Course Material

Engineering Mechanics

I

Dynamics of Rigid Bodies

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I. Kinematics of Rigid Bodies

I.1 Introduction

In this topic ,we study the characteristics of motion of a rigid body and its related kinematic equations to obtain displacement, velocity and acceleration.

Rigid Body: A rigid body is a combination of a large number of particles occupying fixed positions with respect to each other. A rigid body being defined as one which does not deform.

2.0 Types of Motions

1. **Translation** : A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It can also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are straight lines. The motion is said to be a *rectilinear translation (Fig 1)*; If the paths are curved lines, the motion is a *curvilinear translation*.

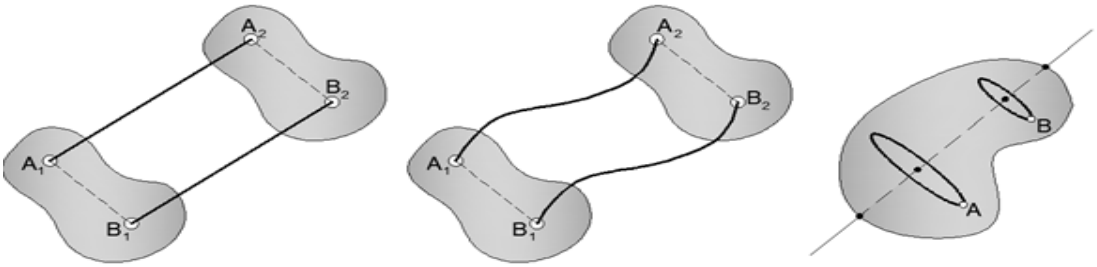


Fig.1 Three types of Motion

2. **Rotation about a Fixed Axis**. In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis (Fig 1). If this axis, called the axis of rotation, intersects the rigid body. The particles located on the axis have zero velocity and zero acceleration.

   Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Fig 2(a) is in curvilinear translation, with all its particles moving along parallel circles, while the plate shown in Fig 2(b) is in rotation, with all its particles moving along concentric circles. In the first case, any given straight line drawn on the plate will maintain the same direction, whereas in the second case, point O remains fixed.Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a plane motion.

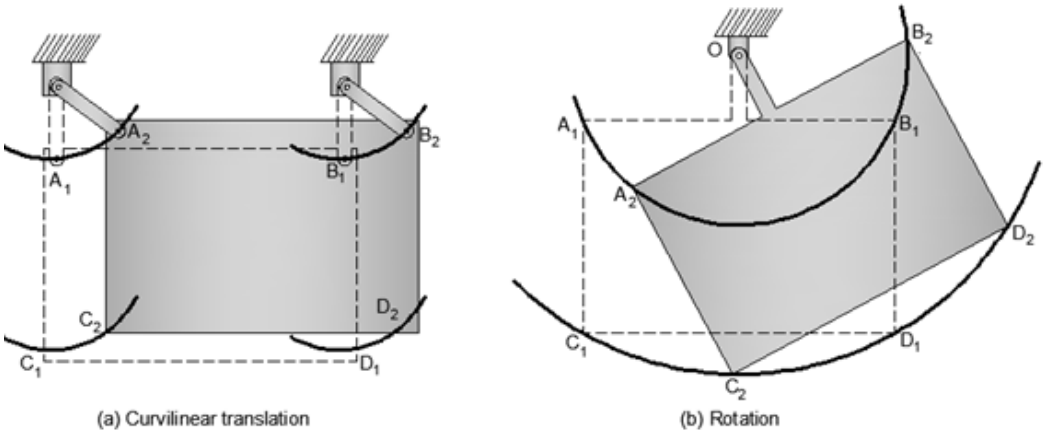


Fig.2 A plate in a curvilinear translation & Rotation Motion.

3. **General plane motion***:* There are many other types of plane motion, that is, motions in which all the particles of the body move in parallel planes. Any plane motion which is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Fig3.

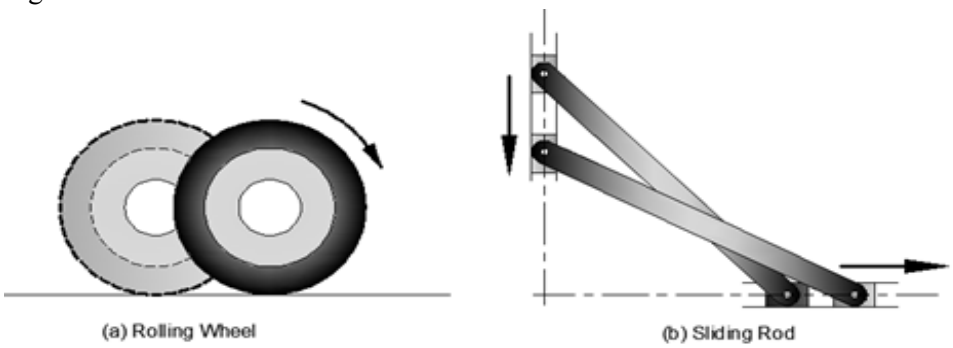


Fig.3. General Plane motion

4. ***Motion about a fixed point***. The three-dimensional motion of a rigid body attached at a fixed point O, for example the motion of a top on a rough floor (Fig. 5) is known as motion about a fixed point.

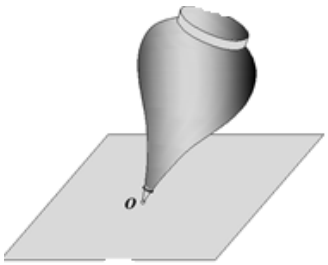


Fig.5. Motion about fixed point

3.0 Rotation of a rigid body about a fixed axis

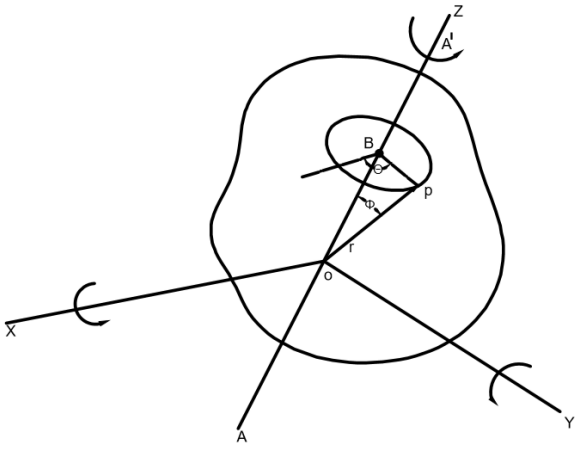


Fig.6 Rotation of a rigid body about a fixed axis

The motion of a rigid body rotating about a fixed axis AA' is said to be *known* when its angular coordinate can be expressed as a known function of *t*. In practice, however the rotation of a rigid body is seldom defined by a relation between More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, α may be given as a function of *t*, as a function of, or as a function of ω.

- ----(1)

-------(2)

-------(3)

Two particular cases of rotation are frequently encountered:

1. *Uniform Rotation.* This case is characterized by the fact that the angular acceleration is zero. **The angular velocity is thus constant**, and the angular coordinate is given by the formula

   ----(4)

2. *Uniformly Accelerated Rotation*. In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time can then be derived in a manner similar to that described. The similarity between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is apparent.

   -------(5)

It should be emphasized that formula (4) can be used only when α= 0, and formulas (5) can be used only when α=constant. In any other case, the general formulas (1) and (3) should be used.

4.0 General Plane Motion

A general plane motion can always be considered as the sum of a translation and a rotation.

Example 1. Consider, for example, a wheel rolling on a straight track (Fig. 7). Over a certain interval of time, two given points A and B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B'_1 (the line AB remaining vertical), followed by a rotation about A bringing B into B_2. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion can be exactly duplicated by a combination of simultaneous translation and rotation.

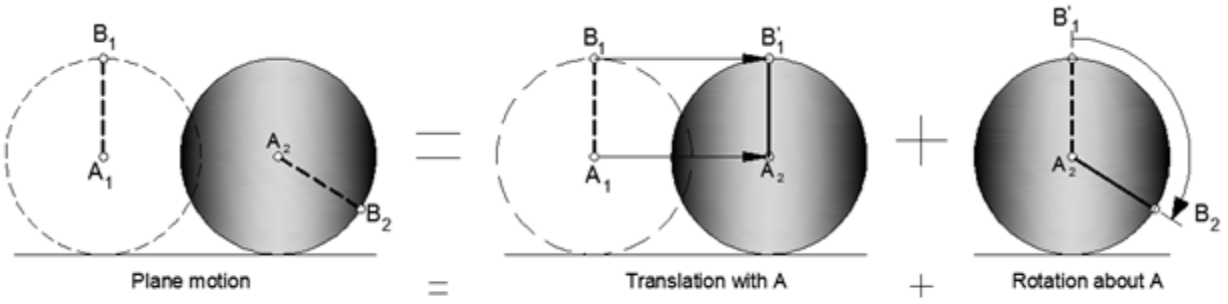


Fig.7 A General plane motion ,a combination of Translation & Rotation

Another example of plane motion is given in Fig. 8, which represents a rod whose extremities slide along a horizontal and a vertical track, respectively. This motion can be replaced by a translation in a horizontal direction and a rotation about A, in fig 8(a) or by a translation in a vertical direction and a rotation about B in fig 8(b).

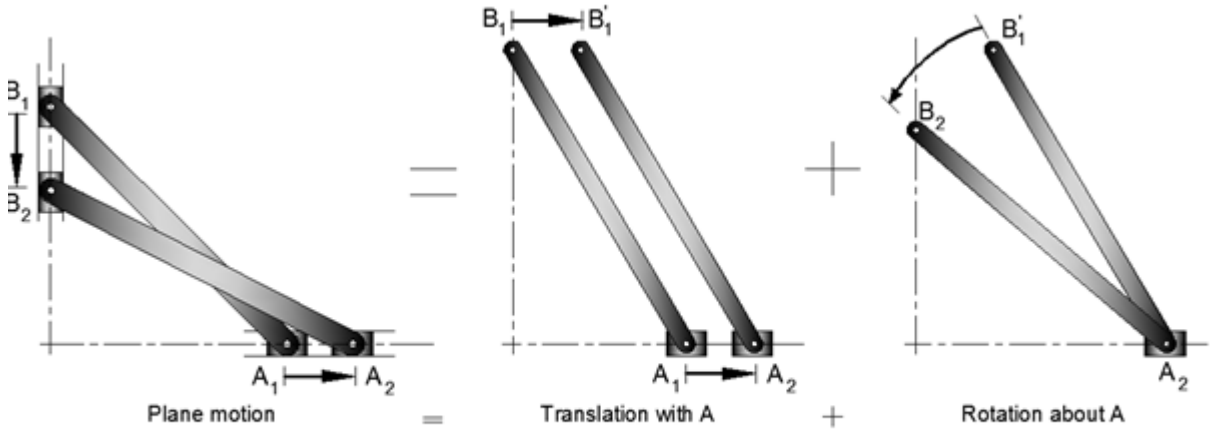


Fig.8(a) A General plane motion ,with sliders

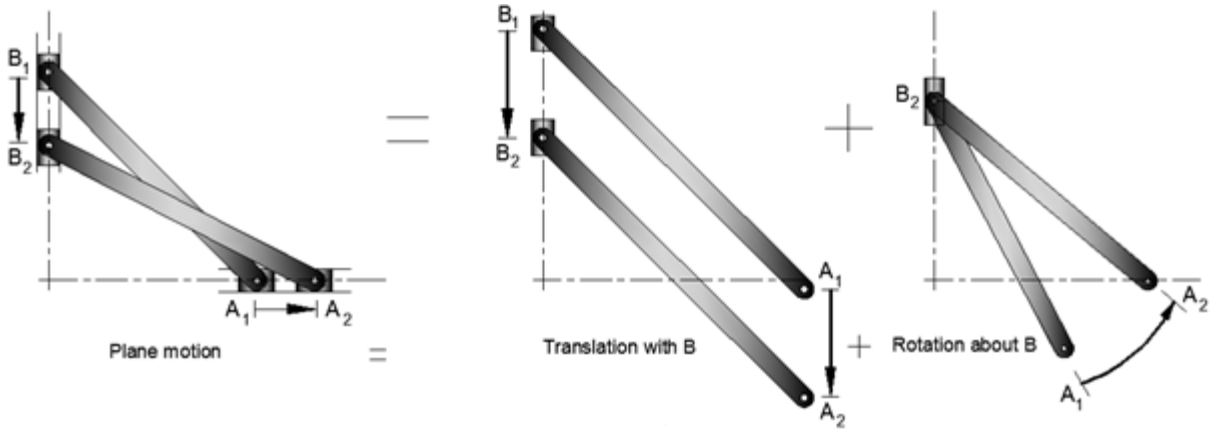


Fig.8(b) A General plane motion ,with sliders

5.0 Absolute and Relative Velocity in Plane Motion

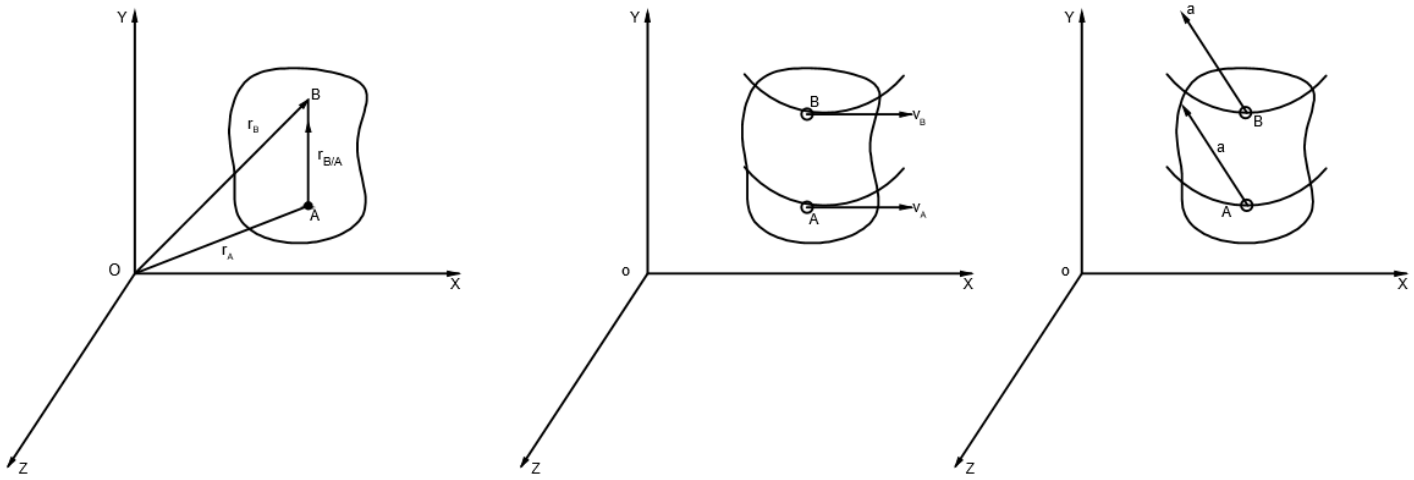


Fig.9. Rigid body in Translation

Translation:

Let A and B be two particles on a rigid body with r_A and r_B as position vectors of A & B with respect to a fixed frame of reference as shown in figure9 and $r_{B/A}$ is the vector joining A and B.

$$r_B = r_A + r_{B/A} \qquad \text{------(6)}$$

From the very definition of a translation, the vector $\mathbf{r_{B/A}}$ must maintain a constant direction; its magnitude must also be constant, since A and B belong to the same rigid body.While differentiating the above relation,the derivative of $r_{B/A}$ is zero.

$$V_B = V_A \qquad \text{-------(7)}$$

Differentiating once more,

$$\mathbf{a_B = a_A} \qquad \text{----------(8)}$$

Note: 1. When a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant.

2. In case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant.

3. In case of rectilinear translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

Motion relative to a frame in Translation:

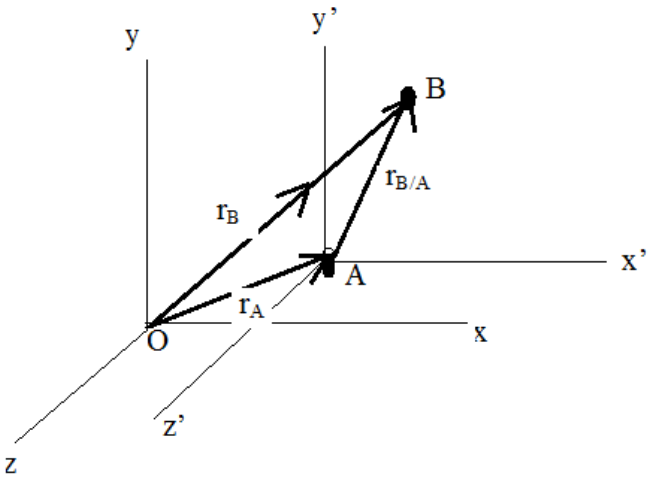


Fig.10. Motion relative to fixed Frame

Consider two particles A and B moving in space.Let the fixed frame of reference be OXYZ and another system of axes x'y'z' centered at A and parallel to the x,y,z axes.While the origin of these axes move,their orientation remains the same;the frame of reference Ax'y'z' is in translation with respect to Oxyz.

The vector $\mathbf{r}_{B/A}$ joining A and B defines the position of B relative to the moving frame Ax'y'z'(i.e., position of B relative to A).Then,

$$r_B = r_A + r_{B/A} \qquad \text{---------(9)}$$

Here V_A & V_B are the velocities of the particles A and B.Since Ax'y'z' is in translation,the derivative of $\mathbf{r}_{B/A}$ represents the rate of change of $\mathbf{r}_{B/A}$ with respect to the frame of Ax'y'z' as well as with the fixed frame.This derivative ,therefore, defines the velocity $\mathbf{V}_{B/A}$ of B relative to the frame Ax'y'z' i.e. Velocity of B relative to A.

--------(10)

Similarly further differentiation results in

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \text{----------(11)}$$

Therefore, the motion of B with respect to the fixed frame oxyz is referred to as the absolute motion of B.

The absolute motion of B can be obtained by combining the motion of A and the relative motion of B with respect to the moving frame attached to A.
$\mathbf{V}_B$ is obtained by adding vectorially the velocity of A and the velocity of B relative to the frame Ax'y'z'.
Acceleration also exhibits similar property.

Equations related to Absolute and relative parameters in plane motion:

The velocity V_A corresponds to the translation of the slab with A, while the relative velocity $V_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation. Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, and by $\boldsymbol{\omega}$ the angular velocity of the slab with respect to axes of fixed orientation.

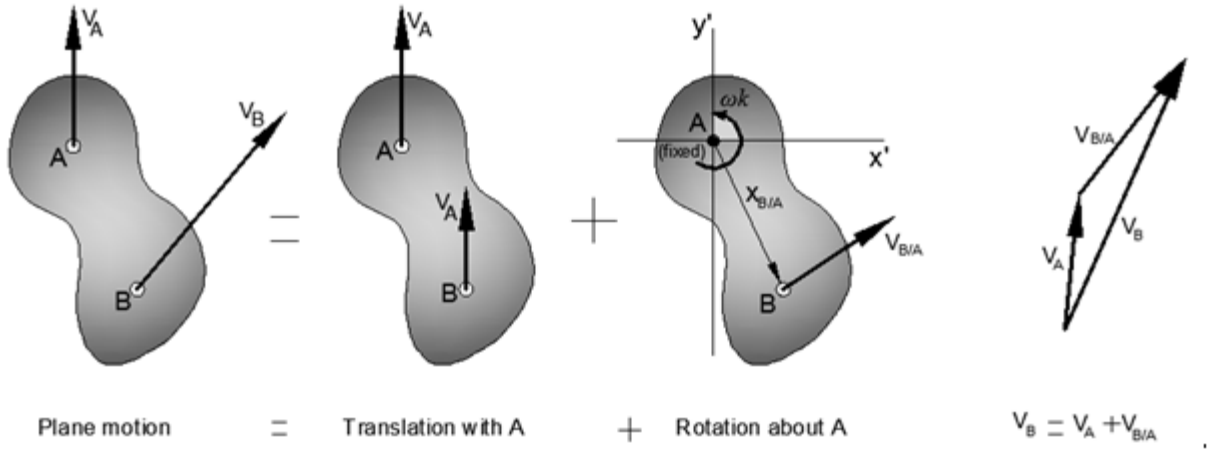


Fig.11 Velocity terms in Plane Motion.

Where r is the distance from A to B. Substituting for $V_{B/A}$

----------(12)

Example: As an example, let us again consider the rod AB of Fig.12. Assuming that the velocity V_A of end A is known, we propose to find the velocity V_B of end B and the angular velocity of the rod, in terms of the velocity V_A, the length l, and the angle Θ. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a simultaneous rotation about A. The absolute velocity of B must therefore be equal to the vector sum

We note that while the direction of $V_{B/A}$ is known, its magnitude lω is unknown. However, this is compensated for by the fact that the direction of V_B is known. We can therefore complete the diagram of 15.16. solving for the magnitudes v_B and ω, we write

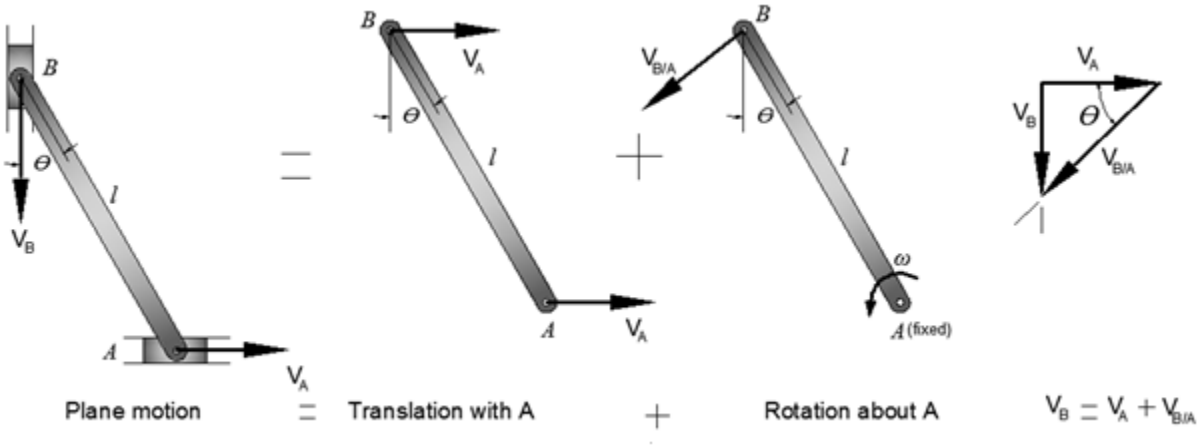


Fig 12 Velocity vector for Sliders

The same result can be obtained by using B as a point of reference. Resolving the given motion into a translation with B and a simultaneous rotation about B, we write the equation

Which is represented graphically in Fig. 13. we note that VA/B and VB/A have the same magnitude lω but opposite sense. The sense of the relative velocity depends, therefore, upon the point of reference which has been selected and should be carefully ascertained from the appropriate diagram (Fig. 12 or 13)

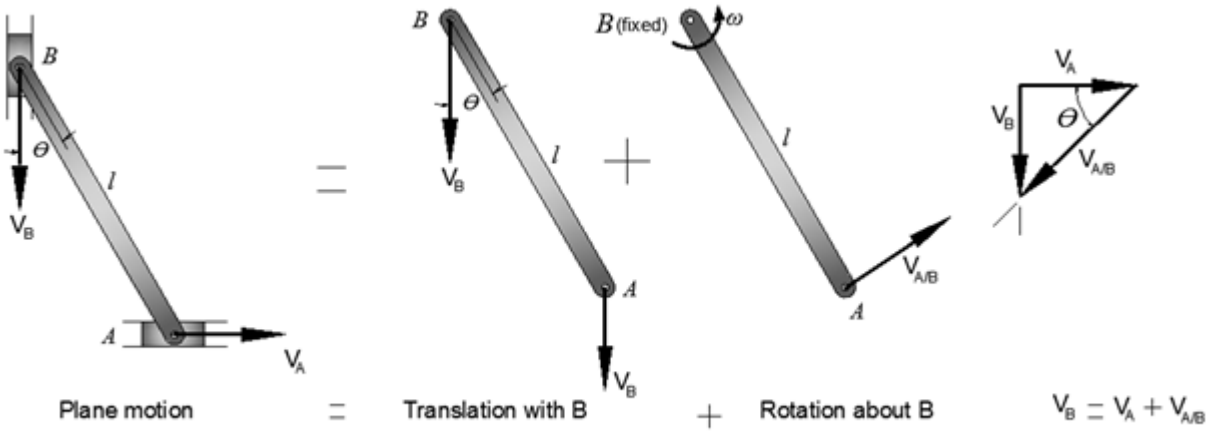


Fig 13 Velocity vector for Sliders

Finally we observe that the angular velocity ω of the rod in its rotation about B is the same as in its rotation about A. It is measured in both cases by the rate of change of the angle Θ. This result is quite general; we should therefore bear in mind that the angular velocity ω of a rigid body in plane motion is independent of the reference point.

6.0 Instantaneous Centre

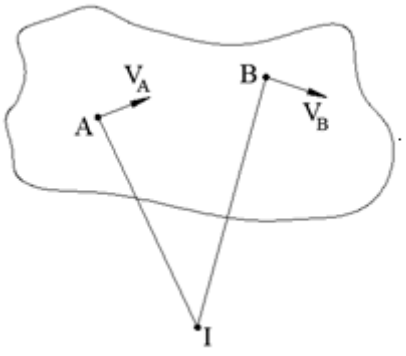


Fig.14 Instantaneous Centre for a a rigid body

Instantaneous centre is useful in analyzing the motion of a rigid body. If the directions of velocity of two points on a rigid body are known, the instantaneous centre can be located. The instantaneous centre is a point about which the body can be considered to have pure rotation at that instant. If the direction of velocity VA of point A is known and also the direction of velocity VB is known in Fig. 14, the intersection of lines drawn perpendicular to VA and VB fixes the instantaneous centre I. At that instant the body can be taken as rotating about that point. Analysis using instantaneous centre reduces the plane motion of a body to a problem of rotation about a fixed point I. Instantaneous centre is useful in finding the velocities of different points on the rigid body if one velocity is known magnitude in addition to its direction. In Fig 15, some examples are shown.

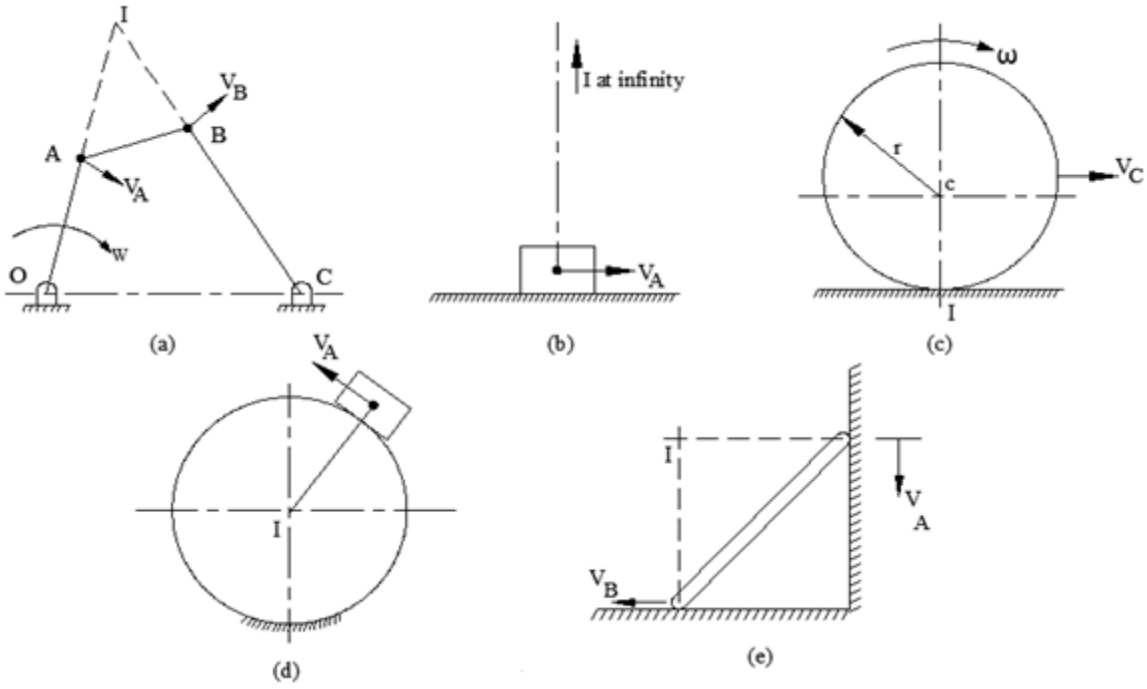


Fig.15 Instantaneous centers on different Mechanisms

Consider a slider crank mechanism shown in Fig.15a to illustrate the application of instantaneous centre in the analysis of velocities. OC is the crank rotating at an angular velocity of ω rad/sec. the slider P is connected to the crank by the link CP. All points are pin connected. The instantaneous centre I is shown in Fig 14.

Velocity of point C=VC=OC·ω and its direction is shown in Fig. 3. Since the link CP can be considered to be rotating about instantaneous centre I at that instant,

Angular velocity of the link CP = ωCP= = .ω

Velocity of slider P = IP. ω_{CP} = IP. ( ).ω

The velocity of slider P, $\mathbf{V_P}$ is obtained by knowing the lengths OC, IC and IP in addition to the angular velocity ω of the crank.

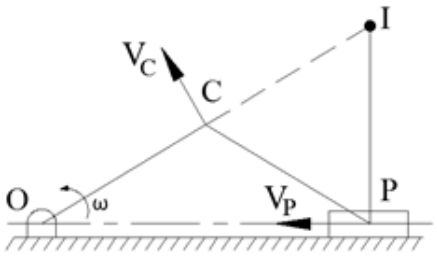


Fig.16 Instantaneous location on a single slider mechanism

7.0 Absolute and Relative Acceleration in plane motion.

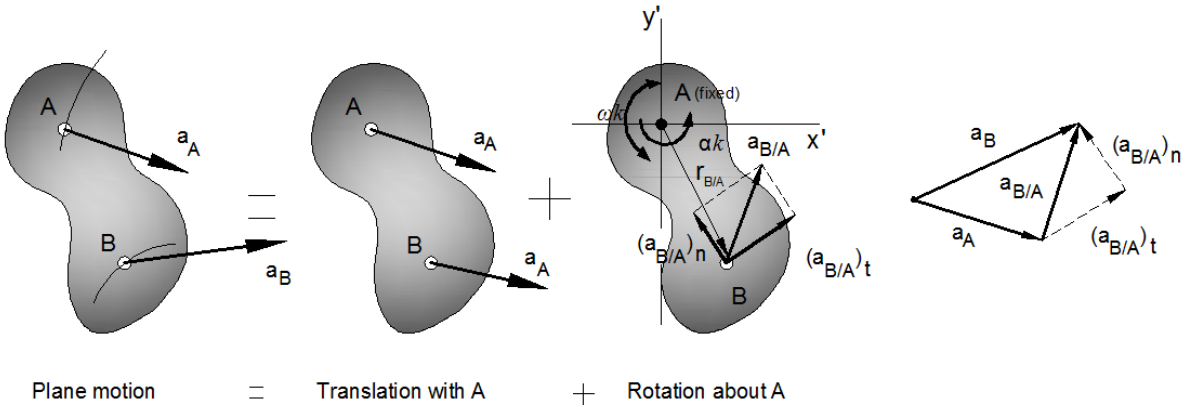


Fig.17 Acceleration reprentation for a plane motion

$$\mathbf{a_B = a_A + a_{B/A}}$$

Here the relative acceleration $a_{B/A}$ can be resolved into two components,a tangential component

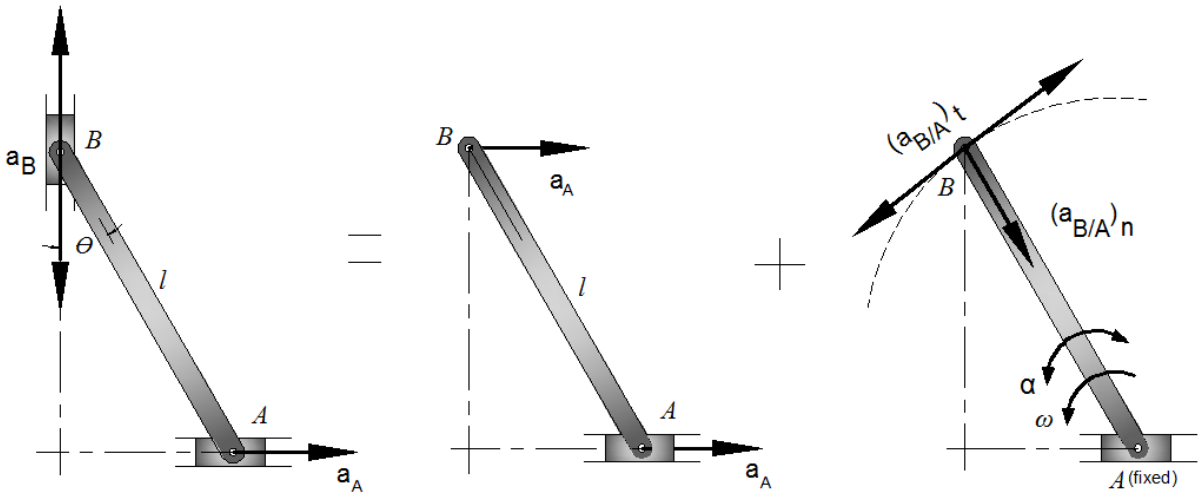
$(a_{B/A})_t$, perpendicular to the line AB and a normal component, $(a_{B/A})_n$ directed towards A.

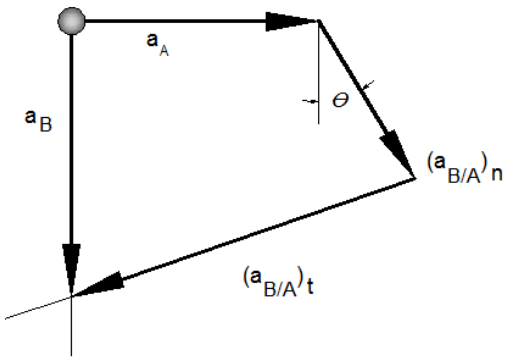
Where , $(\mathbf{a_{B/A}})_t = \mathbf{r\alpha}$ & $(\mathbf{a_{B/A}})_n = \mathbf{r\omega^2}$

Therefore, $\mathbf{a_B = a_A + a_{B/A}}$

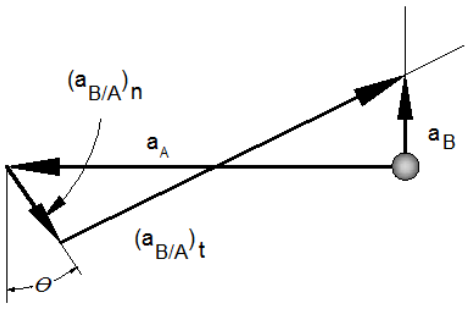
Can be rewritten as, $\mathbf{a_B = a_A + (a_{B/A})_n + (a_{B/A})_t}$

$$\mathbf{a_B = a_A + r\omega^2 + r\alpha} \qquad \text{-------(13)}$$





(a)



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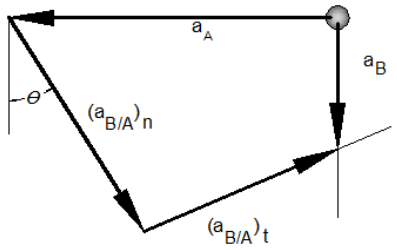

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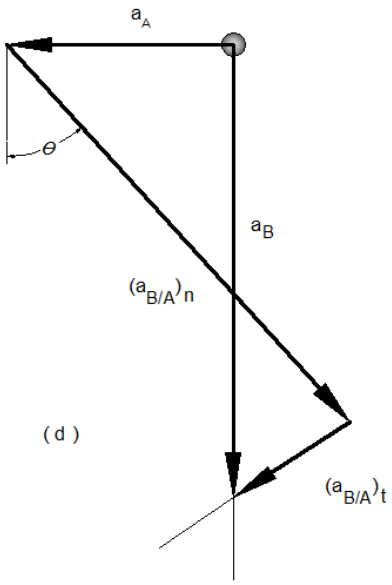
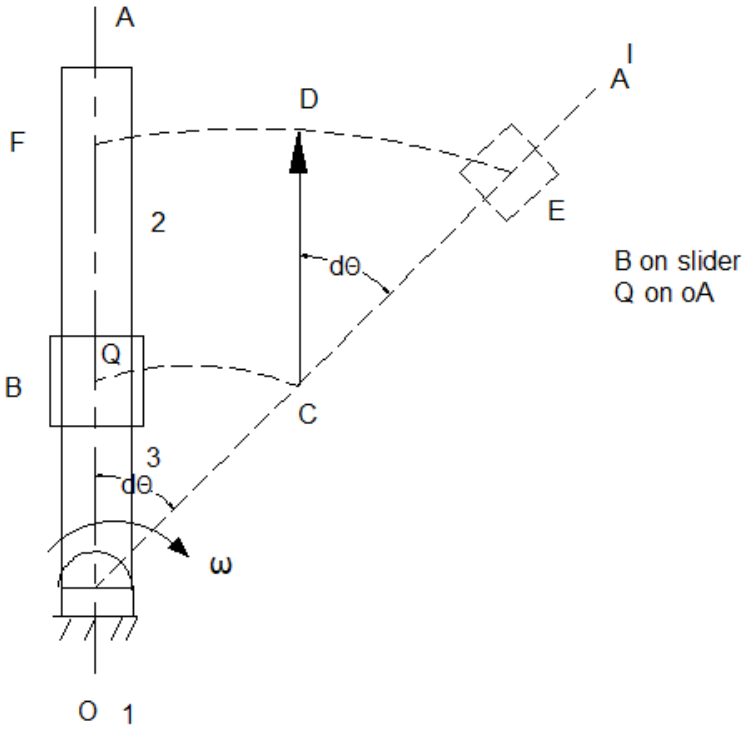


(d)

8.0 Coriolis Acceleration



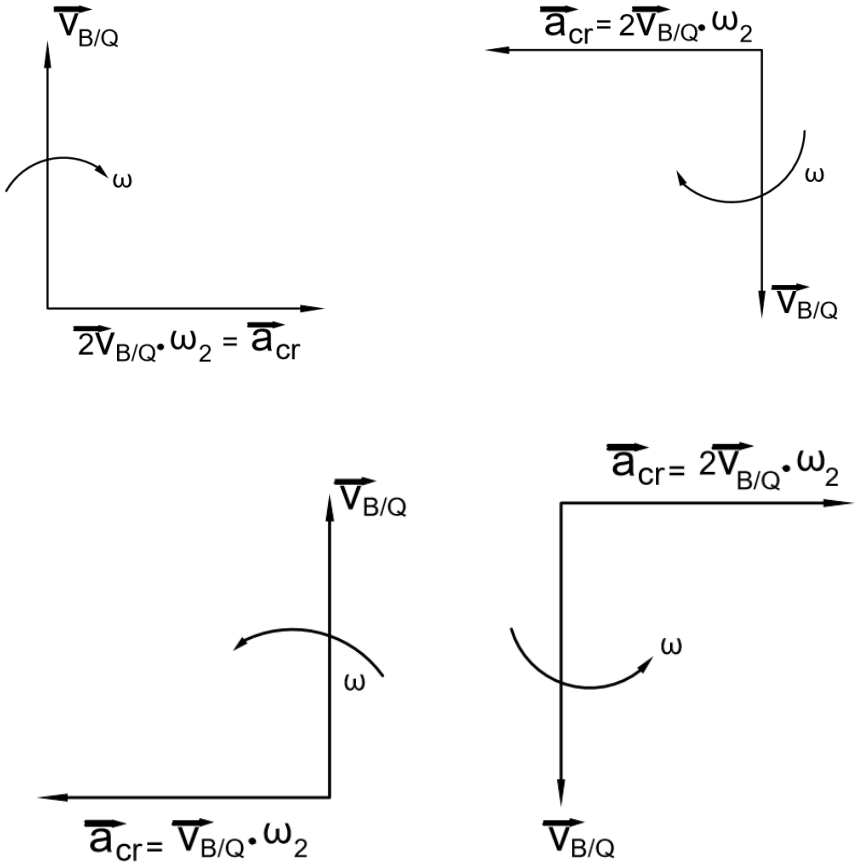
The total acceleration of a point with respect to another point in arigid link is the vector sum of its normal and tangential components.This holds true when the distance between 2 points is fixed and the relative acceleration of the 2 points on amoving rigid link has been considered. If the distance between 2 points varies,i.e., the

second point which was stationery,now slides,the total acceleration will contain additional component,known as Coriolis component.

The magnitude of coriolis component is given by,

$$\mathbf{a_{cr} = 2.V_{B/Q}\ \omega_2} \qquad \text{-------(14)}$$

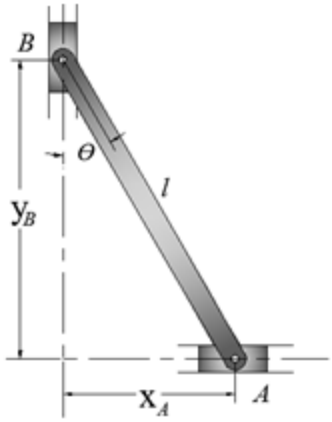
Direction of coriolis component



9.0 Analysis of Plane Motion in terms of a Parameter

In the case of certain mechanisms, it is possible to express the coordinates x and y of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It is sometimes advantageous in such a case to determine the absolute velocity and the absolute acceleration of the various points of the mechanism directly, since the components of the velocity and of the acceleration of a given point can be obtained by differentiating the coordinates x and y of that point.

Let us consider again the rod AB whose extremities slide, respectively, in a horizontal and vertical track (Fig. 13). The coordinates x_A and y_B of the extremities of the rod can be expressed in terms of the angle



the rod forms with the vertical:

Differentiating Eqs. Above twice with respect to t, we write

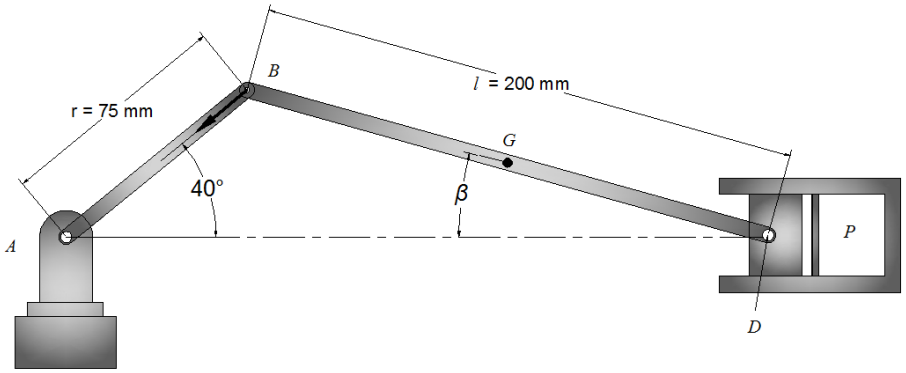
Recalling that and, we obtain

We note that a positive sign for v_A or a_A indicates that the velocity V_A or the acceleration a_A is directed to the right; a positive sign for v_B or a_B is directed upward. Equations (15.25) can be used, for example to determine v_B and ω when v_A and θ are known. Substituting for ω in (15.26), we can then determine a_B and α if a_A is known.

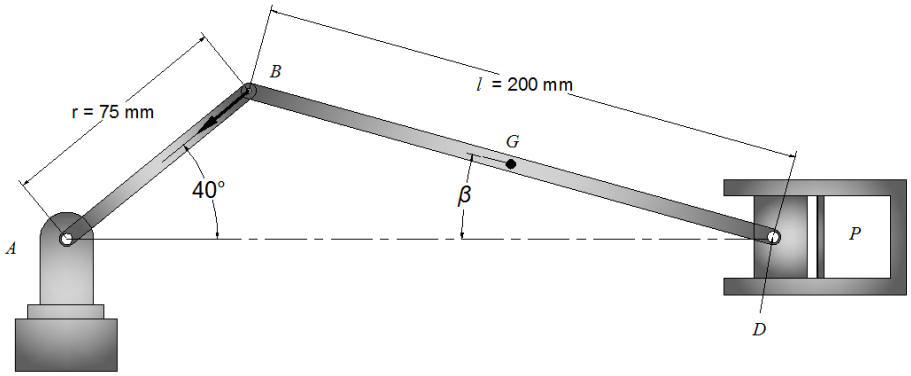
Tutorials

Problem 1: **ABSOLUTE AND RELATIVE VELOCITY IN PLANE MOTION**

In the engine system shown, the crank AB has a constant clockwise angular velocity of 2000rpm.For the crank position indicated,determine a) angular velocity of the connecting rod BD b) the velocity of piston P.



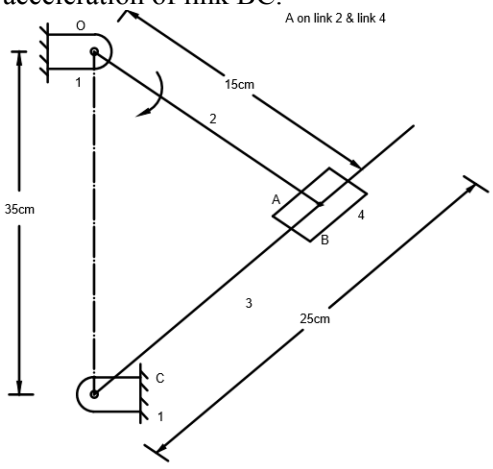
Problem2: ABSOLUTE AND RELATIVE Acceleration IN PLANE MOTION

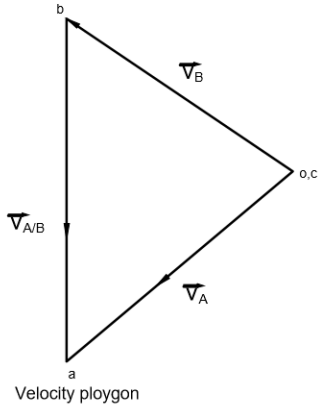


Determine the angular acceleration of the connecting rod BD and the acceleration of point D for the given single slider mechanism.

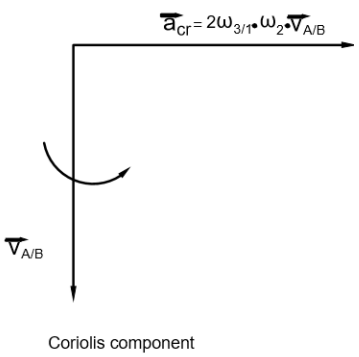
Problem 3. Coriolis Acceleration.

A quick return mechanism is shown in figure. Link2 rotates uniformly at 20 rad/sec. Determine the angular acceleration of link BC.

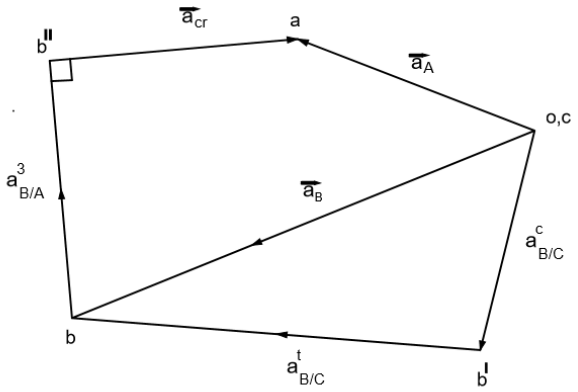




Velocity ploygon



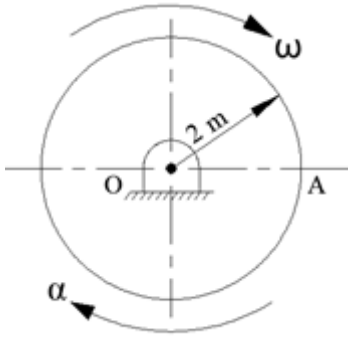
Coriolis component



Acceralation polygon

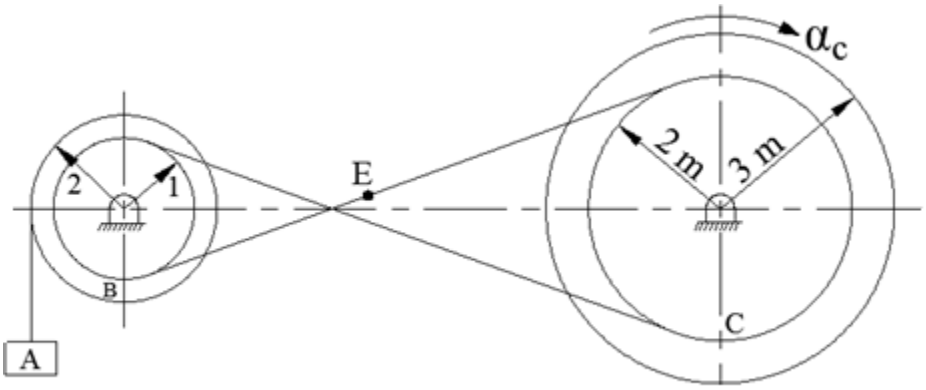
Problem 4:

When the angular velocity of a 4m diameter pulley is 3 rad/sec and total acceleration of a point on its rim is 30 m/sec^2. Determine the angular acceleration of the pulley at this instant.



Problem.5

Two stepped pulleys shown in Fig.5 are connected by a crossed belt. What time is required for A to travel by 64 m from rest if the angular acceleration α_c is 2 rad/sec^2? What is the distance travelled by D while A moves by 100m?



II.Kinetics of Rigid Bodies

11. Introduction

The rigid bodies are assumed to be symmetrical with respect to the plane in which the centre of mass of the body moves. When we say the plane motion of a rigid body the motion include translation and rotation. **The kinetics study deals with the relation between the forces and resulting acceleration of the body.**

12. Analysis of plane motion

Since the plane motion rigid body comprises translation and rotation, the dynamic equations have to be derived for translation and rotation.

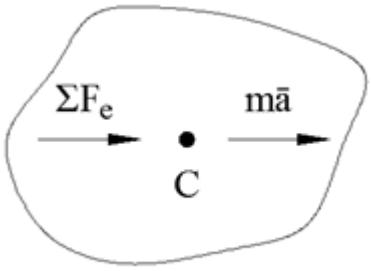
12.1 Equations of Motions

To derive the basic equation of kinetics of a particle or rigid body moving with translation motion Newton's laws of motion provide the basis.

Newton's laws of Motion of a particle:

1. **Newton First Law**: A particle acted upon by a resultant force results in change of state of rest or its motion.
2. **Newton Second Law:**The resultant force acting on the particle causes acceleration in the direction of the resultant force and this force is directly proportional to acceleration.
3. **Newton Third Law**: Action and reaction between two particles are always equal and oppositely directed.

12.2 Translation



Let the resultant force acting on the body which is the vector sum of all external forces F_e acting on the body be denoted by F. By Newton's second law of motion,

F= = m ----------(15)

Where Mass of the body is m and ā is the acceleration of the center of mass.

The above equation may be rewritten as

+(-ma) =0

The force (-ma) is known as inertia force F_i. the iertia has the magnitude equal to the product of mass and acceleration but directed opposite to the direction of acceleration.The alternate form of the above equation is

+ F_i=0 ----------(16)

- The above equation indicates that the body is in equilibrium when the sum of external forces and the inertia force acting on the body is equal to zero.
- This is known as D'Alembert's principle which helps to reduce dynamic problem to an equivalent static one by applying inertia force to the existing system of external forces.

For solving the problem of kinetics involving translation of a rigid body the following guidelines may be useful.

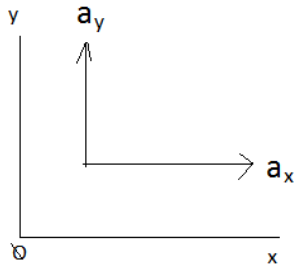
i) Acceleration analysis of the rigid body in a coordinate system which simplifies the solution must be performed.

Coordinate systems:

a) *Rectangular coordinate system*: Scalar equations of the motion

= ma_x & = ma_y

where a_x & a_y are the components of acceleration in the X & Y directions.



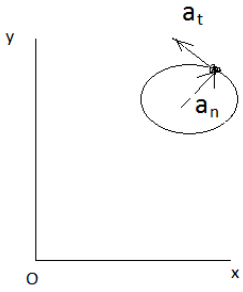
b) *Normal and tangential coordinates*:

= ma_n & = ma_t

where a_n is normal component (centripetal) of acceleration pointing towards the centre of curvature. $a_n = v^2 / \rho$

a_t is the tangential component of acceleration and is given as

c) Polar coordinates

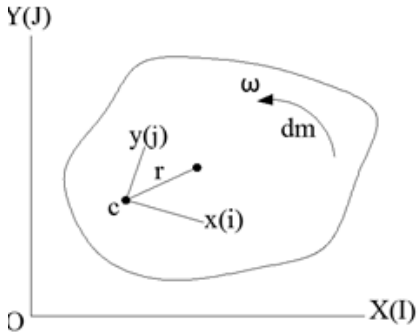


where is the radial component of acceleration,

a_θ is the traverse component of acceleration $a_\theta = r + 2$

12.3 Rotation

For deriving dynamic equations, let us consider a mass dm within the body rotating with angular velocity ω with respect to the fixed coordinate system XOY. Let I, J be unit vectors along X and Y axes respectively. Let *xcy* be the moving coordinate system in the plane fixed with the body with the origin at mass centre c.



The unit vectors i,j, are along *x* and *y* axis respectively.

Linear momentum of the body about the mass centre c

=dm = dm

Angular momentum of the body is moment of linear momentum about c

Angular momentum = =dm

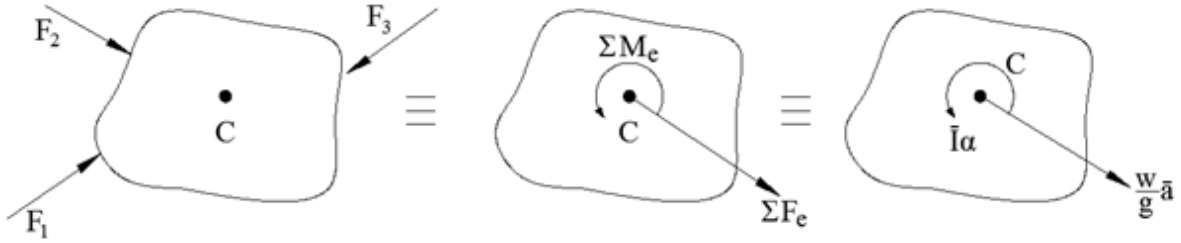
=dm

Mass moment of inertia about centre of mass = = dm

Hence Angular momentum = =

Taking moments about c of all external forces acting on the body and equating to the rate of change of angular momentum

= =( )



Effect of Forces on the system

If the angular momentum is referred to fixed XOY axes

+ x)

+ x) ----------(17)

The term (ωx) is a cross product of angular velocity vector and ( ) is angular momentum vector. Since we are considering plane motion, ω vector and ( ) vector are both perpendicular to the plane of the paper i.e., act along z axis resulting in cross product of these terms zero.

For rotational motion in plane the equation 12.6 simplifies to

----------(18)

Where α is angular acceleration.

It is obvious that two basic equations are important in plane motion

= m and ----------(19)

The effect of resultant force-couple acting on the body is equivalent to the linear motion of mass centre of the body and rotational about the mass centre.

Using D'Alemberts principle

+ (-m) = 0

$+_i = 0$

Where F_i is inertia force + (-) = 0

And $+M_i = 0$

Where M_i is inertia torque or moment.

13. Fixed axis Rotation

Let us consider a rigid body rotating about fixed axis passing through centre of mass.

13.1 Centroidal Axis of rotation

This is pure rotation of body about C where T is the external moment (torque) applied and T_i is the inertia torque. (Fig. 19)

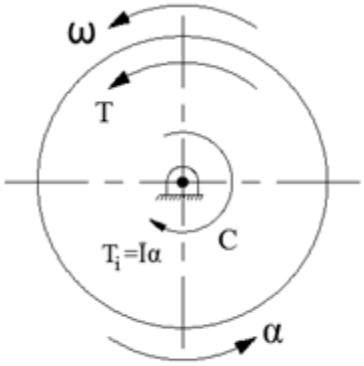


Figure 19. Pure body rotation

Equation of motion is

= 0

)

----------(20)

13.2 Non-Centroidal Axis of rotation

Consider a body rotating about an axis passing through point A and point C is centre of mass. F_1 is the externally applied force.

AC= r

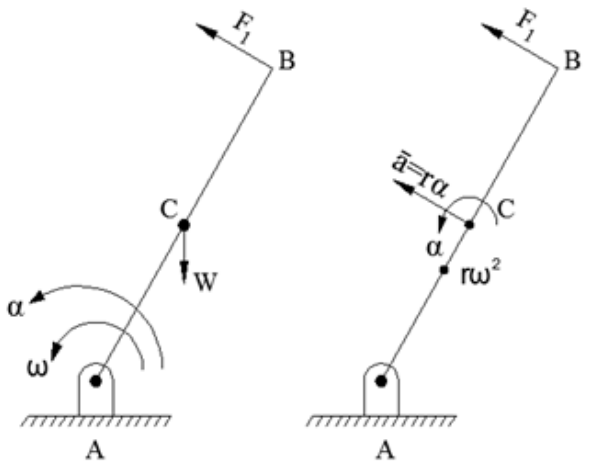


Fig. 20 Effect of Accelerations

Forces and torques are shown on the free body diagrams

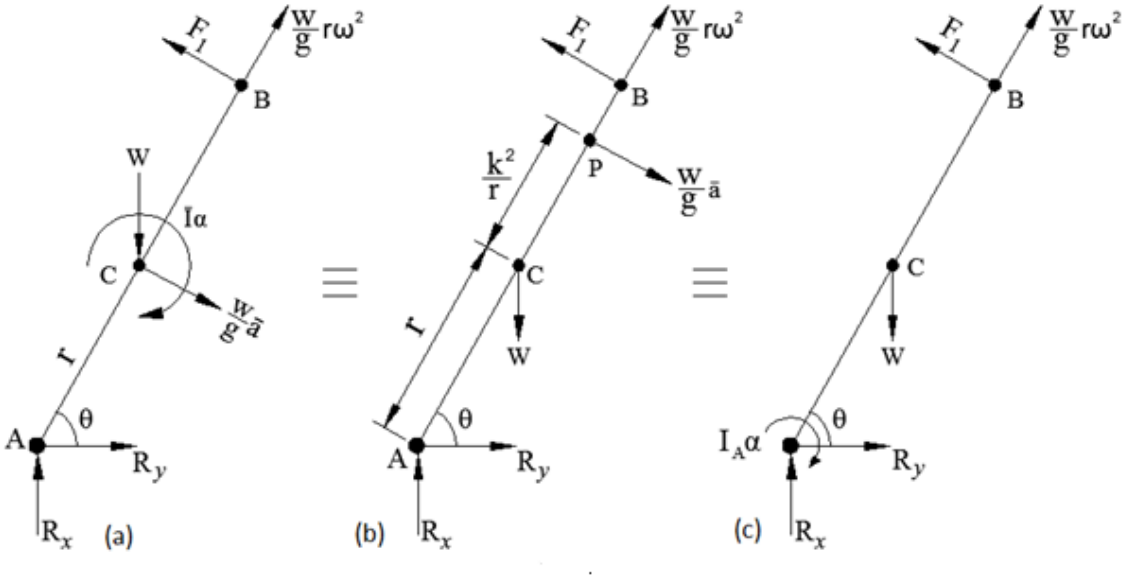


Fig.21 Free-body diagrams.

Three free body diagrams are shown and anyone can be used for analysis in fig 21(a) inertia force and inertia torque $\bar{I}\alpha$ are shown at point C in addition to centrifugal force [2].

In Fig 21(b), instead of considering inertia torque $\bar{I}\alpha$ separately, the inertia force is shifted to a parallel position P so that inertia torque $\bar{I}\alpha$ is eliminated. The point P is called the centre of percussion which is at a distance of (r +) where k is radius of gyration. There is an advantage to use this because only forces are involved in the free body diagram.

In Fig. 21(c), we have included only mass moment of inertia I_A about the point A since the body rotating about A

$$= + \qquad \text{----------(21)}$$

Which is obtained by using parallel axis theorem

14. Rolling Motion

Consider a body rolling without slipping on a surface shown in fig 22. Let P be the externally applied force.

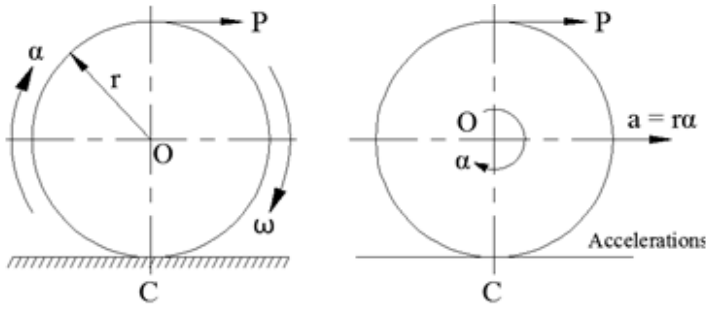


Fig.22 Rolling of a Wheel

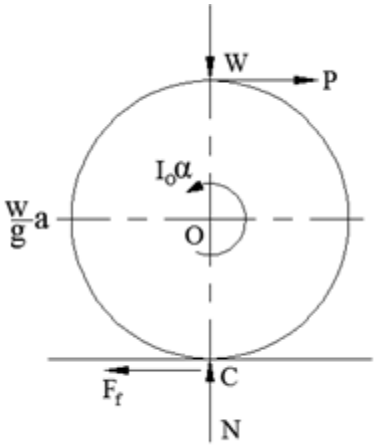


Fig.23 Free Body diagram

Rolling of the wheel shown in Fig 22 is effected by linear acceleration of rα of centre of wheel O and also angular acceleration α about O. Point C is the instantaneous centre.

F_r is the frictional resistance to resist slipping on the surface. Since F_f is unknown, moment about C is considered first

$$= 0$$

$$2P(r) =(r +$$

P = +

= 0

N = W

=0 P =+

Frictional force - ----------(22)

If the body rolls on the surface without slipping then only the relation **a = rα** holds good.

References:

1. Ferdinand L. Singer, "Engineering Mechanics-Statics & Dynamics", Harper International Edition.
2. J.L.Meriam, L.G.Kraige, "Engineering Mechanics-Statics", John Wiley & Sons Pte Ltd.
3. Beer & Johnston," Vector Mechanics for Engineers-Statics & Dynamics". TATA Mc Graw Hill international Editions.