where the Integral taken anticlockwise direction.

Generalization of cauchy Integral formula

If $f(z)$ is analytic. function everywhere on and with in a closed curve 'c'. If $z = a$. is a point with in 'c'. Then $f^{(n)}(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$.

where on and

11) Evaluate $\oint_C \frac{z^2+4}{z-3}\,dz$ where c is $|z|=5$.

Ans:- The singular point of integrand function $z=3$ lies inside the curve $|z|=5$.

$\left(\because z=3 \text{ sub in } |z|=5 \Rightarrow |3|<5 \Rightarrow 3<5\right)$

Let $f(z)=z^2+4$.

By cauchy's integral formula,

$$f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

$$\therefore \int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \int_C \frac{z^2+4}{z-3}\,dz = 2\pi i\, f(3)$$

$$= 2\pi i\left[3^2+4\right] \quad (\because f(z)=z^2+4)$$

$$= 26\pi i$$

2) Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is the circle $|z|=3$.

Ans:- The singular points are $z=1,2$ are lies inside the circle $|z|=3$. $\left(\because |1|<3,\ |2|=2<3\right)$

Let $f(z)=e^{2z}$.

$z-2-z+1$

consider $\frac{1}{(z-1)(z-2)} = \frac{1}{-1}\left[\frac{1}{z-1} - \frac{1}{z-2}\right]$

$= \frac{1}{z-2} - \frac{1}{z-1}$

By cauchy's integral formula,

$$\int_c \frac{f(z)}{(z-a)}\,dz = 2\pi i\, f(a)$$

$$\therefore \int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_c e^{2z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]dz$$

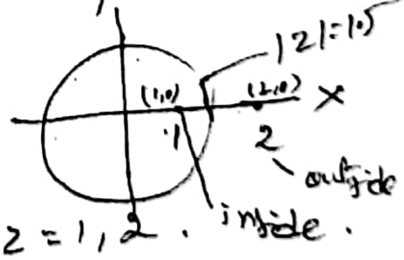
$$= \int_c \frac{e^{2z}}{z-2}\,dz - \int_c \frac{e^{2z}}{z-1}\,dz$$

$$= 2\pi i\, f(2) - 2\pi i\, f(1)$$

$$= 2\pi i\, e^{4} - 2\pi i\, e^{2} \quad (\because f(z) = e^{2z})$$

$$= 2\pi i\left[e^{4} - e^{2}\right]$$

3) Evaluae $\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is The circle $|z| = 1.5$.



Ans:- The singular points are $z = 1, 2$.

$z=1$ lies inside $|z| = 1.5$ $(\because |1| = 1 < 1.5)$

$z=2$ lies outside $|z| = 1.5$ $(\because |2| = 2 > 1.5)$

Let $f(z) = \frac{e^{2z}}{z-2}$

By cauchy's Integral formula,

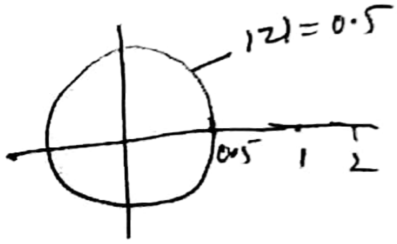
$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \int_C \frac{e^{2z}/z-2}{z-1}\,dz = 2\pi i\, f(1)$$

$$= 2\pi i \left[\frac{e^2}{1-2}\right] \quad \left(\because f(z) = \frac{e^{2z}}{z-2}\right)$$

$$= -2\pi i\, e^2$$

4) Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is the circle $|z| = 0.5$.



Ai:- The singular points are $z = 1, 2$ are lies outside the circle $|z| = 0.5$.

By cauchy's theorem, $f(z)$ is everywhere analytic within & on $|z| = 0.5$.

$$\therefore \int_C f(z)\,dz = 0.$$

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)} = 0.$$

5) using cauchy's Integral formula,

Evaluate $\int_C \frac{2z+1}{z^2+z}\,dz$ where c is the circle $|z| = 1/2$.

Sol: ... $z^2+z=0$

$z(z+1)=0 \Rightarrow z=0,-1$

The singular points are $z=0,-1$.

$z=0$ lies inside $|z|=\frac{1}{2}$ $\left(|0|=0<\frac{1}{2}\right)$

$z=-1$ lies outside $|z|=\frac{1}{2}$ $\left(|-1|=1>\frac{1}{2}\right)$

Let $f(z)=\dfrac{2z+1}{z+1}$.

By Cauchy's Integral formula,

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

$$\int_c \frac{2z+1/z+1}{z}\,dz = 2\pi i\, f(0)$$

$$= 2\pi i\left(\frac{0+1}{0+1}\right) \quad \left(\because f(z)=\frac{2z+1}{z+1}\right)$$

$$= 2\pi i$$

6) Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_c \frac{e^{zt}}{z^2+1}\,dz,\ t>0$ where c is represents the circle $|z|=3$.

Sol:- consider $z^2+1=0$

$z^2=-1 \Rightarrow z=\sqrt{-1}=\pm i = i,-i$

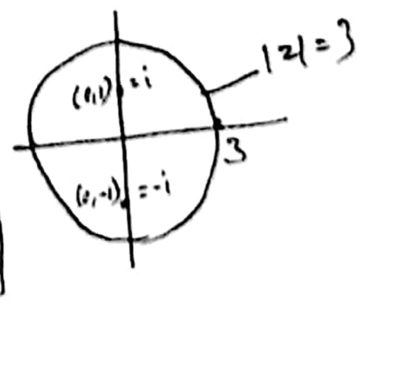
$z=i,-i$ lies inside the circle $|z|=3$.

$\left(\because |i|=\sqrt{0^2+1^2}=\sqrt{1}=1<3\right.$

$\left.|-i|=\sqrt{0+(-1)^2}=\sqrt{1}=1<3\right)$

consider $\dfrac{1}{z^2+1}=\dfrac{1}{(z+i)(z-i)}$

$$=\frac{1}{2i}\left[\frac{1}{z-i}-\frac{1}{z+i}\right]$$



Let $f(z)=e^{zt}$.

By Cauchy's Integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

$$\therefore \int_C \frac{e^{zt}}{z^2+1}\,dz = \int_C e^{zt}\left[\frac{1}{2i}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]\right] dz$$

$$= \frac{1}{2i}\left[\int_C \frac{e^{zt}}{z-i}\,dz - \int_C \frac{e^{zt}}{z+i}\,dz\right]$$

$$= \frac{1}{2i}\left[2\pi i\, f(i) - 2\pi i\, f(-i)\right]$$

$$= \frac{2\pi i}{2i}\left[f(i) - f(-i)\right]$$

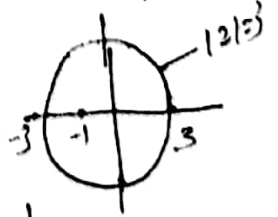
$$= \frac{2\pi i}{2i}\left[e^{it} - e^{-it}\right]$$

$$\therefore \int_C \frac{e^{zt}}{z^2+1}\,dz = \frac{2\pi i}{2i}\left[e^{it} - e^{-it}\right]$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}\,dz = \frac{e^{it} - e^{-it}}{2i} = \sin t$$

7) Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$, where c is the circle $|z| = 3$.

8) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$ where the path of integration c is $|z| = 3$.



Ai:- Consider $z+1=0 \Rightarrow z=-1$

$z=-1$ lies inside $|z|=3$ $(\because |-1| = 1 < 3$

Let $f(z) = e^{2z}$

By cauchy's general integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$$

$$\therefore \int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!} f^{(3)}(-1)$$

$$= \frac{2\pi i}{6}\left[\frac{d^3}{dz^3}(e^{2z})\right]_{z=-1}$$

$$= \frac{\pi i}{3}(8e^{2z})_{z=-1}$$

$$= \frac{8\pi i}{3}e^{-2}$$

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i}{3e^2}$$

9) Find the value of $\int_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz$, if c is the circle $|z|=1$.

Sol:- consider $z-\frac{\pi}{6}=0 \Rightarrow z=\pi/6$.

$z=\frac{\pi}{6}$ lies inside $|z|=1$ $\left(\because \left|\frac{\pi}{6}\right| = \left|\frac{22/7}{6}\right| < 1\right)$

Let $f(z) = \sin^6 z$.

By cauchy's general integral formula,

$$\int_c \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

$$\int_c \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{2!} f^{(2)}\left(\frac{\pi}{6}\right)$$

$$= \pi i \left[\frac{d^2}{dz^2} (\sin^6 z) \right]_{z=\pi/6}$$

$$= \pi i \left[\frac{d}{dz} (6 \sin^5 z \cos z) \right]_{z=\pi/6}$$

$$= 6\pi i \left[\sin^5 z (-\sin z) + \cos z \cdot 5 \sin^4 z \cos z \right]_{z=\pi/6}$$

$$= 6\pi i \left[-\sin^6 z + 5 \sin^4 z \cos^2 z \right]_{z=\pi/6}$$

$$= 6\pi i \left[-\left(\frac{1}{2}\right)^6 + 5\left(\frac{1}{2}\right)^4 \left(\frac{\sqrt{3}}{2}\right)^2 \right]$$

$\sin 30 = \frac{1}{2}$

$\cos 30 = \frac{\sqrt{3}}{2}$

$$= 6\pi i \left[-\frac{1}{64} + \frac{15}{64} \right]$$

$$= {}^{3}\cancel{6}\pi i \left[\frac{\cancel{14}^{7}}{\cancel{64}_{32\,16}} \right] = \frac{21\pi i}{16}.$$

$$\therefore \int_c \frac{\sin^6 z}{\left(z - \frac{\pi}{6}\right)^3} dz = \frac{21\pi i}{16}$$

(10) Evaluate $\int_c \frac{z-3}{z^2+2z+5} dz$, where c is

circle (i) $|z| = 1$

(ii) $|z+1-i| = 2$

(iii) $|z+1+i| = 2$.

Ans:- consider $z^2 + 2z + 5 = 0$.

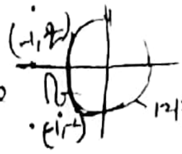
$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$\therefore z = -1+2i, -1-2i$

(i) The points $z = -1+2i, -1-2i$ are both outside the circle $|z|=1$.



$\left(\because |-1+2i| = \sqrt{1+4} = \sqrt{5} > 1\right.$

$\left.|-1-2i| = \sqrt{(-1)^2+(-2)^2} = \sqrt{5} > 1\right)$

$\therefore f(z)$ is analytic everywhere within and on $|z|=1$.

By cauchy Theorem, $\int_C f(z)\,dz = 0$.

(ii) $z = -1+2i$ sub in $|z+1-i| = |-1+2i+1-i|$

$= |i|$

$= \sqrt{0+1} = 1 < 2$.

$z = -1+2i$ lies <u>inside</u> $|z+1-i| < 2$.

$z = -1-2i$ sub in $|z+1-i| = |-1-2i+1-i|$

$= |-3i|$

$= \sqrt{0+(-3)^2} = 3 > 2$.

$\therefore z = -1-2i$ lies outside the circle

$|z+1-i| < 2$.

Let $f(z) = \dfrac{z-3}{z+1+2i}$ $\qquad z^2+2z+5 = (z+1-2i)(z+1+2i)$

By cauchy's Integral formula,

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

$$\int_C \frac{z-3/z+1+2i}{z+1-2i}\,dz = 2\pi i\, f(-1+2i)$$

$$\int_C \frac{z-3}{z^2+2z+5}\,dz = 2\pi i\left[\frac{-1+2i-3}{-1+2i+1+2i}\right]$$

$$= 2\pi i\left[\frac{2i-4}{4i}\right]$$

$$= \pi(i-2)$$

(iii) Students practice this problem.

Ans:- $-\pi(i+2)$

(11) Evaluate $\int_C \frac{z^2-4}{z(z^2+9)}\,dz$ where c is the circle $|z|=1$.

Ans:- $-8\pi i/9$

12) Evalue $\int_C \frac{dz}{z(z+i\pi)}$, where c is circle $|z+3i|=1$ by using cauchy's Integral formula.

Ans:- 0

13) By using cauchy's Integral formula, Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$, where c is $|z-i|=2$.

Ans:- $-2\pi i/9$

14) Evaluate $\oint_C \frac{3z^2+z}{z^2-1}\,dz$, where c is the circle $|z-1|=1$.

Ans:- $4\pi i$

15) Evaluate $\oint_C \frac{e^{-z}}{z^2}\,dz$, where c is $|z|=1$.

Ans:- $-2\pi i$

16) Evaluate $\oint_C \frac{3z^2+7z+1}{z+1}\,dz$, where c is the circle

(i) $|z|=1.5$ → Ans:- $-6\pi i$

(ii) $|z+i|=1$. → Ans:- 0

17) Evaluate $\int_C \frac{z\,dz}{(9-z^2)(z+i)}$ where c is the circle $|z|=2$ by cauchy's Integral formula.

Ans:- $\pi/5$