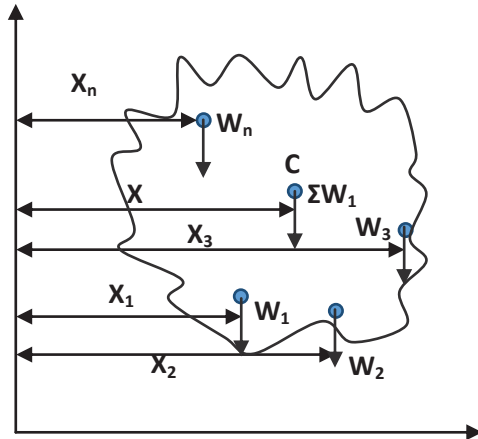# UNIT 3
Centroids and Centers of Gravity

CENTROID AND CENTER OF GRAVITY
UNIT III

Centre of Gravity

- It is defined as an imaginary point on which entire, length, area or volume of body is assumed to be concentrated.
- It is defined as a geometrical centre of object.

- The weight of various parts of body, which acts parallel to each other, can be replaced by an equivalent weight. This equivalent weight acts a point, known as centre of gravity of the body
- The resultant of the force system will algebraic sum of all parallel forces, there force

  $$R = W_1 + W_2 + \ldots\ldots + W_n$$

- It is represented as weight of entire body.

$$W = R = \sum_{i=i}^{n} wi$$

- The location of resultant with reference to any axis (say $y - y$ axis) can be determined by taking moment of all forces & by applying varignon's theorem,
- Moment of resultant of force system about any axis = Moment of individual force about the same axis

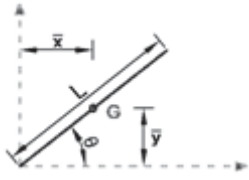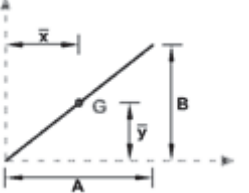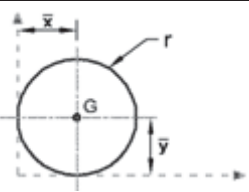$$R.\bar{x} = W_1 x_1 + W_2 x_2 + \ldots\ldots + W_n x_m$$
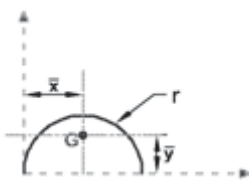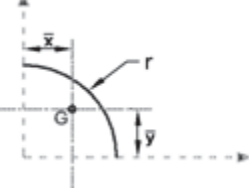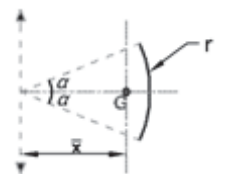
we can write,

$$\bar{x} = \frac{W1x1 + W2x2 + \cdots\ldots\ldots + Wnxm}{N} = \frac{w_i x_i}{w_i}$$

$$\bar{x} = \frac{\int x \, dw}{\int dw}$$

Similarly,

$$\bar{y} = \frac{\sum w_i y_i}{\sum w_i}$$

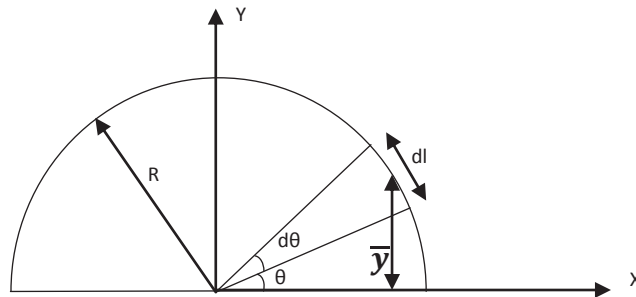| Line Element Centroid – Basic Shape				
Element name	Geometrical Shape	Length	$\bar{x}$	$\bar{y}$
Straight line		L	$\dfrac{L}{2}\cos\theta$	$\dfrac{L}{2}\sin\theta$
Straight line		$\sqrt{A^2 + B^2}$	$\dfrac{A}{2}$	$\dfrac{B}{2}$
Circular wire		$2\pi r$	r	r
Semi-circular		πr	r	$\dfrac{2r}{\pi}$
Quarter circular		$\dfrac{\pi r}{2}$	$\dfrac{2r}{\pi}$	$\dfrac{2r}{\pi}$
Circular arc		$2r\alpha$ (α in radian)	$\dfrac{r\sin\alpha}{\alpha}$	On Axis of Symmetry

Here,

$$\bar{x} = \frac{l_1 x_1 + l_2 x_2 + \dots + l_n x_n}{l_1 + l_2 + l_3 + \cdots l_n} = \frac{\sum l_i x_i}{\sum l}$$

$$\bar{y} = \frac{\sum l_i y_i}{\sum l}$$

Centroid of semi – circular arc

> A semi-circular arc be uniform thin wire or a thin road, place it in such a way that y – axis is the axis of symmetry with this symmetry we have $\bar{x}=0$.

Here
$$\frac{y}{R} = sin\theta$$

$$\therefore Y = sin\theta R$$

$$\frac{dl}{R} = d\theta$$

$$dl = R.\ d\theta$$

> Consider length of element is dl at an angle of θ as shown in fig.

$$\bar{y} = \frac{\int y\,dl}{\int dl} = \frac{\int R\sin\theta\ R\,d\theta}{\int R\,d\theta}$$
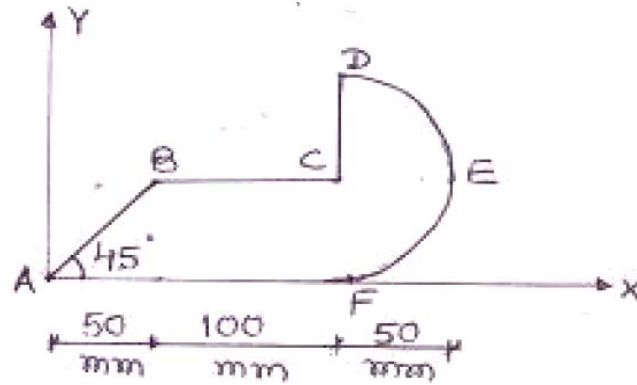
$$= \frac{R\int \sin\theta\ d\theta}{d\theta}$$

$$= \frac{\int_0^\pi \sin\theta\ d\theta}{\int_0^\pi d\theta}$$

$$\bar{y} = \frac{2R}{\pi}$$

Example: 1. Determine the centroid of bar bent in to a shape as shown in figure.

Answer:

For finding out the centroid of given bar, let's divide the bar in to 4 – element as AB, BC, CD, DEF

Member	Length	x mm	Y mm	lx(mm^2)	ly(mm^2)
AB	l_1 $= \sqrt{50^2 + 50^2}$ $= 70.71$	$x_1 = (50/2) = 25$	$y_1 = (50/2)$ $= 25$	$l_1x_1 = 1249.98$	$l_1y_1 = 1249.98$
BC	$l_2 = 100$	$x_2 = (100/2) + 50$ $= 100$	$y_2 = 50$	$l_2x_2 = 10000$	$l_2y_2 = 5000$
CD	$l_3 = 50$	$x_3 = 50 + 100$ $= 150$	$y_3 = (50/2)$ $+ 50 = 75$	$l_3x_3 = 7500$	$l_3y_3 = 3750$
DEF	$l_4 = \pi r = 157.08$	$x_4 = 50 + 100 +$ $(2r/\pi)$ $= 181.83$	$y_4 = r = 50$	$l_4x_4 = 28561.85$	$l_4y_4 = 7853.95$

$$\bar{x} = \frac{l_1x_1 + l_2x_2 + \dots + l_n x_n}{l_1 + l_2 + l_3 + \dots l_n} = \frac{47311.8}{377.79} = 125.23 \text{ mm}$$

$$\bar{y} = \frac{l_1y_1 + l_2y_2 + \dots + l_n y_n}{l_1 + l_2 + l_3 + \dots l_n} = \frac{17853.9}{377.79} = 47.25 \text{ mm}$$

Example-2. Calculate length of part DE such that it remains horizontal when ABCDE is hanged through as shown in figure.
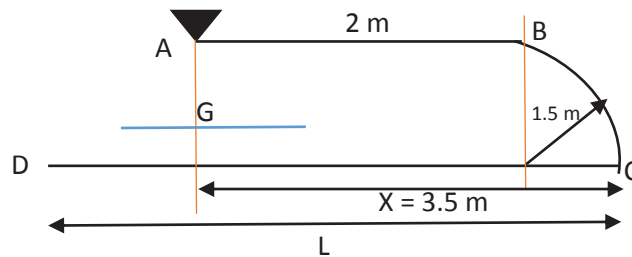
ANSWER :

- here, we want to determine length of DC = l such that DC remains horizontal, for that centroidal axisis passes through "A".
- Reference axis is passing through c as shown in figure.

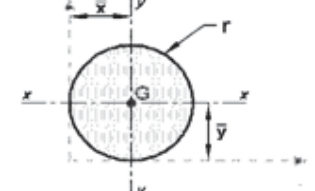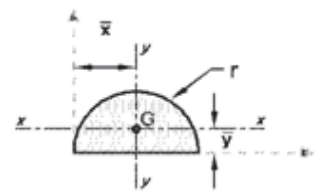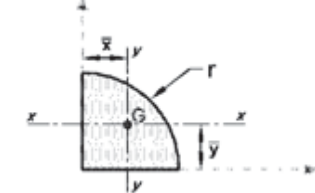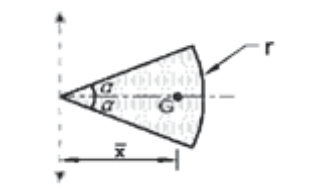| Part	Shape	Length	x mm	$lx(\text{m}^2)$
AB	Straight line	$l_1 = 2$	x_1 $= 1.5 - \dfrac{2}{2}$	$l_1 x_1 = 5$
BC	Semi-circular arc	$l_2 = \dfrac{2\pi r}{4}$	x_2 $= 1.5 - \dfrac{2r}{\pi}$	$l_2 x_2$ $= 1.284$
CD	Straight line	$l_3 = l$	$x_3 = \dfrac{l}{2}$	$l_3 x_3 = \dfrac{l^2}{2}$
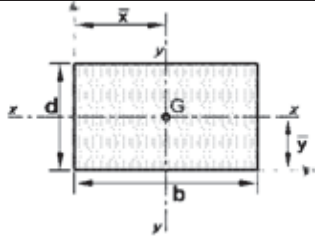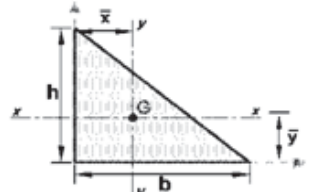
$$\bar{x} = \frac{\sum lx}{\sum l} = \frac{0.5l^2 + 6.284}{4.356 + l} = 3.5$$

∴ $15.246 + 3.5\ l = 0.5l^2 + 6.284$

∴ $0.5l^2 - 3.5\ l - 8.962 = 0$

∴ $l = 8.993\text{m}$
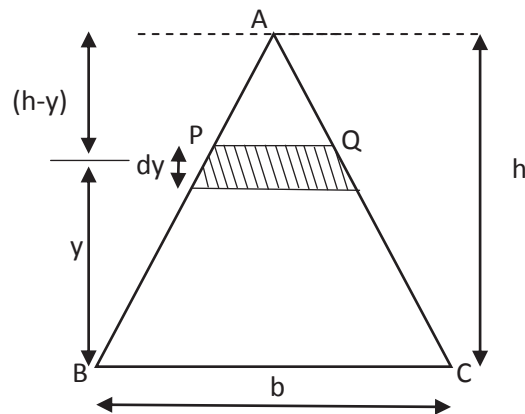
Area(Lamina) Element Centroid– Basic Shape				
Element name	**Geometrical Shape**	**Area**	$\bar{x}$	$\bar{y}$
Rectangle		bd	$\dfrac{b}{2}$	$\dfrac{d}{2}$
Triangle		$\dfrac{1}{2}bh$	$\dfrac{b}{3}$	$\dfrac{h}{3}$
Circle		πr^2	r	r
Semicircle		$\dfrac{\pi r^2}{2}$	r	$\dfrac{4r}{3\pi}$
Quarter circle		$\dfrac{\pi r^2}{4}$	$\dfrac{4r}{3\pi}$	$\dfrac{4r}{3\pi}$
Circular segment		αr^2 (α in radian)	$\dfrac{2\,r\sin\alpha}{3\;\alpha}$	On Axis of Symmetry

Centroid of a triangle area

- Place one side of the triangle on any axis, say $x - x$ axis as shown in fig.
- Consider a differential strip of width 'dy' at height y, by similar triangles ΔABC & ΔCDB

$$\frac{DE}{AB} = \frac{h - Y}{h}$$

$$\therefore \ DE = (1 - \frac{Y}{h})b$$

$$= (b - \frac{Y}{h}b)$$

- Now, area of strip,

$$dA = (b - \frac{Y}{h}b) \ dy$$

- Now, we have

$$\bar{y} = \frac{\int y dA}{dA} = \frac{\int y dA}{A}$$

$$\therefore \ A\bar{y} = \int_0^h y \ dA$$
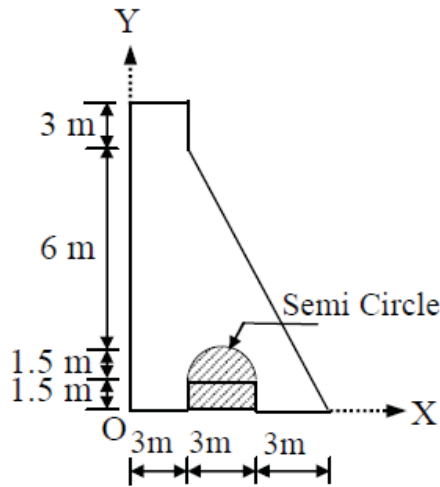
$$= \int_0^h y(by - \frac{b}{h} \ y2) dy$$

$$\frac{1}{2} \times b \times h \times \bar{y} = \frac{bh2}{2} - \frac{bh2}{3}$$

$$\bar{y} = \frac{h}{3}$$

Example-3. **Determine co-ordinates of centroid with respect to 'o' of the section as shown in figure.**

Answer:

Let divide the given section in to 4 (four) pare

(1) : Rectangular (3 X 12)
(2) : Triangle (6 x 9)
(3) : Rectangular (3 x 1.5)
(4) : Semi – circular (r = 1.5m)

| Sr. no	Shape	Area (m²)	x (m)	Y(m)	Ax (m³)	Ay (m³)
1	Rectangle	$A_1 = 12X3$ =36	$x_1 = \dfrac{3}{2}$ $= 1.5$	$y_1 = \dfrac{12}{2}$ $= 6$	$A_1 x_1$ $= 54$	$A_1 y_1 = 216$
2	Triangle	$A_1 = \dfrac{1}{2} X6X9$ =27	x_2 $= 3 + \dfrac{6}{3}$ $= 5$	$y_2 = \dfrac{9}{3}$ $= 3$	$A_2 x_2$ $= 135$	$A_2 y_2 = 81$
3	Rectangle	$A_3 = -3X1.5$ $= -4.5$	x_3 $= 3 + 1.5$ $= 4.5$	$y_3 = \dfrac{1.5}{2}$ $= o.75$	$A_3 x_3$ $= -20.25$	$A_3 y_3$ $= -3.375$
4	Semi-circle	$A_4 = -\dfrac{\pi r2}{2}$ $=- 3.53$	x_4 $= 3 + 1.5$ $= 4.5$	$y_4 = 1.5 + \dfrac{4r}{3\pi}$ $= 2.134$	$A_4 x_4$ $= -15.885$	$A_4 y_4$ $= -7.53$

$$\bar{x} = \frac{\sum Ax}{\sum A} = \frac{A_1 x_1 + A_2 x_2 + \dots + A_n x_n}{A_1 + A_2 + A_3 + \dots + A_n} = 2.78 \text{ mm}$$

$$\bar{Y} = \frac{\sum AY}{\sum A} = \frac{A_1 y_1 + A_2 y_2 + \dots + A_n y_n}{A_1 + A_2 + A_3 + \dots + A_n} = 5.20 \text{ mm}$$

DEPARTMENT OF MECHANICAL ENGINEERING

Example 4 A lamina of uniform thickness is hung through a weight less hook at point B such that side AB remains horizontal as shown in fig. determine the length AB of the lamina.
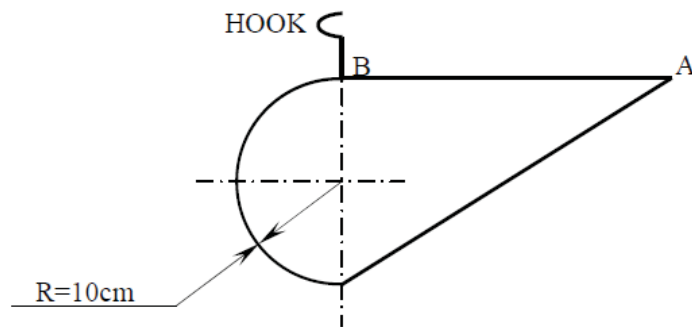
Answer:

Let, length AB=L, for remains horizontal of given lamina moment of areas of lamina on either side of the hook must be equal.
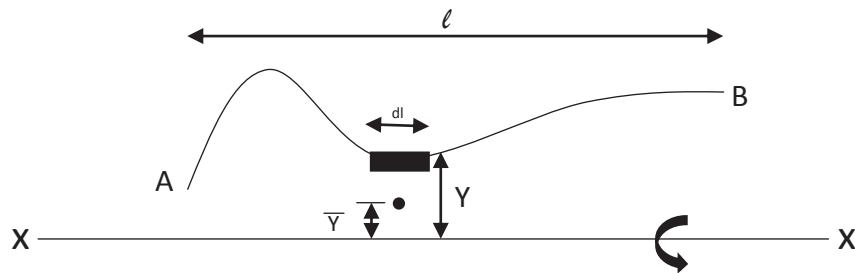
$$\therefore A_1 x_1 = A_2 x_2$$

$$\therefore (\frac{1}{2} \; x \; L \; x \; 20) (\frac{1}{3} \; x \; L) = (\frac{10^2}{2} \; x \; \pi) (\frac{4 \; X \; (r=10)}{3\pi})$$

$$\therefore \frac{20l^2}{6} = 157.08 \; x \; 4.244$$

$$\therefore L = 14.14 \; cm$$

Pappus Guldinus first theorem

➢ This theorem states that, "the area of surface of revolution is equal to the product of length of generating curves & the distance travelled by the centroid of the generating curve while the surface is being generated".

➢ As shown in fig. consider small element having length dl & at 'y' distance from $x - x$ axis.

➢ Surface area dA by revolving this element dA= 2πy.dl (complete revolution)

➢ Now, total area,

$\therefore$ A = $\int dA$ = $\int 2\pi y dl$ = 2π$\int y dl$

$\therefore$ A = $2\pi \overline{y} l$

➢ When the curve rotate by an angle 'θ'

$\therefore$ A = $2\pi \overline{y} l \frac{\theta}{2\pi}$ = $\theta \overline{y} l$

Pappus guldinus second theorem

➢ This the rem states that, "the volume of a body of revolution is equal to the product of the generating area & distance travelled by the centroid of revolving area while rotating around its axis of rotation.

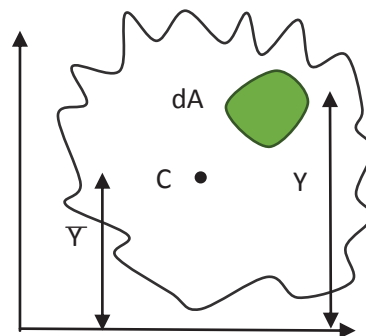➢ Consider area 'dA' as shown in fig. the volume generated by revolution will be

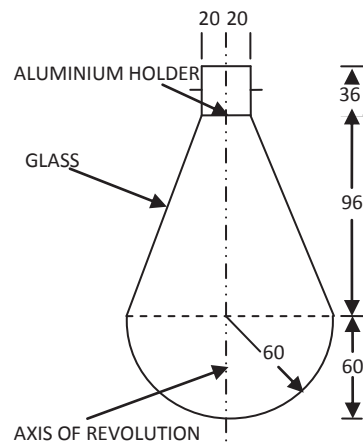   dv= Q π Y .dA

➢ Now, the total volume generated by lamina,

      V=$\int dv$ =$\int 2\pi y d$A

           = $2\pi \overline{y}$A (completed revolution)

➢ When the area revolves about 'θ' angle volume will be

      V=$2\pi \overline{y} A \frac{\theta}{2\pi}$ = $\theta \overline{y} A$

Example-5. Find surface area of the glass to manufacture an electric bulb shown in fig using first theorem of Pappu's Guldinus.

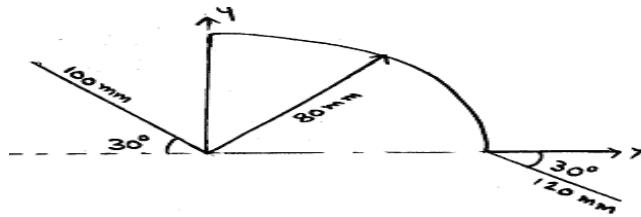| Line	length	x mm	$lx(\text{mm}^2)$
AB	L1=20	$x_1 = \dfrac{20}{2} = 10$	200
BC	L2=36	$x_1 = 20$	720
CD	L3=$\sqrt{40^2 + 96^2}$ =104	$x_3 = 20 + \dfrac{40}{2} = 40$	4160
DE	L4=$\dfrac{\pi R}{2}$ =94.25	$x_4 = \dfrac{2r}{\pi} = 38.20$	36000

$$\overline{x} = \frac{\sum L\,x}{L} = 34.14\text{mm}$$

Surface area = L$\theta\,\overline{x}$ = 254.25 x 2π x34.14
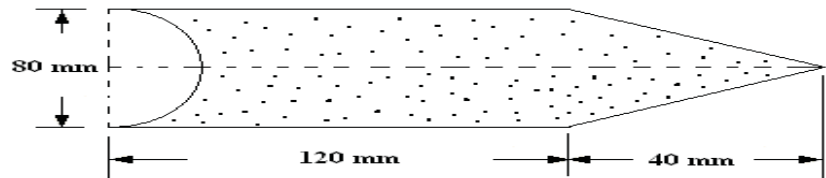
$$= 54510.99\text{mm}^2$$

Tutorial Questions

1. Locate the centroid of the wire bent as shown in figure

2. Find the Centroid for the shaded area about y – axis. As shown in the fig.

3. State and prove Pappus theorem

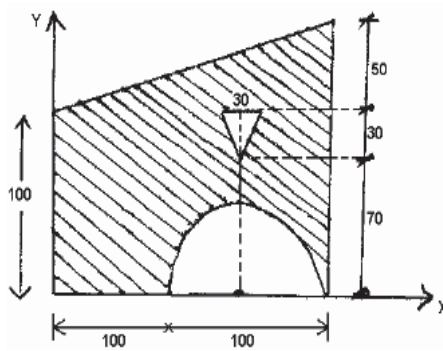4. Locate the centroid of the shaded area shown in figure

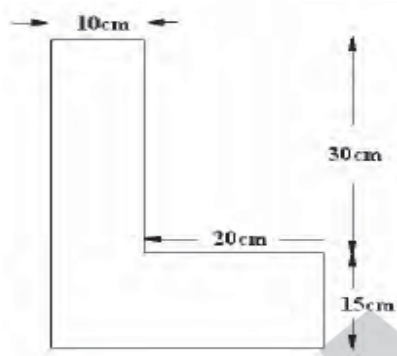5. Find the centroid of Quarter circle having the radius R
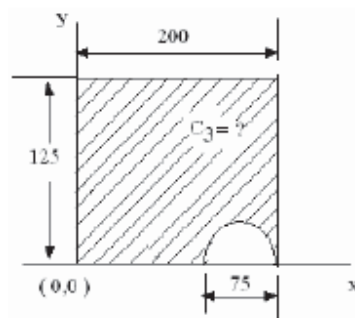
Assignment Questions

1. Determine the centre of gravity of solid cone of base Radius 'R' and height 'h'

2. Locate the centroid of the shaded area and also find the moment of inertia about horizontal centroidal axis shown in figure. All dimensions in mm.

3. Determine the centroid of the figure

4. Determine the centroid of the shaded area as shown in figure

5. Determine the centre of gravity of right solid circularcine of radius R and height h

DEPARTMENT OF MECHANICAL ENGINEERING