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UNIT-III

HYDRAULIC SIMILITUDE

Learning Objectives:

At the end of this topic, you will be able to:

→ Discuss the limitations of dimensional analysis
→ List out the benefits of dimensional analysis
→ Understand the types of dimensions
→ Explain dimensional homogeneity
→ Explain the methods of dimensional analysis
→ Discuss the method of selecting repeating variables
→ Understand the model analysis
→ Explain the types of similarity
→ Dimensionless numbers
→ Describe the model laws & similarity laws
→ Explain the classification of models
→ Describe scale effect
→ Illustrate the uses and limitations of hydraulic similitude

Learning Outcomes:

By the end of this topic, you will be able to:

→ Analyze the ability to use dimensional analyses in solving fluid problems and plan hydraulic similitude studies.
→ Analyze the benefits of dimensional analysis
→ Differentiate the types of various dimensions.
→ Describe the functions of dimensional Homogeneity.
→ Describe the different methods of dimensional analysis.

→ Select the method for repeating variables

→ Analyse the models which used for the various practical experiments.

→ Define the various types of similarities

→ How to find ~~that~~ that the dimensionless number is more adequate in the overall dimensional analysis method

→ Differentiate the various model laws which used in different kind of process.

Introduction:-

→ Any phenomenon is physical sciences and engineering can be described by the fundamental dimensions mass, length, time and temperature

→ Till the rapid development of science and technology the engineers and scientists depend upon the experimental data.

→ But the rapid development of science and technology has created new mathematical methods of solving complicated problems, which could not have been solved completely by analytical methods and could have consumed enormous time.

→ This mathematical method of obtaining the equations governing certain natural phenomenon by balancing the fundamental dimensions is called (Dimensional Analysis)

→ Dimensional analysis is a means of simplifying a Physical problem by appealing to dimensional homogeneity to reduce the number of relevant variables.

→ In the study of fluid mechanics the dimensional ~~analysis~~ analysis has been found to be useful in both analytical and experimental investigations. Some of the uses of dimensional analysis are as narrated below

I. Testing the dimensional homogeneity of any equation of fluid motion.

II. Deriving equations expressed in terms of non-dimensional parameters to show the relative significance of each parameter.

III. Planning model tests and presenting experimental results in a systematic manner in terms non-dimensional parameters; thus making it possible to analyze the complex fluid flow phenomenon

In order to facilitate the study of dimensional analysis and its application to practical problems a preliminary discussion about dimensions is given below.

Benefits of Dimensional Analysis:

One obvious goal of our efforts should be obtain the most information from the fewest experiments.

Dimensional analysis (DA) is an important tool that often helps us to achieve this goal

The benefits of this dimensionless representation are

1. We do not have to vary each variable: It is enough that we vary only Reynolds number (Re) and measure (C_D). As a result, we have enormous time and money saving.
2. Dimensional analysis provides "scaling laws" which can Convert data from a cheap small model to provide information about an expensive large prototype.
3. Dimensional analysis helps our thinking and planning of an experiment or a theory
   * It suggests dimensionless ways of writing equations
   * One can find variables which can be discarded.

As a result, we can avoid to waste money and time while we use experimental and Computational resources.

Types of Dimensions

Secondary dimensions (or) derived equations:

→ Such as area, volume, ~~velo~~ velocity, acceleration, force, energy, power etc., are termed as secondary dimensions & derived quantities, because they can be expressed interms of the primary quantities.

For example :

① Velocity is denoted by distance per unit time $\frac{L}{T}$.

② Acceleration is denoted by distance per unit time square $\frac{L}{T^2}$.

③ Density is denoted by mass per unit volume $\frac{M}{L^3}$.

→ Since velocity, density and acceleration involve more than one fundamental quantities so these are called derived quantities.

Dimensional Homogeneity

→ An equation is called "dimensionally homogeneous" if the fundamental dimensions have identical powers of (L T M) (i.e Length, time and Mass) on both sides

→ Such an equation be independent of the system of measurement (i.e. metric, English, or S I).

Let consider the common equation of Volumetric flow rate $V = \sqrt{2gH}$

Dimension of L.H.S $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}}$

$= \frac{L}{T} = LT^{-1}$

$\therefore$ Dimension of L.H.S = Dimension of R.H.S $= LT^{-1}$

$\therefore V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

→ We see, from the above equation that both right and left hand sides of the equation have the same dimensions, and the equation is therefore dimensionally homogeneous.

Methods of Dimensional Analysis:

→ Dimensional analysis, which enables the variables in a ~~phenomenon~~ problem to be grouped into form of dimensionless groups

→ If the number of variables involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods.

1. Rayleigh's method
2. Buckingham's π-Theorem.

Rayleigh's method :

→ The method proposed by Lord rayleigh in 1899 was initially for determining the relationship of temperature on viscosity of fluid. In rayleigh method, the functional relationship of some of variables is expressed in the form of some exponential equations which must be dimensionally homogeneous. If 'X' is a function of some variables $x_1, x_2, x_3, \ldots x_n$, the functional equation is written as

$$X = f(x_1, x_2, x_3, \ldots x_n) \longrightarrow (1)$$

→ Here X is dependent variables and $x_1, x_2, x_3, \ldots, x_n$ are independent variables. The equation (1) may be written as

$$X = C\left(x_1^{a}, x_2^{b}, x_3^{c}, \ldots x_n^{n}\right) \longrightarrow (2)$$

→ Here, c is dimensionless constant, and exponents a, b, c, -- n are evaluated on the basis that the equation (2) is dimensionally homogeneous.

Buckingham π- theorem

→ The Buckingham π-theorem was first proposed by Vaschy (1892) and proved in increasing generality by buckingham and therefore, it is known as Buckingham π-theorem. The theorem states that if there are 'n' dimensional variables involved to describe a physical phenomenon.

→ Which can be completely described by m fundamental quantities or dimensions such as mass (M), Length (L) and time (T). Then the relationship among the n quantities can be expressed interms of exactly (n-m) dimensionless and independent π terms. Mathematically, If a variable x_1 depends on independent variables $x_2, x_3, x_4, \ldots, x_n$ the functional relationship may be written as

$$x_1 = f(x_1, x_2, x_3, --, x_n) \longrightarrow (3)$$

→ Eq (3) may be transformed to another relationship as

$$f_1(x_1, x_2, x_3, x_4, -- x_n) = C \longrightarrow (4)$$

→ Here C is a dimensionless constant.

According to π-theorem, a non-dimensional equation can be written as in the following form

$$f_2(\pi_1, \pi_2, \pi_3, \pi_4, \ldots, \pi_n) = C_c \longrightarrow (5)$$

→ Each π-term in eq (5) is ~~been~~ formed by combining m variables (i.e. three variables) with one of the $(n-m)$ terms. The three variables which appear repeatedly in all the π-terms are called repeating variables

→ These three repeating variables are selected as: one from fluid characteristics, another from flow characteristics and third may be from Length characteristics.

i.e. (~~i.e.~~ ρ, μ, L & ρ, V, L & μ, V, L) (ρ, g, D) etc.

→ Let, in equation (3), x_2, x_3, x_4 be the repeating variables. Each π-terms are written as

$$\pi_1 = x_2^{a_1}, x_3^{b_1}, x_4^{c_1}, x_1$$

$$\pi_2 = x_2^{a_2}, x_3^{b_2}, x_4^{c_2}, x_5$$

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$$\pi_{n-m} = x_2^{a(n-m)}, x_3^{b(n-m)}, x_4^{c(n-m)}, x_n \longrightarrow (6)$$

→ The values of exponents $a^{a_1}, b^{b_1}, c^{c_1}, a^{a_2}, b^{b_2}, c^{c_2} \ldots$ are obtained by equating ~~eq~~ values of exponents of LHS and RHS in M, L and T and thus $\pi_1, \pi_2, \pi_3 \ldots \pi_n$ are obtained.

→ The final equation of the phenomenon is obtained by expressing any one of the π – terms as a function of the other terms as

$$\pi_1 = \phi(\pi_2, \pi_3, \pi_4, \ldots \pi_{n-m}) \longrightarrow (7)$$

$$\pi_2 = \phi(\pi_1, \pi_3, \pi_4, \ldots \pi_{n-m})$$

→ This Buckigham π – theorem can be nicely demonstrated by the following examples.

Problem 1

Using Buckingham's π theorem, shown that the discharge Q consumed by an oil ring is given by

$$Q = Nd^3 \phi\left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{w}{\rho N^2 d}\right]$$

where d is the internal diameter of the ring, N is rotational speed, ρ is density, μ is viscosity σ is surface tension and w is the specific weight of oil.

Given Data:

Discharge, $Q = f(d, N, \rho, \mu, \sigma, w)$

$$f_1(Q, d, N, \rho, \sigma, w) = 0$$

Number of variables, $n = 7$

Data to be calculated

Discharge, Q

Formula used

$$M^0 L^0 T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} . L^3 T^{-1}$$

$$M^0 L^0 T^0 = L^{a_2} (T^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1} T^{-1}$$

$$M^0 L^0 T^0 = L^{a_3} (T^{-1})^{b_3} . (ML^{-3})^{c_3} . MT^{-2}$$

$$M^0 L^0 T^0 = L^{a_4} (T^{-1})^{b_4} . (ML^{-3})^{c_4} . ML^{-2} T^{-2}$$

Solution:

Given $Q = f(d, N, \rho, \mu, \sigma, w) = 0$

$\therefore$ Total number of variables are

$Q = L^3 T^{-1}$, $d = L$, $N = T^{-1}$, $\rho = ML^{-3}$,

$\mu = ML^{-1} T^{-1}$, $\sigma = MT^{-2}$, $w = ML^{-2} T^{-2}$.

and

$\therefore$ Total number of fundamental dimensions, $m = 3$

Total number of π-terms $= n - m = 7 - 3 = 4$.

$\therefore$ Equation (1) becomes as $f_1(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \rightarrow$ (2)

Choosing d, N, ρ as repeating variables, the π-terms are

$\pi_1 = d^{a_1} . N^{b_1} . \rho^{c_1} . Q$

$\pi_2 = d^{a_2} . N^{b_2} . \rho^{c_2} . \mu$

$\pi_3 = d^{a_3} . N^{b_3} . \rho^{c_3} . \sigma$

$\pi_4 = d^{a_4} . N^{b_4} . \rho^{c_4} . w$

First π - term

Substituting dimensions on both sides

$$M^0 L^0 T^0 = L^{a_1} (T^{-1})^{b_1} . (ML^{-3})^{c_1} . L^3 T^{-1}$$

Equating the powers of M, L, T on both sides,

Power of M, $0 = c_1$, $\therefore c_1 = 0$

Power of L, $0 = a_1 - 3c_1 + 3$, $\therefore a_1 = 3c_1 - 3 = 0 - 3 = 3$

Power of T, $0 = -b_1 - 1$ $\therefore b_1 = -1$.

Substituting a_1, b_1, c_1 in π_1,

$$\pi_1 = d^{-3}, N^{-1} . \rho^0, \; Q = \frac{Q}{d^3 N}.$$

Second π - term $\pi_2 = d^{a_2} . N^{b_2} . \rho^{c_2} . \mu$

Substituting dimensions on both sides

$$M^0 L^0 T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1} T^{-1}.$$

Equating the powers of m, L, T on both sides

Power of M, $0 = c_2 + 1$, $\therefore c_2 = -1$

Power of L, $0 = a_2 - 3c_2 - 1$, $\therefore c_2 = -1$

$\therefore$ $a_2 = 3c_2 + 1 = -3 + 1 = -2$

Power of T, $0 = -b_2 - 1$, $\therefore b_2 = -1$

Substituting a_2, b_2, c_2 in π_2,

$$\pi_2 = d^{-2} N^{-1} . \rho^{-1} . \mu$$

$$\pi_2 = \frac{\mu}{d^2 N \rho} \text{ or } \frac{\mu}{\rho d^2 N}$$

Third π-term $\pi_3 = d^{a_3} \cdot N^{b_3} \cdot \rho^{c_3} \cdot \sigma$

Substituting dimensions on both sides.

$$M^0 L^0 T^0 = L^{a_3} (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot MT^{-2}$$

Equating the powers of M, L, T on both sides.

Power of M, $0 = c_3 + 1$, $\therefore c_3 = -1$

Power of L, $0 = a_3 - 3c_3$, $\therefore a_3 = 3c_3 = -3$

Power of T, $0 = -b_3 - 2$, $\therefore b_3 = -2$

Substituting a_3, b_3, c_3 in π_3

$$\therefore \pi_3 = d^{-3} \cdot N^{-2} \cdot \rho^{-1} \cdot \sigma$$

$$\pi_3 = \frac{\sigma}{d^3 N^2 \rho}.$$

Fourth π-term $\pi_4 = d^{a_4} \cdot N^{b_4} \cdot \rho^{c_4} \cdot w$

Substituting dimensions on both sides

$$M^0 L^0 T^0 = L^{a_4} \cdot (T^{-1})^{b_4} \cdot (ML^{-3})^{c_4} \cdot MT^{-2} L^{-2}$$

Equating the powers of M, L, T on both sides.

Power of M, $0 = c_4 + 1$, $\therefore c_4 = -1$

Power of L, $0 = a_4 - 3c_4 - 2$, $\therefore a_4 = 3c_4 + 2$

$$= -3 + 2$$

$$= -1$$

Power of T, $0 = -b_4 - 2$

$\therefore b_4 = -2$

Substituting a_4, b_4, c_4 in $\pi_4 \Rightarrow \pi_4 = \frac{w}{d N^2 \rho}.$

Substituting the values of $\pi_1, \pi_2, \pi_3, \pi_4$ in (2)

$$f\left(\frac{Q}{d^3 N}, \frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 \rho} \cdot \frac{\omega}{d N^2 \rho}\right) = 0$$

$$\frac{Q}{d^3 N} = f_1\left(\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 \rho}, \frac{\omega}{d N^2 \rho}\right)$$

$$Q = d^3 N \phi \left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 d}, \frac{\omega}{d N^2 \rho}\right]$$

Result: Discharge, $Q = d^3 N \phi \left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{d^3 N^2 d}, \frac{\omega}{d N^2 \rho}\right]$

Method of selecting Repeating variables

→ The dependent variable should not be selected as repeating variable.

→ The dependent variables should be chosen in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Example

* Geometric properties: Length, diameter, height
* Flow properties : Velocity, Acceleration
* Fluid properties : Viscosity, density

→ The repeating variable should not form a dimensionless group

→ The repeating variables should have the same number of fundamental dimensions.

→ No two repeating variables should have the same fundamental dimensions.

Model Analysis

→ The model is a small scale of actual structures. Machines, reservoirs, rivers, dams, weirs, spillways. Stilling basins, harbours, ships, submarines, water-sheds. Submerged objects and some objects in the field of engineering and science, while actual one is called proto-type.

→ The engineers and scientists are always associated with design, construction and efficient functioning of prototype. In order to predict the behaviour of prototypes, models are made and experiments are done.

→ from the result obtained from the models, actual behaviour of ~~pro~~-types may be known in advance

→ Sometimes, investigators researchers and designers may numerically develop some solutions of actual problems & equations whose exact solutions are impossible to arrive at through exact mathematics.

→ Again, to compare the results of numerical solutions and predicting the applicability, the model analysis & investigation is very useful as they help predict the outcome of prototype

→ However, the model results are useful only if there is a complete similarity between model and prototype.

Types of similarity

→ The similitude is defined as the similarity between the model and the prototype.
i.e., a model and a prototype have similar properties & similarities.
The following three types of similarities must be maintained between the model and prototype to facilitate useful study of model investigations

Geometric Similarity:

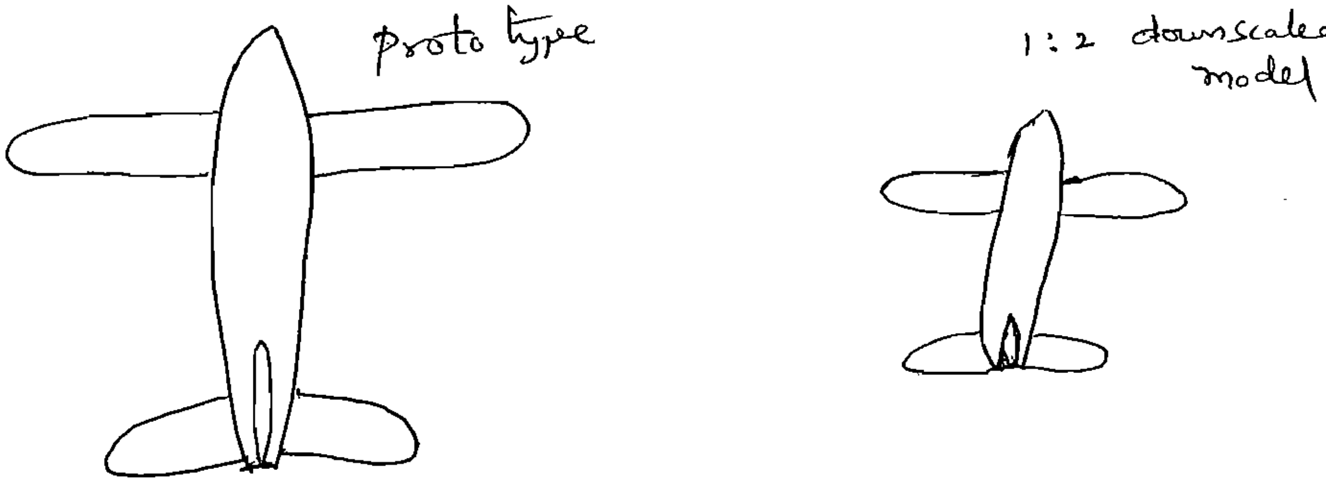
→ The geometric similarity between a model and a prototype is said to exist if the linear dimensions between the two are equal

→ If $L_m, L_p, B_m, B_p, D_m, D_p, V_m, V_p, A_m, A_p, d_m, d_p$, are the length, breadth, diameter, volume, area, and depth of model with Subscript m and prototype with Subscript p. respectively, then

$$\frac{L_m}{L_p} = \frac{D_m}{D_p} = \frac{B_m}{B_p} = L_r \text{ (Length ratio)}$$

$$\frac{A_m}{A_p} = \frac{L_m}{L_p} \times \frac{B_m}{B_p} = L_r \times L_r = L_r^2 \text{ (Area ratio)}$$

$$\frac{V_m}{V_p} = \frac{L_m}{L_p} \times \frac{B_m}{B_p} \times \frac{d_m}{d_p} = L_r \times L_r \times L_r = L_r^3 \text{ (Volume ratio)}$$

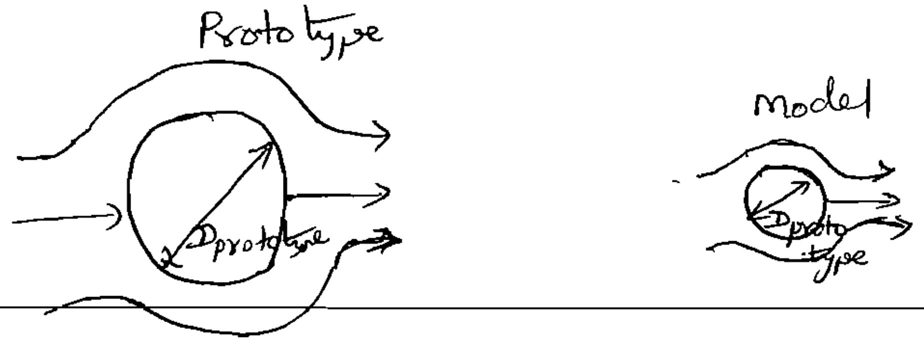


→ In which subscripts m and P are correspond to model and prototype respectively.

Physical quantities	Symbols	Dimensions
Geometric		
Area	A	L^2
Volume	V	L^3
Curvature	C	$1/L$
Slope	S	$M^0 L^0 T^0$
Angle	α, θ, ϕ	$M^0 L^0 T^0$
Shape factor	η	$M^0 L^0 T^0$

Kinematic Similarity :

→ If the kinematic parameters such as velocity, acceleration, time and discharge at the corresponding points in the model and prototype are same, it is said that there is a kinematic similarity between them.



→ Since velocity and acceleration are vector quantities, the directions of velocity and acceleration at the corresponding points in the model and prototype should also be parallel.

$$\frac{T_m}{T_p} = T_r \text{ (Time ratio)}$$

$$\frac{V_m}{V_p} = V_r \text{ (Velocity scale ratio)}$$

$$\frac{A_m}{A_p} = A_r = \frac{\frac{L_m}{T_m^2}}{\frac{L_p}{T_p^2}} = \frac{L_r}{T_r} \text{ (Acceleration scale ratio)}$$

$$\frac{Q_m}{Q_p} = Q_r = \frac{\frac{L_m^3}{T_m}}{\frac{L_p^3}{T_p}} = \left[\frac{L_m}{L_p}\right]^3 \frac{T_p}{T_m} \text{ (Discharge scale ratio)}$$

→ The kinematic similarity is attained if the flownets formed by the streamlines and equipotential lines for model and prototype are geometrically similar.

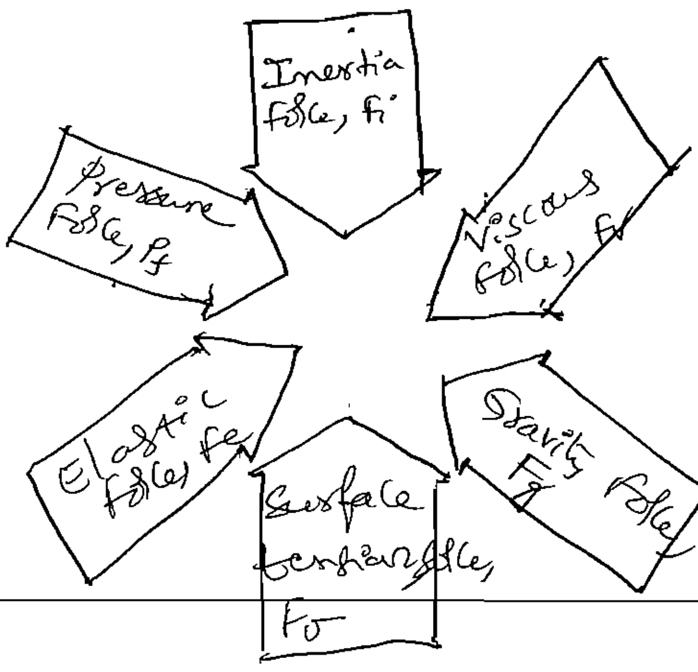
Physical quantities	Symbols	Dimensions
Kinematic		
Time	T, t	T
velocity	V, v	LT^{-1}
Angular velocity	ω	T^{-1}

Physical quantities	Symbols	Dimensions
Acceleration	a, g	LT^{-2}
Angular acceleration	α	T^{-2}
Discharge	Q	$L^{3}T^{-1}$

Dynamic Similarity:

→ The dynamic similarity is said to exist if both the geometrical and kinematical similarities between model and prototype are attained.

→ This means that ratio of all forces acting on homologous points of the model and prototypes are equal. The forces acting on the fluid system are



Modulus of elasticity	K, E	$ML^{-1}T^{-2}$
Surface tension	σ	MT^{-2}
Work, energy	W, E	$ML^{+2}T^{-2}$
Power	P	MLT^{-3}
Torque	T	$ML^{2}T^{-2}$
Momentum	M	MLT^{-1}
Pressure gradient	$\frac{\Delta P}{L}$	$ML^{2}T^{2}$

Dimensionless Numbers:

→ Dimensionless numbers are those numbers which are obtained by dividing the inertia force by viscous force or gravity force or, pressure force or surface tension force & elastic force.

→ As this is a ratio of one force to the other force, it will be a dimensionless number.

The Following are the important dimensionless numbers:

1. Reynold's number
2. Froude's Number
3. Euler's Number
4. Weber's Number
5. Mach's Number

These dimensionless numbers are also called non-dimensional parameters.

Physical Quantities	Symbol	Dimensions
Dimensionless		
Reynold's Number	Re	$M^0 L^0 T^0$
Froude's Number	Fr	$M^0 L^0 T^0$
Euler Number	Eu	$M^0 L^0 T^0$
Weber number	We	$M^0 L^0 T^0$
Mach number	Ma	$M^0 L^0 T^0$

1. Reynold's number :

It is defined as the ratio of inertia force of a flowing fluid and the viscous force of the fluid. The expression for Reynold's number is obtained as

Inertia force (F_i) = Mass × Acceleration of flowing fluid

$$= \rho \times \frac{\text{Volume}}{\text{Time}} \times \text{velocity}$$

$$= \rho \times AV \times V$$

[∵ Volume per sec = Area × velocity = A × V]

$$= \rho \times AV^2$$

Viscous force (F_v) = Shear stress × Area

$$= \tau \times A$$

$\left(\because \tau = \mu \cdot \frac{du}{dy}\right.$, Force $= \tau \times$ Area$\left.\right)$

$$= \left(\mu \cdot \frac{du}{dy}\right) A$$

$\therefore$ Viscous force $(F_v) = \mu \cdot \frac{V}{L} \cdot A$

By definition, Reynold's Number, $\left[\because \frac{du}{dy} = \frac{V}{L}\right]$

$$Re = \frac{F_i}{F_v} = \frac{\rho A V^2}{\mu \times \frac{V}{L} \times A} = \frac{\rho V L}{\mu}$$

$$Re = \frac{V \times L}{\left(\frac{\mu}{\rho}\right)} = \frac{V \times L}{\nu} \quad \left[\because \frac{\mu}{\rho} = \nu = \text{kinematic viscosity}\right]$$

Incase of pipe flow, the linear dimension L is taken as diameter, d. Hence Reynold's number for pipe flow, $Re = \frac{V \times d}{\nu}$ or $\frac{\rho V d}{\mu}$.

2. Froude's number:

The Froude's number is defined as the square root of the ratio of inertia force of a flowing fluid to the gravity force. Mathematically, it is expressed as Froude's number, $F_e = \sqrt{\frac{F_i}{F_g}}$

Where F_i from above equations $= \rho A V^2$

F_g = Force due to gravity

= Mass x acceleration due to gravity

$= \rho \times \text{Volume} \times g$

$$\therefore F_g = \rho \times L^2 \times L \times g \qquad \left[\because \text{Volume} = L^3\right.$$

$$= \rho \times A \times L \times g \qquad \left.\therefore \text{volume} = L^2 \times L = A \times L\right]$$

$$\therefore F_e = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\cancel{\rho A} V^2}{\cancel{\rho A} L g}} = \sqrt{\frac{V^2}{Lg}} = \frac{V}{\sqrt{Lg}}$$

3. Euler's number (E_u):-

It is defined as the square root of the ratio of the inertia force of a flowing fluid to the pressure force. Mathematically, it is expressed as

$$\text{Euler's number}, E_u = \sqrt{\frac{F_i}{F_p}}$$

where F_p = Intensity of pressure × Area

$$= p \times A$$

$$F_i = \rho A V^2$$

$$\therefore E_u = \sqrt{\frac{\rho \cancel{A} V^2}{p \times \cancel{A}}} = \sqrt{\frac{V^2}{\frac{p}{\rho}}} = \frac{V}{\sqrt{\frac{p}{\rho}}}$$

4. Weber's number (W_e)

It is defined as the square root of the ratio of the inertia force of flowing fluid to the surface tension force.

Mathematically, it is expressed as

Weber's number, $We = \sqrt{\frac{F_i}{F_s}}$

Where F_i = Inertia force $= \rho A V^2$

F_s = Surface tension force

= Surface tension per unit length x length

$= \sigma \times L$

$$\therefore We = \sqrt{\frac{\rho A V^2}{\sigma \times L}} = \sqrt{\frac{\rho \times L^2 \times V^2}{\sigma \times L}} \quad (\because A = L^2)$$

$$We = \sqrt{\frac{\rho L V^2}{\sigma}} = \sqrt{\frac{V^2}{\frac{\sigma}{\rho L}}} = \frac{V}{\sqrt{\frac{\sigma}{\rho L}}}$$

5) Mach's number: Mach's number is defined as the square root of the ratio of the inertia force of a flowing fluid to the elastic force. mathematically, it is defined as

$$M = \sqrt{\frac{\text{Inertia force}}{\text{Elastic force}}} = \sqrt{\frac{F_i}{F_e}}$$

Where $F_i = \rho A V^2$

F_e = Elastic force = Elastic stress x Area

$$\therefore F_e = K \times A \qquad [\because K = \text{Elastic stress}]$$

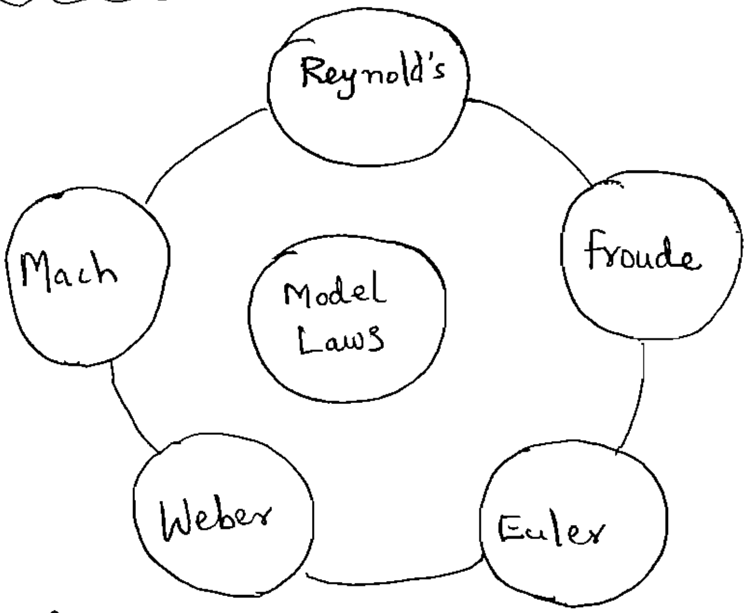
$$F_e = K \times L^2$$

$$\therefore M_a = \sqrt{\frac{\rho A V^2}{K L^2}} = \sqrt{\frac{\rho \cancel{L^2} \times V^2}{K \cancel{L^2}}} = \sqrt{\frac{V^2}{\frac{K}{\rho}}} = \frac{V}{\sqrt{\frac{K}{\rho}}}$$

But $\sqrt{\frac{K}{\rho}} = C$ = velocity of the sounding the ~~fluid~~ fluid

$$\therefore M_a = \frac{V}{C}$$

Model Laws or Similarity Laws :



→ In the similarity between the model and prototype, a ratio of the corresponding forces acting at the corresponding points in the model and prototype should be equal.

→ The ratio of the forces are dimensionless numbers. It means for dynamic similarity between the model and prototype the dimensionless numbers should be same for model and prototype.

But it is quite difficult to satisfy the condition that all the dimensionless numbers (i.e, Re, F, W, E and M) are the same for the model and prototype.

→ Hence models are designed on the basis of the force, which is dominating in the phenomenon.

→ The Laws on which the models are designed for dynamic similarity are called model laws or, Laws of similarity. The given flow chart which describes the model laws.

<u>Reynolds model law</u>:

→ Viscous force is the only other predominant force to the similarity of flow in the model and its prototype can be established.

→ Reynold's number for model must be equal to the Reynolds's number for the prototype. So this type of model is well known as Reynolds model law.

Let

ρ_m = Density of fluid in model

V_m = Velocity of flow in model

L_m = Length or linear dimension of the model

μ_m = Viscosity of fluid in model

$$(Re)_{model} = (Re)_{prototype}$$

$$\frac{\rho_m V_m L_m}{\mu_m} = \frac{\rho_p V_p L_p}{\mu_p}$$

$$\frac{\rho_r V_r L_r}{\mu_r} = 1 \longrightarrow (1)$$

$$\frac{V_r L_r}{\nu_r} = 1 \longrightarrow (2)$$

Where the various quantities with subscript 'r' represent the corresponding scale ratio's.

Eq(1) & Eq(2) may be used to obtain the scale ratios for various other physical quantities on the basis of Reynolds model law. Some of these scale ratios are shown in table.

Scale ratios for models governed by Reynolds and Froude model laws Scale ratios

Description of Quantities	Reynold's Law	Froude Law
Length	L_r	L_r
Velocity	m_r/L_r	$L_r^{1/2} g_r^{1/2}$
Time	$L_r^2 \rho_r/\mu_r$	$L_r^{1/2} g_r^{1/2}$
Acceleration	$\mu_r^2/\rho_r^2 L_r^3$	g_r
Discharge	$L_r \mu_r/\rho_r$	$L_r^{5/2} g_r^{1/2}$
Force	μ_r^2/ρ_r	$\rho_r L_r^3/g_r$
Work, Energy and Torque	$\mu_r^2 L_r/\rho_r$	$\rho_r L_r^4/g_r$
Pressure intensity	$\mu_r^2/L_r^2 \rho_r$	$\rho_r g_r L_r$
Power	$\mu_r^3/\rho_r^2 L_r$	$\rho_r L_r^{7/2} g_r^{3/2}$

Problem 2

⇒ A solid sphere of diameter 100 mm moves in water at 5 m/s. It experiences a drag of magnitude 19.62 N [2 kgf]. What would be the velocity of 5 m diameter sphere moving in air in order to ensure similarity? What will be the drag experienced by it? State which law governs the similarity.

Take $\rho_{water} = 1000\ kg/m^3$, $\rho_{air} = 1.2\ kg/m^3$, $\mu_{air} = 13\ \mu_{water}$

Given data

Diameter (d) = 100 mm

Magnitude drag = 19.62 N

Data to be calculated

Velocity of sphere moving in an air

Formula used

$$V_m = V_p \times \frac{L_p}{L_m} \times \frac{\mu_m}{\mu_p} \quad \Bigg| \quad \frac{F_m}{F_p} = \frac{\rho_m}{\rho_p} \times \left[\frac{L_m}{L_p}\right]^2 \times \left[\frac{V_m}{V_p}\right]^2$$

Solution:

Since the spheres in both the cases are moving in a submerged state, Reynolds model law governs the similarity, according to which

$$* \quad V_m = V_p \times \frac{L_p}{L_m} \times \frac{\mu_m}{\mu_p} \Rightarrow \therefore V_p = V_m \times \frac{L_m}{L_p} \times \frac{\mu_p}{\mu_m}$$

$$V_p = 5 \times \frac{5 \times 1000}{100} \times \frac{1}{13} = 19.23\ m/s$$

∴ The ratio of the drag for the model and prototype is

$$\frac{F_m}{F_p} = \frac{\rho_m}{\rho_p} \times \left(\frac{L_m}{L_p}\right)^2 \times \left(\frac{V_m}{V_p}\right)^2$$

$$= \frac{1000}{1.2} \times \left(\frac{100}{5\times1000}\right)^2 \times \left(\frac{5}{19.23}\right)^2$$

$$= 0.0225$$

$$\therefore F_p = \frac{19.62}{0.0225} = 872\ N$$

$$F_p = \frac{2}{0.0225} = 88.89\ kg\ f,$$

Result :

Velocity of air, $V_p = 19.23\ m/s$

Magnitude of drag, $F_p = 88.89\ kg\ f,$

Problem 3

⇒ A ship 200m long moves in sea-water, whose density is 1080kg/m³. A 1:100 model of this ship is to be tested in a wind tunnel. The velocity of air in the wind tunnel around the model is 25m/s and the resistance of the model is 70N. Determine the velocity of ship in sea water and also the resistance ~~of the ship in sea-water. The density of air is~~

given as 1.34 kg/m^3. Take the kinematic viscosity of sea-water and air as 0.010 stokes and 0.028 Stokes respectively.

Given data:

Length, $L_m = \frac{1}{100} \times 200 = 2m$, Length, $L_p = 200m$

Velocity, $V_m = 25 \ m/s$, Fluid = Sea-water

Resistance, $F_m = 70N$, Density of water, $\rho_p = 1080 \frac{kg}{m^3}$

Density of air, $\rho_m = 1.34 \ kg/m^3$, Velocity of ship $= V_p$, Resistance $= F_p$

Kinematic viscosity of air, $\nu_m = 0.028$ stokes

$= 0.028 \times 10^{-4} \ m^2/s$

Kinematic viscosity of sea water $\nu_p = 0.010$ stokes

$= 0.010 \times 10^{-4} \ m^2/s$

Data to be calculated:

Velocity and Resistance of Ship in sea-water.

Solution:- For dynamic similarity between the prototype and its model, the Reynolds number for both of them should be equal.

$$V_p = \frac{\nu_p}{\nu_m} \times \frac{L_m}{L_p} \times V_m$$

$$V_p = \frac{0.010 \times 10^{-4}}{0.028 \times 10^{-4}} \times \frac{2}{200} \times 25 = 0.08 \ m/s$$

v = velocity
V = volume

$$\text{Resistance} = \text{Mass} \times \text{Acceleration}$$

$$= \rho V \times \frac{v}{t} = \rho \times \frac{V}{t} \times v$$

$$= \rho \times Q \times v$$

$$= \rho \times A \times v \times v$$

$$= \rho A v^2$$

$$= \rho \times L^2 v^2$$

$$\therefore \frac{F_p}{F_m} = \frac{(\rho L^2 v^2)_p}{(\rho L^2 v^2)_m} = \frac{\rho_p}{\rho_m} \times \left(\frac{L_p}{L_m}\right)^2 \times \left(\frac{v_p}{V_m}\right)^2$$

But
$$\frac{\rho_p}{\rho_m} = \frac{1080}{1.34}$$

$$\therefore \frac{F_p}{F_m} = \frac{1080}{1.34} \times \left(\frac{200}{2}\right)^2 \times \left(\frac{0.08}{25}\right)^2 = 82.531$$

$$F_p = 82.531 \times F_m$$

$$= 82.531 \times 70$$

$$F_p = \underline{\underline{5777.19\ N}}$$

Result:

Velocity of ship, $V_p = 0.08$ m/s

Resistance " , $F_p = 5777.19$ N

Froude Model Law.

→ Froude model law is the law in which the models are based on froude number which means for dynamic similarity between the model and prototype, the froude number for both of them should be equal.

Froude model law is applicable when the gravity force is only predominant force which controls the flow in addition to the force of inertia. Froude model law is applied in the following fluid flow problems:

→ Free surface flows such as Flow over spillways, weirs, sluices, channels etc..

→ Flow of jet from an orifice & nozzle

→ where waves are likely to be formed on surface

→ where fluids of different densities flow over one another

$$(F_r)_{model} = (F_r)_{prototype}$$

$$\frac{V_m}{\sqrt{g_m L_m}} = \frac{V_p}{\sqrt{g_p L_p}} \longrightarrow (3)$$

$$\frac{V_r}{\sqrt{g_r L_r}} = 1 \quad ; \& \quad V_r = \sqrt{g_r L_r} \longrightarrow (4)$$

→ ~~Since in most of the cases~~ $g_r = 1$, as the Value of 'g' at the site of model testing will be practically the same as at the site of the proposed Prototype, equation then becomes

$$\frac{V_r}{\sqrt{L_r}} = 1 \quad \& \quad V_r = \sqrt{L_r} \longrightarrow (5)$$

→ Equation (4) or (5) may be used to obtain the scale ratios for various other physical quantities on the basis of froude model law. Some of these scale ratios are also shown in the previous table.

<u>Problem 4</u>

⇒ The performance of a spillway of a power project is to be studied by means of a model constructed to a scale of 1 : 7. Neglecting the viscous and surface tension effect, determine :

(i) Rate of flow in the model for a prototype discharge of 850 m^3/s

(ii) The dissipation of energy in the prototype hydraulic jump, if the jump in the model studies dissipates 294.2 watts (0.4 metric horse power).

<u>Given data:</u>

Discharge, Q_p = 850 m^3/s

Energy, P_m = 294.2 watts

Data to be calculated:

→ Flow rate for model

→ Dissipation of energy in the prototype

Formula used:

$$Q_r = L_r^{5/2} \, g_r^{1/2}$$

$$P_r = \rho_r \, L_r^{7/2} \, g_r^{3/2}$$

Solution: In this case since gravity is the predominant force, according to froude model law, we have

$$Q_r = L_r^{5/2} \, g_r^{1/2}$$

$$L_r = \frac{1}{7} \quad \text{and} \quad g_r = 1$$

$$\therefore \; Q_r = \left(\frac{1}{7}\right)^{5/2} \cdot (1)^{1/2} = \frac{1}{130}$$

Since $Q_r = \dfrac{Q_m}{Q_p}$; and $Q_p = 850 \, m^3/s$

$$\therefore Q_m = Q_p \times Q_r = 850 \times \frac{1}{130} = 6.538 \, m^3/s$$

Again according to Froude model law the scale ratio for power is given by

$$P_r = \rho_r \, L_r^{7/2} \cdot g_r^{3/2}$$

$$P_r = \left(\frac{P_m}{P_p}\right) ; \quad \rho_r = 1 , \quad g_r = 1$$

$P_m = 294.2$ watts

By substitution, we get

$$\frac{294.2}{P_p} = 1 \times \left(\frac{1}{7}\right)^{7/2} \times (1)^{3/2}$$

$$P_p = 294.2 \times (7)^{7/2} \text{ watts}$$

$$P_p = 266.984 \text{ kilowatts}$$

If $P_m = 0.4$ metric horse power

then $$\frac{0.4}{P_p} = 1 \times \left(\frac{1}{7}\right)^{7/2} \times (1)^{3/2}$$

$$P_p = 0.4 \times (7)^{7/2} = 362.9 \text{ metric horse power}$$

Result:

Rate of flow, $Q_m = 6.538\ m^3/s$

Dissipation of energy, $P_p = 362.9$ metric horse power.

Euler model Law:

→ In a fluid system where supplied ~~supplied~~ pressures are the controlling forces in addition to the inertia force and other forces are either entirely absent or insignificant, the dynamic similarity is obtained by equating the Euler number for both the model and its prototype and this is known as Euler model law.

$$(E_u)_{model} = (E_u)_{prototype}$$

$$\frac{V_m}{\left(\frac{P_m}{\rho_m}\right)^{1/2}} = \frac{V_p}{\left(\frac{P_p}{\rho_p}\right)^{1/2}} \longrightarrow (6)$$

$$\frac{V_r}{\left(\frac{P_r}{\rho_r}\right)^{1/2}} = 1 \longrightarrow (7)$$

→ Equation (7) represents the primary relationship for the Euler model law which may be used to evaluate the scale ratios for various other physical quantities in accordance with the Euler model law.

→ Euler model law may be contemplated as an essential requirement for establishing dynamic similarity in an enclosed fluid system where the turbulence is fully developed, so that the viscous forces are insignificant and also the forces of gravity and surface tension are entirely absent.

Mach model Law:

→ Mach model law is the law in which models are designed on mach number, which is the ratio of the square root of inertia force to elastic force of a fluid

→ Hence where the forces due to elastic compression predominate in addition to inertia force, the dynamic similarity between the model and its prototype is obtained by equating the mach number of the model and its prototype. Hence according to this law.

$$(M)_{model} = (M)_{prototype}$$

Where M = Mach number $= \dfrac{V}{\sqrt{\frac{k}{\rho}}}$

V_m = Velocity of fluid in model

k_m = Elastic stress for model

ρ_m = Density of fluid in model

and V_p, k_p and ρ_p = corresponding values for prototype

Then according to Mach Law

$$\frac{V_m}{\sqrt{\frac{K_m}{\rho_m}}} = \frac{V_p}{\sqrt{\frac{K_p}{\rho_p}}} \longrightarrow (8)$$

$$\frac{V_r}{\sqrt{K_r/\rho_r}} = 1 \longrightarrow (9)$$

The expression represented by eq. (9) may be considered as the basic relationship for Mach model law from which the scale ratios for the other physical quantities may be derived.

→ The Similitude based on Mach model Law finds extensive application in aerodynamic testing a phenomena involving velocities exceeding the speed of sound. In addition to this, it is also applied in hydraulic model testing for the cases of unsteady flow, especially water hammer problems.

Weber Model Law:

→ Weber model law is the law in which models are based on weber's number, which is the ratio of the square root of inertia force to surface tension forces.

→ Hence where surface tension effects predominate in addition to inertia force, the dynamic similarity between the model and prototype is obtained by equating the weber number of the model and its prototype.

Hence according to this law

$$(We)_{model} = (We)_{prototype}$$

Where We = Weber ~~number~~ number $= \dfrac{V}{\sqrt{\dfrac{\sigma}{\rho L}}}$

V_m = velocity of fluid in model

σ_m = Surface tensile force in model

ρ_m = Density of fluid in model

L_m = Length of surface in model

and V_p, σ_p, ρ_p and L_p = Corresponding values for prototype.

According to weber law, we have

$$\frac{V_m}{\sqrt{\frac{\sigma_m}{\rho_m L_m}}} = \frac{V_p}{\sqrt{\frac{\sigma_p}{\rho_p L_p}}}$$

$$\frac{V_r}{\sqrt{\frac{\sigma_r}{\rho_r L_r}}} = 1 \longrightarrow (10)$$

The expression represented by eq (10) may be considered as the basic relationship for Weber model law from which the scale ratios for the other physical quantities may be derived.

Weber model law is applied in the following cases:-

i) Capillary rise in narrow passages

ii) Capillary movement of water in soil

iii) Capillary waves in channels.

iv) Flow over weirs for small heads.

Classification of Models:

Undistorted model:

→ In this type of models, which are geometrically similar to their prototype. In other words, the scale ratio for the linear dimensions of the model and its prototype are the same.

→ Since the basic condition of perfect similitude is satisfied, prediction in the case of such models is relatively easy and many of the results obtained from the model tests can be transferred directly to the prototype.

Distorted Model:

→ In this type of models, which are geometrically not similar to its prototype. In other words the scale ratio for the linear dimensions of the model and its prototype are not same.